Quantum Field Theory as Dynamical System

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Quantum Field Theory as Dynamical System

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Abstract:
A review on the representations of symmetries in quantum field theory of local observables will be given. One part is concerned with the interplay of the locality condition in configuration space (Einstein causality) and the spectrum condition in momentum space (positive energy). Another part uses Tomita’s theory of modular Hilbert algebras (Tomita-Takesaki theory) to induce symmetries for the quantum field theory. The review ends with the discussion of some problems.

1. Introduction

There exists an extended literature on the so called dynamical systems. Many of these articles treat $C^*$-dynamical systems. They consist of triples

\[ (\mathcal{A}, G, \alpha), \]

where $\mathcal{A}$ is a $C^*$-algebra, $G$ a topological group (discrete topology not excluded) and $\alpha$ a group homomorphism of $G$ into the automorphisms of $\mathcal{A}$,

\[ \alpha : G \rightarrow \text{Aut}(\mathcal{A}). \quad (1.1) \]

A covariant representation $(\pi, U)$ of $(\mathcal{A}, G, \alpha)$ consists of a non-degenerate representation $\pi$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and a continuous unitary representation $U(g)$ implementing the automorphisms $\alpha_g$ on the same Hilbert space,

\[ U(g)\pi(A)U^*(g) = \pi(\alpha_g A); \quad A \in \mathcal{A}, \quad g \in G. \quad (1.2) \]

If one is dealing with a covariant representation, then the automorphisms $\alpha_g$ extend to the von Neumann algebra $\pi(\mathcal{A})^\prime$, and one becomes a $W^*$-dynamical system. This is a triple

\[ (\mathcal{M}, G, \alpha) \]
where one has replaced the $C^*$-algebra $\mathcal{A}$ by a von Neumann algebra $\mathcal{M}$.

Dealing with a $C^*$- or $W^*$-dynamical system there are three families of questions:

- Find conditions implying that the symmetry group can be implemented.

This kind of investigations does not require a special structure of the $C^*$- or $W^*$-algebra. Usually special requirements on the group and eventually also on the kind of the group representation are made. This is the plain "cinematic" situation.

- Derive a detailed structure of the group representation.

For this family of problems it is necessary that the $C^*$- or $W^*$-algebra has a sub-structure, and that the considered automorphisms act covariant on this sub-structure. This contains also the case, where the generator of a subgroup is given in terms of concrete elements of the algebra of observables. Here I will treat the case of quantum field theory of local observables.

- Does the detailed sub-structure of the algebra with some requirement on its representation imply the existence of additional symmetry groups?

Such questions are of interest in connection with the discussion of "broken symmetries".

These three problems shall be discussed in this talk. At the end some comments on open problems, connected with dynamical systems in quantum field theory, are added.

2. On cinemtical problems:

In the special situation that:

1) The group $G$ is locally compact.

2) The group of automorphisms, $a_g, g \in G$, act strongly continuous, i.e. for every $A \in \mathcal{A}$ the function

$$g \rightarrow a_g(A)$$

is continuous in the norm topology of $\mathcal{A}$.

Then one can form the crossed product of the group with the algebra

$$G \times_\alpha \mathcal{A}.$$ 

Sometimes this crossed product is called the covariance algebra. Doplicher, Kastler and Robinson [DKR66] have shown the following result:

**2.1 Theorem:**

*The covariant representations of the system $(\mathcal{A}, G, \alpha)$ are in one-to-one correspondence with the representations of $G \times_\alpha \mathcal{A}$.*

For an extensive discussion of crossed products see G.K. Pedersen [Ped79]. This theorem does not answer all questions, e.g.:

(i) The requirement that the group acts strongly continuous cannot be fulfilled in many cases. Such situation appears if the algebra contains projections.
(ii) Not every group appearing in physics is locally compact. E.g., gauge groups in a field theory containing zero mass particles, diffeomorphisms in general relativity.

(iii) When can a given representation of the $C^*$-algebra of a dynamical system be extended to a covariant representation?

If one wants to extend a given representation $\pi$ on $\mathcal{H}$ to a covariant representation then there are two obstructions:

(a) The multiplicity problem.
An example shall demonstrate the situation. One can represent the algebra $\mathcal{L}^\infty(\mathbb{R})$ on $\mathcal{L}^2(\mathbb{R}^+) \otimes \mathcal{F}^2 \oplus \mathcal{L}^2(\mathbb{R}^-)$, but the change of the multiplicity is an obstruction for the representation of the translations.

(b) It can happen that the automorphisms are represented by a projective representation.
Quantum mechanics is such a case.

Both obstructions can be cured by passing from a given representation to a quasi-equivalent one.

2.2 Definition:

(1) Two representations $\pi_1, \pi_2$ of $\mathcal{A}$ are called quasi-equivalent if the von Neumann algebras $\pi_1(\mathcal{A})^\ast$ and $\pi_2(\mathcal{A})^\ast$ are isomorphic. This is equivalent to the requirement that $\pi_1(\mathcal{A})$ and $\pi_2(\mathcal{A})$ have the same set of normal states. This family is called the folium $F(\pi)$ of $\pi$.

(2) Let $\pi$ be a representation of the algebra of the triple $(\mathcal{A}, G, a)$, then $\pi$ is called quasi-covariant if there exists a covariant representation $(\pi_1, U)$ such that $\pi$ and $\pi_1$ are quasi-equivalent.

With this notation one obtains:

2.3 Theorem:

Let $(\mathcal{A}, G, a)$ be a $C^*$-dynamical system and $\pi$ be a representation of $\mathcal{A}$. Then $\pi$ is quasi-covariant iff

1) The folium $F(\pi)$ is invariant under the transposed action $a'_\phi$.

2) For every $\omega \in F(\pi)$ the function

$$ g \rightarrow a'_\phi \omega $$

is continuous in the norm-topology of $\mathcal{A}^\ast$. ($\mathcal{A}^\ast$ is the topological dual of $\mathcal{A}$.)

In the case of a $W^\ast$-dynamical system $(\mathcal{M}, G, a)$ one has to replace the dual space by the pre-dual space $\mathcal{M}_\ast$. This result was shown [Bch69] for locally compact groups. For arbitrary groups it is a consequence of the Tomita-Takesaki-theory [Ta70]. It is clear that the passage to a quasi-equivalent representation can cope with the multiplicity problem. That it also handles the problem of projective representations can be seen as follows: Assume $(\pi, U)$ is a representation on $\mathcal{H}$ such that $U$ is a ray representation with multiplier $m(g_1, g_2)$. Then exists a projective representation $\tilde{U}$ such that its multiplier is the conjugate complex of the first one. Then on $\mathcal{H} \otimes \mathcal{H}$ one sets

$$ \pi_1(\mathcal{A}) = \pi(\mathcal{A}) \otimes 1, \quad U_1(g) = U(g) \otimes \tilde{U}(g). $$
Since the multipliers are phase-factors one obtains a true representation. The same remains valid if the multiplier is center-valued.

For obtaining a continuous unitary representation it was necessary and sufficient that the group acts strongly continuous on the folium of the representation. Therefore, the following space is of interest:

**2.4 Definition:**

By \( \mathcal{A}_c^\ast \) we denote the set of \( \phi \in \mathcal{A}^\ast \), such that for every \( \epsilon > 0 \) exists a neighbourhood \( \mathcal{U} \) of the identity of \( G \) such that
\[
\| \phi \circ a_g - \phi \| \leq \epsilon
\]
holds for \( g \in \mathcal{U} \).

Some properties of this set are described in the following:

**2.5 Proposition:**

Let \( (\mathcal{A}, G, \alpha) \) be a \( C^\ast \)-dynamical system and assume \( G(\tau) \) is a topological group, then the space \( \mathcal{A}_c^\ast \) has the following properties:

(i) \( \mathcal{A}_c^\ast \) is a linear norm-closed space.

(ii) \( \mathcal{A}_c^\ast \) is invariant under the action of the group i.e., \( \phi \in \mathcal{A}_c^\ast \) implies \( \phi \circ a_g \in \mathcal{A}_c^\ast \) for every \( g \in G \).

(iii) With \( \phi \in \mathcal{A}_c^\ast \) one finds also that \( \phi^* \) and \( |\phi| \) belong to \( \mathcal{A}_c^\ast \). \( \mathcal{A}_c^\ast \) is generated by its positive elements.

Using Connes' characterization of von Neumann algebras on \( \mathcal{H} \) by self-dual, oriented cones in a Hilbert space [Co74] one can show [Bch83]

**2.6 Theorem:**

Let \( (\mathcal{A}, G, \alpha) \) be a \( C^\ast \)-dynamical system, with \( G \) a topological group. Then \( \mathcal{A}_c^\ast \) is the predual of a von Neumann algebra \( \mathcal{M}_c \subset \mathcal{A}^\ast \). This von Neumann algebra coincides with \( \mathcal{M}_c^{(red)} \) where
\[
\mathcal{M}_c = \{ A \in \mathcal{A}^\ast ; A\omega \in \mathcal{A}_c^\ast, \omega A \in \mathcal{A}_c^\ast, \forall \omega \in \mathcal{A}_c^\ast \}.
\]

The suffix "red" means division by the two-sided annihilator of \( \mathcal{A}_c^\ast \).

Of great interest in physics is the case where \( G \) is the translation group of \( \mathbb{R}^d \) with the requirement that the spectrum of the translations \( T(a) \) is contained in a cone \( C \subset \mathbb{R}^d \). This is the positive energy condition. The first result in this direction is due to Kadison [Kad66] and Sakai [Sa66]. They showed: If a one-parametric symmetry group \( a_t \) acts norm-continuous on \( \pi(\mathcal{A}) \) then there exists a bounded self-adjoint operator \( H \in \pi(\mathcal{A})^\ast \) with
\[
e^{im} \pi(\mathcal{A})e^{-im} = \pi(a_t A).
\]

This result has been generalized [Bch66] to one-parametric groups with semi-bounded spectrum. It took a long time to extend this to the higher dimensional situation with an arbitrary convex cone [Bch96b].
2.7 Theorem:
Let \((\mathcal{A}, \mathbb{R}^d, \alpha)\) be a C*-dynamical system and \(C\) be a proper, convex, closed cone in \(\mathbb{R}^d\) with interior points. Let \(\pi\) be a representation of \(\mathcal{A}\) and assume on \(\mathcal{H}_\pi\) exists a continuous \(\\pi(a)\) of the translation group satisfying:

(i) The spectrum of \(V(a)\) is contained in the cone \(C\).
(ii) \(V(a)\) implements the automorphisms \(\alpha_a\).

Then there exists a continuous unitary representation \(T(a)\) satisfying

(a) \(T(a) \in \pi(\mathcal{A})^\circ\).
(b) \(T(a)\) fulfills the conditions (i) and (ii).

\(V(a)\) and \(T(a)\) can be analytically extended into the tube \(\tau(C) = \{z; \exists m \in C\}\) where \(\hat{C}\) is the (open) dual cone of \(C\). This has several consequences:

If one adds to the definition of \(\mathcal{A}^r\) also the requirement of the spectrum condition, then exists a projection \(P \in \mathcal{Z}(\mathcal{A}^r)\) such that

\[\mathcal{M}^{\mathcal{A}^r} = P \mathcal{A}^{**}\]

holds. Moreover, this set \(\mathcal{A}^r\) can be characterized by analyticity properties of a dense set of elements in \(\mathcal{A}_c^r\). For details see [Bch96b].

3. Quantum field theory and dynamical consequences

Relativistic quantum field theory exists in two settings: Wightman field theory (see e.g. Streifer and Wightman [SW64]) which uses unbounded operators, and the theory of local observables of Araki, Haag and Kastler (see Haag [Ha92]), where bounded operators are used. It is widely believed that both theories coincide in their consequences for physics. Equivalence in the mathematical sense has not been shown. There exist several conditions on Wightman fields which allow to pass to the net of local algebras, see [BY92] for a review. Some results are also known for the construction of Wightman fields from local rings. These are due to Fredenhagen and Hertel [FH81] and to Bostelmann [Bo00].

Here the field theory of local rings will be used. Quantum field theory of local observables in the sense of Araki, Haag and Kastler [Ha92] is concerned with \(C^*-\)algebras \(\mathcal{A}(O)\), associated with bounded open regions \(O \subset \mathbb{R}^d\), which have a common unit. These algebras shall fulfill isometry, i.e. \(O_1 \subset O_2\) implies \(\mathcal{A}(O_1) \subset \mathcal{A}(O_2)\), and locality, i.e. if \(O_1\) and \(O_2\) are space-like separated, then the algebras \(\mathcal{A}(O_1)\) and \(\mathcal{A}(O_2)\) commute element-wise. In order that this statement makes sense the space \(\mathbb{R}^d\) must be furnished with the Minkowski metric. The global algebra \(\mathcal{A}\) is defined as the \(C^*-\)inductive limit of \(\{\mathcal{A}(O)\}\). We also assume that the net \(\{\mathcal{A}(O)\}\) is covariant under a representation of the translation group of \(\mathbb{R}^d\) by automorphisms, i.e. to \(a \in \mathbb{R}^d\) exists an automorphism \(\alpha_a\) with

\[\alpha_a \mathcal{A}(O) = \mathcal{A}(O + a)\]  

A covariant representation \(\pi\) of \(\mathcal{A}\) in the representation space \(\mathcal{H}\) is such that there exists a continuous unitary representation \(T(a)\) of the translation group of \(\mathbb{R}^d\) implementing \(\alpha_a\)

\[T(a)\pi(A)T(-a) = \pi(\alpha_a(A)), \quad A \in \mathcal{A}\]
In addition $T(a)$ shall fulfil the spectrum condition, i.e. spectrum $T(\mathbb{R}^d)$ is contained in the closed forward light-cone. Sometimes (not here) also the covariance under the Poincaré group $\mathcal{P}_d^+$ is required. From Thm. 2.7 it follows, that $T(a)$ can be chosen in $\pi(A)^\ast$. The multiplication of $T(a)$ by a phase-factor $e^{i\langle a, p_0 \rangle}$ corresponds to a shift of the spectrum of $T(a)$ by the value $p_0$. ($p_0$ might be $\pi(A)^\ast$-center valued.) If this vector belongs to the forward light-cone than $e^{i\langle a, p_0 \rangle}T(a)$ fulfills again the spectrum condition. Now one can ask for the converse problem: Does there exist a representation such that the spectrum of $T(a)$ is minimal? The existence of a minimal representation in one dimension is obvious, by pushing the spectrum as far down as possible without violating the spectrum condition. This has been used by D. Olesen and G. K. Pedersen [OP74].

If the spectrum is in a cone, then no general result is possible since there is more than one direction in which one can push to the tip of the cone. (If the base of the cone is a simplex, then the argument of Olesen and Pedersen can be repeated.) One has to use the detailed structure of the local algebras. Let $T(a), T_1(a)$ both be representations of the translations implementing $a_a$ belonging to $\pi(A)^\ast$. In this situation exists an element $P_0$ affiliated with the center of $\pi(A)^\ast$ with

$$T_1(a) = e^{i\langle a, p_0 \rangle}T(a).$$

Assume the spectrum of $T(a)$ is contained in $C$, then $T(a)$ is called the unique minimal representation if for every $T_1(a)$ which also has its spectrum in $C$ it follows that the spectrum of $e^{i\langle a, p_0 \rangle}T(a)$ is contained in $C$. In the case of quantum field theory of local observables the existence of a unique minimal representation can be shown. An indirect argument is given in [BB85] and a direct proof can be found in [Bch87].

3.1 Theorem:

Let $(\{A(O)\}, A, \mathbb{R}^d, \alpha)$ be a theory of local observables and $(\pi, T(a))$ be a covariant representation with $T(a)$ fulfilling the spectrum condition and $T(a) \in \pi(A)^\ast$. Then exists a unique minimal representation.

If one is working with a vacuum representation, and such representations which are generated from the vacuum by charged fields, then the translations one gets are automatically the minimal one.

In general theories one cannot expect that the representations obtained from a vacuum sector are all representations fulfilling the spectrum condition. In the presence of zero-mass particles the investigations of Doplicher and Spera [DS82], Borek [Bk82] in the non-interacting, and of Buchholz and Doplicher [BD84] in the interacting situation show, that most positive energy representations with no gap are not connected with vacuum representations. These situations are sometimes called infra-vacua, and Kunhardt [Ku01] has shown that the theory of super-selections can be performed starting from these infra-vacua situation provided the global representation is of type I. A similar situation might hold also in a gauge field theory. Therefore, Thm. 3.1 and the coming results are of interest. Knowing the existence of a unique minimal representation has the consequence, that in arbitrary covariant representations with spectrum condition one can define energy and momentum unambiguously. Therefore, every result about the spectrum of the translations
(the minimal representation) has a universal meaning. The following result can be found in [BB85] and [Bch85]:

3.2 Theorem:

Let \((A(\mathcal{O}), A, \mathbb{R}^d, a)\) be a theory of local observables and \((\pi, T(a))\) be a covariant representation such that \(T(a)\) fulfills the spectrum condition and \(T(a) \in \pi(A)_a^\vee\). Moreover, let \(T(a)\) be the minimal representation. Then the support of the spectrum is invariant under Lorentz transformations.

It is remarkable that Lorentz covariance of the theory has not been used. Necessary for this result is the locality property in configuration space, which often is called Einstein causality, together with the spectrum condition. This makes it understandable why also for one electron the formula \(p^2 = m^2\) is correct although the Lorentz covariance is broken in the presence of the electro-magnetic field. See Fröhlich, Morchio and Strocchi [FMS79], or Buchholz [Bu86].

Up to now we have looked at general properties of the spectrum of the translations. There are several results about the details of the structure of the spectrum, the first result is due to Wightman [Wi64].

3.3 Theorem:

Let \((\pi_0, T(a))\) be a vacuum representation of a quantum field theory of local observables \((A(\mathcal{O}), A, \mathbb{R}^d, a)\), this means \(T(a)\) does not only fulfill the spectrum condition, but \(\pi_0(A)\) is generated by a translational invariant vector \(\Omega\). Assume that, up to a scale-factor, \(\Omega\) is the only translational invariant vector. Then the support of the spectrum of \(T(a)\) is additive. If \(S\) denotes the support of the spectrum then there holds

\[
S + S \subset S.
\]

This result is based on the cluster property, which is a consequence of the locality and the uniqueness of the vacuum vector. It also works if the operators anti-commute. Therefore, Eq. (3.1) can be generalized to the presence of charged fields. In field theory factor representations are characterized by a charge-quantum-number \(Q\). If the representations are generated from the vacuum by charged fields, and if the support of the spectrum of the translations in that representation space is \(S_Q\), then Eq. (3.1) can be generalized to

\[
S_{Q_1} + S_{Q_2} \subset S_{Q_1 + Q_2},
\]

which holds for additive charges. See [Bch65] and [DHR69a,b].

For factor representations which are not generated from the vacuum a result like Eq. (3.2) does not exist. The following relation has been conjectured in [Bch95]

\[
S + S + S \subset S.
\]

The argument is the following: One believes that for particle and anti-particle representations the spectra of the translations are the same. This is true for representations
generated by charged fields from the vacuum. Moreover, if a representation (sector) describes a charge \( Q \) then one remains in the same sector by adding a neutral pair. A particle plus its anti-particle is such a neutral combination. If the support of the spectrum of the translations for a particle is \( S \), then that of the neutral pair formed by this particle and its anti-particle should be \( S + S \). Therefore, if one starts from a sector characterized by \( Q \) we do not know of any other neutral pair than that of \( (Q - Q) \). This implies the relation. Unfortunately this has not been proved. But there exist two results showing into the direction of the conjecture. If the spectrum starts at \( m_1 \) and if (3.3) is correct then the continuous spectrum has to start at least at \( 3m_1 \). This is the content of the first result [Bch86].

3.4 Theorem:
Let \( \pi \) be a translation covariant factor representation of the theory of local observables fulfilling the spectrum condition, and let \( T(\alpha) \) be the unique minimal representation of the translations. Assume the spectrum of \( T(\alpha) \) consists of two parts:

(a) Isolated hyperbolas with the masses \( m_1 < m_2 < ... < m_i < ... \)

(b) The rest starting at \( m_c > m_i \) where \( m_c \) denotes the beginning of the continuous spectrum or the first accumulation point of the \( m_i \)'s.

Then

\[ 3m_1 > m_c. \]

One would like to show that above \( 3m_1 \) there is no gap in the spectrum. Unfortunately this has not been done. Since one knows from examples that there can be a gap between \( m_1 \) and \( 3m_1 \) one needs a special technique for showing that result. Such method of proof must be able to distinguish between the regions below \( 3m_1 \) and above \( 3m_1 \). Up to now such technique does not exist. Therefore, only the following result has been shown [Bch01]:

3.5 Theorem:
Let \( \pi \) be a factor representation of the theory of local observables covariant under translation fulfilling the spectrum condition, and let \( T(\alpha) \) be the unique minimal representation of the translations. Let the spectrum start at \( m_1 \), then there is no gap with width larger than \( 2m_1 \).

From this theorem the following result holds:

3.6 Corollary:
With the assumption of the last theorem assume that the spectrum starts at \( m_1 = 0 \). Then there is no gap in the spectrum.

4. Symmetries as consequences of the dynamics

In Thm. 3.2 it was shown that the spectrum of the minimal representation of the translations is located on a Lorentz invariant set. This is a consequence of the dynamical requirements. Therefore, one might hope to derive the symmetry itself, if one imposes
slight improved dynamical requirements. This is indeed the case if the representations are in the vacuum sector. All these results are based on the Tomita-Takesaki theory, which must be described first.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a von Neumann algebra, acting on this space, with commutant $\mathcal{M}'$. A vector $\Omega$ is cyclic and separating for $\mathcal{M}$, if $\mathcal{M}\Omega$ and $\mathcal{M}'\Omega$ are dense in $\mathcal{H}$. If these conditions are fulfilled, then a modular operator $\Delta$ and a modular conjugation $J$ are associated with the pair $(\mathcal{M}, \Omega)$, such that
(i) $\Delta$ is self-adjoint, positive and invertible,
$$\Delta \Omega = \Omega, \quad J\Omega = \Omega.$$ (ii) The operator $J$ is a conjugation, i.e., $J$ is anti-linear, $J^* = J$, $J^2 = 1$, and $J$ commutes with $\Delta^\Delta$. This implies the relation
$$\text{Ad} \, J \Delta = \Delta^{-1}.$$ (iii) For every $A \in \mathcal{M}$ the vector $A\Omega$ belongs to the domain of $\Delta^\Delta$, and
$$J \Delta^\Delta A \Omega = A^* \Omega =: S A \Omega.$$ (iv) The unitary group $\Delta^t$ defines a group of automorphisms of $\mathcal{M}$,
$$E \Delta^t \mathcal{M} = \mathcal{M}, \quad \text{for all } t \in \mathbb{R}.$$ (v) $J$ maps $\mathcal{M}$ onto its commutant
$$E \, J \mathcal{M} = \mathcal{M}'.$$ These results apply in particular to the theory of local observables in a vacuum representation. Together with some weak additivity property the Reeh-Schlieder theorem [RS61] implies that the vacuum vector is cyclic and separating for every algebra $\mathcal{M}(G)$, where $G$ is any domain which has a space-like complement with interior points.

The matrix elements of the modular group have the following important analyticity properties: If $A, B \in \mathcal{M}$ and $\sigma^t (A) := E \Delta^t A$ then the continuous function
$$F(t) = (\Omega, B \sigma^t (A) \Omega)$$ has bounded analytic continuation into the strip $S(-1,0) = \{ z \in \mathbb{C} : -1 < \Im z < 0 \}$. At the lower boundary one finds
$$F(t-i) = (\Omega, \sigma^t (A) B \Omega). \quad (4.1)$$ This relation is called the KMS-condition.

A domain of special importance is the wedge. Such a domain can be characterized in two ways:
(a) First characterization: Let $t, s$ be two perpendicular vectors in $\mathbb{R}^d$, i.e. $(t, s) = 0,$
such that $t^2 = 1$ and $t$ belongs to the forward light cone and $s^2 = -1$ is space-like. In this situation one defines

$$W(t, s) := \{ a \in \mathbb{R}^d; \langle a, t \rangle < -(a, s) \}. \quad (4.2)$$

If, for instance, $t$ is the time direction and $s$ is the 1-direction then this becomes $W_R = \{ a; |a| < a_1 \}$.

(3) Second characterization: Every two-plane containing a time-like direction must cut the boundary of the forward light cone in two light rays. Let these light rays be described by the two light-like vectors $\ell_1, \ell_2$ belonging to the boundary of the forward light cone. These non-zero vectors are different. Now define:

$$W(\ell_1, \ell_2) := \{ \lambda_1 \ell_1 - \lambda_2 \ell_2 + \hat{a}; \lambda_1, \lambda_2 > 0, (\hat{a}, \ell_i) = 0, i = 1, 2 \}. \quad (4.3)$$

It is easy to see that the two definitions result in the same set of wedges. The two definitions coincide if $\{t, s\}$ and $\{\ell_1, \ell_2\}$ span the same two-plane and if $s = \lambda_1 \ell_1 - \lambda_2 \ell_2$ with positive coefficients. The two-plane spanned by $\{\ell_1, \ell_2\}$ is called the characteristic two-plane of the wedge.

If $\pi$ is a positive energy representation and $\psi \in \mathcal{H}_*$ a vector analytic for the energy, then this is cyclic and separating for every $\pi(A(W))$ (without weak additivity) because every bounded open region can be shifted into $W$.

The first explicit determination of a modular group is due to Bisognano and Wichmann [BW75], [BW76]. They assumed that the local algebras are generated by Lorentz covariant Wightman fields over the four-dimensional Minkowski space, and that the Lorentz transformations act on the indices of the fields by finite dimensional representations of the Lorentz group, i.e.

$$U(\Lambda)A\ell(x)U^*(\Lambda) = \sum_j D^j(\Lambda)A_j(\Lambda x),$$

where $D^j(\Lambda)$ is the direct sum of finite dimensional representations. These fields are defined in the vacuum sector, and the vacuum vector $\Omega$ is a cyclic and separating vector. For simplicity the result of Bisognano and Wichmann will be formulated for one neutral scalar field only.

### 4.1 Theorem:

Let $A(x)$ be a neutral scalar quantum field. Set $\Delta = U(\Lambda(-i/2))$ and $J = \Theta U(s, \pi)$. Then

(a) $JP(W_R)J = P(W_L)$,

(b) $\Delta^2 P(W_{R,L})\Delta^{-2} = P(W_{R,L})$, \quad $t \in \mathbb{R}$

holds.

$P(W_R)$ is the algebra generated by the field operators localized in $W_R$. $\Theta$ is the PCT-operator which exists in this situation by the result of R. Jost [Jø57]. $U(s, \pi)$ represents the rotation around the $s$-axis (in the hyper-plane perpendicular to the time direction) and $\pi$ is the angle of rotation. $s$ is the space-like direction in the first characterization of the wedge. The group $\Lambda_W(t)$ is the group of normalized Lorentz boosts associated with the wedge. In the special case that $(t, s)$ span the $(0, 1)$-plane and that $s$ is the positive
1-direction, these boosts are the transformation

$$\Lambda_\gamma (t) = \begin{pmatrix}
\cosh 2\pi t & -\sinh 2\pi t & 0 & 0 \\
-\sinh 2\pi t & \cosh 2\pi t & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

In particular, $$\Lambda_\gamma (-\frac{1}{2}) = \text{diag}(-1, -1, 1, 1)$$. Moreover, $$R_\gamma (\pi) = \text{diag}(1, 1, -1, -1)$$, so $$\Lambda_\gamma (-\frac{1}{2}) R_\gamma (\pi) = -1$$.

For the result of Bisognano and Wichmann it is important that the Wightman fields transform in the index space according to a finite dimensional representation of the Lorentz group. If one allows infinite dimensional representation then, probably, the Bisognano and Wichmann result can be violated as it is suggested from the example of Oksak and Todorov [OT68].

Passing from Wightman fields to local rings there appears a problem of the size of the commutant. In order to avoid these problems we will work with maximal local von Neumann algebras. They are defined as follows:

$$\mathcal{M}(D) = \bigcap_{D \subseteq W} \mathcal{M}(W).$$

In the future the maximal algebras will be used without further mentioning. Another concept, which often will be used, is the wedge duality. One says a representation $$\pi$$ of the theory of local observables fulfills wedge duality if for every wedge the commutant of $$\mathcal{M}(W)$$ coincides with $$\mathcal{M}(W')$$, where $$W'$$ is the (open) space-like complement of $$W$$. If $$\{0\}$$ is contained in the edge of $$W$$ then $$W' = -W$$ holds.

The result of Bisognano and Wichmann suggests the following notion:

4.2 Definition:
A representation of a quantum field theory of local observables fulfills the Bisognano-Wichmann property, if:

(i) There exists a vector $$\Omega \in H$$ which is cyclic and separating for every wedge algebra $$\mathcal{M}(W)$$.

(ii) The modular group of any wedge acts local like the associated group of Lorentz boosts. This means the modular group acts on the underlying Minkowski space like the Lorentz boosts.

Since in a general quantum field theory of local observables exists no condition implying that the theory is generated from a finite number of Wightman fields, it took several years to find corresponding conditions. The first result can be found in [Bch92]:

4.3 Theorem:
Assume we are dealing with a vacuum representation of a quantum field theory of local observables over the two-dimensional Minkowski space. This theory shall only be covariant under translations. In addition assume that wedge duality holds, i.e.

$$\pi(\mathcal{A}(W))' = \pi(\mathcal{A}(W'))''.$$
Then the theory is covariant under the Poincaré group \( \mathcal{P}_+ \).

In this situation the modular conjugation acts as total reflection. This contains the time-reversal, and hence one obtains two components of the Poincaré group.

In the last result only translational covariance, spectrum condition and wedge duality has been used. Since double cones are defined by intersections of wedges, also covariance of the maximal double cone algebras is implied. A generalization of Thm. 4.3 to higher dimensions is still missing. In order to obtain Poincaré covariance one has to assume that the modular groups of all wedges act local. The following result has been shown by Brunetti, Guido and Longo [BGL94] and by Guido and Longo [GL95]. In the first quoted paper the constructed group could have been the covering of the Poincaré group \( \mathcal{P}_+ \). In the second paper it is shown, that one deals with a true representation. In [Bch96a] a direct proof of the following result was given.

4.4 Theorem:
Assume the Bisognano-Wichmann property, i.e. the modular group of every wedge algebra \( \mathcal{M}(W[\ell_1, \ell_2, a]) \) act on every double cone algebra like the associated group of Lorentz boosts of the wedge \( W[\ell_1, \ell_2, a] \). Then the modular groups \( \{\Delta^\prime[\ell_1, \ell_2, a]\} \) define a representation of the Poincaré group.

\( W[\ell_1, \ell_2, a] \) denotes the shifted wedge \( W[\ell_1, \ell_2] + a \). It should be remarked that the conditions of Thm. 4.4 imply that wedge duality holds. An extension of Thm. 4.3 to higher dimensions has not been achieved. There is a result due to Gai and Yngvason [GY00] showing that such generalization might be possible. For generalized free fields they got the following result:

4.5 Theorem:
For a quantum field theory generated by one neutral generalized free field over the n-dimensional Minkowski space, \( n \geq 4 \), the following statements are equivalent:

(i) Covariance under translation and wedge duality for all wedges.
(ii) Local action of the modular groups of all wedges, (Bisognano-Wichmann property).
(iii) Poincaré covariance of the field.

For models with lower mass-gap the same holds for \( d = 3 \).

In Thm. 4.4 the local action of the modular groups is formulated for all double cones, also for those which do not belong to the wedge. But it is not necessary. It is sufficient to require that the modular groups act local only on sub-domains of the corresponding wedges. This is much more natural, since modular theory treats only the algebra and their commutant and not other sub-algebras of \( \mathcal{B}(\mathcal{H}) \). The reason to allow this restriction is a result of Wiesbrock [Wie93] which permits to construct the translations in the characteristic two-plane and its behavior under action of the modular group. With these translations every double cone can be transported into the wedge, where the modular action is known. This procedure is due to Guido [Gui95].

A different approach to the construction of the Poincaré symmetry is due to Buchholz, Summers and co-workers [BDFS00], [BFS99]. One knows that the Poincaré group \( \mathcal{P}_+ \) is generated by reflections at planes of co-dimension 2. In the situation discussed before
such reflections are due to the modular conjugations of wedge algebras. The results of 
Buchholz, Dreyer, Florig and Summers can also be applied to quantum field theories on 
certain curved globally hyperbolic manifolds, which shall not be discussed here. For the 
program of Buchholz, Summers and co-workers one needs the following concepts:

4.6 Definitions and requirements:
Let $W$ denote the set of all (open) wedges. By $T$ we denote the set of all transformations 
$T_{W} : W \rightarrow W$,
such that $T^{-1}$ exists and together with $T$ fulfil:

(i) Isotony, i.e. $W_1 \subset W_2$ implies $TW_1 \subset TW_2$ and $T^{-1}W_1 \subset T^{-1}W_2$.
(ii) Stability of non-intersection, i.e $W_1 \cap W_1 = \emptyset$ implies $TW_1 \cap TW_1 = \emptyset$ and 
$T^{-1}W_1 \cap T^{-1}W_1 = \emptyset$.
(iii) The group generated by all $T_W$ acts transitive on $W$.
(iv) Let $G$ be the group generated by all $T_W$. $G^1_W$ denotes the subgroup of $G$ containing the 
elements which map $W$ into itself. Let $T_W$ be the image of the conjugation associated 
with $\mathcal{M}(W)$, then

$$T_W T_G = T_G T_W, \quad T_G \in G^1_W$$

holds

In addition the following requirements are needed:

(I) To every $W \in W$ exists a von Neumann algebra $\mathcal{M}(W)$ acting on a Hilbert space $\mathcal{H}$ 
such that the map $W \rightarrow \mathcal{M}(W)$ is an order preserving bijection.

(II) There exists a vector $\Omega \in \mathcal{H}$ such that $\Omega$ is cyclic and separating for the algebra 
$\mathcal{M}(W_1) \cap \mathcal{M}(W_2)$ if $W_1 \cap W_2 \neq \emptyset$.

(III) Conversely if $\Omega$ is cyclic and separating for the algebra $\mathcal{M}(W_1) \cap \mathcal{M}(W_2)$ then one 
has $\mathcal{M}(W_1) \cap \mathcal{M}(W_2) \neq \emptyset$.

(IV) Since $T_W$ represents a conjugation one must have

$$T_W^2 = 1.$$ 

With these concepts one obtains

4.7 Theorem:
Let $T_W$ fulfil the requirements 4.6, then one has $T_W = P_W$ where $P_W$ is the total reflection 
in the characteristic two-plane of $W$. This, in particular, implies the wedge duality

$$T_W(W) = W'.$$

Applied to quantum field theory of local observables one finds
4.8 Theorem:
Assume we are dealing with a quantum field theory of local observables on a Hilbert space \( \mathcal{H} \). Let there be a vector \( \Omega \in \mathcal{H} \) which is cyclic and separating for all wedge-algebras \( \mathcal{M}(W) \). Assume moreover, the modular conjugations \( J_W \) fulfil the relations

\[
J_W \mathcal{M}(W_1) J_W = \mathcal{M}(T_W(W_1)),
J_W J_W = J_{T_W(W_1)}, \quad J_W J_W = \mathbb{1}.
\]

Then the \( J_W \) generate an adjoint representation of the determinant +1 part of the Poincaré group. Let the dimension of the Minkowski space be \( d \), then the “+” part of the Poincaré group induced by the \( J_W \)'s is a true representation.

In conformal field theory, besides the commutation for space-like separations, one also finds time-like commutativity. This implies in particular that the vacuum vector is cyclic and separating for the algebra of the forward light cone. This is in contrast to the massive field theories, where Sadowski and Woronowicz [SW71] have shown that the von Neumann algebra of the forward light cone coincides with the von Neumann algebra generated by all local observables. In some sense time-like commutativity is characteristic for Huygens principle in field theory. As a consequence of a result in [Bch92] one finds

4.9 Theorem:

Let \( (\pi, T(a)) \) be a vacuum representation of a quantum field theory of local observables. Assume the vacuum vector \( \Omega \) is cyclic and separating for the algebra of the forward light cone, then exists on \( \mathcal{H} \) a representation of the dilatations fulfilling

\[
D(\rho) T(a) D^*(\rho) = T(e^{-2\pi i a}).
\]

Moreover, if the commutant of \( \pi(A(V^+)) \) coincides with \( \pi(A(V^-))'' \) then the theory is covariant under dilatations.

Wiesbrock [Wi97] has shown that the theory is covariant under the whole conformal group provided the quantum field theory is living on a light ray.

One important symmetry for physics is the PCT-symmetry. The demonstration of this symmetry has a long history which shall not be explained here. For a review on this subject see Yngvason and the author [BY01].
5. Problems:

We saw that the introduction of dynamical conditions can have drastic consequences. But several questions remain unsolved which shall be discussed here.

1) Working together of the locality condition in configuration space and the spectrum condition in momentum space led to the conclusion (Thm.3.2) that the support of the spectrum of the minimal representation is a Lorentz invariant set. In the special case of the vacuum representation more is known [Bch65]: As well the Lebesgue continuous as the singular part of the spectrum have supports which are invariant under Lorentz transformations. Is this true in general?

2) The generalized additivity property of the spectrum Eq. (3.3) is known for representations generated from the vacuum by charged fields. There are strong hints (Thm. 3.4 and 3.5) that Eq. (3.3) holds for every factor representation. Is it possible to find a proof showing that there is no gap above $3m_1$? $m_1$ denotes the lower boundary of the spectrum (except the point zero).

3) Besides the support property of the spectrum one does not know more about it. Physical intuition implies that there should be no discrete mass inside the continuous spectrum. A discrete mass should either be isolated or at most be at the beginning of the continuum. The difficulty of dealing with this question is the existence of tensor products. The spectrum of the translation of the tensor product of two theories is the sum of the spectra of the two theories, i.e. $P = P_1 \otimes 1_2 + 1_1 \otimes P_2$. Up to now there is no manageable condition implying that a given theory can not be decomposed into tensor products of simpler ones.

4) The result of Gaier and Yngvason (Thm. 4.5) suggests that for vacuum representations of the theory of local observables the following are equivalent:

(i) Covariance under the Poincaré group $P_+$. 
(ii) Translational covariance together with wedge duality for all wedges.

One would like to prove this. (The situation $d = 3$ and $m = 0$ in Thm. 4.5 can probably cured if one passes from the fields to the local von Neumann algebras. See the article of Yngvason [Yng94] as an indication for the possibility.) The problem can be split into two sub-questions.

a. Thm. 4.3 implies that the modular groups of the wedges act on the translations like the Lorentz boosts. This implies the following: If $G$ is the group generated by all the modular groups associated with wedges, then $G$ must have a normal subgroup $N$ such that

$$G/N \cong P_+^\dagger,$$

holds. Does wedge duality imply that the normal subgroup is trivial?

b. Does the triviality of the normal subgroup imply that the modular groups of the wedges act local?

5) The approach of Buchholz, Summers and co-workers led to a representation of the Poincaré group which might not fulfil the spectrum condition. They gave a condition, called “modular stability condition” implying the spectrum condition. This requirement
has the disadvantage that it uses properties of the group to be constructed. Can one find conditions “in advance” leading to the spectrum condition?

6) For the construction of the Poincaré group out of the modular groups Guido [Gui95] showed that it is sufficient to assume local modular action only for domains inside the corresponding wedge. The approach of Buchholz, Dreyer, Florig and Summers needs, up to now, domains which are not subsets of the considered wedge or its commutant. Therefore, one should try to invent an approach to their procedure, which only uses subsets of the wedge. A suggestion would be that $T_W$ maps sets $W \cap W_1$ onto the intersection of two other wedges.

7) Because of Thm 4.1 and 4.3, the result of Bisognano and Wichmann the modular groups of wedges are well understood. Unfortunately hardly anything is known about the modular groups associated with double cones. The only exception is the conformal field theory, where the algebra of the wedge can be transformed onto the algebra associated with a double cone. Therefore, this modular group acts also local as shown by Hislop and Longo [HL82]. Nothing is known about the modular groups of double cones for other theories. The difficulty is the non-locality of the modular action. If one assumes local action for this group, then this can only be the (eventually scaled) Hislop-Longo transformation. This has been shown by Trebels [Tr97].

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References


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