Bijective and General Arithmetic Codings for Pisot Toral Automorphisms

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Bijective and general arithmetic codings for Pisot toral automorphisms *

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Abstract

Let $T$ be an algebraic automorphism of $\mathbb{T}^m$ having the following property: the characteristic polynomial of its matrix is irreducible over $\mathbb{Q}$, and a Pisot number $\beta$ is one of its roots. We define the mapping $\varphi_t$ acting from the two-sided $\beta$-compactum onto $\mathbb{T}^m$ as follows:

$$\varphi_t(\xi) = \sum_{k \in \mathbb{Z}} \xi_k T^{-k} t,$$

where $t$ is a fundamental homoclinic point for $T$, i.e., a point homoclinic to 0 such that the linear span of its orbit is the whole homoclinic group (provided such a point exists). We call such a mapping an arithmetic coding of $T$. This paper is aimed to show that under some natural hypothesis on $\beta$ (which is apparently satisfied for all Pisot units) the mapping $\varphi_t$ is bijective a.e. with respect to the Haar measure on the torus. Besides, we study the case of more general parameters $t$, not necessarily fundamental, and relate the number of pre-images of $\varphi_t$ to certain number-theoretic quantities. We also give several full criteria for $T$ to admit a bijective arithmetic coding and consider some examples of arithmetic codings of Cartan actions. This work continues the study begun in [25] for the special case $m = 2$.

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1 Introduction

Let $T$ be an algebraic automorphism of the torus $\mathbb{T}^m$ given by a matrix $M \in GL(m, \mathbb{Z})$ with the following property: the characteristic polynomial for $M$ is irreducible over $\mathbb{Q}$, and a Pisot number $\beta > 1$ is one of its roots (we recall that an algebraic integer is called a Pisot number if it is greater than 1 and all its Galois conjugates are less than 1 in modulus). Since $\det M = \pm 1$, $\beta$ is a unit, i.e., an invertible element of the ring $\mathbb{Z}[\beta] = \mathbb{Z}[\beta^{-1}]$. We will call such an automorphism a Pisot automorphism. Note that since none of the eigenvalues of $M$ lies on the unit circle, $T$ is hyperbolic.

This definition is invariant in the following sense: if $M$ is irreducible and one of its eigenvalues ($\lambda$, say) lies outside the unit disc in the complex plane and all the other lie inside it, then it is obvious that either $\lambda$ or $-\lambda$ is a Pisot number. In the “inverse” situation (one eigenvalue is inside and the others are outside) it is either $\lambda^{-1}$ or $-\lambda^{-1}$. We will call $T$ that falls into one of this categories a generalized Pisot automorphism. Our model will cover all generalized Pisot automorphisms — see Remark 21.

Our goal is to present a symbolic coding of $T$ which, roughly speaking, reveals not just the structure of $T$ itself but the natural arithmetic of the torus as well. Let us give more precise definitions.

Let $X_\beta$ denote the two-sided $\beta$-compactum, i.e., the space of all admissible two-sided sequences in the alphabet $\{0, 1, \ldots, [\beta]\}$. More precisely, a representation of an $x \in [0, 1)$ of the form

$$x = \pi_\beta(\varepsilon_1, \varepsilon_2, \ldots) := \sum_{k=1}^{\infty} \varepsilon_k \beta^{-k} \quad \text{(1)}$$

is called the $\beta$-expansion of $x$ if the “digits” $\{\varepsilon_k\}_{k=1}^{\infty}$ are obtained by means of the greedy algorithm (similarly to the decimal expansions), i.e., $\varepsilon_1 = \varepsilon_1(x) = [\beta x]$, $\varepsilon_k = \varepsilon_k(x) = [\beta \tau^k(x)]$, where $\tau(x) = \{\beta x\} := \beta x \mod 1$. The set of all possible sequences $\{\{\varepsilon_k(x)\}_{k=1}^{\infty} : x \in [0, 1]\}$ is called the (one-sided) $\beta$-compactum and denoted by $X_\beta^+$. A sequence whose tail is $0^\infty$ will be called finite.

The $\beta$-compactum can be described more explicitly. Let $1 = \sum_{k=1}^{\infty} d^k \beta^{-k}$ be the expansion of 1 defined as follows: $d^1 = [\beta]$, $d^n = [\beta \tau^n]$, $n \geq 2$. If the sequence $\{d^n\}$ is not finite, we put $\bar{d}_n = d^1$. Otherwise let $k = \max \{j : d^j > 0\}$, and $(d_1, d_2, \ldots) := (\bar{d}_1, \ldots, d_{k-1}, d_k - 1)$, where bar stands for the period of a purely periodic sequence.

We will write $\{x_n\}_1^{\infty} \prec \{y_n\}_1^{\infty}$ if $\{x_n\}_1^{\infty} \neq \{y_n\}_1^{\infty}$ and $x_n < y_n$ for the smallest $n \geq 1$ such that $x_n \neq y_n$. Then by definition,

$$X_\beta^+ = \{\{\varepsilon_n\}_1^{\infty} : (\varepsilon_n, \varepsilon_{n+1}, \ldots) \prec (d_1, d_2, \ldots) \text{ for all } n \in \mathbb{N}\}$$

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(see [18]). Similarly, we define the two-sided \( \beta \)-compactum as

\[
X_\beta = \{ (\varepsilon_n)_{n=\infty} : (\varepsilon_n, \varepsilon_{n+1}, \ldots) \prec (d_1, d_2, \ldots) \text{ for all } n \in \mathbb{Z} \}.
\]

Both compacta are naturally endowed with the weak topology, i.e., with the topology of coordinate-wise convergence, as well as with the natural shifts. Let the \( \beta \)-shift \( \sigma_\beta : X_\beta \to X_\beta \) act as follows: \( \sigma_\beta(\varepsilon)_k = \varepsilon_{k+1} \), and \( \sigma_\beta^+ \) be the corresponding one-sided shift on \( X_\beta^+ \). For a Pisot \( \beta \) the properties of the \( \beta \)-shift are well-studied. Its main property is that it is sofic, i.e., is a factor of a subshift of finite type. In fact this is equivalent to \( \{ d_n \}_n^\infty \) being eventually periodic (see, e.g., the review [6]).

We extend the \( \beta \)-expansions to the nonnegative integers in the usual way (similarly to the decimal expansions). There is a natural operation of addition in \( X_\beta \), namely, if both sequences \( \varepsilon \) and \( \varepsilon' \) are finite to the left (i.e., there exists \( N \in \mathbb{Z} \) such that \( \varepsilon_k = \varepsilon'_k = 0 \), \( k \leq N \)), then by definition, \( \varepsilon + \varepsilon' = \varepsilon'' \) such that \( \sum_k \varepsilon''_k \beta^{-k} = \sum_k (\varepsilon'_k + \varepsilon_k) \beta^{-k} \). Later we will show that under some natural assumption on \( \beta \) this operation can be extended to sequences which are not necessarily finite to the left.

**Lemma 1** (see [4], [21]) Any nonnegative element of the ring \( \mathbb{Z}[\beta] \) has an eventually periodic \( \beta \)-expansion if \( \beta \) is a Pisot number.

Let \( \text{Fin}(\beta) \) denote the set of nonnegative \( x \)'s whose \( \beta \)-expansions are finite. Obviously, \( \text{Fin}(\beta) \subset \mathbb{Z}[\beta]_+ := \mathbb{Z}[\beta] \cap \mathbb{R}_+ \), but the inverse inclusion does not hold for an arbitrary Pisot unit.

**Definition 2** A Pisot unit \( \beta \) is called finitary if

\[
\text{Fin}(\beta) = \mathbb{Z}[\beta]_+.
\]

A large class of Pisot numbers considered in [11] is known to have this property. A practical algorithm for checking whether a given Pisot number is finitary was suggested in [1]. Here is a simple example showing that not every Pisot unit is finitary. Let \( r \geq 3 \), and \( \beta^2 = r \beta - 1 \). Then \( X_\beta = \{ \varepsilon : 0 \leq \varepsilon_k \leq r-1, (\varepsilon_k, \ldots, \varepsilon_k+n) \neq (r-1, r-2, \ldots, r-2, r-1), k \in \mathbb{Z}, n \geq 1 \} \) and \( 1-\beta^{-1} = (r-2)\beta^{-1} + (r-2)\beta^{-2} + \ldots, \)

, i.e., \( \beta \) is not finitary.

**Definition 3** A Pisot unit \( \beta \) is called weakly finitary if for any \( \delta > 0 \) and any \( x \in \mathbb{Z}[\beta]_+ \) there exists \( f \in \text{Fin}(\beta) \cap [0, \delta) \) such that \( x + f \in \text{Fin}(\beta) \) as well.
When the present paper was in preparation, the author was told that this condition had in fact been considered in the recent work by Sh. Akiyama [3], in which the author shows that the boundary of the natural sofic tiling generated by a weakly finitary Pisot $\beta$ has Lebesgue measure zero (moreover, these conditions are actually equivalent). The author is grateful to Sh. Akiyama for drawing his attention to this paper and for helpful discussions.

A slightly weaker (but possibly equivalent) condition

$$Z[\beta] = \text{Fin}(\beta) - \text{Fin}(\beta)$$

together with the finiteness of $\{d_n\}$ was used in the Ph.D. Thesis [13] to show that the spectrum of the Pisot substitutional dynamical system

$$0 \rightarrow 10^{d_1}, 1 \rightarrow 20^{d_2}, \ldots, l - 2 \rightarrow (l - 1)0^{d_{l-1}}, l - 1 \rightarrow 0^{d_l},$$

(where $l = \max \{n : d_n \neq 0\}$), is purely discrete. This claim is a generalization of the corresponding result for a finitary $\beta$ from [27] (see also [28]).

**Conjecture 4** Any Pisot unit is weakly finitary.

To support this conjecture, we are going to explain how to verify that a particular Pisot unit is weakly finitary. Firstly, one needs to describe all the elements of the set

$$Z_\beta = \{\alpha \in Z[\beta] \cap [0,1) : \alpha \text{ has a purely periodic } \beta\text{-expansion}\}. \quad (2)$$

**Lemma 5** (see [3]). The set $Z_\beta$ is finite.

**Proof.** The sketch of the proof is as follows: basically, the claim will follow from Lemma 10 (see below), which implies that the denominator of any $\alpha \in P_\beta$ in the standard basis of $Q(\beta)$ is uniformly bounded, whence the period of the $\beta$-expansion of $\alpha$ is bounded as well. ■

Therefore, we have a finite collection of numbers $\{\sum_{j=0}^{m-1} y_j \beta^j : \|y\| \leq q\}$ to "check for periods" (here $q$ is the denominator of $\xi_0$ defined by (7) in the standard basis of the ring). Next, it is easy to see that if suffices to check that Definition 3 holds for any $x = \alpha \in Z_\beta$ (see [3]). Moreover, we can confine ourselves to the case $f \in \text{Fin}(\beta) \cap [\beta^{-2p}, \beta^{-p})$, where $p$ is the period of $\alpha$. Indeed, if such an $f$ exists, $\beta^{-p} f$ will do as well, and we will be able to make $f$ arbitrarily small. As was shown in [3], there exists a "universal neutralizing word" $f$ which will suit for all periods of $Z_\beta$. Note that all known examples of Pisot units prove to be weakly finitary.

We will need the following technical result.
Lemma 6 A Pisot unit is weakly finitary if and only if the following condition is satisfied: there exists $\eta = \eta(\beta) \in (0, 1)$ such that for any $\delta > 0$ and any $x \in \mathbb{Z}[\beta]_+$ there exists $f \in \text{Fin}(\beta) \cap [\eta \delta, \delta)$ such that $x + f \in \text{Fin}(\beta)$ as well.

Proof. It suffices to show that if $\beta$ is weakly finitary, then $\eta$ in question does exist. Let $\beta$ be weakly finitary; then we know that there exists $f_0 \in \text{Fin}(\beta)$ such that for any $\alpha \in \mathbb{Z}_{\beta}$ we have $\alpha + f_0 \in \text{Fin}(\beta)$. Let $\alpha$ has the $\beta$-expansion $(\overline{\alpha_1, \ldots, \alpha_p})_{\beta}$ and $\alpha^* = \sum_1^p \alpha_j \beta^{-j}$. Without loss of generality we may regard $p$ to be greater than the “pre-period” $+ \text{ the period of the sequence } \{d_n\}_1^\infty$ (as $p$ is not necessarily the smallest period of $\alpha$). Since $f_0$ can be made arbitrarily small, we may fix it such that

$$\alpha + f_0 < \alpha^* + \beta^{-p} \alpha$$

(3)

for any $\alpha \in \mathbb{Z}_{\beta}$. Put $\eta := f_0$ (provided (3) is satisfied).

Let $x \in \mathbb{Z}[\beta]_+$. By Lemma 1, the $\beta$-expansion of $x$ is eventually periodic, and splitting it into the pre-periodic and periodic parts, we have $x = x_0 + \beta^{-k} \alpha$, $x_0 \in \text{Fin}(\beta), k \in \mathbb{Z}, \alpha \in \mathbb{Z}_{\beta}$. Let for simplicity of notation $k = 0$ (the whole picture is shift-invariant). It will now suffice to check the condition for $\delta = \delta_n = \beta^{-m}$. Put $f = f_n := \beta^{-m} \eta$. Then

$$x + f = (x_0 + \alpha^* + \beta^{-p} \alpha^* + \cdots + \beta^{-\ell} \alpha^*) + \beta^{-m} (\alpha + \eta).$$

(4)

The first sum in brackets in (4) belongs to $\text{Fin}(\beta)$ and so does the second term. In view of (3) and the definition of $X_{\beta}$, the whole sum in (4) belongs to $\text{Fin}(\beta)$ as well, because by our choice of $p$ we have necessarily $(\alpha_1, \ldots, \alpha_p) \prec (d_1, \ldots, d_p)$. ■

2 Formulation of the main result and first steps of the proof

We recall that the hyperbolicity of $T$ implies that it has the stable and unstable foliation and consequently the set of homoclinic points. More precisely, a point $t \in \mathbb{T}^m$ is called homoclinic to zero or simply homoclinic if $T^n t \to 0$ as $n \to \pm \infty$ (as is well known, the convergence to 0 in this case will be at an exponential rate). In other terms, a homoclinic point $t$ must belong to the intersection of the leaves of the stable foliation $L_s$ and the unstable foliation $L_u$ passing through 0. Let $H(T)$ denote the set of all homoclinic points for $T$; obviously, $H(T)$ is a group under addition. In [30] it was shown that every homoclinic point can be obtained by applying the following procedure: take a point $n \in \mathbb{Z}^m$ and project it onto $L_u$ along $L_s$. Let $s$ denote this projection; finally, project $s$ onto the torus by taking the fractional parts of all its
coordinates. The correspondence \( n \leftrightarrow s \leftrightarrow t \) known to be one-to-one. We will call \( s = s(t) \) the \( \mathbb{R}^m \)-coordinate of a homoclinic point \( t \) and \( n \) the \( \mathbb{Z}^m \)-coordinate of \( t \). Note that since \( T \) is a Pisot automorphism, we have \( \dim L_u = 1 \), \( \dim L_s = m - 1 \).

We wish to find an arithmetic coding \( \varphi \) of \( T \) in the following sense: we choose \( X_\beta \) as a symbolic compact space and impose the following restrictions on a map \( \varphi : X_\beta \to \mathbb{T}^m \):

1. \( \varphi \) is continuous and bounded-to-one;
2. \( \varphi_\sigma = T \varphi \);
3. \( \varphi(\varepsilon + \varepsilon') = \varphi(\varepsilon) + \varphi(\varepsilon') \) for any pair of sequences finite to the left.

In [25] it was shown that if \( m = 2 \), then there exists \( t \in \mathcal{H}(T) \) such that \( \varphi = \varphi_t : X_\beta \to \mathbb{T}^m \):

\[
\varphi_t(\varepsilon) = \sum_{k \in \mathbb{Z}} \varepsilon_k T^{-k} t.
\]

(5)

The proof for an arbitrary \( m \) is basically the same, and we will omit it. Our primary goal is to find an arithmetic coding that is bijective a.e. Let us make some historic remarks.

Note that the idea of using homoclinic points to “encode” ergodic toral automorphisms had been suggested by A. Vershik in [29] for \( M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \) and was later developed for a more general context in numerous works - see [30], [16], [24], [25], [22]. The choice of \( X_\beta \) as a “coding space” is special in the case in question; indeed, the topological entropy of the shift \( \sigma_\beta \) is known to be \( \log \beta \) and so is the entropy of \( T \). In a more general context (for example, if \( M \) has two eigenvalues outside the unit disc) it is still unclear, which compactum might replace \( X_\beta \). Indeed, since \( \varphi \) is bounded-to-one, the topological entropy of the subshift on this compactum must have the same topological entropy as \( T \), i.e., \( \log \prod_{j=1}^{m} \beta_j \), where \( \beta_j \), \( j = 1, \ldots, m \), are the conjugates of \( \beta \), and there is apparently no natural subshift associated with \( \beta \) which has this entropy. However, it is worth noting that the existence of such compacta in different settings has been shown in [30], [15], [22].

Note that if one allows to use slightly altered symbolic transformations (not necessarily shifts), then there is a hope to give a more explicit expression for the codings of non-Pisot automorphisms via coding of the higher-rank actions - see examples at the end of the paper.

Return to our context. The mapping \( \varphi_t \) defined by (5) is indeed well defined and continuous, as the series (5) converges at an exponential rate. Furthermore, since
\( T^k t = \beta t \mod Z^m \), we have by continuity \( \varphi_k \sigma \beta = T \varphi_k \), i.e., \( \varphi_k \) does semiconjugate the shift and a given automorphism \( T \).

We will call \( \varphi_k \) a general arithmetic coding for \( T \) (parameterized by a homoclinic point \( t \)).

**Lemma 7** For any choice of \( t \) the mapping \( \varphi_k \) is bounded-to-one.

**Proof.** Let \( \| \cdot \| \) denote the distance to the closest integer, \( s \) be the \( R^m \)-coordinate of \( t \) and \( \tilde{T} \) denote the linear transformation of \( R^m \) defined by the matrix \( M \). Let \( \varphi_{N,k} \) be the mapping acting from \( X_\beta \) into \( R^m \) by the formula

\[
\varphi_{N,k}(\xi) := \sum_{-N}^{N} \xi_k (\tilde{T})^{-k} s.
\]

Then by (5),

\[
\varphi_k(\xi) = \lim_{N \to +\infty} (\varphi_{N,k}(\xi) \mod Z^m),
\]

where \((x_1, \ldots, x_m) \mod Z^m = (\{x_1\}, \ldots, \{x_m\})\). Therefore, it suffices to show that the diameters of the sets \( \varphi_{N,k}(X_\beta) \) are uniformly bounded for all \( N \). We have (recall that \( 0 \leq \varepsilon_k \leq [\beta] \):

\[
\max \{ \| \varphi_{N,k}(\xi) \| : \xi \in X_\beta \} \leq [\beta] \left\| \sum_{-N}^{N} (\tilde{T})^{-k} s \right\| \leq [\beta] \sum_{-N}^{N} \left\| (\tilde{T})^{-k} s \right\|
\]

\[
\leq \text{const} \cdot \sum_{0}^{N} \theta^k < \infty,
\]

where \( \theta \in (0, 1) \) is the maximum of the absolute values of the conjugates of \( \beta \) that do not coincide with \( \beta \). This proves the lemma. ■

Let the characteristic equation for \( \beta \) be

\[
\beta^m = k_1 \beta^{m-1} + k_2 \beta^{m-2} + \cdots + k_m
\]

(6)

and \( T_\beta \) denote the toral automorphism given by the companion matrix \( M_\beta \) for \( \beta \), i.e.,

\[
M_\beta = \begin{pmatrix}
    k_1 & k_2 & \cdots & k_{m-1} & k_m \\
    1 & 0 & \cdots & 0 & 0 \\
    0 & 1 & \cdots & 0 & 0 \\
    \cdots & \cdots & \cdots & \cdots & \cdots \\
    0 & 0 & \cdots & 1 & 0 \\
\end{pmatrix}.
\]

We first assume the following conditions to be satisfied:

\[
7
\]
1. $T$ is algebraically conjugate to $T_\beta$, i.e., there exists a matrix $C \in GL(m, \mathbb{Z})$ such that $CM = M_\beta C$ (notation: $T \sim T_\beta$).

2. A homoclinic point $t$ is fundamental, i.e., $\langle T^{nt} | n \in \mathbb{Z} \rangle = \mathcal{H}(T)$.

3. $\beta$ is weakly finitary.

The notion of fundamental homoclinic point for general actions of expansive group automorphisms was introduced in [17] (see also [22]).

Remark 8 Note that the second condition implies the first one, as the mere existence of a fundamental homoclinic point means that $T \sim T_\beta$ (see Theorem 30 below). Conversely, if $T \sim T_\beta$, then there is always a fundamental homoclinic point for $T$. Indeed, let $n_0 = (0,0,\ldots,0,1)$ be the $\mathbb{Z}^m$-coordinate of $t_0$. Then $t_0$ is a fundamental for $T_\beta$ and if $CM = M_\beta C$, then $C^{-1}t_0$ is fundamental for $T$.

Now we are ready to formulate the main theorem of the present paper.

**Theorem 9** Provided the above conditions are satisfied, the mapping $\varphi$, defined by (5) is bijective a.e. with respect to the Haar measure on the torus.

Remark. In [25] the claim of the theorem was shown for $m = 2$. We wish to follow the line of exposition of that paper, though it is worth stressing that our approach will be completely different (rather arithmetic than geometric). In [22] this claim was shown for any finitary $\beta$ and it was conjectured that it holds for any Pisot automorphism satisfying conditions 1 and 2 above. We give further support for this conjecture, as Theorem 9 implies that we actually reduced it to a general number-theoretic conjecture verifiable for any given Pisot unit $\beta$ (see Conjecture 4).

The rest of the section as well as the next section will be devoted to the proof of Theorem 9; in the last section we will discuss the case when conditions 1 and 2 are not necessarily satisfied.

We need the following number-theoretic claim. Let

$$\mathcal{P}_\beta := \{ \xi : \|\xi \beta^n\| \to 0, \ n \to +\infty \}.$$ 

It is obvious that $\mathcal{P}_\beta$ is a group under addition.

**Lemma 10** There exists $\xi_0 \in \mathbb{Q}(\beta) \setminus \mathbb{Z}[\beta]$ such that

$$\mathcal{P}_\beta = \xi_0 \cdot \mathbb{Z}[\beta].$$ (7)
Proof. By the well-known result, for any Pisot \( \beta \), \( \xi \in \mathcal{P}_\beta \leftrightarrow \xi \in \mathcal{Q}(\beta) \) and \( Tr(\beta^k \xi) \in \mathbb{Z} \), \( k \geq k_0 \) (where \( Tr(\xi) \) denotes the trace of an element \( \xi \) of the extension \( \mathcal{Q}(\beta) \), i.e., the sum of all its Galois conjugates) — see, e.g., [8]. Since \( \beta \) is a unit, \( Tr(\xi) \in \mathbb{Z} \) implies \( Tr(\beta^{-1} \xi) \in \mathbb{Z} \), whence

\[
\mathcal{P}_\beta := \{ \xi \in \mathcal{Q}(\beta) : Tr(a \xi) \in \mathbb{Z} \ \forall a \in \mathbb{Z}[\beta] \}.
\]

Thus, if we regard \( \mathbb{Z}[\beta] \) as a lattice over \( \mathbb{Z} \), then by (8), \( \mathcal{P}_\beta \) is by definition the dual lattice for \( \mathbb{Z}[\beta] \). Hence by the well known ramification theorem (see, e.g., [10, Chapter III]) the equality (7) follows with \( \xi_0 = 1/g'(\beta) \), where \( g(x) = x^m - k_1 x^{m-1} - \cdots - k_m \).

We are going to carry out the proof of the main theorem in several steps.

**Step 1 (description of the homoclinic group).**

**Lemma 11** Any homoclinic point \( \mathbf{t} \) for \( T_\beta \) has the \( \mathbb{R}^m \)-coordinate

\[
s(\mathbf{t}) = \xi_0 u(1, \beta^{-1}, \ldots, \beta^{-m+1}),
\]

where \( u \in \mathbb{Z}[\beta] \).

Proof. We have \( M_\beta \tilde{v}_\beta = \beta \tilde{v}_\beta \), where \( \tilde{v}_\beta = (1, \beta^{-1}, \ldots, \beta^{-m+1}) \). As was mentioned above, the dimension of the unstable foliation \( L_u \) is 1, whence \( s(\mathbf{t}) = k \tilde{v}_\beta \), and since \( T_\beta^k \mathbf{t} \to \mathbf{0} \), we have \( \| k \beta^u \| \to 0 \), i.e., \( k \in \mathcal{P}_\beta \). Now the claim of the lemma follows from (7).

Let \( \mathcal{U}_\beta \) denote the group of units (= invertible elements) of the ring \( \mathbb{Z}[\beta] \).

**Lemma 12** There is a one-to-one correspondence between the group \( \mathcal{U}_\beta \) and the set of fundamental homoclinic points for \( T_\beta \). Namely, if \( \mathbf{t} \) is fundamental, then \( u \) in (9) is a unit and vice versa.

Proof. Suppose \( \mathbf{t} \) is fundamental. Then the homoclinic point \( \mathbf{t}_0 \) whose \( \mathbb{R}^m \)-coordinate is \( s_0 = (\xi_0, \xi_0 \beta^{-1}, \ldots, \xi_0 \beta^{-m+1}) \) can be represented as a finite linear integral combination of the powers \( T^k \mathbf{t} \), i.e.,

\[
\xi_0 (1, \beta^{-1}, \ldots, \beta^{-m+1}) = \sum_k e_k \beta^k \xi_0 u(1, \beta^{-1}, \ldots, \beta^{-m+1}),
\]

whence \( u \sum_k e_k \beta^k = 1 \). Therefore, \( u \) is invertible in the ring \( \mathbb{Z}[\beta] \).

Conversely, if \( u \in \mathcal{U}_\beta \), then using the same method, we show that the claim of the lemma follows from the fact that the equation \( ux = u' \) always has the solution in \( \mathbb{Z}[\beta] \), namely, \( x = u^{-1} u' \).
Step 2 (reduction to $T = T_\beta$). To prove Theorem 9, we may without loss of
genrearity assume $T = T_\beta$. Indeed, suppose $M = C^{-1}M_\beta C$, where $C \in GL(m, \mathbb{Z})$.
Then there is a natural one-to-one correspondence between $\mathcal{H}(T)$ and $\mathcal{H}(T_\beta)$, namely,
t $\in \mathcal{H}(T) \iff C \cdot t \in \mathcal{H}(T_\beta)$. Furthermore, if $\varphi_t$ is bijective a.e., then so is $\varphi_{C t}$, as
$\varphi_{C t} = C \varphi_t$.

So, let $T = T_\beta$, and $t$ be a general fundamental homoclinic point for $T_\beta$ given by
(9). In this case the formula (5) turns into the following one:

$$
\varphi_t(\bar{e}) = \lim_{N \to +\infty} \frac{\sum_{k=-N}^{+\infty} \bar{e}_k \beta^{-k} \begin{pmatrix}
\xi_0 u \\
\xi_0 u \beta^{-1} \\
\vdots \\
\xi_0 u \beta^{-m+1}
\end{pmatrix}}{\text{mod } \mathbb{Z}^m}. 
\tag{10}
$$

Step 3 (the pre-image of 0). Let $Z_\beta$ be defined by (2).

**Lemma 13** The pre-image of 0 can be described as follows:

$$
\mathcal{O}_\beta := \varphi_t^{-1}(0) = \{ \bar{e} \in X_\beta : \bar{e} \text{ is purely periodic and } \sum_{1}^{\infty} \bar{e}_j \beta^{-j} \in Z_\beta \}.
$$

**Proof.** By Lemma 7, $\mathcal{O}_\beta$ is finite and since it is shift-invariant, it must contain
purely periodic sequences only. Let $\alpha = \sum_{1}^{\infty} e_j \beta^{-j}$. Then by (10), $\| \alpha u \beta^n \| \to 0$ as
$n \to \infty$, whence by (7), $\alpha u \in \mathbb{Z}^\beta$, and $\alpha \in \mathbb{Z}^\beta$, because $u \in U_\beta$. ■

Step 4 (description of the full pre-image of any point of the torus). We are
going to show that $\varphi_t$ is “linear” in the sense that for any two sequences $\bar{e}, \bar{e}' \in \varphi_t^{-1}(x)$
their “difference” will belong to $\mathcal{O}_\beta$. More precisely, let $\bar{e}^{(N)}$ denote the sequence
$(\ldots, 0, 0, \ldots, 0, e_{-N}, e_{-N+1}, \ldots)$ and its “value” $e^{(N)} := \sum_{k=-N}^{+\infty} e_k \beta^{-k}$. There is an
almost one-to-one correspondence between the set of sequences $\{ \bar{e}^{(N)} \}$ and $\mathbb{R}_+$, namely
$e^{(N)} \mapsto e^{(N)}$.

**Lemma 14** If $\varphi_t(\bar{e}) = \varphi_t(\bar{e}')$, then for any $N \geq 1$ there exists $\alpha \in Z_\beta$ such that

$$
|e^{(N)} - (e')^{(N)}| = \beta^N \alpha.
$$

**Proof.** Fix $\bar{e}, \bar{e}' \in \varphi_t^{-1}(\{x\})$ for some $x \in \mathbb{T}^m$ and let $\mathcal{E}$ denote the set of all partial
limits (in $X_\beta$) of the collection of sequences $[\bar{e}^{(N)} - (e')^{(N)}]$, $N \geq 1$, where $|e^{(N)} - (e')^{(N)}|$ is the sequence
$(\ldots, 0, 0, \ldots, 0, \bar{e}^{(N)}_N, \bar{e}^{(N)}_{N+1}, \ldots)$ whose “value” is $|e^{(N)} - (e')^{(N)}|$. It
suffices to show that $\mathcal{E} \subset \mathcal{O}_\beta$. Let $\bar{d} \in \mathcal{E}$; by definition, there exists a sequence
of positive integers $\{N_k\}$ such that $\delta^{(N_k)} = \left| (e')^{(N_k)} - e^{(N_k)} \right|$, $k = 1, 2, \ldots$. Then
$\varphi_t(\bar{d}) = \lim_{k \to \infty} \varphi_t(\bar{d}^{(N_k)}) = 0$, and we are done. ■
Therefore, if $\xi \in \varphi_\lambda^{-1}(\{x\})$ for some $x \in \mathbb{T}^m$, then we know that to obtain any $\xi' \in \varphi_\lambda^{-1}(\{x\})$, one may take one of the partial limits of the sequence $\{\varepsilon^{(N)} + \beta^N \alpha\}$ for $\alpha \in \mathbb{Z}_\beta$, perhaps, depending on $N$. We will write
\[
\xi \sim \xi' \iff \varphi_\lambda(\xi) = \varphi_\lambda(\xi').
\] (11)

**Conclusion.** Thus, we reduced the proof of Theorem 9 to a certain claim about the two-sided $\beta$-compactum.

Roughly speaking, our goal now is to show that the procedure described above will not change an arbitrarily long tail of a generic sequence $\xi \in X_\beta$ and therefore, will not change $\xi$ itself.

## 3 Final steps of the proof and examples

Let $\mu_\beta$ denote the measure of maximal entropy for the shift $(X_\beta, \sigma_\beta)$, and $\mu_\beta^+$ be its one-sided analog. We wish to prove that
\[
\mu_\beta \{\xi \in X_\beta : \#[\xi] = 1\} = 1,
\] (12)
where $[\xi] = \{\xi' \in X_\beta : \xi' \sim \xi\}$.

**Step 5 (estimation of the measure of the “bad” set).** We will need some basic facts about the measure $\mu_\beta$. For technical reasons we prefer to deal with its one-sided analog $\mu_\beta^+$.

**Lemma 15** There exists a constant $C_1 = C_1(\beta) \in (0, 1)$ such that for any $n \geq 2$ and any $(i_1, i_2, \ldots) \in X_\beta^+$,
\[
\mu_\beta^+ (\varepsilon_n = i_n | \varepsilon_{n-1} = i_{n-1}, \ldots, \varepsilon_1 = i_1) \geq C_1.
\]

**Proof.** Let the mapping $\pi_\beta : X_\beta^+ \to [0, 1)$ be given by formula (1) and $m_\beta^+ = \pi_\beta(\mu_\beta^+)$. Let $C_n(\xi) = (\varepsilon_n = i_n, \varepsilon_{n-1} = i_{n-1}, \ldots, \varepsilon_1 = i_1) \subset X_\beta^+$ and $\Delta_n(\xi) = \pi_\beta(C_n(\xi))$. The Garsia Separation Lemma [12] says that there exists a constant $K = K(\beta) > 0$ such that if $\xi$ and $\xi'$ are two sequences in $X_\beta^+$ and $\sum_{k=1}^n \varepsilon_k \beta^{-k} \neq \sum_{k=1}^n \varepsilon'_k \beta^{-k}$, then $|\sum_{k=1}^n (\varepsilon_k - \varepsilon'_k) \beta^{-k}| \geq K \beta^{-n}$. Hence
\[
K \leq \beta^n \mathcal{L}_1(\Delta_n(\xi)) \leq 1,
\]
where $\mathcal{L}_1$ denotes the Lebesgue measure on $[0, 1]$. Since for any $\beta > 1$, $m_\beta^+$ is equivalent to $\mathcal{L}_1$ and the corresponding density is uniformly bounded away from 0 and $\infty$ (see [19]), we have for some $K' > 1$,
\[
1/K' \leq \beta^n m_\beta^+(\Delta_n(\xi)) \leq K',
\]
whence by the fact that $\pi_\beta$ is one-to-one except for a countable set of points,

$$1/K' \leq \beta^n \mu_\beta^+(C_n(\bar{\varepsilon})) \leq K'$$

and the claim of the lemma holds with $C_1 = (\beta K')^{-2}$. ■

There is a natural arithmetic structure on $X^+_\beta$: the sum of two sequences $\bar{\varepsilon}$ and $\bar{\varepsilon}'$ is defined as the sequence equal to the $\beta$-expansion of the sum $\{\sum_{k=0}^\infty (\varepsilon_k + \varepsilon'_k)\beta^{-k}\}$. Let $X^{(n)}_\beta$ denote the set of finite words of length $n$ that are extendable to a sequence in $X^+_\beta$ by writing noughts at all places starting with $n+1$. We will sometimes identify $X^{(n)}_\beta$ with the set $Fin_n(\beta) := \{\bar{\varepsilon} : \varepsilon_k \equiv 0, k \geq n + 1\}$.

By the sum $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) + \varepsilon'$, we will imply $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, 0, 0, \ldots) + \varepsilon'$. In [11] it was shown that there exists a natural $L_1 = L_1(\beta)$ such that if $\bar{\varepsilon} \in Fin_n(\beta)$, $\bar{\varepsilon}' \in Fin_n(\beta)$ and $\bar{\varepsilon} + \bar{\varepsilon}' \in Fin(\beta)$, then $\bar{\varepsilon} + \bar{\varepsilon}' \in Fin_{n+L_1}(\beta)$.

Recall that by Lemma 6 there exists $\eta = \eta(\beta) \in (0, 1)$ such that the quantity $f$ in Definition 3 can be chosen in $(\eta \delta, \delta)$ instead of $(0, \delta)$. We set

$$L_2 := \frac{\log(1/\eta)}{\log \beta}$$

and

$$L := \max \{L_1, L_2\}.$$ 

We can reformulate the hypothesis that $\beta$ is weakly finitary as follows ($\bar{\alpha}$ denotes the $\beta$-expansion of $\alpha$):

for any $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in X^{(n)}_\beta$ there exists $(\varepsilon_{n+1}, \ldots, \varepsilon_{n+L}) \in X^{(L)}_\beta$ such that

$$\begin{equation}
(\varepsilon_1, \ldots, \varepsilon_{n+L}) \in X^{(n+L)}_\beta, \quad \bar{\alpha} + (\varepsilon_1, \ldots, \varepsilon_{n+L}) \in Fin(\beta) \text{ for all } \alpha \in Z_\beta.
\end{equation}$$

A direct consequence of Lemma 15 is

**Corollary 16** For any $(i_1, i_2, \ldots) \in X^+_\beta$,

$$\frac{\mu_\beta^+(\varepsilon_{n+L} = i_{n+L}, \ldots, \varepsilon_1 = i_1)}{\mu_\beta^+(\varepsilon_n = i_n, \ldots, \varepsilon_1 = i_1)} \geq C_2 = C^L_1.$$

**Lemma 17** If $(i_1, \ldots, i_n) \in X^{(n)}_\beta$ and $(j_1, \ldots, j_k) \in X^{(k)}_\beta$, then $(i_1, \ldots, i_n, 0, 0, 0, 0, j_1, \ldots, j_k) \in X^{(n+k+4)}_\beta$. 

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Proof. The claim follows from the definition of $X_{\beta}$ (see Introduction) and the fact that the positive root $\beta_0$ of the equation $x^3 = x + 1$ is the smallest Pisot number [9]. Indeed, $\beta_0^5 = \beta_0^4 + 1$ and $X_{\beta_0}$ is a subshift of finite type, namely,

$$X_{\beta_0} = \left\{ \vec{\varepsilon} \in \prod_{n=-\infty}^{+\infty} \{0, 1\} \mid \varepsilon_n = 1 \Rightarrow \varepsilon_{n+1} = \varepsilon_{n+2} = \varepsilon_{n+3} = \varepsilon_{n+4} = 0 \right\}.$$  

Now the desired claim follows from [18, Lemma 3] asserting that if $\beta' < \beta$, then $(d_1(\beta'), d_2(\beta'), \ldots) \prec (d_1(\beta), d_2(\beta), \ldots)$.  

Let

$$\mathcal{A} = \{ \vec{\varepsilon} \in X^{+}_\beta \mid \exists n \in \mathbb{N} : \forall \alpha \in Z_\beta, \, \vec{\alpha} + (\varepsilon_1, \ldots, \varepsilon_n) \in \text{Fin}(\beta) \},$$

$$\mathcal{A}_n = \{ \vec{\varepsilon} \in X^{+}_\beta \mid \forall \alpha \in Z_\beta, \, \vec{\alpha} + (\varepsilon_1, \ldots, \varepsilon_n) \in \text{Fin}(\beta) \},$$

$$\mathcal{A}' = \{ \vec{\varepsilon} \in X^{+}_\beta \mid \exists n \in \mathbb{N} : \varepsilon_{n+1} = \cdots = \varepsilon_{n+L+4} = 0 \}.$$  

We will write $\text{tail}(\vec{\varepsilon}) = \text{tail}(\vec{\varepsilon}')$ if there exists $n \in \mathbb{N}$ such that $\varepsilon_k = \varepsilon'_k$, $k \geq n$. The meaning of the above definitions consists in the fact that if $\vec{\varepsilon} \in \mathcal{A} \cap \mathcal{A}'$, then $\vec{\varepsilon} \in \mathcal{A}_n \cap \mathcal{A}'$ for some $n \geq 1$ and by the theorem from [11] mentioned above, $(\varepsilon_1, \ldots, \varepsilon_n) + \vec{\alpha} = (\varepsilon'_1, \ldots, \varepsilon'_{n+L})$, whence by Lemma 17,

$$\text{tail}(\vec{\varepsilon} + \vec{\alpha}) = \text{tail}(\vec{\varepsilon})$$

(more precisely, the tail will stay unchanged starting with the $(n + L + 1)$th symbol). It is obvious that $\mathcal{A} = \bigcup_n \mathcal{A}_n$. We wish to prove that $\mu^+_\beta(\mathcal{A} \cap \mathcal{A}') = 1$. By the ergodicity of $(X^+_\beta, \mu^+_\beta, \sigma^+_\beta)$, we have $\mu^+_\beta(\mathcal{A}') = 1$, it suffices to show that $\mu^+_\beta(\mathcal{A}) = 1$. Let $\mathcal{B}_n = X^+_\beta \setminus \mathcal{A}_n$.

**Proposition 18** There exists a constant $\gamma = \gamma(\beta) \in (0, 1)$ such that

$$\mu^+_\beta \left( \bigcap_{k=1}^{n} \mathcal{B}_k \right) \leq \gamma^n. \quad (15)$$

**Proof.** We have

$$\mu^+_\beta (\mathcal{B}_1 \cap \mathcal{B}_2 \cap \ldots \cap \mathcal{B}_n) \leq \prod_{k=2}^{n} \mu^+_\beta (\mathcal{B}_k \mid \mathcal{B}_{k-1} \cap \ldots \cap \mathcal{B}_1).$$
$$
\prod_{j=k-L}^{k} \mu_{j}^{+}(B_j | B_{j-1} \cap \ldots \cap B_1) = \frac{\mu_{j}^{+}(B_k \cap \ldots \cap B_1)}{\mu_{j}^{+}(B_{k-L-1} \cap \ldots \cap B_1)} \leq \frac{\mu_{j}^{+}(B_k \cap B_{k-L-1} \cap B_{k-L-2} \cap \ldots \cap B_1)}{\mu_{j}^{+}(B_{k-L-1} \cap \ldots \cap B_1)} = \mu_{j}^{+}(B_k | B_{k-L-1} \cap B_{k-L-2} \cap \ldots \cap B_1),
$$

we have

$$\mu_{j}^{+}(B_1 \cap B_2 \cap \ldots \cap B_n) \leq \prod_{k=2}^{[n/L]} \mu_{j}^{+}(B_{lk} | B_{lk-L-1} \cap B_{lk-L-2} \cap \ldots \cap B_1). \quad (16)$$

Now by the formula (14), \( \beta \) being weakly finitary (see (13)) and the definition of \( L \) we have

$$\mu_{j}^{+}(A_{k+L} | \varepsilon_k = i_k, \ldots, \varepsilon_1 = i_1) \geq C_2 > 0$$

for any \( k \geq 1 \) and any \( (i_1, \ldots, i_k) \in X_{\beta}^{(k)} \). Hence

$$\mu_{j}^{+}(B_{lk} | B_{lk-L-1} \cap B_{lk-L-2} \cap \ldots \cap B_1) \leq 1 - C_2,$$

and from (16) we finally obtain the estimate

$$\mu_{j}^{+}(B_1 \cap B_2 \cap \ldots \cap B_n) \leq (1 - C_2)^{[n/L]},$$

whence one can take \( \gamma = (1 - C_2)^{1/2L} \), and (15) is proven. \( \blacksquare \)

As a by-product we obtain the following claim about the irrational rotations of the circle by the elements of \( \mathbb{Z}[\beta] \). Let, as above, \( \tilde{a} \) denote the \( \beta \)-expansion of \( \alpha \).

**Theorem 19** For a weakly finitary Pisot unit \( \beta \) and any \( \alpha \in \mathbb{Z}[\beta] \cap [0,1) \) we have

$$\text{tail}(\varepsilon + \tilde{a}) = \text{tail}(\varepsilon)$$

for \( \mu_{j}^{+} \)-a.e. \( \varepsilon \in X_{\beta}^{+} \).

**Proof.** We showed that \( \mu_{j}^{+}(\cap_{i=1}^{n} B_n) = 0 \), whence \( \mu_{j}^{+}(A) = \mu_{j}^{+}(\cup_{i=1}^{n} A_n) = 1 \). \( \blacksquare \)

**Conclusion of the proof of Theorem 9.** Fix \( k \in \mathbb{N} \). To complete the proof of Theorem 9, it suffices to show that the set

$$D^{(k)} = \{ \varepsilon \in X_{\beta} \mid \varepsilon_j \equiv \varepsilon_j', j \geq k \ \forall \varepsilon' \sim \varepsilon \}$$

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has the full measure $\mu_\beta$. By Proposition 18, for
$$
D_N^{(k)} = \{(\varepsilon_N, \varepsilon_{N+1}, \ldots) \in X_\beta^+ \mid (\ldots 0, 0, \varepsilon_N, \varepsilon_{N+1}, \ldots) \in D^{(k)}\},
$$
$$
\mu_\beta^+(D_N^{(k)}) \geq 1 - \gamma^{k-N} \to 1 \text{ as } N \to +\infty. \text{ Hence }
$$
$$
\mu_\beta(D^{(k)}) = \lim_{N \to +\infty} \mu_\beta^+(D_N^{(k)}) = 1,
$$
and therefore
$$
\mu_\beta \left( \bigcap_{k=1}^{\infty} D^{(k)} \right) = 1,
$$
which implies (12). We have thus shown that for $\mu_\beta$-a.e. $\bar{\varepsilon} \in X_\beta$, $\# \varphi^{-1}(\varphi(\bar{\varepsilon})) = 1$.

Let $\mathcal{L}$ denote the image of $\mu_\beta$ under $\varphi$. Since $\mu_\beta$ is ergodic, so is $\mathcal{L}$ and since $h_{\mu_\beta}(\sigma_\beta) = \log \beta$, we have $h_{\mathcal{L}}(T) = \log \beta$ as well. Hence $\mathcal{L} = \mathcal{L}_m$ is the Haar measure on the torus, as it is the unique ergodic measure of maximal entropy. So, we proved that
$$
\mathcal{L}_m \{ x \in \mathbb{T}^m \mid \# \varphi^{-1}_k(x) = 1 \} = 1,
$$
which is the claim of Theorem 9.

As a corollary we obtain the following claim about the arithmetic structure of $X_\beta$ itself.

**Proposition 20** Let $\sim$ denote the equivalence relation on $X_\beta$ defined by (11) and $X'_\beta := X_\beta / \sim$. Then $X'_\beta$ is a group isomorphic to $\mathbb{T}^m$.

Thus, $X_\beta$ is an almost group in the sense that it suffices to “glue” some $k$-tuples (for $k < \infty$) within the set of measure zero to turn the two-sided $\beta$-compactum for a weakly finitary Pisot unit $\beta$ into a group (which will be isomorphic to the torus of the corresponding dimension). Note that in dimension 2 this factorization can be described more explicitly – see [25, Section 1].

**Remark 21** In fact, we covered all generalized Pisot automorphisms (see the beginning of Introduction), i.e., $\lambda = \pm \beta$ or $\pm \beta^{-1}$. Indeed, in the case $\lambda = \beta^{-1}$ the same coding $\varphi$ will conjugate the inverse shift $\sigma_\beta^{-1}$ and $T$. In the case $\lambda = -\beta$ the mapping $\varphi$ conjugates $\sigma_\beta'$ and $T$, where $\sigma_\beta'(\bar{\varepsilon}) = \sigma_\beta(-\bar{\varepsilon})$; it follows from Proposition 20 that the operation $\bar{\varepsilon} \mapsto -\bar{\varepsilon}$ is well defined a.e. on $X_\beta$.

The following claim is a generalization of Theorem 4 from [26]. Let $\mathcal{D}(T)$ denotes the centralizer for $T$, i.e.,
$$
\mathcal{D}(T) = \{ A \in GL(m, \mathbb{Z}) : AT = TA \}. 
$$
Proposition 22. For a Pisot automorphism whose matrix is algebraically conjugate
to the corresponding companion matrix there is a one-to-one correspondence between
the following sets:

1. the fundamental homoclinic points for $T$;
2. the bijective arithmetic codings for $T$;
3. the units of the ring $\mathbb{Z}[\beta]$;
4. the centralizer for $T$;

Proof. We already know that any bijective arithmetic coding is parameterized
by a fundamental homoclinic point. Let $t_0$ be such a point for $T$; then any other
fundamental homoclinic point $t$ satisfies $s = us_0$, where $s_0$ and $s$ are the corresponding
$\mathbb{R}^m$-coordinates and $u \in \mathcal{U}_\beta$ – the proof is essentially the same as in Lemma 12. On
the other hand, if $\varphi_t$ is a bijective arithmetic coding for $T$, then as easy to see,
$A := \varphi_t^{-1}$ is a toral automorphism commuting with $T$ (this mapping is well defined
almost everywhere on the torus, hence it can be defined everywhere by continuity).
Finally, if $u \in \mathcal{U}_\beta$ and $u = \sum_{j=1}^{m-1} u_j \beta^j$, then $A := \sum_{j=1}^{m-1} u_j M^j$ belongs to $GL(m, \mathbb{Z})$
and commutes with $M$ and vice versa.

Example 1. (see [24]) Let $T$ be given by the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Here $\beta$ is the
golden ratio, $\xi_0 = \frac{1}{\sqrt{5}} = \frac{1 + \sqrt{5}}{5}$ and
\[ \mathcal{U}_\beta = \{ \pm \beta^n, \ n \in \mathbb{Z} \}. \]
Any bijective arithmetic coding for $T$ thus will be of the form
\[ \varphi(\bar{z}) = \lim_{N \to +\infty} \sum_{k=-N}^{+\infty} \varepsilon_k \beta^{-k} \left( \vartheta \beta^n / \sqrt{5} \right) \mod \mathbb{Z}, \]
where $\vartheta \in \{ \pm 1 \}$ and $n \in \mathbb{Z}$.

For more two-dimensional examples see [25].

Example 2. Let $T$ be given by the matrix $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Here $\beta$ is the real
root of the “tribonacci” equation $x^3 = x^2 + x + 1$; as is well known, $\beta$ is finitary in
this case (see, e.g., [11]). We have $\xi_0 = \frac{1}{\sqrt{2} - 2} = \frac{1 + \sqrt{2} - 2}{2}$, and since $\mathbb{Z}[\beta]$ is the
maximal order in the field $\mathbb{Q}(\beta)$ and both conjugates of $\beta$ are complex, again
\[ \mathcal{U}_\beta = \{ \pm \beta^n, \ n \in \mathbb{Z} \} \]
(recall that by Dirichlet’s Theorem, $\mathcal{U}_\beta$ must be “one-dimensional”, see, e.g., [7]). Hence any bijective arithmetic coding for $T$ is of the form
\[
\varphi(\varepsilon) = \lim_{N \to +\infty} \sum_{k=-N}^{+\infty} \varepsilon_k \beta^{-k} \left( \begin{array}{c}
\hat{d} \frac{1-13-21\beta+6\beta^2}{7} \\
\hat{d} \frac{-13-21\beta+6\beta^2}{7} \\
\hat{d} \frac{-13-21\beta+6\beta^2}{7} \\
\end{array} \right) \beta^n \mod \mathbb{Z}^3,
\]
where $\hat{d} \in \{\pm 1\}$ and $n \in \mathbb{Z}$.

**Example 3.** Let $M = \begin{pmatrix} 3 & 4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Here $\beta$ is the positive root of $x^3 = 5x^2 + 4x + 1$.

By the result from [2], $\beta$ is finitary (see Introduction for the definition), and it is easy to guess that the fundamental units of the ring are $\beta$ and $1 + \beta$, i.e.,
\[
\mathcal{U}_\beta = \{ \pm \beta^n, \pm (1 + \beta)^n, \, n \in \mathbb{Z} \}.
\]
Besides, $\xi_0 = \frac{1}{-3\beta^2 + 4\beta^3 - 6\beta^4} = \frac{-13-21\beta+6\beta^2}{7}$. Hence any bijective arithmetic coding is either
\[
\varphi(\varepsilon) = \lim_{N \to +\infty} \sum_{k=-N}^{+\infty} \varepsilon_k \beta^{-k} \left( \begin{array}{c}
\hat{d} \frac{-13-21\beta+6\beta^2}{7} \\
\hat{d} \frac{-13-21\beta+6\beta^2}{7} \\
\hat{d} \frac{-13-21\beta+6\beta^2}{7} \\
\end{array} \right) \beta^n \mod \mathbb{Z}^3
\]
or
\[
\varphi(\varepsilon) = \lim_{N \to +\infty} \sum_{k=-N}^{+\infty} \varepsilon_k \beta^{-k} \left( \begin{array}{c}
\hat{d} \frac{-13-21\beta+6\beta^2}{7} (1 + \beta)^n \\
\hat{d} \frac{-13-21\beta+6\beta^2}{7} (1 + \beta)^{n-1} \\
\hat{d} \frac{-13-21\beta+6\beta^2}{7} (1 + \beta)^{n-2} \\
\end{array} \right) \beta^n \mod \mathbb{Z}^3,
\]
where $\hat{d} \in \{\pm 1\}$ and $n \in \mathbb{Z}$.

**Example 4.** Finally, let $M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Here $\beta$ satisfies $x^4 = x^3 + 1$. Let us show that $\beta$ is weakly finitary. A direct inspection shows that the only nonzero period for the positive elements of $\mathbb{Z}[\beta]$ is $100000$. Hence $Z_\beta = \{ 0, \beta^{-2}, \beta^{-3}, \beta^{-4}, \beta^{-5}, \beta^{-6}, \beta^{-7} \}$. Let, for example, $x = \beta^{-2} + \beta^{-3}$; since $x + \beta^{-7} = \beta^{-1} + \beta^{-3} = \beta^{-1} + \beta^{-4} + \beta^{-7} = 1 + \beta^{-7} \in \text{Fin}(\beta)$, we have by periodicity $x + \beta^{-5n} \in \text{Fin}(\beta)$ for any $n \geq 1$. The other cases of $\alpha \in Z_\beta$ are similar. Hence $\beta$ is weakly finitary and we can apply Theorem 9. It suffices to compute $\mathcal{U}_\beta$; by the Dirichlet Theorem, it must be “two-dimensional” and it is easy to guess that the second fundamental unit (besides $\beta$ itself) is $1 + \beta$. Hence $\mathcal{U}_\beta = \{ \pm \beta^n, \pm (1 + \beta)^n, \, n \in \mathbb{Z} \}$ and the formula for a bijective arithmetic coding can be derived similarly to the previous examples in view of $\xi_0 = \frac{1}{-3\beta^2 + 4\beta^3 - 6\beta^4} = \frac{-13-15\beta^2 + 8\beta^3}{283}$.
4 General arithmetic codings and related algebraic results

In this section we will present some results for the case when \( t \) is not necessarily fundamental or \( T \) is not algebraically conjugate to the companion matrix automorphism. We will still assume \( \beta \) to be weakly finitary. Let us begin with the case \( T = T_\beta \) with a general \( t \). We recall that there exists a natural isomorphism between the homoclinic group \( \mathcal{H}(T) \) and the group \( \mathcal{P}_\beta \), i.e., \( t \leftrightarrow \xi \). Let \( \varphi_\xi : X_\beta \to \mathbb{T}^m \) be defined as above:

\[
\varphi_\xi(\bar{z}) = \varphi_t(\bar{z}) = \sum_{k \in \mathbb{Z}} \varepsilon_k T_\beta^{-k} t = \lim_{N \to +\infty} \left( \sum_{k=-N}^{\infty} \varepsilon_k \beta^{-k} \right) \left( \begin{array}{c} \xi \\ \xi \beta^{-1} \\ \vdots \\ \xi \beta^{-m+1} \end{array} \right) \mod \mathbb{Z}^m,
\]

where \( \xi = \xi(t) \in \mathcal{P}_\beta \). The question is, what will be the value of \( \# \varphi^{-1}_\xi(x) \) for a \( \mathcal{L}_m \)-generic \( x \in \mathbb{T}^m \)?

The next assertion answers this question; it is a generalization of the corresponding result proven in [25] for \( m = 2 \) and for a finitary \( \beta \) in [23]. Let \( D = D(\beta) \) denote the discriminant of \( \beta \) in the field extension \( \mathbb{Q}(\beta)/\mathbb{Q} \), i.e., the product \( \prod_{i \neq j} (\beta_i - \beta_j)^2 \), where \( \{\beta_1 = \beta, \beta_2, \ldots, \beta_m\} \) are the Galois conjugates of \( \beta \).

**Theorem 23** For an a.e. \( x \in \mathbb{T}^m \) with respect to the Haar measure,

\[
\# \varphi^{-1}_\xi(x) \equiv |DN(\xi)|,
\]

where \( N(\cdot) \) denotes the norm of an element of the extension \( \mathbb{Q}(\beta)/\mathbb{Q} \).

**Proof.** Let \( \varphi_0 \) denote the bijective arithmetic coding for \( T_\beta \) parameterized by \( \xi_0 \) and \( \ell := \xi/\xi_0 \in \mathbb{Z}[\beta] \). If \( \ell = \sum_{i=0}^{m-1} c_i \beta^i \), then one can consider the mapping \( A_\ell := \varphi_\xi \varphi_0^{-1} : \mathbb{T}^m \to \mathbb{T}^m \); it will be well defined on the dense set and we can extend it to the whole torus. By the linearity of both maps, \( A_\ell \) is a toral endomorphism. Thus, we have

\[
\varphi_\xi = A_\ell \varphi_0.
\]  

Let \( A_\ell \) is given by the formula \( A_\ell' = \sum_{i=0}^{m-1} c_i T_\beta^i \). For the basis sequence \( f^{(0)} = \)
$(\ldots, 0, 0, \ldots, 0, 1, 0, \ldots, 0, 0, \ldots)$ with the unity at the first coordinate we have

$$
(A_{\xi} \phi_0)(f^{(0)}) = A_{\xi}(\xi_0, \xi_0 \beta^{-1}, \ldots, \xi_0 \beta^{-m+1}) \mod \mathbb{Z}^m
$$

$$
= \sum_{i=0}^{m-1} c_i T^i(\xi_0, \xi_0 \beta^{-1}, \ldots, \xi_0 \beta^{-m+1}) \mod \mathbb{Z}^m
$$

$$
= \sum_{i=0}^{m-1} c_i \beta^i(\xi_0, \xi_0 \beta^{-1}, \ldots, \xi_0 \beta^{-m+1}) \mod \mathbb{Z}^m
$$

$$
= (\xi, \xi \beta^{-1}, \ldots, \xi \beta^{-m+1}) \mod \mathbb{Z}^m = \varphi_{\xi}(f^{(1)}).
$$

Therefore, by the linearity and continuity, we have $A_{\xi} = A'_{\xi} = \sum_{i=0}^{m-1} c_i T^i_{\beta}$. As $\varphi_0$ is 1-to-1 a.e., $\varphi_{\xi}$ will be $K$-to-$1$ a.e. with $K = |\det A_{\xi}|$. By definition, $N(\ell)$ is the determinant of the matrix of the multiplication operator $x \mapsto \ell x$ in the standard basis of $\mathbb{Q}(\beta)$, whence $N(\ell) = \det A_{\xi}$, because $T_{\beta}$ is given by the companion matrix. Finally, $N(\ell) = N(\xi)/N(\xi_0) = DN(\xi)$, as by the result from [20, Section 2.7], $N(\xi_0) = 1/D$ whenever $\xi_0$ is as in formula (7). $\blacksquare$

Note that historically the first attempt to find an arithmetic coding for a Pisot automorphism (actually, even for a Pisot endomorphism, i.e., an endomorphism of a torus with the same property as a Pisot automorphism) was undertaken in [5]. The author considered the case $T = T_{\beta}$ and $t$ given by the $\mathbb{R}^m$-coordinate $s = (1, \beta^{-1}, \ldots, \beta^{-m+1})$. From the above theorem follows

**Corollary 24** The mapping

$$
\varphi_t(\xi) = \lim_{N \to +\infty} \sum_{k=-N}^{+\infty} \varepsilon_k \beta^{-k} \begin{pmatrix}
1 \\
\beta^{-1} \\
\vdots \\
\beta^{-m+1}
\end{pmatrix}
$$

is $|D|$-to-$1$ a.e.

Suppose now $T$ is not necessarily algebraically conjugate to $T_{\beta}$. Let $M$ be, as usual, the matrix of $T$, and for $n \in \mathbb{Z}^m$ the matrix $B_M(n)$ be defined as follows (we write it column-wise):

$$
B_M(n) = (Mn, (M^2 - k_1 M)n, (M^3 - k_1 M^2 - k_2 M)n, \ldots, \\
M^{m-1} - k_1 M^{m-2} - \cdots - k_{m-2} M)n, k_m n).
$$

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Lemma 25 Any integral square matrix satisfying the relation

$$BM_{\beta} = MB$$  \hspace{1cm} (18)

is $B = B_M(n)$ for some $n \in \mathbb{Z}^m$.

Proof. Let $B$ be written column-wise as follows: $B = (n_1, \ldots, n_m)$. Then by (18) and the definition of $M_{\beta}$,

$$(k_1n_1 + n_2, k_2n_1 + n_3, \ldots, k_{m-1}n_1 + n_m, k mn_1) = (Mn_1, \ldots, Mn_m),$$

whence by the fact that $k_m = \pm 1$, we have $B = B_M(n)$ for $n = k_m n_m$. □

Definition 26 The integral $m$-form of $m$ variables defined by the formula

$$f_M(n) = \det B_M(n)$$

will be called the form associated with $T$.

Proposition 27 Let $t \in \mathcal{H}(T)$. Then there exists $n \in \mathbb{Z}^m$ such that

$$\#\varphi^{-1}_t(x) \equiv |f_M(n)|$$

for $\mathcal{L}_m$-a.e. point $x \in \mathbb{T}^m$.

Proof. Let $\tilde{B} := \varphi_t \varphi_0^{-1}$, where $\varphi_0$ is a certain bijective arithmetic coding for $T_\beta$. Then $\tilde{B}$ is a linear mapping from $\mathbb{T}^m$ onto itself defined a.e.; let the same letter denote the corresponding toral endomorphism. Then $\tilde{B}T_{\beta} = \varphi_t \varphi_0^{-1}T_\beta = \varphi_t \sigma_\beta \varphi_0^{-1} = T \varphi_t \varphi_0^{-1} = T \tilde{B}$. Therefore the matrix $B$ of the endomorphism $\tilde{B}$ satisfies (18), whence by Lemma 25, $B = B_M(n)$ for some $n \in \mathbb{Z}^m$. Hence $\varphi_t = B_M(n) \varphi_0$, and we are done. □

As a consequence we obtain

Theorem 28 The minimum of the number of pre-images for an arithmetic coding of a given automorphism $T$ equals the arithmetic minimum of the associated form $f_M$.

Remark 29 It would be helpful to know whether there is any relationship between the $n$ in the proposition and the $\mathbb{Z}^m$-coordinate of $t$.

Theorem 30 The following conditions are equivalent:

1. A Pisot automorphism $T$ admits a bijective arithmetic coding.
2. $T$ is algebraically conjugate to $T_{\beta}$.

3. The equation
   \[ f_M(n) = \pm 1 \]
   has a solution in $n \in \mathbb{Z}^m$.

4. There exists a homoclinic point $\mathbf{t}$ such that for its $\mathbb{Z}^m$-coordinate $\mathbf{n}$,
   \[ \langle M^k \mathbf{n} \mid k \in \mathbb{Z} \rangle = \mathbb{Z}^m. \]

**Proof.** (2)$\Rightarrow$(1): see Remark 8;
(1)$\Rightarrow$(2): see the Proposition 27;
(2)$\Leftrightarrow$(3): also follows from Proposition 27;
(2)$\Leftrightarrow$(4): it is obvious that $M_{\beta}$ satisfies this property (take $\mathbf{n} = (0, 0, \ldots, 0, 1)$).

Hence so does any $M$ which is conjugate to $M_{\beta}$.  

Recall that a matrix $M \in GL(m, \mathbb{Z})$ is called *primitive* if there is no matrix $K \in GL(m, \mathbb{Z})$ such that $M = K^n$ for $n \geq 2$. Following [25], we ask the following question: can a Pisot toral automorphism given by a non-primitive matrix admit a bijective arithmetic coding?

Note first that one can simplify the formula for $f_M$. Namely, since the determinant of a matrix stays unchanged if we multiply one column by some number and add to another column, we have

\[ f_M(n) = \pm \det(n, Mn, \ldots, M^{m-1}n). \quad (19) \]

**Proposition 31** There exists a sequence of integers $N_n(\beta)$ such that

\[ f_M^* = \pm N_n(\beta) \cdot f_M. \]

More precisely,

\[ N_n(\beta) = \det \begin{pmatrix} a_n^{(1)} & \cdots & a_n^{(m)} \\ a_n^{(1)} & \cdots & a_n^{(m)} \\ \vdots & \ddots & \vdots \\ a_1^{(m)} & \cdots & a_1^{(m)} \end{pmatrix}, \]

where $\{a^{(j)}_n\}_{j=1}^m$ are defined as the coefficients of the equation

\[ \beta^n = a_n^{(1)}\beta^{m-1} + a_n^{(2)}\beta^{m-2} + \cdots + a_n^{(m-1)}\beta + a_n^{(m)} \]

derived from (6).
Proof. By (19), the definition of $a_n^{(j)}$ and the Hamilton-Cayley Theorem,
\[
f_{M^n}(n) = \pm \det (n, M^n n, M^{2n} n, \ldots, M^{(m-1)n} n) \\
= \pm \det \left( n, \left( \sum_{j=1}^{m} a_n^{(j)} M^{m-j} \right) n, \ldots, \left( \sum_{j=1}^{m} a_{(m-1)n}^{(j)} M^{m-j} \right) n \right) \\
= \pm \mathcal{N}_n(\beta) \cdot \det (n, M^r n, M^2 n, \ldots, M^n n).
\]

\[\Box\]

Corollary 32 A non-primitive matrix $M^n \in GL(m, \mathbb{Z})$ is algebraically conjugate to the corresponding companion matrix if and only if so is $M$, and $\mathcal{N}_n(\beta) = \pm 1$.

Let us deduce some corollaries for smaller dimensions.

Corollary 33 (see [25]) For $m = 2$ the automorphism given by a non-primitive matrix $M^n$, $n \geq 1$ admits a bijective arithmetic coding if and only if $n = 2$ and $Tr(M) = \pm 1$.

Corollary 34 For $m = 3$ the matrix $K = M^2, M \in GL(3, \mathbb{Z})$, is algebraically conjugate to the corresponding companion matrix if and only if $\beta$ satisfies one of the following equations:

1. $\beta^3 = r \beta^2 + 1$, $r \geq 1$;
2. $\beta^3 = r \beta^2 - 1$, $r \geq 3$;
3. $\beta^3 = 2\beta^2 - \beta + 1$.

Proof. We have $\mathcal{N}_2(\beta) = \det \begin{pmatrix} 1 & k_1^2 + k_2 \\ 0 & k_1 k_2 + k_3 \end{pmatrix} = k_1 k_2 + k_3 = \pm 1$. The case $k_3 = +1$ thus leads to subcases 1 and 3 and $k_3 = -1$ yields subcase 2. $\Box$

Note that if $M$ is the matrix for the “tribonacci automorphism” (see Example 2), then apparently the only power of $M$ greater than 1 that is algebraically conjugate to the corresponding companion matrix, is the cube! Indeed, $\mathcal{N}_2(\beta) = 2, \mathcal{N}_3(\beta) = -1, \mathcal{N}_4(\beta) = -8, \mathcal{N}_5(\beta) = 29$, etc. It seems to be an easy exercise to prove this rigorously; we leave it to the reader.

Example 5. Let $M = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Here $\beta$ satisfies $x^3 = 5x^2 - 4x + 1$ and the form associated with $M$ is (we write $n = (x, y, z)'$)

\[
f_M(x, y, z) = x^3 + 2x^2 z - xy^2 - xyz + 3xz^2 + y^3 - 3yz^2 + 2yz^2 + z^3.
\]
Obviously, the Diophantine equation \( f_M(x, y, z) = \pm 1 \) has a solution, namely, \( x = 1, y = z = 0 \). Hence by Theorem 30, \( M \) is algebraically conjugate to \( M_\beta \); for example, \( B = B_M(1, 0, 0) = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \) conjugates them. To show that \( T \) admits a bijective arithmetic coding, it suffices to check that \( \beta \) is weakly finitary. A direct inspection shows that the set of periods for the elements of \( Z_\beta \) here is \( \{0, 1, 2, 3\} \) and the “universal neutralizing word” (see Introduction) is 13. The author would like to thank Sh. Akiyama for this computation.

In [25] the author together with A. Vershik considered the case \( m = 2 \). Here if \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), then for \( \sigma = \det M = \pm 1 \),

\[
 f_M(x, y) = \begin{vmatrix} ax + by & -\sigma x \\ cx + dy & -\sigma y \end{vmatrix} = \sigma (cx^2 - (a - d)xy - by^2),
\]

and we related the problem of arithmetic codings to the classical theory of binary quadratic forms. In particular, \( T \) admits a bijective arithmetic coding if and only if the Diophantine equation

\[
 cx^2 - (a - d)xy - by^2 = \pm 1
\]
is solvable.

The theory of general \( m \)-forms of \( m \) variables does not seem to be well developed; nonetheless, we would like to mention a certain algebraic result which looks relevant. Recall that two integral forms are called equivalent if there exists a unimodular integral change of variables turning one form into another.

**Proposition 35** Let \( M_1, M_2 \) in \( GL(m, \mathbb{Z}) \) be algebraically conjugate, and

\[
 AM_1 A^{-1} = M_2,
\]

where \( A \in GL(m, \mathbb{Z}) \). Then \( f_{M_1} \) is equivalent either to \( f_{M_2} \) or to \(-f_{M_2} \), and moreover,

\[
 A'^T f_{M_2} A = \det A \cdot f_{M_1},
\]

where \( A' \) is the transpose of \( A \) (we identify a form with the symmetric matrix which defines it).

**Proof.** Since \( M_1 \) and \( M_2 \) are conjugate, they have one and the same characteristic polynomial. By the definition of \( f_M \) we have

\[
 f_{M_2}(A\mathbf{v}) = \det(M_2 A\mathbf{v}, (M_2^2 - k_1 M_2)A\mathbf{v}, \ldots, A\mathbf{v}) = \det(A M_1 \mathbf{v}, A(M_1^2 - k_1 M_1)\mathbf{v}, \ldots, A\mathbf{v}) = \det A \cdot f_{M_1}(\mathbf{v}),
\]

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which is equivalent to (20). ■

In the end of the paper we would like to present two examples of arithmetic codings for higher-rank actions on tori that can be obtained as a consequence of the main construction of this paper. We refer the reader to the recent work [14] for the necessary definitions and references.

Let the automorphism $T$ of $\mathbb{T}^3$ is given by the matrix from Example 3, namely, $M = \begin{pmatrix} 3 & 4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Let $\beta$ denote the leading eigenvalue. Then the group of units for $\mathbb{Z}[\beta]$ is generated by $\beta$ itself and $u = \beta + 1$ (which is also a Pisot unit). Let $T'$ be given by $M' = M + E = \begin{pmatrix} 4 & 4 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in GL(3, \mathbb{Z})$; it is also a Pisot matrix algebraically conjugate to the corresponding companion matrix. Then the (Cartan) action generated by $M$ and $M'$ can be encoded by a certain action on $X_\beta$; namely, let $\varphi : X_\beta \rightarrow \mathbb{T}^3$ denote a bijective arithmetic coding of $T$ and $\sigma'_\beta : X_\beta \rightarrow X_\beta$ be given by the formula

$$\sigma'_\beta(\xi) := \sigma_\beta(\xi) + \xi.$$

It is easy to see that $\sigma'_\beta$ is well defined for almost all the sequences in $X_\beta$ - as $\beta$ is finitary, $\sigma'_\beta$ will be well defined for any sequence having the block of $L$ zeroes infinitely many times to the left, where $L$ is large enough. Besides, $\varphi$ conjugates the action $(\sigma_\beta, \sigma'_\beta)$ on $X_\beta$ and the Cartan action $(T, T')$ on the 3-torus. It is worth noting that for this example the homoclinic groups for $T$ and $T'$ coincide or, in terms of the ring, $Fin(\beta) = Fin(u)$.

One may argue that both generators of this action are both generalized Pisot automorphisms. This is true but let us give another example, this time of a Cartan action on $\mathbb{T}^4$.

Namely, let $M = \begin{pmatrix} 4 & 0 & -3 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Note $M$ is a companion matrix, and its spectrum is purely real. Now take the action generated by $M_1 = M, M_2 = M + E$ and $M_3 = M - E$. It is easy to check that they all belong to $GL(4, \mathbb{Z})$ and that this will yield a Cartan action on $\mathbb{T}^4$ as well as the fact that the dominant eigenvalue $\beta$ of $M$ is indeed weakly finitary. We leave the details to the reader. Therefore, the usual encoding mapping $\varphi$ conjugates the action $(\sigma_\beta, \sigma_\beta + id, \sigma_\beta - id)$ on the compactum $X_\beta$ and the Cartan action $(T_1, T_2, T_3)$. However, $T_3$ has two eigenvalues inside the unit disc and two outside. Perhaps, this is the first ever explicit bijective a.e. encoding of a non-generalized Pisot automorphism (though not by means of a shift).
We believe that the underlying ideas of these examples can be extended to more general Cartan actions; however, there are obvious problems that arise in doing so. They are as follows:

1. to show that any Pisot unit with the real conjugates is weakly finitary (see Introduction);

2. to find out whether a given Cartan action contains a Pisot automorphism whose matrix is conjugate to its companion matrix (of course, every Cartan action is known to contain some Pisot automorphism);

3. if so, find it in such a way that a given element of $\alpha$ is a linear integral combination of the powers of this Pisot automorphism. For instance, this is true if the ring $\mathbb{Z}[\beta]$ is the maximal order of the field $\mathbb{Q}(\beta)$, where $\beta$ is, as usual, the dominant eigenvalue of the matrix of the Pisot automorphism in question – see [14].

We plan to develop this direction elsewhere.

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