Groups of Hierarchomorphisms of Trees
and Related Hilbert Spaces

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0.1. Hierarchomorphisms (spheromorphisms). The Bruhat–Tits tree $T_p$ is an infinite tree such that any vertex belongs to $(p+1)$ edges. As was observed by Cartier [3], the groups $\text{Aut}(T_p)$ of automorphisms of the trees $T_p$ are analogues of real and $p$-adic groups of rank 1 (as $\text{SL}_2(\mathbb{R})$, $\text{SL}_2(\mathbb{C})$, $O(1,n)$, $\text{SL}_2(\mathbb{Q}_p)$ etc.). The representation theory of $\text{Aut}(T_p)$ was developed in Cartier’s [3] and Olshansky’s [25] papers. In fact, the group $\text{Aut}(T_p)$ is essentially simpler than the rank 1 groups on locally compact fields, but many nontrivial phenomena related to rank 1 groups survive for the group of automorphisms of Bruhat–Tits trees.

The absolute of the Bruhat–Tits tree is an analogue of the boundaries of rank 1 symmetric spaces, in particular, the absolute is an analogue of the circle. The group of hierarchomorphisms $\text{Hier}(T_p)$ (defined in [20]) is a tree analogue of the group $\text{Diff}(S^1)$ of diffeomorphisms of the circle. The group $\text{Hier}(T_p)$ consists of homeomorphisms of the absolute of $T_p$ that can be extended to the whole Bruhat–Tits tree except a finite subtree. It turns out to be ([20], [21]), that the representation theory of $\text{Diff}(S^1)$ partially survives for the groups $\text{Hier}(T_p)$.

In fact, the group $\text{Hier}(T_p)$ contains the group of locally analytic diffeomorphisms of $p$-adic line (see [21]), and this partially explains the similarity of $\text{Diff}(S^1)$ and $\text{Hier}(T_p)$.

The following facts 1°-4° are known about the groups $\text{Hier}(T_p)$. The phenomena 1°-3° are an exact reflection of the representation theory of $\text{Diff}(S^1)$, the last phenomenon now does not have a visible real analogue.

1°. ([20], [21]) Denote by $O(\infty)$ the group of all orthogonal operators in a real Hilbert space $H$. Denote by $\text{GLO}(\infty)$ the group of all invertible operators in $H$ having the form $A = B + T$, where $B \in O(\infty)$ and $T$ has finite rank. Denote by $H_{\mathbb{C}}$ the complexification of $H$. Denote by $UO(\infty)$ the group of all unitary operators in $H_{\mathbb{C}}$ having the form $A = B + T$, where $B \in O(\infty)$ and $T$ has finite rank. There exist some series of embeddings

$$\text{Hier}(T_p) \to \text{GLO}(\infty); \quad \text{Hier}(T_p) \to UO(\infty).$$

This allows to apply the second quantization machinery (see [27], [22], [24]) for obtaining unitary representations of $\text{Hier}(T_p)$.

2°. Embeddings $\text{Hier}(T_p) \to \text{GLO}(\infty)$ allow to develop a theory of fractional diffusions with a Cantor set time (the Cantor set appears as the absolute of the

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2° In [21], there was proposed the term ‘ball-morphisms’, which is difficult for pronunciation. In English translation, it was replaced by ‘spheromorphisms’. I want to propose the neologism ‘hierarchomorphisms’; this is a map regarding hierarchy of balls on the absolute; see below Subsection 5.1.

3° Another heuristic explanation can be obtain by the monstrous degeneration construction from [17], chapter 9; the Lobachevsky plane can be degenerated to the universal $\mathbb{R}$-tree.
I never wrote a text on this topic, but, on the whole, the picture here is quite parallel to fractional diffusions with real time (see, [23]).

3°. (Kapoudjian, [13], [16]) There exists a $\mathbb{Z}/2\mathbb{Z}$-central extension of $\text{Hier}(T_p)$.

4°. (Kapoudjian, [15]) Consider the dyadic Bruhat–Tits tree $T_2$. There exists a canonical action of the group $\text{Hier}(T_2)$ on the inductive limit of the Deligne–Mumford [5] moduli spaces $\lim_{n \to \infty} \mathcal{M}_{n,2^n}$ of $2^n$ point configurations on the Riemann sphere. This construction also has two versions over $\mathbb{R}$. The first variant is an action on the inductive limit of Stasheff associahedrons ([35]). The second variant is an action on the inductive limit of the spaces constructed by Davis, Janiszewicz, Scott ([4]). The last case is most interesting, since this real space has an interesting topology.

0.2. The purposes of this paper. This paper has two purposes. The first aim is to construct a new series of embeddings of the groups of hierarchomorphisms to the group $\text{GLO}(\infty)$. By the Feldman-Hajek theorem (see [34]), this gives constructions of unitary representations of groups of hierarchomorphisms, but we do not discuss this subject.

There exists the wide and nice theory of actions of groups on trees (see [31], [32], [33], [17]). It is clear that a hierarchomorphism type extension can be constructed for any group $\Gamma$ acting on a tree (and even on an $\mathbb{R}$-tree), it is sufficient to allow to cut a finite collection of edges. The second purpose of this paper is to understand, is this "hierarchomorphization" of arbitrary group $\Gamma$ a reasonable object?

One example of such "hierarchomorphization" is quite known, this is the Richard Thompson group $[18]$, which firstly appeared as an counterexample in theory of discrete groups. Later it became clear, that this group is not a semipathological counterexample, but a rich and unusual object (see works of Greenberg, Ghys, Sergiescu, Penner, Freyd, Heller and others [10], [11], [28], [9], [1] see also [2]), relation of the hierarchomorphisms and the Thompson group was observed by Sergiescu).

If the group $\Gamma$ is discrete, then the corresponding group of hierarchomorphisms is a discrete Thompson-like group. If the group $\Gamma$ is locally compact, then the group of hierarchomorphisms (see some examples in [21]) is an "infinite dimensional group" (or, better, "large group") similar to the group of diffeomorphisms of the circle or diffeomorphisms of $p$-adic line.

0.3. The structure of the paper. Sections 1-2 contain preliminary definitions and examples.

In Section 3, we define the groups of hierarchomorphisms of tree (this definition can be adapted also for $\mathbb{R}$-tree, but nontrivial constructions of Sections 4-6 do not survive in this case).

In Section 4, we discuss a family of Hilbert spaces $\mathcal{H}_\lambda(J)$, where $0 < \lambda < 1$, associated with a tree $J$. The space $\mathcal{H}_\lambda(J)$ contains the (nonorthogonal) basis $e_a$ enumerated by vertices $a$ of the tree, and the inner products of the vectors

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*see also the recent preprint of Nekrashevich [19].*
\( \epsilon_a, \epsilon_b \) are given by

\[ \langle \epsilon_a, \epsilon_b \rangle = \lambda \text{distance between } a \text{ and } b \]

We show that the group of hierarchomorphisms of \( J \) acts in \( \mathcal{H}_\lambda(J) \) by operators of the class \( \text{GLO}(\infty) \).

In Section 5, for sufficiently large \( \lambda \) we construct an operator of the 'restriction to the absolute' in the space \( \mathcal{H}_\lambda \).

In Section 6, we discuss the action of the group of hierarchomorphisms in spaces of functions (distributions) on the absolute.

The results of Sections 4–5 are 'new' for the groups of hierarchomorphisms of the Bruhat–Tits trees. The construction of Section 6 in for Bruhat–Tits trees coincides with [20].

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1. Notation and terminology

1.1. Simplicial trees. A simplicial tree \( J \) is a connected graph without circuits.

By \( \text{Vert}(J) \) we denote the set of vertices of \( J \).

By \( \text{Edge}(J) \) we denote the set of edges of \( J \).

We say that two vertices \( a, b \in \text{Vert}(J) \) are adjacent, if they are connected by an edge. We denote this edge by \([a, b] \).

We assume that the sets \( \text{Vert}(J), \text{Edge}(J) \) are countable or finite. A simplicial tree is locally finite if any vertex \( a \) belongs to finitely many edges. We admit non locally finite trees.

A way in \( J \) is a sequence of distinct vertices

\[ \ldots, a_1, a_2, a_3, \ldots \]

such that \( a_j, a_{j+1} \) are adjacent. A way can be finite, or infinite to one side, or infinite to the both sides.

For vertices \( a, b \) there exists a unique way \( a_0 = a, a_1, \ldots, a_k = b \) connecting \( a \) and \( b \). We say that \( k \) is the simplicial distance between \( a \) and \( b \). We denote the simplicial metrics by

\[ d_{\text{symp}}(a, b) \]

A subtree \( I \subset J \) is a connected subgraph in the tree \( J \).

The boundary \( \partial I \) of a subtree \( I \subset J \) is the set of all \( a \in \text{Vert}(I) \) such that there exists an edge \([a, b] \) with \( b \notin \text{Vert}(I) \).

A subtree \( I \subset J \) is right, if the number of edges \([a, b] \in \text{Edge}(J) \) such that \( a \in I, b \notin I \) is finite.

A subtree \( I \subset J \) is a branch if there is a unique edge \([a, b] \in \text{Edge}(J) \) such that \( a \in \text{Vert}(I), b \notin \text{Vert}(I) \), see Picture 1. The vertex \( a \) is called a root of the branch. If we delete an edge of the tree \( J \), then we obtain two branches.
A subtree \( I \subset J \) is a \textit{bush} if its boundary contain only one point \( a \) (a root) and number of edges \([a, b] \in \text{Edge}(J)\) such that \( b \notin I \) is finite, see Figure 1.

\textbf{Lemma 1.1.} \ \textit{a)} The intersection of a finite family of right subtrees is a right subtree.

\textit{b)} For a right subtree \( I \subset J \), there exists a finite collection of edges \( \ell_1, \ldots, \ell_k \in \text{Edge}(I) \) such that \( I \) without \( \ell_1, \ldots, \ell_k \) is a union of bushes.

\textbf{Proof.} The statement \textit{a)} is obvious.

The statement \textit{b)}. Let \( a_1, \ldots, a_k \) be the boundary of \( I \). Let \( L \subset I \) be the minimal subtree containing the vertices \( a_1, \ldots, a_k \). It is sufficient to delete all edges of \( L \).

\[ \square \]

\begin{figure}%
\centering
\includegraphics[width=\textwidth]{tree_diagram.png}
\caption{A branch and A bush}
\end{figure}

We say that a tree \( J \) is \textit{perfect} if any vertex of \( J \) belongs to \( \geq 3 \) edges. Obviously, perfect trees are infinite.

\textbf{1.2. Actions of groups on simplicial trees.} A bijection \( \text{Vert}(J) \to \text{Vert}(J) \) is an \textit{automorphism} of a simplicial tree \( J \) if the images of adjacent vertices are adjacent vertices.

An action of a group \( \Gamma \) on a simplicial tree is an embedding of \( \Gamma \) to its group of automorphisms.

\textbf{1.3. Absolute.} The absolute \( \text{Abs}(J) \) of a tree is the set of points of the tree on infinity. Let us give the formal definition.

We say that a \textit{ray} is an infinite way \( a_1, a_2, \ldots \). We say that rays \( a_1, a_2, \ldots \) and \( b_1, b_2, \ldots \) are equivalent if there exist \( k \) and a sufficiently large \( N \) such that \( b_j = a_{j+k} \) for all \( j \geq N \).

A point of an absolute is a class of equivalent ways.
1.4. **Metric trees.** Let $J$ be a simplicial tree. Let us assign a positive number $\rho(a, b)$ to each edge $[a, b]$. Let $a, c$ be arbitrary vertices of $J$, let $a_0 = a, a_1, \ldots, a_k = c$ be the way connecting $a$ and $c$. We assume

$$\rho(a, c) = \sum_{j=1}^{k} \rho(a_{j-1}, a_j)$$

Obviously, $\rho$ is a metric on Vert$(J)$. We call by metric trees the **countable** spaces Vert$(J)$ equipped with the metrics $\rho$.

Obviously, the edges of $J$ can be reconstructed using the metric $\rho$. Hence we prefer to think that the edges are present in a metric tree as an additional (combinatorial) structure.

**Remark.** We also can assume that lengths of all edges is 1, and thus a simplicial tree is a partial case of metric trees.

**Remark.** In literature, sometimes the term metric tree is used in the quite different sense (for $\mathbb{R}$-trees).

A metric tree $J$ is **locally finite** if it is locally finite as a simplicial tree and for any $a \in$ Vert$(J)$ and each $C > 0$ the set of vertices $b$ satisfying $\rho(a, b) < C$ is finite.

1.5. **Actions of groups on metric trees.** Let $J$ be a metric tree. A bijection Vert$(J) \to$ Vert$(J)$ is an **automorphism** of $J$ if it preserves the distance (hence it automatically preserves the structure of simplicial tree).

An action of group $\Gamma$ on a metric tree $J$ is an embedding of $\Gamma$ to the group of automorphisms of $J$.

2. **Examples of actions of groups on trees.**

The purpose of this Section is to give a collection of examples for abstract constructions given in Sections 3-6 (all these examples are standard). For algebraic and combinatorial theory of actions of groups on trees, see [31], [32], [33].

2.1. **Bruhat–Tits trees.** The Bruhat–Tits tree $T_p$ is the tree, in which each vertex belongs to $(p + 1)$ edges. The group Aut$(T_p)$ of automorphisms of $T_p$ is a locally compact group. This group is similar to rank 1 groups over $\mathbb{R}$ and over $p$-adic fields. The representation theory of Aut$(T_p)$ and related harmonic analysis are well understood, see [3], [25], [6], [7].

2.2. **The tree $T_\infty$.** We denote by $T_\infty$ the simplicial tree, in which each vertex belongs to a countable set of edges. At first sight, the group Aut$(T_\infty)$ seems pathological. Nevertheless, it is a useful object as one of the simplest examples of infinite-dimensional groups, see [26], [24]. This group is an imitation of the group $O(1, \infty)$.

2.3. **The tree of free group.** Denote by $F_2$ the free group with two generators $a, \beta$. Vertices of the tree $J(F_2)$ are numerated by elements of the group $F_2$. Vertices $v_p, v_q$ are connected by an edge if

$$p = qa^{\pm 1} \quad \text{or} \quad p = q\beta^{\pm 1}.$$
Obviously, \( J(F_2) \) is the Bruhat–Tits tree \( \mathcal{T}_3 \). The group \( F_2 \) acts on the tree \( J(F_2) \) by the transformations

\[
 r : \quad v_p \mapsto r v_p \quad \text{where} \quad r \in F_2.
\]

Fix \( l_1, l_2 > 0 \). Assign the length \( l_1 \) to any edge \([v_p, v_{p\alpha}]\), and the length \( l_1 \) to any edge \([v_p, v_{p\beta}]\). Thus we obtain a metric tree with an action of \( F_2 \).

2.4. Another tree of free group. Let us contract all the edges of the type \([v_p, v_{p\alpha}]\) of the tree \( J(F_2) \) described in 2.3. Thus, we obtain the action of \( F_2 \) on \( \mathcal{T}_\infty \).

2.5 Dyadic intervals. Vertices \( V_{u,n} \) of the tree \( J_2(\mathbb{R}) \) are enumerated by segments in \( \mathbb{R} \) having the form

\[
 s_{u,n} = \left[ \frac{u}{2^n}, \frac{u + 1}{2^n} \right], \quad \text{where} \quad u \in \mathbb{Z}, n \in \mathbb{Z}.
\]

We connect \( V_{u,n} \) and \( V_{u,n-1} \) by an edge if \( s_{u,n-1} \supset V_{u,n} \).

Obviously, we obtain the simplicial tree \( \mathcal{T}_2 \).

2.6. Balls on \( p \)-adic line. Denote by \( \mathbb{Q}_p \) the field of \( p \)-adic numbers, denote by \( \mathbb{Z}_p \) the \( p \)-adic integers. Denote by \( B_{a,k} \) the ball

\[
 |z - a| < p^{-k}.
\]

Remark. The radius \( p^{-k} \) is determined by the ball. But \( B_{a,k} = B_{e,k} \) for any \( e \in B_{a,k} \).

2.7. Tree of lattices. Consider the \( p \)-adic plane \( \mathbb{Q}_p^2 \) equipped with a skew symmetric bilinear form \( A(v, w) \). Denote by \( Sp_2(\mathbb{Q}_p) \) the group of linear transformations preserving the form \( A(v, w) \).

A lattice in \( \mathbb{Q}_p^2 \) is a compact subset \( R \subset \mathbb{Q}_p^2 \) having the form

\[
 \mathbb{Q}_p v \oplus \mathbb{Q}_p w; \quad \text{where} \quad v, w \text{ are not proportional}.
\]

We say that a lattice \( R \) is self-dual if

1. \( A(v, w) \in \mathbb{Z}_p \) for all \( v, w \) in \( R \)
2. if \( h \in \mathbb{Z}_p^2 \) satisfies \( A(h, v) \in \mathbb{Z}_p \) for all \( v \in R \), then \( h \in R \).

Vertices of the tree \( \mathcal{T}(\mathbb{Q}_p^2) \) are self-dual lattices. Two vertices \( R, S \) are connected by an edge if

\[
 \text{volume of } R \cap S = \frac{1}{p} \text{volume of } R.
\]

It can be shown that \( \mathcal{T}(\mathbb{Q}_p^2) \) is the Bruhat–Tits tree \( \mathcal{T}_p \). Obviously, the group \( Sp_2(\mathbb{Q}_p) \) acts on our tree by automorphisms.

2.8. Modular tree. Consider the following standard picture from arbitrary textbook on complex analysis. Consider the Lobachevsky plane \( L : \text{Im} \ z > 0 \) and the triangle \( \Delta \) with three vertices 0, 1, \( \infty \) on the absolute \( \text{Im} \ z = 0 \). Consider
the reflections of $\Delta$ with respect to the sides of $\Delta$. We obtain 3 new triangles $\Delta_1$, $\Delta_2$, $\Delta_3$. Then we consider the reflections of $\Delta_j$ with respect to their sides etc. We obtain a tiling of $L$ by infinite triangles (with vertices in rational points of the absolute $\text{Im} \ z = 0$).

Vertices of the modular tree are enumerated by the triangles of the tiling. Two vertices are connected by an edge, if the corresponding triangles have a common side.

The group $\text{SL}_2(\mathbb{Z})$ acts on the modular tree in the obvious way.

2.9 Tree of pants. Let $R$ be a compact Riemann surface. Fix a collection $C_1, \ldots, C_k$ of closed mutually disjoint geodesics on $R$. The universal covering of $R$ is the Lobachevsky plane.

The coverings of the cycles $C_j$ are geodesics on $L$. Thus we obtain the countable family of mutually disjoint geodesics on $L$. They divide $L$ into the countable collection of domains.

Now we construct a tree. Vertices of the tree are enumerated by the domains on $L$ obtained above. Two vertices are connected by an edge, if the corresponding domains have a common side.

The fundamental group $\pi_1(R)$ of the surface $R$ acts on this tree in the obvious way.

3. Hieromorphisms

3.1. Large group of hieromorphisms. Consider a group $\Gamma$ acting on a simplicial (or metric) tree $J$. Consider a partition of $J$ into a finite collection of right subtrees $S_1, \ldots, S_k$, i.e., the subtrees $S_j$ are mutually disjoint, and $\text{Vert}(J) = \bigcup \text{Vert} (S_j)$. Let

$$g_1 : S_1 \to J, \ldots, g_k : S_k \to J$$

be a collection of embeddings such that

1) the subtrees $g_j(S_j)$ are mutually disjoint;
2) $\bigcup \text{Vert}(g_j(S_j)) = \text{Vert}(J)$.

Thus we obtain the bijection

$$g = \{g_j, S_j\} : \text{Vert}(J) \to \text{Vert}(J)$$

given by

$$g(a) = g_j(a) \quad \text{if} \quad a \in \text{Vert}(S_j)$$

We call such maps hieromorphisms, see Picture 2. Denote the group of all such hieromorphisms by $\text{Hier}^*(J, \Gamma)$.

3.2. Action of hieromorphisms on absolute. Consider a hieromorphism $g = \{g_j, S_j\}$. Let $\omega \in \text{Abs}(J)$. Let $a_1, a_2, \ldots$ be a way leading to $\omega$. For a sufficiently large $N$ and for some $S_j$, we have $a_N, a_{N+1}, \ldots \in S_j$. Hence $g_j(a_N), g_j(a_{N+1}), \ldots \in g_j(S_j)$ is a way leading to some point

$$\nu \in \text{Abs} ((g_j(S_j)) \subset \text{Abs}(J).$$
Picture 2. An example of hierarchomorphism: a re-glueing of two branches

We assume
\[ v = g(\omega). \]

Fix a point \( \xi \in \text{Vert}(J) \). Under the previous notation, consider the sequence
\[ n_M = \rho(\xi, aM) - \rho(\xi, g(aM)). \]

This sequence becomes a constant after a sufficiently large \( M \). We denote this constant (the pseudoderivative) by
\[ n(g, \omega) = n_\xi(g, \omega). \]

The following statement is obvious.

**Proposition 3.1.** For \( g, h \in \text{Hier}^*(J, \Gamma), \omega \in \text{Abs}(J) \),
\[ n(gh, \omega) = n(h, \omega) + n(g, h\omega). \] (3.1)

### 3.3. Small group of hierarchomorphisms.

Denote by \( \text{Hier}(J, \Gamma) \) the group of transformations of the absolute induced by elements \( g \in \text{Hier}^*(J, \Gamma) \).

The kernel of the canonical map
\[ \text{Hier}^*(J, \Gamma) \rightarrow \text{Hier}(J, \Gamma) \]

consists of finite permutations of the set \( \text{Vert}(J) \).

Obviously, the pseudoderivative \( n(g, \omega) \) is well defined for \( g \in \text{Hier}(J, \Gamma) \).

### 3.4. A variant: planar hierarchomorphisms.

Assume a simplicial tree \( J \) be planar (this means, that for each vertex \( a \) we fix the cyclic order on the set of edges containing \( a \); it is the case in some of our examples. Then also we have a canonical cyclic order on the absolute.
Now we can consider the group of hierarchomorphisms that preserves the cyclic order on the absolute.

4. Hilbert spaces $\mathcal{H}_\lambda(J)$

4.1. Definition. Let $J$ be a metric tree, let $0 < \lambda < 1$. Denote by $\mathcal{H}_\lambda(J)$ the real Hilbert space spanned by the formal vectors $e_a$, where $a$ ranges in $\text{Vert}(J)$, with inner products given by

$$\langle e_a, e_b \rangle = \lambda^{\theta(a,b)}, \quad \forall a, b \in \text{Vert}(J). \quad (4.1)$$

We must show that a system of vectors with inner products (4.1) can be realized in a Hilbert space.

4.2. Existence of $\mathcal{H}_\lambda(J)$. Let $a$ be a vertex of $J$. Let $b_1, b_2, \ldots$ be the vertices adjacent to $a$. Consider an arbitrary unit vector $e_a$ in a real infinite dimensional Hilbert space $\mathcal{H}$. Consider a collection $L_{b_1}, L_{b_2}, \ldots$ of pairwise perpendicular two-dimensional planes\(^5\) containing $e_a$. For each plane $L_{b_k}$, we draw a vector $e_{b_k} \in L_{b_k}$ such that

$$\langle e_{b_k}, e_a \rangle = \lambda^{\theta(a,b_k)},$$

see Picture 3.

![Picture 3](image)

By the perpendicularity,

$$\langle e_{b_k}, e_{b_{k'}} \rangle = \langle e_{b_k}, e_a \rangle \cdot \langle e_a, e_{b_{k'}} \rangle = \lambda^{\theta(b_k,b_{k'})}.$$

Then we apply the following inductive process. Assume that for a subtree $S$ the required embedding $\text{Vert}(S) \to \mathcal{H}$ is constructed, i.e., we have a subspace $\mathcal{H}_\lambda(S) \subset \mathcal{H}$. Let $b \in \text{Vert}(J)$, and $c \notin \text{Vert}(J)$ be adjacent to $b$. Consider the two-dimensional plane $L_c \subset \mathcal{H}$ that contains $e_b$ and is perpendicular to $\mathcal{H}_\lambda(S)$. Let us draw a unit vector $e_c \in L_c$ such that

$$\langle e_c, e_b \rangle = \lambda^{\theta(b,c)}.$$

Thus we obtained the required embedding $\text{Vert}(S) \cup \{b\} \to \mathcal{H}.$

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\(^5\)Subspaces $M_1, M_2$ in a Hilbert space are perpendicular if there is an orthogonal system of vectors $u_1, u_2, \ldots, w_1, w_2, \ldots$ such that $M_1$ is spanned by the vectors $u_i, u_j$, and $M_2$ is spanned by the vectors $w_i, v_j$.\(^6\)
There is a sufficient place in the Hilbert space, and thus we obtain the embedding $\text{Vert}(J) \to \mathcal{H}$.

Remark. This geometric picture is especially pleasant, if lengths of all edges are equal.

4.3. More formal description of $\mathcal{H}_x(J)$. Consider an affine real infinite dimensional Hilbert space $\mathcal{K}$, i.e., a Hilbert space, where the origin of coordinates is not fixed. Denote by $\| \cdot \|$ the length in $\mathcal{K}$. Consider a collection of points $N_a \in \mathcal{K}$, where $a \in \text{Vert}(J)$, such that

1) if $[a, b], [c, d]$ are different edges of $J$, then $N_a N_b \perp N_c N_d$;
2) for $[a, b] \in \text{Edge}(J)$,

$$\|N_a N_b\|^2 = \rho(a, b).$$

The existence of such embedding is obvious.

By the Pythagoras theorem,

$$\|N_b N_c\|^2 = \rho(b, c) \quad \forall b, c \in \text{Vert}(J).$$

Now let us apply the following standard Fock-Schoenberg construction ([8], [29]). For an affine Hilbert space $\mathcal{K}$, there exists a linear Hilbert space $\text{Exp}(\mathcal{K})$ and an embedding $\phi : \mathcal{K} \to \text{Exp}(\mathcal{K})$ such that for all $X, Y \in \mathcal{K}$

$$\langle \phi(X), \phi(Y) \rangle = \exp(-\|XY\|^2).$$

Fix any origin of the coordinates in $\mathcal{K}$. We can assume that $\text{Exp}(\mathcal{K})$ is the direct sum of all symmetric powers of $\mathcal{K}$

$$\text{Exp}(\mathcal{K}) = \mathbb{R} \oplus \mathcal{K} \oplus S^3 \mathcal{K} \oplus S^6 \mathcal{K} \oplus \ldots,$$
and 
\[ \phi(X) = e^{-\|X\|^2} \left[ \frac{1}{1!} X_1 \oplus \frac{X_2^2}{2!} \oplus \frac{X_3^3}{3!} \oplus \ldots \right]. \]

It remains to apply the Fock–Schoenberg construction to the space \( \mathcal{K} \) constructed above. The vectors \( \phi(N_a) \) satisfy the relations (4.1).

Remark. The spaces \( \mathcal{H}_\lambda \) associated with a tree are present in Olshansky’s paper [26]. In an implicit form, they are present in [12] (without a tree).

4.4. Action of the group of hierarchomorphisms in \( \mathcal{H}_\lambda(J) \). Let a group \( \Gamma \) acts on \( J \) by isometries. Then \( \Gamma \) acts in \( \mathcal{H}_\lambda(J) \) by the orthogonal operators\(^6\) of the Hilbert space \( \mathcal{H}_\lambda(J) \) by the formula

\[ U(\gamma)e_a = e_{\gamma a}. \]  

Let now \( g \in \text{Hier}^*(J, \Gamma) \) be a hierarchomorphism. Define the operators \( U(g) \) by the same formula (4.2).

Theorem 4.1. a) The operators \( U(g) \) are well defined and bounded.

b) Each operator \( U(g) \) can be represented in the form \( U(g) = A(1 + R) \), where \( A \) is an orthogonal operator and \( R \) is an operator of finite rank.

The theorem is proved below in 4.6.

4.5. The subspaces \( \mathcal{H}_\lambda(S) \). Let \( S \) be a subtree in \( J \). Denote by \( \mathcal{H}_\lambda(S) \) the subspace in \( \mathcal{H}_\lambda(J) \) generated by the vectors \( e_c \), where \( c \in \text{Vert}(S) \). Denote by \( P(S) \) the operator of projection \( \mathcal{H}_\lambda(J) \to \mathcal{H}_\lambda(S) \).

Lemma 4.2. Let \( S_1, S_2 \) be two disjoint subtrees in \( J \). Let \( b \in \text{Vert}(S_1) \), \( c \in \text{Vert}(S_2) \) be the nearest vertices of the subtrees \( S_1, S_2 \).

a) The sum \( \mathcal{H}_\lambda(S_1) + \mathcal{H}_\lambda(S_2) \) is a topological direct sum in \( \mathcal{H}_\lambda(J) \).

b) Let \( Q : \mathcal{H}_\lambda(S_1) \to \mathcal{H}_\lambda(S_2) \) be the restriction of the projection operator \( P(S_2) \) to \( \mathcal{H}_\lambda(S_1) \). Then the image of \( Q \) is the line spanned by \( e_c \), and the kernel of \( Q \) is the orthocomplement in \( \mathcal{H}_\lambda(S_1) \) to \( e_b \).

Proof. Let \( h_1 \in \mathcal{H}_\lambda(S_1), h_2 \in \mathcal{H}_\lambda(S_2) \) be unit vectors. The both statements are corollaries of the following inequalities

\[ \langle h_1, h_2 \rangle \leq \langle h_1, e_c \rangle; \quad \langle h_1, h_2 \rangle \leq \langle e_b, h_2 \rangle. \]

4.6. Proof of Theorem 4.1. Let \( g = \{g_j, S_j\} \in \text{Hier}^*(J) \) be a hierarchomorphism. Without loss of generality (see Lemma 1.1), we can assume that \( S_j \) are bushes or single-point sets.

By Lemma 4.2, the decomposition

\[ \mathcal{H}_\lambda(J) = \bigoplus_j \mathcal{H}_\lambda(S_j) \]

is a topological direct sum.

\(^6\)An orthogonal operator is an invertible operator in a real Hilbert space preserving the inner product.
Consider the bilinear form

\[ Q(h_1, h_2) = \langle U(g)h_1, U(g)h_2 \rangle - \langle h_1, h_2 \rangle \]

on \( \mathcal{H}_\lambda (J) \times \mathcal{H}_\lambda (J) \). It is sufficient to prove that \( Q \) is a bounded form on \( \mathcal{H}_\lambda (J) \times \mathcal{H}_\lambda (J) \) and the rank of \( Q \) is finite.

The matrix of \( Q \) in the basis \( e_a \) is

\[ Q(e_a, e_b) = \langle e_{ga}, e_{gb} \rangle - \langle e_a, e_b \rangle = \lambda^a(ga, gb) - \lambda^a(a, b) \]

The matrix \( Q(e_a, e_b) \) has the natural block decomposition corresponding to the partition

\[ \text{Vert}(J) = \bigcup \text{Vert}(S_j) \]

It is sufficient to prove that each block has finite rank.

Thus, let \( a \) ranges in \( S_i \), \( b \) ranges in \( S_j \). If \( S_i \) is an one-point space, then the required statement is obvious.

Thus, we assume that \( S_i, S_j \) are bushes (see 1.1). Let \( u_i, u_j \) be their roots. If \( S_i = S_j \), then \( Q(e_a, e_b) \) is the identical zero.

Thus, assume \( S_i \neq S_j \). Then

\[ \rho(a, b) = \rho(a, u_i) + \rho(u_i, u_j) + \rho(u_j, b); \]
\[ \rho(ga, gb) = \rho(ga, gu_i) + \rho(gu_i, gu_j) + \rho(gu_j, gb) = \]
\[ = \rho(a, u_i) + \rho(gu_i, gu_j) + \rho(u_j, b). \]

Thus,

\[ Q(e_a, e_b) = \left[ \lambda^a(gu_i, gu_j) - \lambda^a(u_i, u_j) \right] \cdot \lambda^a(u_i, u_j) \cdot \lambda^a(b, u_j) = \]
\[ = \text{const} \cdot \langle e_{u_i}, e_a \rangle \cdot \langle e_{u_j}, e_b \rangle. \]

Thus we obtain that the bilinear form \( Q \) on \( \mathcal{H}_\lambda (S_i) \times \mathcal{H}_\lambda (S_j) \) is given by the formula

\[ Q(h_1, h_2) = \text{const} \cdot \langle e_{u_i}, h_1 \rangle \cdot \langle e_{u_j}, h_2 \rangle \]

Thus the form \( Q \) on \( \mathcal{H}_\lambda (S_i) \times \mathcal{H}_\lambda (S_j) \) is of rank \( \leq 1 \). This finishes the proof.

4.7. **Remark. Spaces \( \mathcal{H}_\lambda \) associated with \( \mathbb{R} \)-trees.** Let we have a countable family \( J_1, J_2, \ldots \) of metric trees and let we have isometric embeddings \( i_k : J_k \to J_{k+1} \):

\[ \cdots \overset{i_k}{\longrightarrow} J_k \overset{i_k}{\longrightarrow} J_{k+1} \overset{i_{k+1}}{\longrightarrow} J_{k+2} \overset{i_{k+2}}{\longrightarrow} \cdots \]

Let \( J \) be the direct limit (the union) of \( J_k \). Such spaces are called \( \mathbb{R} \)

- trees.\(^7\)

Obviously, we have the chain of inclusions

\[ \cdots \subset \mathcal{H}_\lambda (J_k) \subset \mathcal{H}_\lambda (J_{k+1}) \subset \mathcal{H}_\lambda (J_{k+2}) \subset \cdots \]

Denote the inductive limit of this chain by \( \mathcal{H}_\lambda (J) \). Thus the Hilbert space \( \mathcal{H}_\lambda \) survives for \( \mathbb{R} \)-trees. Nethertheless, the analogue of Theorem 4.1 is wrong.

\(^7\)up to a minor variation of terminology
5. Boundary spaces

In this Section, we construct some spaces $\mathcal{E}_\lambda$ of ‘distributions’ on the absolute of a metric tree. These spaces can be considered as an analogue of the Sobolev spaces on the spheres. For the Bruhat–Tits trees, the spaces $\mathcal{E}_\lambda$ are well-known, see [3]. We also construct the operator $\mathcal{H}_\lambda \to \mathcal{E}_\lambda$ of restriction of a “function on tree” to the absolute.

In this Section, $J$ is a locally finite perfect metric tree.

5.1. Balls in absolute. Let $S$ be a branch of $J$. A ball $B[S] \subset \text{Abs}(J)$ is the absolute of the branch $S$. If we delete the root of the $S$ and all edges containing the root, then $S$ will be disintegrated into the finite collection of branches $S^{(1)}, S^{(2)}, \ldots, S^{(k)}$. Hence the ball $B[S]$ admits the canonical partition

$$B[S] = B[S^{(1)}] \cup \cdots \cup B[S^{(k)}].$$

(5.1)

into the balls $B[S^{(k)}]$.

We define the topology on $\text{Abs}(J)$ by the assumption that all the balls $B[S]$ are open-and-closed subset in $\text{Abs}[S]$. Obviously, $\text{Abs}(J)$ is a completely discontinuous compact set.

Remark. Obviously, hierarchomorphisms locally preserve hierarchy of balls on the absolute\footnote{Firstly, this hierarchy structure on $p$-adic manifolds was mentioned in Addendum in Serre’s book [31].}. Obviously, spheromorphisms are homeomorphisms of the absolute. But preserving of the hierarchy of balls is a very rigid condition on a homeomorphism.

5.2. New notation in the space $\mathcal{H}_\lambda(J)$. Let us fix a vertex $\xi \in \text{Vert}(J)$. Let $a, b \in \text{Vert}(J)$. Consider the way $a_0 = a, a_1, \ldots, a_i = b$ connecting $a, b$. Assume

$$\theta(a, b) = 2\min_\rho(\xi, a_j).$$

We emphasis that this function has sense also if $a$ or $b$ are points of the absolute, and the value $\theta(a, b)$ is finite except the case $a = b \in \text{Abs}(J)$.

For $a \in \text{Vert}(J)$, consider the vector $f_a \in \mathcal{H}_\lambda(J)$ given by

$$f_a = \lambda^{-\rho(\xi, a)}e_a.$$ 

Then

$$\langle f_a, f_b \rangle = \lambda^{-\theta(a, b)}.$$ 

Remark. Let $S$ be a subtree in $J$ containing $\xi$. For $c \in \text{Vert}(J)$, consider the nearest vertex $b \in \text{Vert}(S)$. Then the projection of $f_c$ to $\mathcal{H}_\lambda(S)$ is $f_b$.

5.3. Measures on $\text{Abs}(J)$ and compatible systems of measures on $\text{Vert}(J)$. Let $R \subset J$ be a subtree. We say that $R$ is complete if any $a \in \text{Vert}(R)$ satisfies one of two following conditions (see Picture 5).
1. Any vertex $b$ of $J$ adjacent to $a$ is contained in $R$.
2. Only one vertex of $J$ adjacent to $a$ is contained in $R$.

Let $\partial R$ denote the boundary of $R$, i.e., the set of all vertices of the second type.

We also assume $\xi \in \text{Vert}(R) \setminus \partial R$.

Consider a real-valued measure (charge) $\mu$ of finite variation on $\text{Abs}(J)$. Recall that any measure $\mu$ of finite variation admits the canonical representation

$$\mu = \mu^+ - \mu^-,$$

where $\mu^+$ are nonnegative finite measures, and for some (noncanonical) Borel subset $U \subset \text{Abs}$,

$$\mu^+(U) = 0; \quad \mu^+(\text{Abs} \setminus U) = 0.$$

The variation of the measure $\mu$ is

$$\text{var}(\mu) = \mu^+(U) + \mu^-(\text{Abs} \setminus U).$$

For a complete subtree $R$, denote by $u_1, u_2, \ldots$ the points of $\partial R$. For any $u_k$, there exists a unique branch $S_{u_k} \subset J$ such that $u_k$ is the root of $S_{u_k}$ and $\xi \notin S_{u_k}$.

Consider the measure $\mu_R$ defined on the finite set $\partial R$ by

$$\mu_R(u_j) = \mu(B[S_{u_j}]).$$

Consider also the vector

$$\Psi[\mu|R] = \sum_{u_j \in \partial R} \mu(B[S_{u_j}]) f_{u_j}.$$

Let $R_2 \supset R_1$ be complete subtrees. Then we have the obvious retraction

$$\eta_{R_2}^{R_1} : \text{Vert}(R_2) \to \text{Vert}(R_1) :$$

if $a \in \text{Vert}(R_2)$, then $\eta_{R_2}^{R_1}(a)$ is the nearest vertex of $R_1$. 

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Lemma 5.1. a) \( \mu \circ \eta \) is the image of \( \mu \) under the retraction \( \eta^R \).

b) The vector \( \Psi[\mu|R] \) is the projection of \( \Psi[\mu|R_2] \) to the space \( \mathcal{H}(R_1) \). In particular,

\[
\|\Psi[\mu|R_1]\| \leq \|\Psi[\mu|R_2]\|.
\]

Proof. Assertion a) is obvious, and assertion b) follows from the last remark from 5.2.

Conversely, consider a family of complete subtrees

\[
R_1 \subset R_2 \subset R_3 \subset \ldots
\]
such that \( \bigcup R_j = J \). Let for each \( j \) we have a measure \( \nu_j \) on \( \partial R_j \), and \( \eta_{R_j}^R \nu_j = \nu_j \) for all \( j \). If \( \sup \text{var}(\nu_j) < \infty \), then there exists a unique measure \( \nu \) on \( \text{Abs} \) such that \( \nu_j = \nu(R_j) \).

5.4. Boundary spaces \( \mathcal{E}_\lambda \subset \mathcal{H}_\lambda \). Let \( R_1 \subset R_2 \subset \ldots \) be a sequence of complete subtrees in \( J \), and \( \bigcup R_k = J \) (the construction below does not depend on choice of the sequence).

Let \( \mu \) be a measure of finite variation on \( \text{Abs}(J) \). We say that \( \mu \) belongs to the class \( \mathcal{E}_\lambda = \mathcal{E}_\lambda(J) \) if

\[
\lim_{j \to \infty} \|\Psi[\mu|R_j]\|_\mathcal{H}_\lambda < \infty.
\]

Proposition 5.2. For \( \mu, \mu' \in \mathcal{E}_\lambda \), the following statements hold.

a) There exists the following limit in the space \( \mathcal{H}_\lambda(J) \)

\[
\Psi[\mu] := \lim_{j \to \infty} \Psi[\mu|R_j].
\]

b) \( \|\Psi[\mu]\|_\mathcal{H}_\lambda = \lim_{j \to \infty} \|\Psi[\mu|R_j]\|_\mathcal{H}_\lambda \).

c) \( \langle \mu \rangle_{\mathcal{E}_\lambda(J)} = \lim_{j \to \infty} \langle \mu[R_j], \mu'[R_j] \rangle_{\mathcal{H}_\lambda} \).

Proof. All statements follow from Lemma 5.1. \( \square \)

Thus we obtain the embedding \( \mathcal{E}_\lambda(J) \hookrightarrow \mathcal{H}_\lambda(J) \) given by \( \Psi : \mu \to \Psi[\mu] \). We define the inner product in \( \mathcal{E}_\lambda(J) \) by

\[
\langle \mu_1, \mu_2 \rangle_{\mathcal{E}_\lambda(J)} := \langle \mu_1, \mu_2 \rangle_{\mathcal{H}_\lambda(J)}.
\]

Denote by \( \mathcal{E}_\lambda \subset \mathcal{H}_\lambda \) the image of the embedding \( \Psi \). Denote by \( \overline{\mathcal{E}_\lambda} \) the closure of \( \mathcal{E}_\lambda \) in \( \mathcal{H}_\lambda \), and also denote by \( \overline{\mathcal{E}_\lambda} \) the completion of the space \( \mathcal{E}_\lambda \) with respect to the norm (5.3).

5.5. More direct description of \( \mathcal{E}_\lambda \). We can write formally

\[
\|\mu\|_{\mathcal{E}_\lambda}^2 = \int_{\text{Abs} \times \text{Abs}} \lambda^{-\theta(\omega_1, \omega_2)} d\mu(\omega_1, \omega_2) d\mu(\omega_2); \tag{5.5}
\]

\[
\langle \mu_1, \mu_2 \rangle_{\mathcal{E}_\lambda} = \int_{\text{Abs} \times \text{Abs}} \lambda^{-\theta(\omega_1, \omega_2)} d\mu_1(\omega_1, \omega_2) d\mu_2(\omega_2). \tag{5.6}
\]
These integrals are very simple, since the integrand \( \lambda^{-\theta(\omega_1, \omega_2)} \) has only countable set of values. Nevertheless, generally (even for the Bruhat–Tits tree \( T_B \)) for \( \mu_1, \mu_2 \in \mathcal{E}_\lambda \), these integrals diverge as Lebesgue integrals.

Our limit procedure is equivalent to the Riemann improper integration in the following sense. Consider a complete subtree \( R \subset J \) such that \( \xi \in R \). Then \( J \setminus R \) is a union of disjoint branches \( S_1, \ldots, S_k \). Thus

\[
\text{Abs}(J) = B[S_1] \cup \cdots \cup B[S_k].
\]

Let us define the Darboux sum

\[
S_R(\mu_1, \mu_2) = \sum_{i,j} \left\{ \min_{\omega_1 \in B[S_i]} \min_{\omega_2 \in B[S_j]} \lambda^{-\theta(\omega_1, \omega_2)} \right\} \mu_1(B[S_i]) \mu_2(B[S_j]).
\]

**Remark.** If \( i \neq j \), then the value \( \lambda^{-\theta(\omega_1, \omega_2)} \) is a constant on \( B[S_i] \times B[S_j] \).

We have seen, that

\[
R_2 \supset R_1 \quad \Rightarrow \quad S_{R_1}(\mu, \mu) \leq S_{R_2}(\mu, \mu). \tag{5.7}
\]

Now we can define the integral (5.5) as the limit of these Darboux sums under refinement of the partition. A measure \( \mu \) is contained in \( \mathcal{E}_\lambda \) iff the Riemann integral (5.5) is finite.

After this, we can define the inner product in \( \mathcal{E}_\lambda \) as the Riemann improper integral (5.6).

Nevertheless, the space \( \mathcal{H}_\lambda \) was essentially used in the justification of this construction, since the convergence of Darboux sums and positivity of the integral (5.5) are not obvious.

**5.6. Non-emptiness of \( \mathcal{E}_\lambda \).**

**Theorem 5.3.** a) There exists \( \sigma \), which belongs to \( 0 \leq \sigma \leq 1 \), such that the space \( \mathcal{E}_\lambda \) is zero for \( \lambda < \sigma \) and the space \( \mathcal{E}_\lambda \) is not zero for \( \lambda > \sigma \).

b) If lengths of edges of \( J \) are bounded, then \( \sigma < 1 \).

c) Let \( J \) contain a subtree \( I \) that is isomorphic to the Bruhat–Tits tree \( T_B \) as a simplicial tree, and lengths of all edges of \( I \) are \( \leq \tau \). Then \( \sigma \leq 1/\sqrt{\tau} \).

d) Assume lengths of edges of \( J \) are bounded away from zero. Let the number \( s(N) \) of \( a \in \text{Vert}(J) \), satisfying \( d_{\text{symp}}(\xi, a) \leq N \), has exponential growth, i.e., \( s(N) \leq \exp(a N) \) for some constant \( a \). Then \( \sigma > 0 \).

The proof of the Theorem is contained below in 5.7–5.11

**5.7. Expansion of \( \|\Psi\|^2 \) into series with positive terms.** Let \( R_1 \subset R_2 \subset R_3 \subset \ldots \) be a sequence of complete subtrees in \( J \), and \( \bigcup R_m = J \). We say that the sequence \( R_m \) is **incompressible** if

1. \( R_0 \) consists of the vertex \( \xi \);
2. for each \( m \), there exists \( u \in \partial R_m \) such that \( \text{Vert}(R_{m+1}) \setminus \text{Vert}(R_m) \) consists of vertices adjacent to \( u \).

Fix a measure (charge) \( \mu \) on Abs.
Obviously, $\Psi[\mu|R_0] = \mu(\text{Abs})f_\xi$, and hence
\[
\|\Psi[\mu|R_0]\|^2 = \mu(\text{Abs})^2.
\]

Let us evaluate
\[
z^{(m)}(\lambda) = \|\Psi[\mu|R_{m+1}]\|^2_{\mathcal{H}_\lambda} - \|\Psi[\mu|R_m]\|^2_{\mathcal{H}_\lambda}.
\]

Let $u$ be the vertex defined in $2^n$. Let $v_1, \ldots, v_n \in \partial R_{m+1}$ be the vertices adjacent to $u$, see Picture 6.

\begin{center}
\begin{tikzpicture}
\node [scale=1] at (0,0) {\footnotesize $\xi$};
\node [scale=1] at (2,0) {\footnotesize $u$};
\node [scale=1] at (4,0) {\footnotesize $v_k$};
\draw [->] (0,0) -- (2,0);
\draw [->] (0,0) -- (4,0);
\draw [->] (2,0) -- (4,0);
\end{tikzpicture}
\end{center}

Picture 6.

Let $\mu_{R_{m+1}}(v_k) = t_k$ (these numbers can be negative), respectively $\mu_{R_m}(u) = t_1 + \cdots + t_n$. It is readily seen that
\[
z^{(m)}(\lambda) = \left(\lambda^{-2\rho(\xi,u)} \sum_{k \neq 1} t_k + \lambda^{-2\rho(\xi,u)} \sum_{k} \lambda^{-2\rho(u,v_k)} t_k^2\right) - \lambda^{-2\rho(\xi,u)} \left(\sum t_k\right)^2 = \\
= \lambda^{-2\rho(\xi,u)} \left(\sum_{k} \lambda^{-2\rho(u,v_k)} - 1\right) t_k^2. \quad (5.8)
\]

First, we observe that this expression is completely determined by the measure $\mu$ and the vertex $u$. The subtrees $R_m$, $R_{m+1}$ are nonessential. Hence it is natural to denote $z^{(m)}(\lambda)$ by $z_u(\lambda)$.

Thus,
\[
\|\Psi[\mu]\|^2 = \mu(\text{Abs})^2 + \sum_{m=1}^{\infty} z^{(m)}(\lambda) = \\
= \mu(\text{Abs})^2 + \sum_{u \in \text{Ver}(\tilde{I}), u \neq \xi} z_u(\lambda). \quad (5.9)
\]

We emphasis that
a) all summands of this series are positive;
\[b) \text{all summands } z_u(\lambda) \text{ are decreasing functions on } \lambda \text{ for } 0 \leq \lambda \leq 1.\]

5.8. Existence of $\sigma$. The Statement a) of Theorem 5.4 follows from the last observation of previous subsection.

5.9. Existence of $\mathcal{E}_I$. It is sufficient to prove c), since b) is a corollary of c). Furthermore, it is sufficient to prove nontriviality of $\mathcal{E}_I(I)$ for the subtree $I$. Denote by $R_k$ the subtree of $I$, consisting of all vertices $a \in I$ such that the simplicial distance $d_{\text{simp}}(\xi, a) \leq k$. 

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Consider the uniform measure $\mu_R$ on $\partial R_k$, i.e., the measure of each point is $1/(p^{k-1}(p+1))$. Obviously, the measures $\mu_k$ form a compatible system of the measures, denote by $\mu$ the inverse limit of the measures $\mu_R$.

Let us estimate

$$\Psi[\mu|R_k]|^2 = \frac{1}{(p+1)^2 p^{2(k-1)}} \sum_{a \in \partial R_k} |f_a|^2$$

$$= \frac{1}{(p+1)^2 p^{2(k-1)}} \sum_{a,b \in \partial R_k} \lambda^{-2p(a,b)} \leq$$

$$\leq \frac{1}{(p+1)^2 p^{2(k-1)}} \sum_{a,b \in \partial R_k} \lambda^{-2\tau j} \cdot \left\{ \text{number of pairs } (a, b) \in \partial R_k \right\} =$$

$$= \frac{1}{(p+1)^2 p^{2(k-1)}} \left[ (p+1)p^{2k-1} \sum_{j=1}^{k-1} \lambda^{-2\tau j} + \sum_{j=1}^{k-1} \lambda^{-2\tau j} \left( p+1 \right)^{k-1} (p+1)^{k-1} \lambda^{-2\tau j} \leq \sum_{j=0}^{k} \lambda^{-2\tau j} p^{-j} \right.$$}

If $\lambda^{2\tau p} > 1$, then these sums are uniformly bounded in $k$; hence $\mu \in \mathcal{E}_\lambda(I) \subset \mathcal{E}_\lambda(J)$.

### 5.10. Localization.

**Lemma 5.4.** Let $\mu \in \mathcal{E}_\lambda$, and let $B[S] \subset \text{Abs}$ be a ball. Let $\nu$ be the restriction of $\mu$ to $B[S]$ (i.e., $\nu(A) = \mu(A \cap B(S])$ for any Borel subset $A \subset \text{Abs}$). Then $\nu \in \mathcal{E}_\lambda$.

**Proof.** We can assume $\xi \notin S$. Denote by $v$ the root of the branch $S$. The quantity $\|\mu\|_{E_\lambda}^2$ is the sum of the series $\sum z_n(\lambda)$ given by (5.8), (5.10). The series for $\|\nu\|_{E_\lambda}^2$ is obtained from the series for $\|\mu\|_{E_\lambda}^2$ by the following operations.

1) For $u$ lying between $\xi$ and $v$, the summants $z_u(\lambda)$ are changed in a non-predictable way.

2) For any $u \in S$, the summand $z_u(\lambda)$ does not change.

3) All other summants become zero.

Obviously, the new series $z_u(\lambda)$ is convergent.

**Remark.** Consider a Borel subset $U$ in the absolute. Let $\nu$ be the restriction of $\mu \in \mathcal{E}_\lambda$ to $U$. Generally, $\nu \notin \mathcal{E}_\lambda$. Also, generally, $\mu^\pm \notin \mathcal{E}_\lambda$.

### 5.11. Lower estimate of $\sigma$.

By Lemma 5.4, if $\mathcal{E}_\lambda \neq \emptyset$, then there exists a measure $\mu \in \mathcal{E}_\lambda$ such that $\mu(\text{Abs}) \neq 0$. For definiteness, assume $\mu(\text{Abs}) = 1$.

Let $\sigma$ be a lower bound for lengths of edges. Consider a complete subtree $R \subset J$ defined by the condition $d_{\text{synp}}(\xi, a) \leq N$. Consider the measure $\mu_R$ on $\partial R$. In notation 5.7,

$$\|\Psi[\mu]\|^2 = 1 + \sum_{v \in \text{Vert} J, v \neq \xi} z_v \geq \sum_{u \in \partial R} z_u \geq$$

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\[ \lambda^{-2N\sigma} (\lambda^{-2\sigma} - 1) \sum_{u \in \partial R} \mu_R(u)^2. \]  

(5.11)

The number of points of \( \partial R \) is less than \( \exp\{aN\} \), where \( a \) is a constant. Furthermore, \( \sum_{u \in \partial R} \mu_R(u) = 1 \), hence the last expression is larger than

\[ \lambda^{-2N\sigma} (\lambda^{-2\sigma} - 1) \exp\{-aN\}. \]

For a sufficiently small \( \lambda > 0 \), the last expression tends to \( \infty \) as \( N \to \infty \), and thus \( \mathcal{E}_\lambda = 0 \).

6. Action of group of hierarchomorphisms in \( \mathcal{E}_\lambda \)

Let \( J \) be a perfect locally finite metric tree. Let \( \mathcal{H}_\lambda(J) \supseteq \mathcal{E}_\lambda(J) \simeq \mathcal{E}_\lambda(J) \) be the same spaces as above. Let a group \( \Gamma \) act on \( J \) by isometries. Let \( \text{Hier}^s(J, \Gamma) \), \( \text{Hier}(J, \Gamma) \) be the corresponding hierarchomorphisms groups. The group \( \text{Hier}^s(J, \Gamma) \) acts in \( \mathcal{H}_\lambda(J) \) by the operators \( U(g) \) given by (4.1).

6.1. Action of hierarchomorphisms in \( \mathcal{E}_\lambda \).

Proposition 6.1 a) The space \( \mathcal{E}_\lambda(J) \subset \mathcal{H}_\lambda(J) \) is invariant with respect to \( \text{Hier}^s(J, \Gamma) \).

b) For \( g \in \text{Hier}^s(J, \Gamma) \), the restriction of the operator \( U(g) \) to \( \mathcal{E}_\lambda \) depends only on the corresponding element \( \tilde{g} \in \text{Hier}(J, \Gamma) \).

c) The action of \( \text{Hier}(J, \Gamma) \) in \( \mathcal{E}_\lambda(J) \simeq \mathcal{E}_\lambda(J) \) is given by

\[ T_\lambda(\tilde{g})\mu(\omega) = \lambda^{n(\tilde{g}, \omega)} \cdot \mu(g\omega), \quad \text{where } g \in \text{Hier}(J, \Gamma), \]  

(6.1)

where the pseudoderivative \( n(g, \omega) = n(\tilde{g}, \omega) \) of a hierarchomorphism on the absolute was defined in 3.2, and \( \mu(g\omega) \) is the image of the measure \( \mu \) under the transformation \( \omega \mapsto g\omega \).

Remark. For \( g \in \Gamma \), the operator \( T_\lambda(g) \) is unitary.

Proof. Fix \( g \in \text{Hier}^s(J, \Gamma) \). Let \( R_0 \subset R_1 \subset \ldots \) be an incompressible sequence of complete subarcs as in 5.7, \( \bigcup R_k = J \). Consider the sequence \( g \cdot \partial R_1, g \cdot \partial R_2, \ldots \). There exists \( l \) such that for all \( k \geq l \)

\[ g \cdot \partial R_k = \partial T_k \quad \text{where } T_k \text{ is a complete subarc.} \]

Hence,

\[ U(g)[\Psi[\mu|R_k]] = \Psi[\nu|T_k]. \]

where \( \nu \) is some measure on \( \text{Aut}(J) \).

We must show that the numbers \( ||\Psi|\nu|T_k|\| \) are bounded. Consider the expansion of \( ||\Psi||\nu|\|^2 \) and \( ||\Psi|\nu|\|_2^2 \) into the series \( \sum z^k(\lambda) \), see (5.9), (5.10). The summands with numbers \( < l \) are essentially different, but this do not influence on the convergence. Other summands are rearranged and multiplied by the factors \( \lambda^{n(g, \omega)} \).

But \( \lambda^{n(g, \omega)} \) has only finite number of values and hence the series \( \sum z^k(\lambda) \) for the measure \( \nu \) is also convergent. Thus \( \nu \in \mathcal{E}_\lambda(J) \).
The statement a) is proved, the statement b) is obvious, and the statement c) follows from the same considerations.

6.2. Almost orthogonality.

Theorem 6.2. Let \( g \in \text{Hier}(J, \Gamma) \). The operators \( T_\lambda(g) \) in \( \mathcal{E}_\lambda(J) \) given by (6.1) admit the representation \( T_\lambda(g) = A(1 + Q) \), where \( A \) is an orthogonal operator and \( Q \) is a finite rank operator.

This statement follows from Theorem 4.1. This can also be proved directly from the explicit formulas (5.6), (6.1).

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