Uniform Estimates of the Resolvent of the Laplace–Beltrami Operator on Infinite Volume Riemannian Manifolds with Cusps.II

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Abstract
We prove uniform weighted high frequency estimates for the resolvent of the Laplace-Beltrami operator on connected infinite volume Riemannian manifolds under some natural assumptions on the metric on the ends of the manifold. This extends previous results by Burq [3] and Vodev [8].

1 Introduction and statement of results

The purpose of this paper is to extend the results in [8] to more general Riemannian manifolds which may have cusps. Let $(M, g)$ be an $n$-dimensional unbounded, connected Riemannian manifold with a Riemannian metric $g$ of class $C^\infty (\overline{M})$ and a compact $C^\infty$-smooth boundary $\partial M$ (which may be empty), of the form $M = X_0 \cup X_1 \cup X_2$, where $X_0$ is a compact, connected Riemannian manifold with a metric $g_{|X_0}$ of class $C^\infty (\overline{X_0})$ with a compact boundary $\partial X_0 = \partial M \cup \partial X_1 \cup \partial X_2$, $\partial M \cap \partial X_1 = \emptyset$, $\partial M \cap \partial X_2 = \emptyset$, $\partial X_1 \cap \partial X_2 = \emptyset$, $X_k = [r_k, +\infty) \times S_k$, $r_k \gg 1$, with metric $g_{|X_k} := dr^2 + \sigma_k (r)$, $k = 1, 2$. Here $(S_k, \sigma_k (r))$, $k = 1, 2$, are $n-1$ dimensional compact Riemannian manifolds without boundary equipped with families of Riemannian metrics $\sigma_k (r)$

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depending smoothly on $r$ which can be written in any local coordinates $\theta \in S_k$ in the form

$$\sigma_k(r) = \sum_{i,j} g^k_{ij}(r,\theta) d\theta_i d\theta_j, \quad g^k_{ij} \in C^\infty(X_k).$$

Denote $X_{k,r} = [r, +\infty) \times S_k$. Clearly, $\partial X_{k,r}$ can be identified with the Riemannian manifold $(S_k, \sigma_k(r))$ with the Laplace-Beltrami operator $\Delta_{\partial X_{k,r}}$ written as follows

$$\Delta_{\partial X_{k,r}} = -p^{-1}_k \sum_{i,j} \partial_{\theta_i} (p_k g^k_{ij} \partial_{\theta_j}),$$

where $(g^k_{ij})$ is the inverse matrix to $(g^k_{ij})$ and $p_k = (\det (g^k_{ij}))^{1/2} = (\det (g^k_{ij}))^{-1/2}$. Let $\Delta_g$ denote the Laplace-Beltrami operator on $(M,g)$. We have

$$\Delta X_k := \Delta_g|_{X_k} = -p^{-1}_k \partial_r (p_k \partial_r) + \Delta_{\partial X_{k,r}} = -\partial_r^2 + \Lambda_{k,r} + q_k(r,\theta),$$

(1.1)

where

$$\Lambda_{k,r} = -\sum_{i,j} \partial_{\theta_i} (g^k_{ij} \partial_{\theta_j}),$$

$$q_k(r,\theta) = (2p_k)^{-2} \left( \frac{\partial p_k}{\partial r} \right)^2 + (2p_k)^{-2} \sum_{i,j} \partial_{\theta_i} \partial_{\theta_j} g^k_{ij} + 2^{-1} p_k \Delta_{X_k}(p_k^{-1}).$$

We make the following assumptions:

$$|q_k(r,\theta)| \leq C, \quad \frac{\partial q_1}{\partial r}(r,\theta) \leq C r^{-1-\delta_0}, \quad -\frac{\partial q_2}{\partial r}(r,\theta) \leq C r^{-1},$$

(1.2)

with constants $C, \delta_0 > 0$. Denote by $h_k$ the principal symbol of $\Delta_{\partial X_{k,r}}$, that is,

$$h_k(r,\theta,\xi) = \sum_{i,j} g^k_{ij}(r,\theta)\xi_i \xi_j, \quad (\theta,\xi) \in T^* S_k.$$

Clearly, $-\partial h_k / \partial r$ can be interpreted as being the second fundamental form of the surface $\partial X_{k,r}$. We suppose that

$$(-1)^i \frac{\partial h_k}{\partial r}(r,\theta,\xi) \geq \frac{C}{r} h_k(r,\theta,\xi), \quad \forall (\theta,\xi) \in T^* S_k,$$

(1.3)

with a constant $C > 0$. This means that $\partial X_{1,r}$ (resp. $\partial X_{2,r}$) is strictly convex (resp. strictly concave) viewed from $X_{1,r}$ (resp. $X_{2,r}$). Denote by $G$ the selfadjoint realization of $\Delta_g$ on the Hilbert space $H = L^2(M, d\text{Vol}_g)$ with Dirichlet or Neumann boundary conditions on $\partial M$. Given $s_1, s_2 \in \mathbb{R}$, choose a real-valued positive function $\chi_{s_1,s_2} \in C^\infty(M)$, $\chi_{s_1,s_2} = 1$ on $M \setminus (X_{1,r_1+1} \cup X_{2,r_2+1})$, $\chi_{s_1,s_2} = r^{-s_k}$ on $X_{k,r_k+2}$. Also, given $a > r_1$ choose a real-valued positive function $\eta_a \in C^\infty(M)$, $\eta_a = 0$ on $M \setminus X_{1,a}$, $\eta_a = 1$ on $X_{1,a+1}$. Our main result is the following.
Theorem 1.1 Under the assumptions (1.2) and (1.3), for every \( s_1 > 1/2, s_2 > 1 \), there exist positive constants \( C_0, C > 0, a > r_1 \) so that for \( z \in \mathbb{R}, z \geq C_0 \), the limit
\[
R_{s_1, s_2}^+(z) := \lim_{\varepsilon \to 0^+} \chi_{s_1, s_2}(G - z + i\varepsilon)^{-1} \chi_{s_1, s_2} : H \to H
\]
exists and satisfies the bounds
\[
\| R_{s_1, s_2}^+(z) \|_{\mathcal{L}(H)} \leq e^{Cz^{1/2}}, \quad (1.4)
\]
\[
\| \eta R_{s_1, s_2}^+(z) \eta \|_{\mathcal{L}(H)} \leq Cz^{-1/2}. \quad (1.5)
\]

Suppose that there exist metrics \( \bar{\sigma}_k(r) \) depending smoothly on \( r \in (-\infty, +\infty) \) such that \( \bar{\sigma}_k(r) = \sigma_k(r) \) for \( r \geq r_k \) and the resolvents (defined for \( \text{Im} z < 0, \text{Re} z > 0 \))
\[
R_{\chi_k}(z) := (\Delta X_k^0 - z)^{-1} : L^2_{\text{comp}}(X_k^0, d\text{Vol}_{g_k}) \to H^2_{\text{loc}}(X_k^0, d\text{Vol}_{g_k}),
\]
where \( X_k^0 = (-\infty, +\infty) \times S_k \) with metric \( g_k^0 = dr^2 + \bar{\sigma}_k(r) \), \( \Delta X_k^0 \) denoting the self-adjoint realization of the Laplace-Beltrami operator on \( X_k^0 \) on the Hilbert space \( L^2(X_k^0, d\text{Vol}_{g_k}) \), extend analytically to \( \text{Im} z \leq -\gamma \| F \|^2 \), \( \gamma > 0 \), and satisfy in this region the bounds (with \( \alpha = 0, 1 \)):
\[
\| \partial^\alpha_z R_{\chi_k}(z) R_{\chi_k}(z) \|_{\mathcal{L}(L^2(X_k^0, d\text{Vol}_{g_k}))} \leq C e^{\gamma |\text{Re} z|^2}, \quad \forall \chi \in C^\infty_0(X_k^0), \quad (1.6)
\]
with some constants \( C, \gamma > 0 \). As a consequence of Theorem 1.1 we get the following

Corollary 1.2 Under the assumptions (1.2), (1.3) and (1.6), the resolvent (defined for \( \text{Im} z < 0, \text{Re} z > 0 \))
\[
R_M(z) := (G - z)^{-1} : L^2_{\text{comp}}(M, d\text{Vol}_g) \to H^2_{\text{loc}}(M, d\text{Vol}_g),
\]
extends analytically to \( \text{Im} z \leq -\gamma |\text{Re} z|^2 \) and satisfies in this region the bound
\[
\| \chi R_M(z) \chi \|_{\mathcal{L}(H)} \leq C e^{\gamma |\text{Re} z|^2}, \quad (1.7)
\]
\( \forall \chi \in C^\infty_0(M) \) of compact support, with some constants \( C, \gamma > 0 \).

Remark. It is easy to see from the proof that the above results hold for more general connected Riemannian manifolds of the form
\[
M = X_0 \cup X_1 \cup \ldots \cup X_I \cup X_1 \cup \ldots \cup X_1, \quad I \geq 0, J \geq 1,
\]
with \( X_i \) like \( X_1, \) \( X_2 \) like \( X_2, \) and \( X_0 \) being a compact Riemannian manifold with boundary \( \partial X_0 = \partial M \cup \partial X_1 \cup \ldots \cup \partial X_J \cup \partial X_1 \cup \ldots \cup \partial X_1 \cup \partial M \cap \partial X_1 \cap \partial X_1 = \emptyset, \) \( \partial M \cap \partial X_1 \cap \partial X_1 = \emptyset, \partial X_1 \cap \partial X_1 = \emptyset, \partial X_1 \cap \partial X_1 = \emptyset, \)
\( \partial X_1 = X_0, j_1 \neq j_2, \partial X_1 \cap \partial X_1 = \emptyset, i_1 \neq i_2. \)

This corollary can be derived from the bounds (1.4) and (1.6) in precisely the same way as in the proof of Theorem 1.2 of [8] and this is why we omit the proof.

Another consequence of the above theorem is that we get uniform high frequency resolvent estimates for long-range perturbations of the Euclidean metric. Let \( \mathcal{O} \subset \mathbb{R}^n, n \geq 2, \) be a
bounded domain with a $C^\infty$-smooth boundary $\Gamma$ and a connected complement $\Omega = \mathbb{R}^n \setminus \mathcal{O}$. Let $g$ be a Riemannian metric in $\Omega$ of the form

$$g = \sum_{i,j=1}^{n} g_{ij}(x)dx_idx_j, \quad g_{ij}(x) \in C^\infty(\overline{\Omega}).$$

We make the following assumption:

$$|\partial_r^\alpha (g_{ij}(x) - \delta_{ij})| \leq C_\alpha \langle x \rangle^{-\delta_0 - |\alpha|}, \quad (1.8)$$

for every multi-index $\alpha$, with constants $C_\alpha, \delta_0 > 0$, where $\langle x \rangle := (1 + |x|^2)^{1/2}$ and $\delta_{ij}$ denotes the Kronecker symbol. Denote by $\Delta_g$ the corresponding Laplace-Beltrami operator, i.e.

$$\Delta_g = -f^{-1/2} \sum_{i,j=1}^{n} \partial_{x_i}(f^{1/2} g^{ij} \partial_{x_j}),$$

where $(g^{ij})$ is the inverse matrix to $(g_{ij})$ and $f = \det(g_{ij})$. Denote by $G$ the self-adjoint realization of $\Delta_g$ on the Hilbert space $H = L^2(\Omega; d\text{Vol}_g)$, $d\text{Vol}_g := f^{1/2} dx$, with Dirichlet or Neumann boundary conditions on $\Gamma$. It is not hard to see (e.g. see the appendix of [3]) for the proof of an analytic version) that under the assumption (1.8), there exists a global smooth change of variables, $(r, \theta) = (r(x), \theta(x))$, for $|x| > 1$, where $r \in [r_0, +\infty)$, $r_0 > 1$, $\theta \in S = \{ y \in \mathbb{R}^n : |y| = 1 \}$, which transforms the metric $g$ in the form

$$dr^2 + \sum_{i,j} h_{ij}(r, \theta) d\theta_i d\theta_j, \quad (1.9)$$

where $h_{ij} \in C^\infty$ satisfy the inequalities

$$|\partial_r^\alpha \partial_\theta^\beta (r^{-2} h_{ij}(r, \theta) - h_{ij}^0(\theta))| \leq C_{\alpha, \beta} r^{-\delta_0 - |\alpha|} \quad (1.10)$$

for all multi-indices $\alpha$ and $\beta$. Here $\sum_{i,j} h_{ij}^0(\theta)d\theta_id\theta_j$ is the metric on $S$ induced by the Euclidean one. The coordinates $(r, \theta)$ are just the normal geodesics coordinates which are well defined outside a sufficiently large compact since the metric $g$ is close to the Euclidean one. In other words, the Riemannian manifold $(\Omega, g)$ is isometric to a connected Riemannian manifold $(M, g)$ of the form $M = Y_0 \cup Y$, where $Y_0$ is a compact connected Riemannian manifold with boundary $\partial Y_0 = \partial M \cup \partial Y$, $\partial M \cap \partial Y = \emptyset$, and $Y = [r_0, +\infty) \times S$, $r_0 > 1$, with metric given by (1.9) and satisfying (1.10). Therefore, $Y$ is a particular case of the manifold $X_1$ above, and we get the following consequence of Theorem 1.1.

**Corollary 1.3** Under the assumption (1.8), for every $s > 1/2$ there exist constants $C_0, C > 0$ and $a > 1$ so that for $z \in \mathbb{R}, z \geq C_0$, the limit

$$R_s^+(z) := \lim_{\varepsilon \to 0^+} \langle x \rangle^{-s}(G - z + i\varepsilon)^{-1}\langle x \rangle^{-s} : H \to H$$

exists and satisfies the bounds

$$\|R_s^+(z)\|_H \leq e^{Cz^{1/2}}, \quad (1.11)$$

$$\|\chi_s R_s^+(z)\chi_s\|_H \leq Cz^{-1/2}, \quad (1.12)$$

where $\chi_s$ denotes the characteristic function of $|x| \geq a$. 

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Remark. It is easy to see that it suffices to have (1.10) for $a + |\beta| \leq 3$.

When $g = g_0$ outside some compact the bound (1.11) follows from the results of Burq [2], where he proved a similar bound for the cutoff resolvent. This was improved in [7] for metrics satisfying $g_{ij} - \delta_{ij} = O(e^{-\varepsilon |z|^2 + \alpha})$, $\epsilon_0 > 0$. Burq [3] has recently extended his result to long-range metric perturbations assuming that $g_{ij}$ admit an analytic extension from $\{x \in \mathbb{R}^n : |x| \geq \rho_0\}$, $\rho_0 \gg 1$, to $\{z \in \mathbb{C}^n : |\Re z| \geq \rho_0, |\Im z| \leq \gamma_0 |\Re z|\}$, $\gamma_0 > 0$. In particular, this implies that if (1.8) holds with $\alpha = 0$, it holds for any $\alpha$. He used the complex scaling method to show that there are no resonances in an exponentially small neighbourhood of the real axis. In particular, it follows from [3] that one has an analogue of (1.11) for the cutoff resolvent, which combined with the result of Bruneau-Petkov [1] imply the bound (1.11) itself in that case. Burq [3] has also proved an analogue of (1.12) with $\chi_a$ replaced by the characteristic function of $a < |x| < b$ with $b > a \gg 1$.

The bound (1.4) is proved in [8] for manifolds which have a similar structure at infinity as the manifold $M$ above, but under the restriction that the metric on the ends $X_k$, $k = 1, 2$, is of the form $dr^2 + p_k(r)^2 \sigma_k$, where $\sigma_k$ does not depend on $r$, and $p_k(r)$ are smooth positive functions satisfying conditions analogous to (1.2) and (1.3) above. The fact that we have a separation of variables was used in an essential way in the methods developed in [8]. In the situation we treat in the present paper we do not have such a separation of variables, which requires a different approach. It is based on an idea of Burq [3] which consists of using Carleman estimates outside a sufficiently large compact with a real-valued phase function, $\varphi(r)$, $\varphi'(r) > 0$, depending on the spectral parameter (in our case $\lambda \gg 1$) such that $\varphi' = O(\lambda^{-1} r^{-1})$ outside another compact in which region the estimates are no longer of Carleman type and the lower order terms of the Laplace-Beltrami operator begin to play an important role. We apply this on the elliptic (infinite volume) end $X_1$ - see Proposition 2.3 which is essentially due to Burq (see Propositions 6.2 and 7.2 of [3]), but here we give a different proof in a little bit more general situation. Moreover, our construction of the phase function $\varphi$ is simpler than that one in [3]. Then the problem is to paste together this estimate with estimates on the compact part of the manifold essentially due to Lebeau-Robbiano [4], [5] (see Proposition 3.2 and also Theorem A.2 of [7]), with weighted estimates at the infinity of $X_1$ (see Proposition 2.4) as well as with weighted Carleman estimates on $X_2$ (see Proposition 3.1). This is carried out in Section 4.

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2 Uniform a priori estimates on $X_1$

We begin this section by constructing a real-valued phase function, $\varphi$, with properties described in Lemma 2.1 below. A similar phase function was first constructed by Burq [3]. Here we simplify this construction (as well as some of his arguments) adapting it to our approach.

Let $\lambda \gg 1$ be a big parameter, let $0 < \delta \ll 1$ be independent of $\lambda$, and let $\gamma_0 > 1$ be independent of $\lambda$ and $\delta$. In what follows, $C$ will denote a positive constant independent of $\lambda$. 


while $C'$ will denote a positive constant independent of $\lambda$ and $\delta$. Define the continuous function $\tilde{\varphi}_1(r)$ so that $\tilde{\varphi}_1(r) = (Ar^{-\delta} - 1)^{1/2}$ for $r_1 \leq r < a_1 = A^{1/\delta}$, $\tilde{\varphi}_1(r) = 0$ for $r \geq a_1$, where $A = (r_1 + 2)^\delta (\gamma_0 + 1)^2/4 + 1$. Choose a real-valued function $\varphi \in C_0^\infty((-1, 1))$ such that $\varphi \geq 0$, $\int \varphi = 1$ and $\varphi'' \leq C_0 \varphi$ with some constant $C_0 > 0$, and set $\tilde{\varphi}_2(r) = \epsilon^{-1} \varphi(r/\epsilon)$, $0 < \epsilon \ll 1$. Let $\zeta \in C_0^\infty(\mathbb{R})$ be a real-valued function, $\zeta \geq 0$, equal to 1 in a small neighbourhood of $a_1$ and to zero outside another small neighbourhood of $a_1$. Then the function

$$\varphi_1 = (1 - \zeta)\tilde{\varphi}_1 + \varphi \ast (\zeta\tilde{\varphi}_1)$$

belongs to $C^\infty([r_1, \infty))$ and vanishes for $r \geq a_1 + 1$. Moreover, since $\varphi'\varphi_1 \to \varphi'_1\tilde{\varphi}_1 = -2^{-1}\delta Ar^{1-\delta}$ if $r < a_1$ and to zero if $r > a_1$ as $\epsilon \to 0$, taking $\epsilon > 0$ small enough we can arrange

$$-\varphi'_1(r)\varphi_1(r) \leq C'\delta r^{-1}, \quad \forall r \geq r_1. \quad (2.1)$$

Also, the choice of $\varphi$ guarantees the bound

$$\varphi'(r)^2 \leq C\varphi_1(r), \quad \forall r \geq r_1. \quad (2.2)$$

Define a real-valued function $\varphi \in C^\infty([r_1, +\infty))$ such that $\varphi(r_1) = -1$ and

$$\varphi'(r) = \varphi_1(r) + \lambda^{-1/2}r^{-1} \varphi_2(r)(1 + \lambda^{-1/2}\varphi_3(r))^{-1},$$

where $\varphi_j \in C^\infty([r_1, +\infty))$, $j = 2, 3$, are real-valued functions independent of $\lambda$, $0 \leq \varphi_j(r) \leq 1$, $\varphi'(r) \geq 0$, $\forall r$, chosen so that $\varphi_2 = 0$ for $r \leq a_1'$, $\varphi_2 = 1$ for $r \geq a_2''$, $r_1 + 2 < a_1' < a_2'' \in \text{supp} (1-\zeta)$, $\varphi_3 = 0$ for $r \leq a_2'$, $\varphi_3 = 1$ for $r \geq a_2''$, $a_1 + 1 < a_2' < a_2''$. We also require that

$$r\varphi_2'(r) \leq \frac{1}{4}, \quad \forall r. \quad (2.3)$$

Moreover, near $a_1'$ we choose $\varphi_2$ in the form $\varphi_2(r) = \exp((a_1' - r)^{-1})$ if $r > a_1'$, which guarantees the inequality

$$\varphi_2'(r)^2 \leq C\varphi_2(r), \quad \forall r \geq r_1. \quad (2.4)$$

It is easy also to see that we have the inequalities

$$|\varphi_j'(r)| + |\varphi_j''(r)| + |\varphi_j'''(r)| \leq C r^{-1} \varphi_2(r), \quad j = 1, 3, \quad |\varphi_2'(r)| + |\varphi_2''(r)| + |\varphi_2'''(r)| \leq C \varphi_1(r). \quad (2.5)$$

Note that the choice of the constant $A$ guarantees that $\varphi(r_1 + 2) \geq \gamma_0$.

**Lemma 2.1** The following inequalities hold for $\lambda \geq \lambda_0(\delta) \gg 1$ and $\forall r \geq r_1$:

$$C\lambda^{-1}r^{-1} \leq \varphi'(r) \leq Cr^{-1}, \quad (2.6)$$

$$-\varphi'(r)\varphi''(r) \leq C'\delta r^{-1}, \quad (2.7)$$

$$|\varphi''(r)| \leq C\lambda^{1/2}r^{-1} \varphi'(r), \quad |\varphi''(r)| \leq C\lambda^{1/2}r^{-1} \varphi'(r), \quad (2.8)$$

$$|\varphi'''(r)| \leq C\lambda r^{-1} \varphi'(r), \quad |\varphi'''(r)| \leq C\lambda^{1/2}r^{-1}, \quad (2.9)$$

$$|\varphi^{(4)}(r)| \leq C\lambda^{3/2}r^{-1}, \quad (2.10)$$

$$C'\lambda^{r^{-1}} \varphi'(r) \leq 2\lambda \varphi'(r)^2 + \varphi''(r) \leq C\lambda r^{-1} \varphi'(r). \quad (2.11)$$
\[ C \lambda^{-1} r^{-1} \leq \lambda^{-1} r^{-1} (r \varphi_1(r) + \varphi_2(r)) \leq \varphi'(r) \leq r^{-1} (r \varphi_1(r) + \lambda^{-1/2} \varphi_2(r)) \leq C r^{-1}, \]

which proves (2.6). To prove (2.7) observe that

\[ \varphi''(r) = \varphi'(r) - \lambda^{-1/2} r^{-2} \varphi_2(r)(1 + \lambda^{1/2} \varphi_3(r))^{-1} \]
\[ + \lambda^{-1/2} r^{-1} \varphi''_2(r)(1 + \lambda^{1/2} \varphi_3(r))^{-1} - r^{-1} \varphi_2(r) \varphi''_2(r)(1 + \lambda^{1/2} \varphi_3(r))^{-2}, \]

and hence, in view of (2.1),

\[ - \varphi' \varphi'' = - \varphi_1 \varphi'_1 + \lambda^{-1/2} r^{-2} \varphi_1 \varphi_2(1 + \lambda^{1/2} \varphi_3)^{-1} \]
\[ - \lambda^{-1/2} r^{-1} (\varphi'_1 \varphi_2 + \varphi_1 \varphi'_2)(1 + \lambda^{1/2} \varphi_3)^{-1} \]
\[ + \lambda^{-1/2} r^{-2} \varphi_2^2(1 + \lambda^{1/2} \varphi_3)^{-2} - \lambda^{-1/2} r^{-2} \varphi_2 \varphi'_2(1 + \lambda^{1/2} \varphi_3)^{-2} \]
\[ + \lambda^{-1/2} r^{-2} \varphi_2^3(1 + \lambda^{1/2} \varphi_3)^{-3} \leq C' \delta r^{-1} + C \lambda^{-1/2} r^{-1} \leq 2 C' \delta r^{-1}. \]

Moreover, in view of (2.5) we have

\[ |\varphi''| \leq C \lambda^{-1/2} \varphi_1 + 2 r^{-1} \varphi_2(1 + \lambda^{1/2} \varphi_3)^{-1} + C r^{-1} \varphi_2 \leq C \lambda \varphi'. \]

On the other hand,

\[ \varphi'' \leq 4 \varphi_1^2 + C r^{-1} (\varphi_2 + \varphi_2^2)(1 + \lambda^{1/2} \varphi_3)^{-1}, \]

and hence (2.8) follows in view of (2.2) and (2.4). Furthermore, we have

\[ \varphi''' = \varphi''_1 + 2 \lambda^{-1/2} r^{-3} \varphi_2(1 + \lambda^{1/2} \varphi_3)^{-1} - 2 \lambda^{-1/2} r^{-2} \varphi'_2(1 + \lambda^{1/2} \varphi_3)^{-1} \]
\[ + 2 r^{-2} \varphi''_2(1 + \lambda^{1/2} \varphi_3)^{-2} + \lambda^{-1/2} r^{-1} \varphi''_2(1 + \lambda^{1/2} \varphi_3)^{-1} \]
\[ - r^{-1} \varphi''_2(1 + \lambda^{1/2} \varphi_3)^{-2} + 2 \lambda^{1/2} r^{-2} \varphi''^3(1 + \lambda^{1/2} \varphi_3)^{-3}, \]

and hence \(|\varphi'''| \leq C \lambda r^{-1} \varphi'). On the other hand, in view of (2.5) we have

\[ |\varphi'''| \leq C r^{-1} \varphi_2 + \lambda^{-1/2} \varphi_1 + C \lambda^{1/2} r^{-1} \varphi_2(1 + \lambda^{1/2} \varphi_3)^{-1} \leq C \lambda \varphi', \]

which proves (2.9). In the same way,

\[ |\varphi^{(4)}| \leq |\varphi'''| + C \lambda r^{-2} \varphi_2(1 + \lambda^{1/2} \varphi_3)^{-1} + C \lambda^{-1/2} (|\varphi'_2| + |\varphi''_2| + |\varphi'''_2|) \leq C \lambda^{3/2} r^{-1} \varphi'. \]

To prove (2.11) observe that

\[ 2 \lambda \varphi'^2 + \varphi'' \leq 4 \lambda \varphi_1^2 + 4 r^{-2} \varphi''_2(1 + \lambda^{1/2} \varphi_3)^{-2} + \lambda^{-1/2} r^{-1} \varphi'_2 \leq C \lambda r^{-1} \varphi', \]

and

\[ 2 \lambda \varphi'^2 + \varphi'' \geq 2 \lambda \varphi_1^2 + 2 r^{-2} \varphi''_2(1 + \lambda^{1/2} \varphi_3)^{-2} + \varphi'_1 \]
\[ - \lambda^{-1/2} r^{-2} \varphi_2(1 + \lambda^{1/2} \varphi_3)^{-1} - r^{-1} \varphi_2 \varphi'_2(1 + \lambda^{1/2} \varphi_3)^{-2}. \]

For \( r \geq a + 1 \) we have \( \varphi_1 = \varphi'_1 = 0 \) and hence, in view of (2.3),

\[ 2 \lambda \varphi'^2 + \varphi'' \geq (1 - 2 r \varphi_3 - \lambda^{-1/2}) r^{-2} (1 + \lambda^{1/2} \varphi_3)^{-2} \geq C' r^{-1} \varphi'. \]
For \( r < a_1 + 1 \) we have \( \varphi_3 = 0 \) and hence
\[
2\lambda \varphi'^2 + \varphi'' \geq 2\lambda \varphi_1^2 + \varphi'_1 + r^{-2} \varphi_2^2.
\]
Since \( \varphi'_1(a_1 + 1) = 0 \), there exists \( a_0 < a_1 + 1 \) such that \( |\varphi'_1| \leq (2r)^{-2} \varphi_2^2 \) for \( a_0 \leq r \leq a_1 + 1 \).
Hence, for \( a_0 \leq r \leq a_1 + 1 \),
\[
2\lambda \varphi'^2 + \varphi'' \geq \lambda \varphi'^2 \geq C\lambda^{1/2} r^{-1} \varphi'.
\] (2.12)

For \( r_1 \leq r \leq a_0 \), we have \( |\varphi'_1| \leq C \varphi_1^2 \), which again implies (2.12).

Throughout this section \( \|\cdot\| \) and \( \langle \cdot, \cdot \rangle \) will denote the norm and the scalar product on \( L^2(S_1) \), while the Sobolev space \( H^1(X_1, d\text{Vol}_g) \) will be equipped with the semiclassical norm given by
\[
\|u\|_{H^1(X_1, d\text{Vol}_g)}^2 = \|u\|_{L^2(X_1, d\text{Vol}_g)}^2 + \|D_r u\|_{L^2(X_1, d\text{Vol}_g)}^2 + \int_{r_1}^{\infty} \left( \|u(r, \cdot)\|^2 + \|D_r u(r, \cdot)\|^2 + \sum_{i,j} \langle g_{ij} D_{\delta_i} u(r, \cdot), D_{\delta_j} u(r, \cdot) \rangle \right) dr,
\]
where \( D_r = (i\lambda)^{-1} \partial_r \), \( D_{\delta_j} = (i\lambda)^{-1} \partial_{\delta_j} \). Denote by \( L^2(X_1) \) and \( H^1(X_1) \) the spaces equipped with the norms
\[
\|u\|_{L^2(X_1)}^2 = \int_{r_1}^{\infty} \|u(r, \cdot)\|^2 dr,
\]
\[
\|u\|_{H^1(X_1)}^2 = \int_{r_1}^{\infty} \left( \|u(r, \cdot)\|^2 + \|D_r u(r, \cdot)\|^2 + \sum_{i,j} \langle g_{ij} D_{\delta_i} u(r, \cdot), D_{\delta_j} u(r, \cdot) \rangle \right) dr.
\]
It is easy to see that
\[
\|u\|_{L^2(X_1, d\text{Vol}_g)} = \|p^{1/2} u\|_{L^2(X_1)}, \quad \|u\|_{H^1(X_1, d\text{Vol}_g)} \simeq \|p^{1/2} u\|_{H^1(X_1)}.
\]
Finally, given an \( a \geq r_1 \) and functions \( u(r, \theta), v(r, \theta) \), we denote
\[
\|u\|_{L^2(\partial X_1, a)} := \|u(a, \cdot)\|, \quad \langle u, v \rangle_{L^2(\partial X_1, a)} := \langle u(a, \cdot), v(a, \cdot) \rangle,
\]
\[
\|u\|_{H^1(\partial X_1, a)}^2 := \|u(a, \cdot)\|^2 + \sum_{i,j} \langle g_{ij} D_{\delta_i} u(a, \cdot), D_{\delta_j} u(a, \cdot) \rangle.
\]
It is clear from the definition of the function \( \varphi \) above that there exists an \( a \geq r_1 \) such that \( \varphi'(r) = \lambda^{-1} r^{-1} \) for \( r \geq a \). The main result in this section is the following

**Theorem 2.2** Let \( u \in H^2(X_1, d\text{Vol}_g) \), \( u = 0 \) on \( \partial X_1 \), be such that \( r^s (\Delta_{X_1} - \lambda^2 + i\varepsilon) u \in L^2(X_1, d\text{Vol}_g) \) for \( \lambda > 0 \), \( 0 < \varepsilon \leq 1 \) and \( 0 < s - 1/2 \ll 1 \). Then, for a suitable choice of the parameter \( \delta > 0 \), there exist constants \( C_1, C_2, \lambda_0 > 0 \) (independent of \( \lambda \) and \( \varepsilon \)) so that for \( \lambda \geq \lambda_0 \) we have
\[
\|e^{\lambda(r - \varphi(r))} u\|_{H^1(X_1 \setminus X_1, d\text{Vol}_g)}^2 + \|r^{-s} u\|_{H^1(X_1 \setminus X_1, d\text{Vol}_g)}^2 \leq C_1 \lambda^{-2} \|e^{\lambda(r - \varphi(r))} (\Delta_{X_1} - \lambda^2 + i\varepsilon) u\|_{L^2(X_1 \setminus X_1, d\text{Vol}_g)}^2
\]
\[
+ C_1 \lambda^{-2} \|r^s (\Delta_{X_1} - \lambda^2 + i\varepsilon) u\|_{L^2(X_1 \setminus X_1, d\text{Vol}_g)}^2 - C_2 \lambda^{-1} \text{Im} \langle \partial_r u, u \rangle_{L^2(\partial X_1 \setminus X_1)}.
\] (2.13)
Proof. Denote

\[ P = p_1^{1/2}(\lambda^{-2}\Delta_X - 1 + \varepsilon)p_1^{-1/2} = D_r^2 + L_r - 1 + V + \varepsilon, \]

where \( 0 < \varepsilon = O(\lambda^{-2}) \), \( L_r = \lambda^{-2}A_{1,r} \), \( V = \lambda^{-2}q_1 \), and

\[ P\varphi = e^{\lambda\varphi}Pe^{-\lambda\varphi} = P - \varphi'(r)^2 + \lambda^{-1}\varphi''(r) + 2i\varphi'(r)D_r. \]

We will first prove the following

**Proposition 2.3** Let \( u \in H^2(X_1 \setminus X_{1,\varepsilon}) \), \( u = 0 \), \( \partial_r u = 0 \) on \( \partial X_1 \cup \partial X_{1,\varepsilon} \). Then, there exist constants \( C, \lambda_0 > 0 \) (independent of \( \lambda \) and \( \varepsilon \)) so that for \( \lambda \geq \lambda_0 \) we have

\[ \left\| \left( \varphi' / r \right)^{1/2} u \right\|^2_{L^2(X_1 \setminus X_{1,\varepsilon})} \leq C\lambda \left\| P\varphi u \right\|^2_{L^2(X_1 \setminus X_{1,\varepsilon})}. \]  

(2.14)

Proof. Let \( \psi(r) \in C^\infty([r_1, a]) \) be a real-valued function. Integrating by parts one can easily get the identity

\[ \text{Re} \langle \psi P\varphi u, v \rangle_{L^2(X_1 \setminus X_{1,\varepsilon})} = \langle \psi D_r u, D_ru \rangle_{L^2(X_1 \setminus X_{1,\varepsilon})} + \langle \psi L_r u, u \rangle_{L^2(X_1 \setminus X_{1,\varepsilon})} - \langle \left( \psi + \varphi \varphi' - \lambda^{-2}q_1 - \lambda^{-1}\varphi' + 2^{-1}\lambda^{-2}\varphi'' \right) u, u \rangle_{L^2(X_1 \setminus X_{1,\varepsilon})}. \]  

(2.15)

Set

\[ F(r) = -\langle \left( L_r - 1 + W \right) u(r, \cdot), u(r, \cdot) \rangle + \left\| D_r u(r, \cdot) \right\|^2, \]

where \( W = \lambda^{-2}q_1 - \varphi'^2 + \lambda^{-1}\varphi'' \). We have

\[ \begin{align*}
F'(r) & = -2\text{Re} \langle L_r u(r, \cdot), u'(r, \cdot) \rangle - 2\text{Re} \langle D_r^2 u(r, \cdot), u'(r, \cdot) \rangle + 2\text{Re} \langle \left( 1 - W \right) u(r, \cdot), u'(r, \cdot) \rangle \\
& \quad - \langle \left[ \partial_r, L_r \right] u(r, \cdot), u(r, \cdot) \rangle - \langle W'u(r, \cdot), u(r, \cdot) \rangle \\
& = -2\text{Re} \langle P\varphi u(r, \cdot), u'(r, \cdot) \rangle + 4\lambda\varphi' \left\| D_r u(r, \cdot) \right\|^2 - 2\varepsilon \text{Im} \langle u(r, \cdot), u'(r, \cdot) \rangle \\
& \quad - \langle \left[ \partial_r, L_r \right] u(r, \cdot), u(r, \cdot) \rangle - \langle W'u(r, \cdot), u(r, \cdot) \rangle.
\end{align*} \]

Multiplying this identity by \( \varphi' \) and integrating with respect to \( r \) lead to

\[ \int_{r_1}^a \varphi' F'dr = -2\text{Re} \int_{r_1}^a \langle \varphi' P\varphi u, u' \rangle dr + 4\lambda \int_{r_1}^a \left\| \varphi' D_r u \right\|^2 dr \\
-2\varepsilon \text{Im} \int_{r_1}^a \langle \varphi' u, u' \rangle dr - \int_{r_1}^a \langle \varphi' \left[ \partial_r, L_r \right] u, u \rangle dr - \int_{r_1}^a \langle \varphi' W'u, u \rangle dr. \]  

(2.16)

On the other hand, we have

\[ \int_{r_1}^a \varphi' F'dr = -\int_{r_1}^a \varphi'' F'dr \\
= \text{Re} \int_{r_1}^a \langle \varphi'' L_r u, u \rangle dr - \int_{r_1}^a \langle \varphi'' D_r u, D_r u \rangle dr - \int_{r_1}^a \langle \varphi'' (1 - W) u, u \rangle dr \\
= \text{Re} \int_{r_1}^a \langle \varphi'' P\varphi u, u \rangle dr - 2\int_{r_1}^a \langle \varphi'' D_r u, D_r u \rangle dr \\
+ \int_{r_1}^a \langle (\lambda^{-1}\varphi'' + \lambda^{-1}\varphi'\varphi'' + 2^{-1}\lambda^{-2}\varphi''')u, u \rangle dr, \]  

(2.17)

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where we have used (2.15) with \( \psi = \varphi'' \). Combining (2.16) and (2.17) we get the identity
\[
2 \int_{r_1}^{a} \left\langle (2\lambda \varphi'^2 + \varphi'') D_r u, D_r u \right\rangle dr - \int_{r_1}^{a} \left\langle \varphi' (\partial_r, L_r) u, u \right\rangle dr
\]
\[
= 2 \text{Re} \int_{r_1}^{a} \left\langle \varphi' P_u u', u' \right\rangle dr + \text{Re} \int_{r_1}^{a} \left\langle \varphi'' P_u u, u \right\rangle dr + 2 \varepsilon \text{Im} \int_{r_1}^{a} \left\langle \varphi' u, u' \right\rangle dr
\]
\[
+ \int_{r_1}^{a} \left\langle (-2 \varphi'^2 \varphi'' + \lambda^{-1} \varphi'^2 - 2 \lambda^{-1} \varphi' \varphi'' + 2 \lambda^{-1} \varphi' \varphi'' + \lambda^{-2} \varphi'^2) ) u, u \right\rangle dr.
\]  
(2.18)

It is easy to see that (1.3) implies
\[
-[\partial_r, L_r] \geq \frac{C}{r} L_r, \quad C > 0,
\]  
(2.19)

and hence in view of (1.2) and Lemma 2.1 we conclude from (2.18)
\[
\int_{r_1}^{a} \left\| (\varphi'/r)^{1/2} D_r u \right\|^2 dr + \int_{r_1}^{a} \left\| (\varphi'/r)^{1/2} L_r u \right\|^2 dr
\]
\[
\leq O(\lambda) \int_{r_1}^{a} \left\| P_u u \right\|^2 dr + C \delta \int_{r_1}^{a} \left\| (\varphi'/r)^{1/2} u \right\|^2 dr.
\]  
(2.20)

for \( \lambda \geq \lambda_0 (a, \delta) \gg 1 \), where \( C > 0 \) does not depend on \( \lambda, \delta \) and \( a \). On the other hand, by (2.15) used with \( \psi = r^{-1} \varphi' \) we have
\[
\int_{r_1}^{a} \left\langle \left( r^{-1} \varphi' (1 + \varphi'^2 + \lambda^{-1} \varphi'' - \lambda^{-1} r^{-1} \varphi' + \lambda^{-2} r^{-2} - \lambda^{-2} \varphi_1) \right) u, u \right\rangle dr
\]
\[
= \int_{r_1}^{a} \left\langle r^{-1} \varphi' D_r u, D_r u \right\rangle dr + \int_{r_1}^{a} \left\langle r^{-1} \varphi' L_r u, u \right\rangle dr - \text{Re} \int_{r_1}^{a} \left\langle P_u u, r^{-1} \varphi' u \right\rangle dr,
\]
and hence, in view of Lemma 2.1 and (1.2), we get
\[
\frac{1}{4} \int_{r_1}^{a} \left\| (\varphi'/r)^{1/2} u \right\|^2 dr \leq \int_{r_1}^{a} \left\| (\varphi'/r)^{1/2} D_r u \right\|^2 dr + \int_{r_1}^{a} \left\| (\varphi'/r)^{1/2} L_r u \right\|^2 dr + \int_{r_1}^{a} \left\| P_u u \right\|^2 dr.
\]  
(2.21)

Now (2.14) follows from (2.20) and (2.21). \( \square \)

**Proposition 2.4** Let \( u \in H^2(X_{1,a}) \) be such that \( r^s P u \in L^2(X_{1,a}) \) for \( 1/2 < s \leq (1 + \delta_0)/2 \). Then, \( \forall \gamma > 0 \) there exist constants \( C_1, C_2, \lambda_0 > 0 \) (which may depend on \( \gamma \) but are independent of \( \lambda \) and \( \varepsilon \)) so that for \( \lambda \geq \lambda_0 \) we have
\[
\| r^{-s} u \|_{H^1(X_{1,a+1})}^2 \leq C_1 \lambda^{2} \| r^s P u \|_{L^2(X_{1,a})}^2 - C_2 \lambda^{-1} \text{Im} \left\langle \partial_r u, u \right\rangle_{L^2(\partial X_{1,a})} + \gamma \| u \|_{H^1(X_{1,a}\setminus X_{1,a+1})}^2
\]  
(2.22)

**Proof.** Choose a real-valued function \( \phi \in C^\infty(\mathbb{R}) \), \( 0 \leq \phi \leq 1 \), such that \( \phi(r) = 0 \) for \( r \leq a + 1/2 \), \( \phi(r) = 1 \) for \( r \geq a + 2/3 \) and \( \phi'(r) \geq 0 \), \( \forall r \). Integrating by parts we get
\[
\left\langle r^{-2s} (L_r - 1 + V) \phi u, \phi u \right\rangle_{L^2(X_{1,a})} + \| r^{-s} D_r (\phi u) \|_{L^2(X_{1,a})}^2
\]
\[ = \text{Re} \langle r^{-2s} P(\phi u), \phi u \rangle_{L^2(X_{1,a})} + 2 \lambda^{-2} \text{Re} \langle r^{-2s-1} (\phi u)', \phi u \rangle_{L^2(X_{1,a})}, \]

and hence
\[
\left| \langle r^{-2s} (L_r - 1 + V)\phi u, \phi u \rangle_{L^2(X_{1,a})} + \| r^{-s} D_r (\phi u) \|_{L^2(X_{1,a})}^2 \right| \leq O(\lambda) \| P(\phi u) \|_{L^2(X_{1,a})}^2 + O\left( \left( \| r^{-s} \phi u \|_{L^2(X_{1,a})}^2 + \| r^{-s} D_r (\phi u) \|_{L^2(X_{1,a})}^2 \right) \right). \tag{2.23}
\]

We also have
\[
\varepsilon \| u \|_{L^2(X_{1,a})}^2 = \text{Im} \langle P u, u \rangle_{L^2(X_{1,a})} - \lambda^{-2} \text{Im} \langle \langle u', u \rangle_{L^2(\partial X_{1,a})} \leq \gamma^{-1} \lambda \| u \|_{H^1(X_{1,a})}^2 + \gamma \lambda^{-1} \| \phi u \|_{H^1(X_{1,a})}^2 - \lambda^{-2} \text{Im} \langle \langle u', u \rangle_{L^2(\partial X_{1,a})},
\forall \gamma > 0, \text{ and }
\| D_r (\phi u) \|_{L^2(X_{1,a})} \leq 2 \| \phi u \|_{L^2(X_{1,a})}^2 + \| P(\phi u) \|_{L^2(X_{1,a})}^2 \leq 2 \| u \|_{L^2(X_{1,a})}^2 + \| P u \|_{L^2(X_{1,a})}^2 + O(\lambda^{-2}) \| \phi u \|_{H^1(X_{1,a})}^2,
\]

where \( \phi_1 \in C_0^\infty ([a, a + 1]) \), \( \phi_1 = 1 \) on \([a + 1/3, a + 3/4]\). Hence,
\[
\varepsilon \lambda \left( \| \phi u \|_{L^2(X_{1,a})}^2 + \| D_r (\phi u) \|_{L^2(X_{1,a})}^2 \right) \leq O_\gamma(\lambda^2) \| u \|_{L^2(X_{1,a})}^2 + \gamma \| r^{-s} \phi u \|_{H^1(X_{1,a})}^2 - \lambda^{-2} \text{Im} \langle \langle u', u \rangle_{L^2(\partial X_{1,a})}, \tag{2.24}
\]

\( \forall \gamma > 0 \). Set
\[
E(r) = -\langle (L_r - 1 + V)\phi u(r, \cdot), \phi u(r, \cdot) \rangle = \| D_r (\phi u)(r, \cdot) \|^2.
\]

We have
\[
E'(r) = -\langle [\phi, L_r] \phi u(r, \cdot), \phi u(r, \cdot) \rangle - \langle V' \phi u(r, \cdot), \phi u(r, \cdot) \rangle - 2\varepsilon \text{Im} \langle \phi u(r, \cdot), (\phi u)'(r, \cdot) \rangle - 2\lambda \text{Im} \langle P(\phi u)(r, \cdot), D_r (\phi u)(r, \cdot) \rangle \]
\[
= -\langle [\phi, L_r] \phi u(r, \cdot), \phi u(r, \cdot) \rangle - \langle V' \phi u(r, \cdot), \phi u(r, \cdot) \rangle - 2\varepsilon \text{Im} \langle \phi u(r, \cdot), (\phi u)'(r, \cdot) \rangle - 2\lambda \text{Im} \langle P \phi u(r, \cdot), D_r (\phi u)(r, \cdot) \rangle \]
\[
- 2\lambda \text{Im} \langle [P, \phi] u(r, \cdot), D_r (\phi u)(r, \cdot) \rangle = 2\lambda \text{Im} \langle [P, \phi] u(r, \cdot), [D_r, \phi] u(r, \cdot) \rangle.
\]

Since
\[
[P, \phi] = [D_r^2, \phi] = -\lambda^{-2} \phi'' - 2i\lambda^{-1} \phi' D_r,
\]
we obtain in view of (2.19),
\[
E'(r) \geq \frac{C}{r} \langle L_r (\phi u)(r, \cdot), \phi u(r, \cdot) \rangle - \varepsilon \lambda \left( \| \phi u(r, \cdot) \|^2 + \| D_r (\phi u)(r, \cdot) \|^2 \right) - O(\gamma) r^{-2s} \left( \| \phi u(r, \cdot) \|^2 + \| D_r (\phi u)(r, \cdot) \|^2 \right) - O(\lambda^{-1}) \left( \| \phi u(r, \cdot) \|^2 + \| D_r (\phi u)(r, \cdot) \|^2 \right) + 4\phi'' \| D_r u(r, \cdot) \|^2 - O_\gamma(\lambda^2) r^{2s} \| P u(r, \cdot) \|^2.
\]

Since \( \phi \phi' \geq 0 \), we deduce
\[
E'(r) \geq \frac{C}{r} \langle L_r (\phi u)(r, \cdot), \phi u(r, \cdot) \rangle - \varepsilon \lambda \left( \| \phi u(r, \cdot) \|^2 + \| D_r (\phi u)(r, \cdot) \|^2 \right)
- O(\gamma) r^{-2s} \left( \| u(r, \cdot) \|^2 + \| D_r u(r, \cdot) \|^2 \right) - O_\gamma(\lambda^2) r^{2s} \| P u(r, \cdot) \|^2, \tag{2.25}
\]

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Integrating (2.25) from $t \geq a$ to $+\infty$ and using that $L_r \geq 0$ and (2.24), we get

$$E(t) \leq O(\gamma)\|r^{-s}u\|_{H^1(X_1, a)}^2 + O(\gamma^2)\|r^sPu\|_{L^2(X_1, a)}^2 - 3\lambda^{-1}\text{Im} \langle u', u \rangle_{L^2(\partial X_1, a)},$$  \hspace{1cm} (2.26)

\forall \gamma > 0. Multiplying (2.26) by $t^{-2s}$ and integrating from $a$ to $+\infty$ yield (with a constant $C > 0$):

$$\int_a^\infty r^{-2s}E(r)dr \leq O(\gamma)\|r^{-s}u\|_{H^1(X_1, a)}^2$$

$$+O(\gamma^2)\|r^sPu\|_{L^2(X_1, a)}^2 - C\lambda^{-1}\text{Im} \langle u', u \rangle_{L^2(\partial X_1, a)},$$  \hspace{1cm} (2.27)

\forall \gamma > 0. On the other hand, multiplying (2.25) by $r^{1-2s}$, integrating from $a$ to $+\infty$, using (2.23), (2.24) and the identity

$$\int_a^\infty r^{1-2s}E'(r)dr = (2s - 1)\int_a^\infty r^{-2s}E(r)dr,$$

we obtain (with a new constant $C > 0$):

$$\|r^{-s}L_r^{1/2}(\phi u)\|_{L^2(X_1, a)}^2 \leq O(\gamma)\|r^{-s}u\|_{H^1(X_1, a)}^2$$

$$+O(\gamma^2)\|r^sPu\|_{L^2(X_1, a)}^2 - C\lambda^{-1}\text{Im} \langle u', u \rangle_{L^2(\partial X_1, a)},$$  \hspace{1cm} (2.28)

\forall \gamma > 0. Combining (2.23), (2.27) and (2.28), we get (with possibly a new constant $C > 0$):

$$\|r^{-s}\phi u\|_{H^1(X_1, a)}^2 \leq O(\gamma)\|r^{-s}u\|_{H^1(X_1, a)}^2$$

$$+O(\gamma^2)\|r^sPu\|_{L^2(X_1, a)}^2 - C\lambda^{-1}\text{Im} \langle u', u \rangle_{L^2(\partial X_1, a)},$$  \hspace{1cm} (2.29)

\forall \gamma > 0, which clearly implies (2.22).

Let $u \in H^2(X_1)$, $u = 0$, $\partial_r u = 0$ on $\partial X_1$, be such that $r^sPu \in L^2(X_1)$. Choose a function $\chi \in C^\infty(X_1)$ such that $\chi = 1$ on $X_1 \setminus X_1, a + 2$, $\chi = 0$ on $X_1, a + 3$. Applying Proposition 2.2 to the function $e^{\lambda}\phi u$ (with $a$ replaced by $a + 3$), we get

$$\|e^{\lambda}\phi u\|_{H^1(X_1, a + 2)}^2 \leq O(\lambda^2)\|e^{\lambda}\phi Pu\|_{L^2(X_1, a + 3)}^2 + O(1)\|e^{\lambda}\phi u\|_{H^1(X_1, a + 2)}^2,$$  \hspace{1cm} (2.30)

Since $1 \leq e^{\lambda}(\varphi(r) - \varphi(a)) \leq \text{Const}$ for $a \leq r \leq a + 3$, we deduce

$$\|e^{\lambda}(\varphi(r) - \varphi(a))u\|^2_{H^1(X_1, a)} + \|u\|^2_{H^1(X_1, a + 2)}$$

$$\leq O(\lambda^2)\|e^{\lambda}(\varphi(r) - \varphi(a))Pu\|^2_{L^2(X_1, a)}$$

$$+O(\lambda^2)\|Pu\|^2_{L^2(X_1, a + 3)} + O(1)\|u\|^2_{H^1(X_1, a + 2)}.$$

It is easy to see that (2.13) follows from combining (2.22) and (2.31).
3 Uniform a priori estimates on $X_2$

The purpose of this section is to prove the following

**Proposition 3.1** Let $u \in H^2(X_2, d\text{Vol}_g)$, $u = 0$, $\partial_r u = 0$ on $\partial X_2$. Then $\forall \delta > 0$, $0 < \varepsilon \leq 1$, we have

$$\|r^{-1-\delta} e^{\lambda r^{-2\delta}} u\|_{L^2(X_2, d\text{Vol}_g)} \leq C' \lambda^{3/2} \|e^{\lambda r^{-2\delta}} (\Delta X_2 - \lambda^2 + i\varepsilon) u\|_{L^2(X_2, d\text{Vol}_g)},$$

for $\lambda \geq \lambda_0$ with constants $C', \lambda_0 > 0$ independent of $\lambda, \varepsilon$ and $u$ but depending on $\delta$.

**Proof.** Define the spaces $L^2(X_2)$ and $H^1(X_2)$ analogously to $L^2(X_1)$ and $H^1(X_1)$ introduced in the previous section. Denote $\varphi(r) = r^{-2\delta}, w = e^{\lambda r\varphi}$, and

$$P := p_2^{-1/2} (\lambda^{-2} \Delta X_2 - 1 + i\varepsilon)p_2^{-1/2} = \mathcal{D}_r^2 + L_r - 1 + V + i\varepsilon,$$

$$P_{\varphi} = e^{\lambda r\varphi} p e^{-\lambda r\varphi} = P - \varphi'(r)^2 + \lambda^{-1} \varphi''(r) + 2i\varphi'(r)\mathcal{D}_r,$$

where $0 < \varepsilon = O(\lambda^{-2}), L_r = \lambda^{-2} \Delta_{2r}, V = \lambda^{-2} g_2$. Note that (1.3) implies

$$[\partial_r, L_r] \geq \frac{C}{r} L_r, \quad C > 0.$$

Clearly, (3.1) is equivalent to the estimate

$$\|r^{-1-\delta} w\|_{H^1(X_2)} \leq O(\lambda^{1/2}) \|P_{\varphi} w\|_{L^2(X_2)}.$$ (3.3)

Denote by $P^*$ the adjoint operator of $P_{\varphi}$ with respect to the scalar product in $L^2(X_2)$, and set

$$\text{Re} P_{\varphi} = \frac{P_{\varphi} + P_{\varphi}^*}{2}, \quad \text{Im} P_{\varphi} = \frac{P_{\varphi} - P_{\varphi}^*}{2i}.$$ We have

$$\text{Re} P_{\varphi} = \mathcal{D}_r^2 + L_r - 1 - \varphi'(r)^2 + V, \quad \text{Im} P_{\varphi} = \varphi'(r)\mathcal{D}_r + \mathcal{D}_r \varphi'(r) + \varepsilon.$$ (3.2)

In view of (1.2) and (3.2), and taking into account that

$$\varphi'(r) = -2\delta r^{-2\delta - 1}, \quad \varphi''(r) = 2\delta (2\delta + 1)r^{-2\delta - 2}, \quad \varphi'''(r) = -2\delta (2\delta + 1)(2\delta + 2)r^{-2\delta - 3},$$

it is easy to see that we have, in view of (3.2) and (1.2),

$$\lambda \|P_{\varphi} w\|_{L^2(X_2)}^2 = \lambda \|\text{Re} P_{\varphi} w\|_{L^2(X_2)}^2 + \lambda \|(\text{Im} P_{\varphi}) w\|_{L^2(X_2)}^2 + i\lambda \langle [\text{Re} P_{\varphi}, \text{Im} P_{\varphi}] w, w \rangle_{L^2(X_2)}$$

$$\geq \lambda \|\text{Re} P_{\varphi} w\|_{L^2(X_2)}^2 + \lambda \|(\text{Im} P_{\varphi}) w\|_{L^2(X_2)}^2 + 2 \langle \varphi'' \mathcal{D}_r w, \mathcal{D}_r w \rangle_{L^2(X_2)}$$

$$+ 4 \langle -\varphi' [\partial_r, L_r] w, w \rangle_{L^2(X_2)} + 4 \langle \varphi' \varphi'' w, w \rangle_{L^2(X_2)} - 2 \langle \varphi' V w, w \rangle_{L^2(X_2)}$$

$$- O(\lambda^{-1}) \left( \|r^{-1-\delta} \mathcal{D}_r w\|_{L^2(X_2)}^2 + \|r^{-1-\delta} w\|_{L^2(X_2)}^2 \right)$$

$$\geq C \|r^{-1-\delta} \mathcal{D}_r w\|_{L^2(X_2)}^2 + C \|r^{-1-\delta} L_r^{1/2} w\|_{L^2(X_2)} + O(\lambda^{-1}) \|r^{-1-\delta} w\|_{L^2(X_2)}^2.$$ (3.4)

On the other hand, integrating by parts leads to the identity

$$\text{Re} \langle r^{-2-2\delta} P_{\varphi} w, w \rangle_{L^2(X_2)} = \|r^{-1-\delta} \mathcal{D}_r w\|_{L^2(X_2)}^2$$

$$+ \langle r^{-2-2\delta} (L_r - 1 + V - \varphi'^2 - 4\delta(\delta + 1)\lambda^{-1} r^{-2-2\delta} - (\delta + 1)(2\delta + 3)\lambda^{-2} r^{-2}) w, w \rangle_{L^2(X_2)}.$$
and hence
\[ \frac{1}{2} \| r^{-1-\delta} w \|^2_{L^2(\Omega, X_2)} \leq \| r^{-1-\delta} D_r w \|^2_{L^2(\Omega, X_2)} + \| r^{-1-\delta} L^{1/2}_r w \|^2_{L^2(\Omega, X_2)} + \left( \langle r^{-2-2\delta} P_\varphi w, w \rangle_{L^2(\Omega, X_2)} \right). \]

Since
\[ \left| \langle r^{-2-2\delta} P_\varphi w, w \rangle_{L^2(\Omega, X_2)} \right| \leq \frac{1}{4} \| r^{-1-\delta} w \|^2_{L^2(\Omega, X_2)} + \| P_\varphi w \|^2_{L^2(\Omega, X_2)}, \]
we conclude
\[ \frac{1}{4} \| r^{-1-\delta} w \|^2_{L^2(\Omega, X_2)} \leq \| r^{-1-\delta} D_r w \|^2_{L^2(\Omega, X_2)} + \| r^{-1-\delta} L^{1/2}_r w \|^2_{L^2(\Omega, X_2)} + \| P_\varphi w \|^2_{L^2(\Omega, X_2)} \tag{3.5} \]

Now (3.3) follows from (3.4) and (3.5).

Let now \((M_0, g_0)\) be a compact, connected Riemannian manifold with a \(C^\infty\)-smooth boundary \(\partial M_0\) and a metric \(g_0\) of class \(C^\infty(M_0)\). Denote by \(\Delta_{M_0}\) the (positive) Laplace-Beltrami operator on \((M_0, g_0)\) and let \(U \subseteq M_0, U \neq \emptyset\), be an arbitrary open domain such that \(\partial U \cap \partial M_0 = \emptyset\). Suppose that \(\partial M_0 = \Gamma \cup U, \Gamma \neq \emptyset, \Gamma \neq \emptyset, \Gamma \cap \Gamma = \emptyset\), and given \(0 < \varepsilon_0 \ll 1\) denote \(M_{0, \varepsilon_0} = M_0 \setminus \{ x \in M_0 : \text{dist}(x, \partial M_0) \leq \varepsilon_0 \}, \tilde{M}_{0, \varepsilon_0} = M_0 \setminus \{ x \in M_0 : \text{dist}(x, \Gamma) \leq \varepsilon_0 \}\).

Let \(U \subseteq M_{0, 2\varepsilon_0}\). The following proposition is proved in [8] (see Theorem 3.2 of [8]) and we omit the proof.

**Proposition 3.2** Let \(u \in H^2(M_0)\) satisfy either Dirichlet or Neumann boundary conditions on \(\Gamma\). Then, \(\forall \beta > 0 \ \exists C_\beta, \gamma_\beta > 0\) (independent of \(u\) and \(\lambda\) below but depending on \(U\)) so that we have
\[ \| u \|^2_{H^1(M_{0, \varepsilon_0})} \leq C_\beta e^{\gamma_\beta |M|} \| (\Delta_{M_0} - \lambda^2) u \|^2_{L^2(M_0)} + C_\beta e^{\gamma_\beta |M|} \| u \|^2_{H^1(U)} + e^{-\beta |M|} \| u \|^2_{H^1(M_0 \setminus \tilde{M}_{0, \varepsilon_0})}, \quad \forall \lambda \in \mathbb{C}. \tag{3.6} \]

**4 Proof of Theorem 1.1**

Let \(u \in D(G)\) be such that \(\chi_1^{-1} s_1 u \in L^2(M, d\text{Vol}_\varphi)\), where \(s_1\) and \(s_2\) are as in Theorem 1.1. Let \(\chi_2 \in C^\infty(M)\), \(\chi_2 = 0\) on \(M \setminus X_{2,r_2+1}\), \(\chi_2 = 1\) on \(X_{2,r_2+2}\). Applying Proposition 3.1 (with \(\delta = s_2 - 1\)) to \(\chi_2 u\) yields
\[ \| r^{-s_2} u \|^2_{H^1(X_{2,r_2+2}, d\text{Vol}_\varphi)} \leq e^{\alpha \lambda} \| (\Delta_g - \lambda^2 + i\varepsilon) u \|^2_{L^2(X_2, d\text{Vol}_\varphi)} + e^{-\theta \lambda} \| u \|^2_{H^1(X_{2,r_2+1} \setminus X_{2,r_2+2}, d\text{Vol}_\varphi)}. \tag{4.1} \]

Let \(\chi_1 \in C^\infty(M)\), \(\chi_1 = 1\) on \(M \setminus X_{1,r_1+2}\), \(\chi_1 = 0\) on \(X_{1,r_1+3}\). By Proposition 3.2 applied to the function \(\chi_1 u\) we get
\[ \| \chi_1 u \|^2_{H^1(M \setminus X_{2,r_2+2}, d\text{Vol}_\varphi)} \leq C_\beta e^{\gamma_\beta \lambda} \| (\Delta_g - \lambda^2 + i\varepsilon) \chi_1 u \|^2_{L^2(M \setminus X_{2,r_2+3}, d\text{Vol}_\varphi)} + e^{-\beta \lambda} \| u \|^2_{H^1(X_{2,r_2+2} \setminus X_{2,r_2+3}, d\text{Vol}_\varphi)}, \tag{4.2} \]
\(\forall \beta > 0\) with \(C_\beta, \gamma_\beta > 0\) independent of \(\lambda, \varepsilon\) and \(u\). Hence,
\[ \| u \|^2_{H^1(M \setminus (X_{1,r_1+2} \cup X_{2,r_2+2}), d\text{Vol}_\varphi)} \leq C_\beta e^{\gamma_\beta \lambda} \| (\Delta_g - \lambda^2 + i\varepsilon) u \|^2_{L^2(M \setminus (X_{1,r_1+3} \cup X_{2,r_2+3}), d\text{Vol}_\varphi)} \]

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\[ + C_\beta \lambda^2 e^{\gamma \lambda} \| u \|_{L^2}^2 (X_{r_1, r_2 + 2} \setminus X_{r_1, r_2 + 3}, d\Omega_\beta) + e^{-\beta \lambda} \| u \|_{L^2}^2 (X_{r_2, r_2 + 2} \setminus X_{r_2, r_2 + 3}, d\Omega_\beta), \tag{4.3} \]

\( \forall \beta > 0. \) Combining (4.1) and (4.3), for \( \lambda \gg 1, \) we obtain

\[ \| r^{-s_2} u \|_{L^2}^2 (X_{r_1, r_2 + 2} \setminus X_{r_1, r_3 + 3}, d\Omega) + \| u \|_{L^2}^2 (M \setminus (X_{r_1, r_2 + 2} \cup X_{r_2, r_3 + 2}), d\Omega) \]

\[ \leq e^{-\alpha \lambda} \| (\Delta_g - \lambda^2 + i \varepsilon) u \|_{L^2}^2 (X_{r_2, r_2 + 2}, d\Omega) \]

\[ + e^{s_2 \gamma \lambda} \| (\Delta_g - \lambda^2 + i \varepsilon) u \|_{L^2}^2 (M \setminus \{X_{r_1, r_2 + 3} \cup X_{r_2, r_3 + 3}\}, d\Omega) \]

\[ + e^{s_2 \gamma \lambda} \| u \|_{L^2}^2 (X_{r_1, r_3 + 2} \setminus X_{r_1, r_3 + 3}, d\Omega), \tag{4.4} \]

with a constant \( \gamma_1 > 0 \) independent of \( \lambda, \varepsilon, \) and \( u. \) Let \( r_1 < b_1 < b_2 < r_1 + 1 \) be such that \( \varphi(b_1) < \varphi(b_2) < 0 \) and choose \( \chi_1 \in C(\mathcal{M}), \chi_1 = 0 \) on \( M \setminus X_{b_1, b_2}, \chi_1 = 1 \) on \( X_{b_1, b_2}. \) By Theorem 2.2 applied to \( \chi_1 u \) (with \( \gamma_0 = \gamma_1 + 1, s = s_1 \)), we get

\[ \| e^{\lambda \varphi} u \|_{L^2}^2 (X_{b_1, b_2} \setminus X_{b_1, b_2}, d\Omega) + e^{2\lambda \varphi(s)} \| r^{-s_1} u \|_{L^2}^2 (X_{b_1, b_2}, d\Omega) \]

\[ \leq O (\lambda^{-2}) \| e^{\lambda \varphi} (\Delta_g - \lambda^2 + i \varepsilon) u \|_{L^2}^2 (X_{b_1, b_2}, d\Omega) + O (\lambda^{-2}) e^{2\lambda \varphi(s)} \| r^{-s_1} (\Delta_g - \lambda^2 + i \varepsilon) u \|_{L^2}^2 (X_{b_1, b_2}, d\Omega) \]

\[ - C\lambda^{-1} e^{2\lambda \varphi(s)} \text{Im} (\partial_2 u, u)_{L^2} (X_{b_1, b_2}, d\Omega), \tag{4.5} \]

with some \( c, C > 0. \) Since \( \varphi(r) \geq \gamma_1 + 1 \) for \( r \geq r_1 + 2, \) by combining (4.4) and (4.5) one can absorb the last terms in the right-hand sides and conclude

\[ \| r^{-s_2} u \|_{L^2}^2 (X_{r_1, r_2 + 2} \setminus X_{r_1, r_3 + 3}, d\Omega) + \| u \|_{L^2}^2 (M \setminus (X_{r_1, r_2 + 2} \cup X_{r_2, r_3 + 2}), d\Omega) \]

\[ + e^{\lambda \varphi} u \|_{L^2}^2 (X_{b_1, b_2} \setminus X_{b_1, b_2}, d\Omega) + e^{2\lambda \varphi(s)} \| r^{-s_1} u \|_{L^2}^2 (X_{b_1, b_2}, d\Omega) \]

\[ \leq e^{-\alpha \lambda} \| (\Delta_g - \lambda^2 + i \varepsilon) u \|_{L^2}^2 (X_{r_2, r_2 + 2}, d\Omega) + e^{s_2 \gamma \lambda} \| (\Delta_g - \lambda^2 + i \varepsilon) u \|_{L^2}^2 (M \setminus \{X_{r_1, r_2 + 3} \cup X_{r_2, r_3 + 3}\}, d\Omega) \]

\[ + O (\lambda^{-2}) e^{\lambda \varphi} (\Delta_g - \lambda^2 + i \varepsilon) u \|_{L^2}^2 (X_{r_2, r_2 + 2}, d\Omega) \]

\[ + O (\lambda^{-2}) e^{2\lambda \varphi(s)} \| r^{-s_1} (\Delta_g - \lambda^2 + i \varepsilon) u \|_{L^2}^2 (X_{b_1, b_2}, d\Omega) \]

\[ + C\lambda^{-1} e^{2\lambda \varphi(s)} \text{Im} (\partial_2 u, u)_{L^2} (X_{b_1, b_2}, d\Omega), \tag{4.6} \]

On the other hand, by Green’s formula we have

\[ - \text{Im} (\partial_2 u, u)_{L^2} (\partial X_{b_1, b_2}) = - \text{Im} (\partial_2 u, u)_{L^2} (M \setminus X_{b_1, b_2}, d\Omega) - \varepsilon \| u \|_{L^2}^2 (M \setminus X_{b_1, b_2}, d\Omega) \]

\[ \leq e^{-\beta \lambda} \| \rho_{s_2} u \|_{L^2}^2 (M \setminus X_{b_1, b_2}, d\Omega) + e^{\beta \lambda} \| \rho_{s_2}^{-1} (\Delta_g - \lambda^2 + i \varepsilon) u \|_{L^2}^2 (M \setminus X_{b_1, b_2}, d\Omega), \tag{4.7} \]

\( \forall \beta > 0, \) where \( \rho_0 \in C(\mathcal{M}), \rho_2 = r^{-s_1} \) on \( X_{r_2, r_2 + 1}, \rho_1 = 1 \) on \( M \setminus X_2. \) Combining (4.6) and (4.7) leads to the estimate

\[ e^{-\alpha \lambda} \| \rho_{s_2} u \|_{L^2}^2 (M \setminus X_{b_1, b_2}, d\Omega) + e^{s_2 \gamma \lambda} \| \rho_{s_2}^{-1} (\Delta_g - \lambda^2 + i \varepsilon) u \|_{L^2}^2 (M \setminus X_{b_1, b_2}, d\Omega), \]

\[ \leq e^{s_2 \gamma \lambda} \| \rho_{s_2}^{-1} (\Delta_g - \lambda^2 + i \varepsilon) u \|_{L^2}^2 (M \setminus X_{b_1, b_2}, d\Omega) + O (\lambda^{-2}) \| r^{-s_1} (\Delta_g - \lambda^2 + i \varepsilon) u \|_{L^2}^2 (X_{b_1, b_2}, d\Omega), \tag{4.8} \]

with some constants \( c_1, c_2 > 0. \) Hence,

\| X_{s_1, s_2} u \|_{L^2}^2 (M, d\Omega) \leq C e^{s_1 \lambda} \| (\Delta_g - \lambda^2 + i \varepsilon) u \|_{L^2}^2 (M, d\Omega), \tag{4.9} \]

for \( \lambda \geq \lambda_0, \) with some constants \( C, \lambda_0, \gamma > 0 \) independent of \( \lambda, \varepsilon, \) and \( u, \) which implies the existence of the limit in Theorem 1.1 as well as the bound (1.4) (with \( z = \lambda^2 \)).

Let now \( \Delta_g - \lambda^2 + i \varepsilon \equiv 0 \) in \( M \setminus X_{s_1, s_2}. \) Then (4.8) yields

\[ \| r^{-s_1} u \|_{L^2}^2 (X_{s_1, s_2}, d\Omega) \leq O (\lambda^{-1}) \| r^{s_1} (\Delta_g - \lambda^2 + i \varepsilon) u \|_{L^2}^2 (X_{s_1, s_2}, d\Omega), \tag{4.10} \]

which clearly implies (1.5).
References


