On the Density of States of Periodic Media
in the Large Coupling Limit

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ON THE DENSITY OF STATES OF PERIODIC MEDIA
IN THE LARGE COUPLING LIMIT

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Abstract Let $\Omega_\circ$ be a domain in the cube $(0, 2\pi)^n$, and let $\chi_\tau(x)$ be a function that equals 1 inside $\Omega_\circ$, equals $\tau$ in $(0, 2\pi)^n \setminus \Omega_\circ$, and that is extended periodically to $\mathbf{R}^n$. It is known that, in the limit $\tau \to \infty$, the spectrum of the operator $-\nabla \chi_\tau(x)\nabla$ exhibits the band-gap structure. We establish the asymptotic behavior of the density of states function in the bands.

1. Introduction

Let $\Omega_\circ$ be a connected domain with smooth boundary lying inside of the open cube $(0, 2\pi)^n$ in $\mathbf{R}^n$, and let $\Omega = \{\Omega_\circ + 2\pi m : m \in \mathbf{Z}^n\}$. Throughout the paper, we assume that the complement of $\Omega_\circ$ in $(0, 2\pi)^n$ is connected. For a positive number $\tau$, we define a divergence type operator operator in $\mathbf{R}^n$

$$L_\tau = -\nabla \chi_\tau(x)\nabla$$

where

$$\chi_\tau(x) = \begin{cases} 1, & \text{if } x \in \Omega_\circ; \\ \tau, & \text{otherwise.} \end{cases}$$

The operator $L_\tau$ is understood in the sense of the corresponding quadratic form

$$(L_\tau u, u) = \int \chi_\tau(x)|\nabla u(x)|^2 dx. \quad (1.1)$$

Recently, Hempel and Lienau [HL1] showed that, under some mild assumptions, as $\tau \to \infty$, more and more gaps in the spectrum of the operator $L_\tau$ open. Previously, Figotin and Kuchment [FK1,2] and Hempel [H] proved that spectral gaps form in some operators of this type.

Let $\lambda_j$ be the eigenvalues of the Dirichlet Laplacian in $\Omega_\circ$. For making our exposition simpler, we assume that all these eigenvalues are simple,

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots.$$
This assumption is not, in fact, essential for what follows. It is known ([A], [Mi], [U]) that it is satisfied for a generic domain ("generic" meaning that domains for which the spectrum of the Laplacian is simple form a residual set in the space of all domains). We will use the notation \( \lambda_0 = -\infty \). The second assumption about the domain \( \Omega_0 \) is essential. We assume that

\[
\int_{\Omega_0} u_j(x)dx \neq 0
\]

for every eigenfunction \( u_j(x) \) of the Dirichlet Laplacian in \( \Omega_0 \). This assumption also holds for a generic domain ([HL1]). For the convenience of the reader, we prove this genericity statement in the Appendix.

Now we are ready to formulate a theorem of Hempel and Lienau.

**Theorem 1.** ([HL1]) Assume that the spectrum of the Dirichlet Laplacian in \( \Omega_0 \) is simple and that (1.2) holds. Then there exists a sequence \( \nu_k \) that interlaces with \( \lambda_k \),

\[
0 = \nu_1 < \lambda_1 < \nu_2 < \lambda_2 < \nu_3 < \cdots,
\]

such that, for every \( K > 0 \), and for sufficiently large values of \( \tau \), the spectrum of \( L_\tau \) in \( (-\infty, K) \) is purely absolutely continuous, it consists from a finite number of non-overlapping bands, and, as \( \tau \to \infty \) it converges to the intersection of \( (-\infty, K) \) with the union

\[
\bigcup_{j=1}^{\infty} [\nu_j, \lambda_j].
\]

In fact, Hempel and Lienau treated more general class of operators, and they also treated the case when the eigenvalues of the Dirichlet Laplacian in \( \Omega_0 \) are not simple, and when not all eigenfunctions \( u_j(x) \) satisfy (1.2).

In this paper, we will give an alternative proof of Theorem 1. Here, the emphasis is in showing a mechanism that is different from the one used in [HL1] (though, it may very well be the case that one can make translation from one language to another). In addition, we will derive bounds for the integrated density of states function when \( \tau \) is large.

Let \( m_\tau((-\infty, \lambda]) = m_\tau(\lambda) \) be the integrated density of states for the operator \( L_\tau \). It was noticed in [HL1] that, within an interval \([\nu_j, \lambda_j]\), it accumulates near the right end of the interval, \( \lambda_j \), when \( \tau \) is large. The following theorem quantifies this statement.

**Theorem 2.** If the assumptions of Theorem 1 are met, then for any \( \gamma, 0 \leq \gamma < 1 \), and for any positive constant \( C \)

\[
m_\tau(\lambda_j - C\tau^{-\gamma}) = j - 1 + O(\tau^{-n(1-\gamma)/2}), \quad \tau \to \infty.
\]

We will see that \( m(\lambda_{j-1}) = j - 1 \), so Theorem 2 tells us exactly that, on an interval \((\lambda_{j-1}, \lambda_j)\), the integrated density of states function accumulates near the right end \( \lambda_j \).

In section 2, we introduce the Dirichlet-to-Neumann operators that play a crucial role in our proofs, and we explain the basic mechanism behind the proofs of both theorems. In section 3, we prove Theorem 1, and, in section 4, we prove Theorem 2.
2. The Dirichlet-to-Neumann operators

Let \( C = [0, 2\pi]^n \). By \( L_\tau(k) \) we denote the operator defined by a quadratic form

\[
\int_C \chi_\tau(x)|\nabla u|^2 dx,
\]

with the quasi-periodic boundary conditions

\[
u(x + 2\pi m) = e^{2\pi k \cdot m} u(x), \quad m \in \mathbb{Z}^n,
\]

where \( k \in [-1/2, 1/2]^n \). The Floquet theory tells us that the spectrum of \( L_\tau \) in \([0, K]\) is the union of the spectra of \( L_\tau(k) \). It is an easy exercise to see that the spectral problem for \( L_\tau(k) \) can be formulated as the problem of finding a function \( u(x) \) in \( C \) that satisfies the equations

\[
\begin{cases}
\Delta u(x) + \lambda u(x) = 0, & x \in \Omega_0; \\
\tau \Delta u(x) + \lambda u(x) = 0, & x \in C \setminus \bar{\Omega}_0,
\end{cases}
\]

(2.3)

together with the quasiperiodic boundary condition (2.2) and the transmission boundary conditions

\[
u_+(x) = \nu_-(x), \quad \frac{\partial u}{\partial n_+}(x) + \tau \frac{\partial u}{\partial n_-}(x) = 0, \quad x \in \partial \Omega_0,
\]

(2.4)

where \( u_\pm(x) \) are limiting values of \( u(x) \) from inside/outside of \( \Omega_0 \), and \( n_\pm \) are outward normal vectors to \( \partial \Omega_0 \); the vector \( n_+ \) is an outward vector with respect to \( \Omega_+ = \Omega_0 \), and \( n_- \) is an outward vector with respect to \( \Omega_- = C \setminus \bar{\Omega}_0 \). Note that \( n_- = -n_+ \). From this point, we will usually use the notation \( \Omega_+ \) instead of \( \Omega_0 \). The transmission problems of this type were studied in details in the 1950s, and it was shown then that the variational problem for the functional (2.1) is equivalent to the transmission problem (2.3), (2.4); this is an elliptic problem, and its spectrum is discrete (e.g., see [5]). Let

\[
\mu_1(\tau, k) \leq \mu_2(\tau, k) \leq \mu_3(\tau, k) \leq \cdots
\]

be the eigenvalues of the problem (2.2)-(2.4).

Now we introduce the Dirichlet-to-Neumann operators. Let \( \psi(x) \) be a function defined on \( \Gamma = \partial \Omega_+ \), and let \( v_\pm(x) \) be the solution of the problem \( \Delta v_\pm(x) + \lambda v_\pm(x) = 0 \) in \( \Omega_\pm \) that satisfies the Dirichlet boundary condition on \( \Gamma \), and, in the case of \( v_- \), it satisfies the quasi-periodic boundary conditions (2.2). Then, we set

\[
N_\pm \psi(x) = \frac{\partial v_\pm(x)}{\partial n_\pm}.
\]

The operator \( N_\pm = N_\pm(\lambda) \) depends on \( \lambda \) only, and it is defined for all values of \( \lambda \), except of the points of the Dirichlet spectrum of the Laplacian in \( \Omega_+ \). The operator \( N_- = N_-(\lambda, k) \) depends on both \( \lambda \) and \( k \), and it is defined for all values of \( \lambda \), except of the points of the spectrum of the Laplacian in \( \Omega_- \) with the Dirichlet boundary condition on \( \Gamma \) and the quasi-periodic conditions (2.2). Now, the equations (2.3), together with the transmission
boundary conditions (2.4), can be reformulated in the following way:

\[ N(\tau, \lambda, k) = N_+ (\lambda) + \tau N_-(\tau^{-1} \lambda, k) \]  

(2.5)

has a non-trivial kernel. Moreover, the multiplicity of an eigenvalue \( \lambda \) of \( L_\tau (k) \) equals exactly the dimension of the kernel of \( N(\tau, \lambda, k) \).

It is known that the operators \( N_\pm (\lambda) \) are elliptic pseudo-differential operators of order 1, and their principal symbols equal \( \sqrt{\mu} \) where \( (x', \xi') \) is a point in the cotangent bundle to \( \Gamma \). The operator-valued functions \( N_\pm (\lambda) \) of \( \lambda \) (in the case of \( N_- \), we temporarily suppress the \( k \)-dependence in the notations) are differentiable in the sense that the limit of the difference quotient \( (N_\pm (\mu) - N_\pm (\lambda))/(\mu - \lambda) \) when \( \mu \to \lambda \) exists in the operator norm topology. In fact, let \( P_\pm (\lambda) \) be the Poisson operators (actually, \( P_- = P_- (\lambda, k) \)): they assign to a function \( \psi (x) \) on \( \Gamma \) the solution of the problem \( (\Delta + \lambda) u = 0 \) in \( \Omega_\pm \) that equals \( \psi (x) \) on \( \Gamma \), and let \( j_\pm \) be the operators of taking the normal derivative on \( \Gamma \): \( j_\pm u_\pm (x) = \partial u_\pm /\partial n_\pm \). Clearly, \( N_\pm (\lambda) = j_\pm P_\pm (\lambda) \). An easy computation shows that

\[ \frac{N_\pm (\mu) - N_\pm (\lambda)}{\mu - \lambda} = - j_\pm R_\pm (\mu) P_\pm (\lambda), \]

and the difference quotient converges to \( j_\pm R_\pm (\lambda) P_\pm (\lambda) \). Here \( R_\pm (\lambda) \) is the resolvent of the Laplacian in \( \Omega_\pm \), with the Dirichlet boundary conditions on \( \Gamma \) (and the quasiperiodic conditions (2.2) in the case of \( \Omega_- \)). We have shown that

\[ \hat{N}_\pm (\lambda) = \frac{dN_\pm (\lambda)}{d\lambda} = - j_\pm R_\pm (\lambda) P_\pm (\lambda), \]  

(2.6)

It follows from (2.6), and it was shown in [F1] in the pure Dirichlet case (no quasiperiodic conditions), that the operators \( \hat{N}_\pm (\lambda) \) are negative (when they are defined) in the sense that

\[ (\hat{N}_\pm (\lambda) \psi, \psi) < 0 \]  

(2.7)

for every non-zero function \( \psi (x) \). In fact, let \( u_\pm (x) = P_\pm (\lambda) \psi \) and \( v_\pm (x) = R_\pm (\lambda) u_\pm (x) \). Then

\[ (\hat{N}_\pm (\lambda) \psi, \psi) = \int_{\Gamma} \frac{\partial v_\pm}{\partial n_\pm} \bar{u} dS = - \int_{\Omega_\pm} (\Delta + \lambda) v_\pm \bar{u} dx = - \int_{\Omega_\pm} |u|^2 dx < 0. \]

Let \( \alpha \) be the smallest eigenvalue of the Laplacian in \( \Omega_- \), with the Dirichlet condition on \( \Gamma \) and the Neumann condition on \( \partial C \). Note that \( \alpha > 0 \) and that, for any \( k \), the smallest eigenvalue of the Laplacian in \( \Omega_- \), with the Dirichlet condition on \( \Gamma \) and the transmission condition (2.2) on \( \partial C \), is bigger than \( \alpha \). Therefore \( N_- (\lambda, k) \) is defined and smooth in \( \lambda \) when \( \lambda < \alpha \). For \( \lambda < K \) and \( \tau > K/\alpha \), one derives from (2.5) that

\[ N(\tau, \lambda, k) = N_+ (\lambda) + \tau N_- (0, k) + \lambda \hat{N}_- (0, k) + O(\tau^{-1}), \quad \tau \to \infty, \]  

(2.8)

uniformly in \( \lambda \), \( 0 \leq \lambda < K \), and in \( k, k \in [\alpha, \alpha] \).
Now we can explain the idea of our proofs. The operator \( N_-(0, k) \) is non-negative, and, for \( k \neq 0 \), it is strictly positive. It is easy to see that the smallest eigenvalue of the operator \( N_-(0, k) \) minimizes the Rayleigh quotient

\[
\int_{\Omega^-} |\nabla u(x)|^2 \, dx / \int_{\Gamma} |u(x)|^2 \, dS
\]

over \( H^1(\Omega^-) \)-functions that satisfy the quasi-periodic conditions (2.2). Clearly, the minimum of the Rayleigh quotient is non-negative, and it equals zero only if constants belong to the corresponding space of functions. A (non-zero) constant satisfies (2.2) only if \( k = 0 \).

For non-zero values of \( k \), the second term on the right in (2.8) is bounded from below (as an operator) by \( C\tau \) where \( C \) is a positive constant. As \( \lambda \) approaches \( \lambda_j \), an eigenvalue of the Dirichlet Laplacian in \( \Omega_+ \), from the left, the smallest eigenvalue of \( N_+(\lambda) \) goes to \(-\infty\) as \( C(\lambda - \lambda_j)^{-1} \) ([F1], [M]). We will conclude from these facts that, when \( \lambda \) is close enough to \( \lambda_j \) (actually, \( \lambda \approx \lambda_j - C/\tau \)), the operator \( N(\tau, \lambda, k) \) has a zero mode, or, in other words, the value of \( \lambda \) belongs to the spectrum of \( L_\tau(k) \). Therefore, for large values of \( \tau \), the contribution \( \mu_j(k) \) of the quasi-momenta \( k \neq 0 \) to the spectrum of \( L_\tau \) concentrate more and more around the points \( \lambda_j \), and \( \lambda_j - \mu_j(k) \approx C/\tau \). This is the reason for (1.3).

When \( k = 0 \), the operator \( N_-(0, 0) \) has a simple zero mode 1, and the third term on the right in (2.8) becomes crucial for determining the position of \( \mu_j(0) \).

Before we proceed to the next section, we will say how the numbers \( \nu_j \) are determined.

It is clear from the above discussion that the quasi-momentum \( k = 0 \) should be responsible for determining \( \nu_j \). The function \( (N_+(\lambda)1, 1) \) is decreasing on the intervals where it is defined because of (2.7). By the same reason, the number \( (N_-(0, 0)1, 1) \) is negative. Therefore, the equation

\[
(N_+(\lambda)1, 1) + \lambda(N_-(0, 0)1, 1) = 0
\]

has a discrete set of solutions

\[
0 = \nu_1 < \nu_2 < \nu_3 < \cdots
\]

on \([0, \infty)\). These solutions \( \nu_k \) are exactly the numbers that appear in Theorem 1.

3. Proof of Theorem 1

The proof of Theorem 1 will be broken into several lemmas. These lemmas are also used for proving Theorem 2. We start with

Lemma 1. Assume that the spectrum of the Dirichlet Laplacian in \( \Omega_+ \) is simple and that (1.2) hold. Then the sequences \( \nu_j \) and \( \nu_j \) interlace, i.e. \( \nu_j < \lambda_j < \nu_{j+1} \), \( j = 1, 2, \ldots \).

Proof. We have already mentioned that the function \( (N_+(\lambda)1, 1) \) is strictly decreasing on any interval where it is defined (it follows from (2.7)); it is defined on \((\lambda_j, \lambda_{j+1})\).

A function \( (N_+(\lambda)\psi, \psi) \) stays bounded as \( \lambda \) approaches an eigenvalue \( \lambda_j \) only if \( \psi \) is orthogonal to \( \partial u_j / \partial n_+ \) ([F1], [M]). One has

\[
\int_{\Gamma} \frac{\partial u_j(x)}{\partial n_+} \cdot 1 \, dS = \int_{\Omega_+} \Delta u_j(x) \, dx = -\lambda_j \int_{\Omega_+} u_j(x) \, dx \neq 0.
\]
Therefore,
\[ \lim_{\lambda \to \lambda_j \pm 0} (N_+ (\lambda)1, 1) = \pm \infty. \]

The coefficient in front of \( \lambda \) in (2.9) is negative, so the left-hand side of the equation (2.9), as a function of \( \lambda \), strictly decreases from \( +\infty \) to \( -\infty \) on each interval \( (\lambda_j, \lambda_{j+1}) \), and the equation (2.9) has exactly one solution on each of these intervals. It also decreases on the interval \([0, \lambda_1]\), and the left-hand side of (2.9) equals 0 when \( \lambda = 0 \). Therefore, (2.9) has also one solution \( \nu_1 = 0 \) on the interval \([0, \lambda_1]\).

Q.E.D.

Next, we show that the operator-valued function \( N_-(0, k) \) is differentiable with respect to \( k \) in the strongest possible sense.

**Lemma 2.** There exists an n-tuple of bounded operators \( D_k N_-(0, k) \) such that

\[ N_-(0, k_1) - N_-(0, k_2) = D_k N_-(0, k_2) \cdot (k_1 - k_2) + o(|k_1 - k_2|), \quad k_1 \to k_2 \quad (3.1) \]

in the norm operator topology.

**Proof.** If a function \( u(x) \) satisfies (2.2), then the function \( v(x) = \exp(-ikx)u(x) \) satisfies the periodic boundary conditions on \( \partial C \) (it can be considered as a function on \( T^n \setminus \Omega_+ \) where \( T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n \)). The equation \( \Delta u = 0 \) is equivalent to

\[ \Delta v + 2ik \cdot \nabla v - |k|^2 v = 0, \quad (3.2) \]

and

\[ \frac{\partial v}{\partial n_-} = e^{ikx} \left( \frac{\partial v}{\partial n_-} + ik \cdot n_-(x)v \right). \]

Therefore,

\[ N_-(0, k) = e^{ikx} \tilde{N}(k) e^{-ikx} + ik \cdot n_-(x) \quad (3.3) \]

where \( \tilde{N}(k) \) is the Dirichlet-to-Neumann operator for the equation (3.2) on \( T^n \setminus \Omega_+ \). Let \( \tilde{P}(k) \) be the corresponding Poisson operator: it assigns a solution of (3.2) to its value on \( \Gamma \). Denote by \( \tilde{I}(k) \) the operator that is given by the differential expression in (3.2), together with the Dirichlet boundary conditions on \( \Gamma \). A straightforward computation shows that

\[ \tilde{P}(k_1) - \tilde{P}(k_2) = (k_1 - k_2) \tilde{I}(k_2)^{-1} [-2i\nabla \tilde{P}(k_1) + (k_1 + k_2) \tilde{P}(k_1)]. \]

This formula implies that the operator valued function \( \tilde{N}(k) \) is differentiable in \( k \), and

\[ D_k \tilde{N}(k) = 2j_- \tilde{I}(k)^{-1} (-i\nabla + k) \tilde{P}(k). \quad (3.4) \]

Equations (3.3) and (3.4) already imply the statement of the theorem. We would like to write down the formula for \( D_k N_-(0, k) \) in a convenient form. To do that, we go back to the quasi-periodic conditions (2.2). Let \( P_-(k) = P_-(0, k) \) be the Poisson operator for the Laplacian in \( \Omega_- \) and let \( L_-(k) \) be the Dirichlet Laplacian in \( \Omega_- \), both with the quasi-periodic conditions (2.2). Then

\[ e^{ikx} j_- \tilde{I}(k)^{-1} (-i\nabla + k) \tilde{P}(k) e^{-ikx} = -i j_- L_-(k)^{-1} \nabla P_-(k), \]

where
and the equations (3.3), (3.4) imply that
\[ D_k N_-(0, k) = i(x N_+(0, k) - N_-(0, k)x) - 2i j_\nu N_-(0, k)x - 2i j_\nu L_\nu(k) - \nabla P(k) + in_\nu(x). \] (3.5)

Q.E.D.

Remark. In a similar way, one can show that the operator valued function \( N_-(\lambda, k) \) is differentiable with respect to both \( \lambda \) and \( k \) when \( \lambda < \alpha \) (recall that \( \alpha \) is the smallest eigenvalue of a mixed Dirichlet–Neumann Laplacian in \( \Omega_- \)). The only difference in the proof is that, instead of taking the Poisson operator for the Laplacian, one should take the Poisson operator for \( \Delta + \lambda \). The expressions for the derivatives of \( N_-(\lambda, k) \) show that they are smooth operator valued functions, so is \( N_-(\lambda, k) \) itself.

Now, as a first step toward proving Theorem 1, we will show that, when \( \tau \) is large enough, an interval \( (\lambda_j, \nu_{j+1} - \epsilon) \) does not intersect the spectrum of \( L_\tau \). This follows from

Lemma 3. For every \( j \) and for every \( \epsilon > 0 \) there exists a number \( T = T(j, \epsilon) \) such that the operators \( N(\lambda, \tau, k) \) are strictly positive when \( \lambda \in (\lambda_j, \nu_{j+1} - \epsilon) \) and \( \tau > T \).

Proof. We break the proof into three steps.

1. For any \( \eta > 0 \) there exists \( T_1 = T_1(\eta, j) \) such that the operator \( N(\tau, \lambda, k) \) is positive when \( \lambda \in (\lambda_j, \nu_{j+1}) \), \( \tau > T_1 \), and \( |k| \geq \eta \).

In fact, the operators \( N_+(\lambda) \) are bounded from below, uniformly in \( \lambda \) when \( \lambda \in (\lambda_j, \nu_{j+1}) \) (actually, one can replace \( \nu_{j+1} \) by any number smaller than \( \lambda_{j+1} \)), operators \( N_-(0, k) \) are bounded by a positive constant, uniformly in \( k \), when \( k \in [-1/2, 1/2]^n \), \( |k| \geq \eta \), and the operators \( N_-(0, k) \) are uniformly bounded. Therefore,

\[ N(\tau, \lambda, k) \geq C_1 + C_2 \tau, \quad C_2 > 0, \]

(see (2.8)). Now, one takes \( T_1 = -C_1/C_2 \).

Let \( \psi(x) \) be a (smooth) non-zero function on \( \Gamma \), and let

\[ \psi(x) = a + \phi(x), \quad \text{where} \quad a = \text{const} \quad \text{and} \quad \int_\Gamma \phi(x) dS = 0. \]

By \( \| \cdot \| \) we will denote the \( L^2 \)-norm.

2. For any \( \delta > 0 \) there exists \( T_2 = T_2(\delta, j) \) such that \( (N(\tau, \lambda, k)\psi, \psi) > 0 \) when \( \| \phi \| \geq \delta \| a \| \), \( \lambda \in (\lambda_j, \nu_{j+1}) \), and \( \tau > T_2 \).

From differentiability of \( N_-(0, k) \) with respect to \( k \) (Lemma 2), we conclude that

\[ (N_-(0, k)\psi, \psi) \geq (N_-(0, 0)\psi, \psi) + C_3|k| \| \psi \|^2 \]

for some constant \( C_3 \) (it may be negative). The operator \( N_-(0, 0) \) annihilates constants, and it is positive when restricted to the space of functions with zero average. Therefore,

\[ (N_-(0, 0)\psi, \psi) = (N_-(0, 0)\phi, \phi) \geq C_4 \| \phi \|^2 \geq \frac{C_4 \delta^2}{1 + \delta^2} \| \psi \|^2 \]
where $C_4$ is a positive constant. We choose $\eta > 0$ in such a way that $C_2 = C_3 \eta + C_3 \delta^2 / (1 + \delta^2) > 0$. Then $(N, (0, k)\psi, \psi) \geq C_2 \| \psi \|^2$ when $|k| < \eta$, and, as in the proof of (1),

$$(N(\lambda, \tau, k)\psi, \psi) \geq C_1 + C_2 \tau,$$

so, when $\tau > -C_1 / C_2$, the quadratic form $(N(\lambda, \tau, k)\psi, \psi)$ is positive. When $|k| \geq \eta$, positivity of this form for sufficiently large values of $\tau$ was proven in (1).

(3) There exist numbers $\delta = \delta(j, \epsilon)$, $\eta = \eta(j, \epsilon)$, and $T_3 = T_3(j, \epsilon)$ such that

$$(N(\tau, \lambda, k)\psi, \psi) > 0 \text{ when } \| \phi \| < \delta \| \phi' \|, \; |k| < \eta, \; \lambda \in (\lambda_j, \nu_{j+1} - \epsilon), \text{ and } \tau > T_3. \text{ Here we assume that } \psi \text{ is decomposed as in (3.6).}$$

Operators $N(\tau, \lambda, k)$ are decreasing in $\lambda$ when $\lambda \in (\lambda_j, \lambda_{j+1})$ (see (2.7)), so it is sufficient to prove the statement for $\lambda = \nu_{j+1} - \epsilon$. It follows form (2.8), differentiability of $\hat{N}_\lambda(0, k)$ (see the remark after the proof of Lemma 2), and from non-negativity of $\hat{N}_\lambda(0, k)$ that

$$(N(\tau, \lambda, k)\psi, \psi) \geq ((N_+ (\lambda) + \lambda \hat{N}_\lambda(0, 0))\psi, \psi) - (C_1 \eta + C_2 \tau^{-1})\| \psi \|^2$$

when $|k| < \eta$. Now, by $C_3$ we denote positive constants. One has

$$(N_+ (\nu_{j+1} - \epsilon)\psi, \psi) \geq (N_+ (\nu_j - \epsilon)\psi, \psi) - C_3 \| \phi \|^2 - C_4 \| \phi \| \cdot \| \phi' \| \geq (N_+ (\nu_j - \epsilon)\psi, \psi) - C_5 \| \phi \|^2$$

where $-C_3$ is the smallest eigenvalue of $N_+ (\nu_{j+1})$, and $C_4$ is the maximum of $\| N_+ (\lambda) \phi \|$ when $\lambda \in ((\lambda_j + \nu_{j+1})/2, \nu_{j+1})$ (we assume that $\epsilon < (\nu_{j+1} - \lambda_j)/2$: we do not want $C_4$ to depend on $\epsilon$). The operator $\hat{N}_\lambda(0, 0)$ is bounded, so immediately

$$(\hat{N}_\lambda(0, 0)\phi, \phi) \geq (\hat{N}_\lambda(0, 0)\phi, \phi) - C_5 \| \phi \|^2.$$

The previous inequalities yield

$$(N(\tau, \nu_{j+1} - \epsilon, k)\psi, \psi) \geq ((N_+ (\nu_{j+1} - \epsilon) + (\nu_{j+1} - \epsilon) \hat{N}_\lambda(0, 0))1, 1) \| \phi \|^2$$

when $|k| < \eta$. It was shown in the proof of Lemma 1 that

$$(N_+ (\nu_{j+1} - \epsilon) + (\nu_{j+1} - \epsilon) \hat{N}_\lambda(0, 0))1, 1) > 0.$$

Clearly, $\| \phi \| \geq \| \psi \| / 2$ if $\delta$ is small enough. Therefore,

$$(N(\tau, \nu_{j+1} - \epsilon, k)\psi, \psi) \geq (C_8(\epsilon) - (C_1 \eta + C_2 \tau^{-1} + C_7 \delta)) \| \psi \|^2.$$

Now, one takes $\eta = C_8 / 4C_1$, $\delta = C_8 / 4C_7$, and $T_3 = 4C_2 / C_8$.

The statement of the lemma obviously follows from (1)–(3).

Q.E.D.

Next lemma says that $L_\tau(0)$ has an eigenvalue that is close to $\nu_{j+1}$ if $\tau$ is large enough.

**Lemma 4.** For every $j$ and for every $\epsilon > 0$ there exists $T = T(j, \epsilon)$ such that the operator $N(\tau, \nu_{j+1} + \epsilon, 0)$ has a negative eigenvalue when $\tau > T$.  

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Proof. We use (2.8) and \(N_-(0,0)1 = 0\):
\[
(N(\tau, \nu_{j+1} + \epsilon)1, 1) = ((N_+(\nu_{j+1} + \epsilon) + (\nu_{j+1} + \epsilon)N_-(0,0))1, 1) + O(\tau^{-1}).
\]
The statement of the lemma follows from
\[
((N_+(\nu_{j+1} + \epsilon) + (\nu_{j+1} + \epsilon)N_-(0,0))1, 1) < 0
\]
(see the proof of Lemma 1).

Q.E.D.

Next lemma tells us that, for a non-zero value of \(k\), the operator \(L_\tau(k)\) has an eigenvalue that is close to \(\lambda_j\) when \(\tau\) is large enough.

**Lemma 5.** Let \(k \neq 0\). Then for every \(j\) there exist constants \(C_1, C_2,\) and \(T\) (that depend on \(j\) and \(k\)) such that the operator \(N(\tau, \lambda_j - C_1\tau^{-1}, k)\) is positive and the operator \(N(\tau, \lambda_j - C_2\tau^{-1}, k)\) has a negative eigenvalue when \(\tau > T\).

**Proof.** Once more, we use (2.8). The operator \(N_+(\lambda)\) is bounded from below by \(-C_3(\lambda_j - \lambda)^{-1}\) when \(\lambda \in (\lambda_{j-1}, \lambda_j)\). The operator \(N_-(0, k)\) is bounded from below by a positive constant \(C_4\), and all other operators in the right hand side of (2.8) are uniformly bounded. Therefore,
\[
N(\tau, \lambda, k) \geq -C_3(\lambda_j - \lambda)^{-1} + C_4\tau - C_5.
\]
One can take \(C_1 = 2C_3/C_4\). Then, the corresponding operator is positive when \(\tau > 2C_5/C_4\).

In the opposite direction, take \((N(\tau, \lambda, k)\psi_j, \psi_j)\) where \(\psi_j\) is the normal derivative of the eigenfunctions \(u_j\) of the Dirichlet Laplacian in \(\Omega_+\). Then
\[
(N_+((\lambda_j - \lambda)^{-1}, k)\psi_j, \psi_j) \leq -C_6(\lambda_j - \lambda)^{-1},
\]
and (2.8) yields
\[
(N(\tau, \lambda, k)\psi_j, \psi_j) \leq -C_6(\lambda_j - \lambda)^{-1} + C_7\tau + C_8.
\]
Now, one can take \(C_2 = C_6/2C_7\). Then, for \(\tau > C_8/C_7\), the quadratic form \((N(\tau, \lambda, k)\psi_j, \psi_j)\) is negative.

Q.E.D.

Lemmas 4 and 5 imply that, within each interval \((\lambda_{j-1}, \lambda_j)\), the lower bound of the spectrum of \(L_\tau\) approaches \(\nu_j\), and the upper bound approaches \(\lambda_j\), as \(\tau \to \infty\). What remains to be seen is that, when \(\tau\) is large, there is exactly one band of the spectrum of \(L_\tau\) lying in \((\lambda_{j-1}, \lambda_j)\). To prove this fact, we need the following lemma.

**Lemma 6.** For every \(j, j = 1, 2, \ldots\), there exists \(T = T(j)\) such that the operator \(N(\tau, \lambda, k)\) has not more than one eigenvalue that is smaller than \(1\) when \(\tau > T, \lambda \in (\lambda_{j-1}, \lambda_j)\), and \(k \in [-1/2, 1/2]^n\).

**Proof.** It is sufficient to exhibit a subspace \(\mathcal{L} \subset L^2(\Gamma)\) of codimension 1 such that
\[
(N(\tau, \lambda, k)\psi, \psi) \geq ||\psi||^2
\]
when \(\lambda \in (\lambda_{j-1}, \lambda_j), k\) is arbitrary, and \(\tau\) is large enough. Let \(\psi_j(x)\) be the normal derivative on \(\Gamma\) of the eigenfunction \(u_j(x)\) of the Dirichlet Laplacian in \(\Omega_+\), and let \(\mathcal{L}\) be
the space of functions that are orthogonal to $\psi_j$. Then:

1. The operator $N_+(\lambda)$, when restricted to $\mathcal{L}$ is bounded from below uniformly on the interval $(\lambda_{j-1}, \lambda_j)$; actually, one can define the restriction of $N_+(\lambda_j)$ to $\mathcal{L}$ (see [F1], [M]). So,

$$
(N_+(\lambda)\psi, \psi) \geq C_1 \|\psi\|^2, \quad \lambda \in (\lambda_{j-1}, \lambda_j), \quad \psi \in \mathcal{L}
$$

(3.8)

where $C_1$ is a constant. Here we used the fact that the eigenvalue $\lambda_j$ is simple.

2. The space $\mathcal{L}$ does not contain non-zero constants because

$$
\int_{\Gamma} \psi_j(x) dS = -\frac{1}{\lambda_j} \int_{\Omega^+} u_j(x) dx \neq 0.
$$

The operator $N_-(0,0)$ is a non-negative operator with discrete spectrum, and its kernel consists exactly of constants. Therefore, $(N_-(0,0)\psi, \psi) \geq C_2 \|\psi\|^2$ when $\psi \in \mathcal{L}$. Here $C_2$ is a positive constant. The operators $N_-(0, k)$ form a continuous (even differentiable) in $k$ family, so, if $\eta > 0$ is small enough, $(N_-(0, k)\psi, \psi) \geq (C_2 / 2) \|\psi\|^2$ when $\psi \in \mathcal{L}$ and $|k| < \eta$. On the other hand, when $|k| \geq \eta$, the operators $N_-(0, k)$ have a uniform positive lower bound. Therefore, for some positive constant $C_3$, one has

$$
(N_-(0,k)\psi, \psi) \geq C_3 \|\psi\|^2, \quad \psi \in \mathcal{L}.
$$

(3.9)

Equations (2.8), (3.8), and (3.9) imply

$$
(N(\tau, \lambda, k) \psi, \psi) \geq (C_4 + C_3 \tau) \|\psi\|^2, \quad \lambda \in (\lambda_{j-1}, \lambda_j), \quad \psi \in \mathcal{L}.
$$

For $\tau > (1 - C_4) / C_3$, one gets (3.7).

Q.E.D.

**Proof of Theorem 1.** Let $K$ be an arbitrary number, and let $\tau > K/\alpha$ (see the paragraph preceding (2.8)). For $0 \leq \lambda < K$, we denote by $\beta(\tau, \lambda, k)$ the smallest eigenvalue of the operator $N(\tau, \lambda, k)$; it is defined for all $\lambda$ except of $\lambda = \lambda_j$. Points of the spectrum of the operator $L_\tau(k)$ are determined from the equation

$$
\beta(\tau, \lambda, k) = 0.
$$

(3.10)

Lemma 6 tells us in particular that, if $\tau$ is large enough, an eigenvalue $\beta$ of $N(\tau, \lambda, k)$ that satisfies (3.10) is simple, and therefore, in a neighborhood of a point where (3.10) is satisfied, the function $\beta(\tau, \lambda, k)$ is differentiable, even smooth (see the Remark after the proof of Lemma 2). If $(\tau, \lambda, k)$ is a solution of (3.10), and $\psi$ is a normalized eigenfunction of $N(\tau, \lambda, k)$ that correspond to the eigenvalue $0$, then

$$
\frac{\partial \beta}{\partial \lambda}(\tau, \lambda, k) = (\tilde{N}(\tau, \lambda, k) \psi, \psi) < 0
$$

(see (2.7)). By the implicit function theorem, the equation (3.10) defines a smooth function $\lambda = \mu(\tau, k)$. We conclude that, for sufficiently large values of $\tau$, the part of the spectrum
of the operator $L$, that lies below a level $K$ is the union of ranges of a certain number of smooth functions $\mu(\tau, k)$ of $k \in [-1/2, 1/2]$. 

Each function $\mu(\tau, k)$ takes values within a certain interval $(\lambda_{j-1}, \lambda_j)$. In fact, Lemma 3 says that $\beta(\lambda, \tau, k) > 0$ when $\lambda \in (\lambda_j, \nu_{j+1} - \epsilon)$ and when $\lambda \in (\lambda_{j-1}, \nu_j - \epsilon)$ (if $j = 1$ then $\nu_1 = 0$, and the second part of the statement holds because the operator $N(\tau, \lambda, k)$ is positive for negative values of $\lambda$). Therefore, the equation (3.10) cannot have solutions in these intervals, and the graphs of the functions $\mu(\tau, \lambda)$ cannot cross the levels $\lambda_j$.

Now we will show that only one function $\mu(\lambda)$ can take values within a fixed interval $(\lambda_{j-1}, \lambda_j)$: for a fixed value of $k$ (and, as usual, for sufficiently large values of $\tau$) the operator $L_\tau(k)$ cannot have more than one eigenvalue inside the interval $(\lambda_{j-1}, \lambda_j)$. In fact, assume that it has two eigenvalues $\lambda_{j-1} < \mu < \lambda_j$. If $\mu = \mu'$ (a multiple eigenvalue) then the operator $N(\tau, \mu, k)$ has at least two eigenvalues that equal 0, which contradicts to Lemma 6. In the case when $\mu < \mu'$, the operator $N(\tau, \mu, k)$ has an eigenvalue 0, and the operator $N(\tau, \mu', k)$ has a strictly negative eigenvalue because the operator-valued function $N(\tau, \lambda, k)$ is strictly decreasing in $(\lambda_{j-1}, \lambda_j)$. On the other hand, the operator $N(\tau, \mu', k)$ has a zero eigenvalue, so it has at least two eigenvalues that are smaller than 1. This contradicts to Lemma 6. In particular, all eigenvalues of $L_\tau(k)$, below the threshold $K$, are simple.

Lemma 5 implies, in particular, that, for $k \neq 0$, the equation (3.10) has a solution in the interval $(\lambda_{j-1}, \lambda_j)$, so there exists a function $\mu(\tau, k)$ that takes values in this interval, and this function is unique. In particular, it is not just a function that is defined in $[-1/2, 1/2]^n$, but this is a smooth function, periodic in $k$.

Let us summarize. For sufficiently large values of $\tau$, $\tau > T(K)$, all eigenvalues of $L_\tau(k)$ that are smaller than $K$ are simple, they form a certain number of smooth, periodic with respect to $k$ functions $\mu_j(\tau, k)$, and each function $\mu_j(\tau, k)$ takes values in the interval $(\lambda_{j-1}, \lambda_j)$. The part of the spectrum of $L_\tau$ that lies below $K$ is the union of ranges of these functions.

Let $[c_j(\tau), d_j(\tau)]$ be the range of the function $\mu_j(\tau, \lambda)$. Lemma 3 implies that $c_j(\tau) > \lambda_{j-1}$ and $d_j(\tau) \leq \lambda_j$ (actually, it follows easily from lemmas 4 and 5 that $d_j < \lambda_j$).

Lemma 5 implies that $\lambda_j - d_j(\tau) \leq C/\tau$, so

$$\lim_{\tau \to \infty} d_j(\tau) = \lambda_j.$$ 

Lemma 4 implies that, when $j > 1$,

$$\lim_{\tau \to \infty} c_j(\tau) = \nu_j.$$

For $j = 1$, one has $c_1(\tau) = 0 = \nu_1$. In particular, if $\tau$ is large enough, the functions $\mu_j(\tau, k)$ are not constants as functions of $k$. This implies (see [Th]) that the part of the spectrum of the operator $L_\tau$ below the level $K$ is absolutely continuous.

This completes the proof of Theorem 1.

Q.E.D.
4. Proof of Theorem 2

The Floquet theory (e.g., see [K]), together with the structure of the spectrum of $L_r$ that was established in the proof of Theorem 1, tells us that the integrated density of states function $m_r(\lambda)$ is determined by the formula

$$m_r(\lambda) = j - 1 + \text{vol}\{ k \in \mathbb{R}^n / \mathbb{Z}^n : \mu_j(\tau, k) \leq \lambda \}, \quad \lambda \in (\lambda_{j-1}, \lambda_j). \quad (4.1)$$

Lemma 5 implies that $\lambda_j - \mu_j(\tau, k) = O(1/\tau)$ for $k \neq 0$. Therefore, for proving (1.3), one has to examine what happens when the quasi-momentum $k$ is close to 0. The proof of Theorem 2 is based on the following lemma.

**Lemma 7.** Let $\omega(k)$ be the smallest eigenvalue of the operator $N_+(0, k)$. The function $\omega(k)$ is smooth in a neighborhood of $k = 0$, the point $k = 0$ is a point of minimum for $\omega(k)$, and the Hessian $(\partial^2 \omega / \partial k_j \partial k_p)(0)$ is strictly positive.

First, we derive the statement of the theorem from Lemma 7. The condition $\mu_j(\tau, k) \leq \lambda$ is equivalent to the fact that the operator $N(\tau, \lambda, k)$ has a non-positive eigenvalue, because the operator valued function $N(\tau, \lambda, k)$ strictly decreases in $\lambda$. Let us fix the value of $\lambda$, $\lambda_{j-1} < \lambda < \lambda_j$. Denote by $E(\tau, \lambda)$ the set of all $k$ for which there exists a non-zero function $\psi$, for which $(N(\tau, \lambda, k)\psi, \psi) \leq 0$. Then, the equality (4.1) can be rewritten as

$$m_r(\lambda) = j - 1 + \text{vol}(E(\tau, \lambda)), \quad \lambda \in (\lambda_{j-1}, \lambda_j). \quad (4.2)$$

Let $\eta$ be a positive number. Lemma 7, together with positivity of the operator $N_+(0, k)$ for non-zero values of $k$, implies that

$$(N_+(0, k)\psi, \psi) \geq C_1\eta^2\|\psi\|^2, \quad \text{when} \quad |k| \geq \eta.$$  

We recall that $N_+(\lambda) \geq -C_2(\lambda_j - \lambda)^{-1}$ when $\lambda \in (\lambda_{j-1}, \lambda_j)$, so one derives from (2.8) that

$$(N(\tau, \lambda, k)\psi, \psi) \geq [-C_2(\lambda_j - \lambda)^{-1} - C_3 + C_1\tau\eta^2]\|\psi\|, \quad |k| \geq \eta.$$  

Therefore, the operator $N(\tau, \lambda, k)$ is positive when

$$|k|^2 > \frac{1}{C_1\tau} \left( C_3 + \frac{C_2}{\lambda_j - \lambda} \right). \quad (4.3)$$

The set $E(\tau, \lambda)$ is contained in the ball centered at $k = 0$, the radius of which equals the square root of the expression on the right in (4.3), and the volume of this ball is bounded by $C_4\tau^{-n(1-\gamma)/2}$ when $\lambda = \lambda_j - C\tau^{-\gamma}$. Note that the constant $C_4$ can be computed more or less effectively.

Q.E.D.

**Proof of Lemma 7.** The only thing that has to be proven, is the statement about the Hessian: $\omega(0) = 0$ is a simple eigenvalue of a smooth family of operators, so $\omega(k)$ is smooth in a neighborhood of $k = 0$; all operators $N_+(0, k)$ are strictly positive when $k \neq 0$, so $k = 0$ is the point of global minimum of $\omega(k)$. In the proof of Lemma 7, we will use the
notations from the proof of Lemma 2. In addition, $D_j = \partial/\partial k_j$, $D_{j,p} = \partial^2/\partial k_j \partial k_p$. To simplify notations, we will assume that the area of $\Gamma$ equals 1, so that 1 is a normalized function. Clearly, this assumption is not essential.

Let $\psi(k)$ be a normalized eigenfunction of $N_-(0,k)$,

$$N_-(0,k)\psi(k) = \omega(k)\psi(k), \quad \psi(0) = 1.$$  

We differentiate this equation with respect to $k_j$:

$$(D_j N_-(0,k))\psi(k) + N_-(0,k)\psi_j(k) = \omega_j(k)\psi(k) + \omega(k)\psi_j(k). \quad (4.4)$$

Here $\psi_j = D_j \psi$, $\omega_j = D_j \omega$. We differentiate (4.4) with respect to $k_p$, and set $k = 0$:

$$\omega_{j,p}(0) = (D_{j,p} N_-(0,0))1 + (D_j N_-(0,0)\psi_p(0) + (D_p N_-(0,0))\psi_j(0)$$
$$+ N_-(0,0)\psi_{j,p}(0). \quad (4.5)$$

Here we used $\omega(0) = \omega_j(0) = \omega_p(0) = 0$. Take the scalar product of both sides of (4.5) with 1, and use $N_-(0,0)1 = 0$, together with self-adjointness of the operators $N_-(0,0)$, $D_j N_-(0,0)$ and $D_p N_-(0,0)$, to get

$$\omega_{j,p}(0) = ((D_{j,p} N_-(0,0))1, 1) + (\psi_p(0), (D_j N_-(0,0))1) + (\psi_j(0), (D_p N_-(0,0))1). \quad (4.6)$$

We start with the sum of the second and the third term on the right in (4.6). Set $k = 0$ in (4.5) and apply the operators on both sides to the function 1:

$$(D_j N_-(0,0))1 = -iN_-(0,0)x_j + in_{-j}(x) \quad (4.7)$$

where $n_{-j}(n)$ is the $j$-th component of the normal vector $n_-$. Here, we used $N_-(0,0)1 = 0$, and we also used $P_-(0)1 = 1$, so $\partial x, P_-(0)1 = 0$. Of course, one can replace $j$ by $p$ in (4.7).

The second term from the right-hand side of (4.6) can be written in the form

$$((\psi_p(0), (D_j N_-(0,0))1) = i(\psi_p(0), N_-(0,0)x_j) - i(\psi_p, (0)n_{-j}(x))$$
$$= i(N_-(0,0)\psi_p(0), x_j) - i(\psi_p(0), n_{-j}(x)). \quad (4.8)$$

Set $k = 0$ in (4.4) (with $j$ replaced by $p$), and use (3.5) to get

$$N_-(0,0)\psi_p(0) = -(D_p N_-(0,0))1 = iN_-(0,0)x_p - in_{-p}(x), \quad (4.9)$$

The space

$$\mathcal{M} = \left\{ f(x) : \int_{\Gamma} f(x)dS = 0 \right\}$$

of functions on $\Gamma$ is invariant under the action of $N_-(0,0)$, and the restriction of $N_-(0,0)$ to $\mathcal{M}$ is strictly positive. By $N_-(0,0)^{-1}$ we will denote the inverse of the restriction of $N_-(0,0)$ to $\mathcal{M}$. Note that the functions $\psi_p(0)$ and $n_{-p}$ belong to $\mathcal{M}$. We derive from (4.9) that

$$\psi_p(0) = i\tilde{x}_p - iN_-(0,0)^{-1}n_{-p} \quad (4.10)$$
where $\hat{x}_p$ is the projection of the function $x_p$ to $\mathcal{M}$. We use (4.10) and the second equality (4.9), with $p$ replaced by $j$, to write down the second term from the right hand side of (4.6) in the form

$$(i\hat{x}_j - i\mathcal{N}_-(0,0)^{-1}n_{-,\beta}(x), -i\mathcal{N}_-(0,0)x_j + in_{-,\beta}(x)) = -(\mathcal{N}_-(0,0)x_j, x_j)$$

$$- (\mathcal{N}_-(0,0)^{-1}n_{-,\beta}(x), n_{-,\beta}(x)) + (x_j, n_{-,\beta}(x)) + (n_{-,\beta}(x), x_j).$$

Here, we used $\mathcal{N}_-(0,0)\hat{x}_p = \mathcal{N}_-(0,0)x_p$. Now,

$$(x_j, n_{-,\beta}(x)) = -(x_j, n_{+,\beta}(x)) = -\int_{\Omega^+} (\Delta x_j \cdot x_j + \nabla x_j \cdot \nabla x_j) dx = -|\Omega^+|\hat{\rho}_{j,\beta}$$

where $|\Omega^+| = \text{vol}(\Omega^+)$. Finally, the second term from the right hand side of (4.6) equals

$$-(\mathcal{N}_-(0,0)x_j, x_j) - (\mathcal{N}_-(0,0)^{-1}n_{-,\beta}(x), n_{-,\beta}(x)) - 2|\Omega^+|\hat{\rho}_{j,\beta}.$$ 

This expression is symmetric in $(j, p)$, so the third term on the right in (4.6) has the same value, and

$$\omega_{j,\beta}(0) = ((D_j, \mathcal{N}_-(0,0))1, 1) - 2(\mathcal{N}_-(0,0)x_j, x_j) - 2(\mathcal{N}_-(0,0)^{-1}n_{-,\beta}, n_{-,\beta})$$

$$- 4|\Omega^+|\hat{\rho}_{j,\beta}. \quad (4.11)$$

We proceed to computing $((D_j, \mathcal{N}_-(0,0))1, 1)$. One has

$$-2iL_-^k \nabla P_-^k = 2e^{ikx}L_-^k(\nabla P_-^k = 2e^{ikx}L_-^k(1 - i\nabla + k)\hat{P}_-^k e^{-ikx}.$$ 

(For notations, see the proof of Lemma 2.) The functions from the image of $L_-^k(1)$ equal 0 on $\Gamma$, so

$$-2iL_-^k \nabla P_-^k = 2e^{ikx}L_-^k(\nabla P_-^k = 2e^{ikx}L_-^k(1 - i\nabla + k)\hat{P}_-^k e^{-ikx},$$

and the $j$-th component of (3.5) can be written down as

$$D_j \mathcal{N}_-(0,0) = i(x_j \mathcal{N}_-(0,0) - \mathcal{N}_-(0,0)x_j) + 2e^{ikx}L_-^k(\nabla P_-^k = 2e^{ikx}L_-^k(1 - i\nabla + k)\hat{P}_-^k e^{-ikx} + in_{-,\beta}(x).$$

Here $\partial_j = \partial / \partial x_j$. We differentiate the last equality with respect to $k_p$ and set $k = 0$:

$$D_j \mathcal{N}_-(0,0) = i(x_j D_p \mathcal{N}_-(0,0) - D_p \mathcal{N}_-(0,0)x_j)$$

$$+ 2i(x_j (-i\partial_j \mathcal{N}_-(0,0)^{-1}\partial_j P_-^0) - (-i\partial_j \mathcal{N}_-(0,0)^{-1}(\partial_j P_-^0))x_j)$$

$$+ 2\partial_j (D_p \hat{L}_p(0)) - (\partial_j \hat{P}_-^0) + 2\delta_{j,\beta} \partial_j \hat{L}_p(0) - \hat{P}_-^0$$

$$+ 2j \hat{L}_p(0) - (\partial_j \hat{P}_-^0)(D_p \hat{P}_-^0). \quad (4.12)$$

Note that

$$-i\partial_j \hat{P}_-^0 1 = -i\partial_j 1 = 0,$$
so the third term from the right hand side of (4.12), when applied to 1, gives 0. The computations from the proof of Lemma 2 (see the formula preceding (3.4)) show that

\[ D_p \tilde{P}(0) = -2i \tilde{L}(0)^{-1} \partial_p \tilde{P}(0), \]

so the last term from the right hand side of (4.12), when applied to 1, also gives 0. In addition, \( \tilde{L}(0) = L(0) \), \( \tilde{P}(0) = P(0) \), and the first part of the second term in (4.12) gives 0, when applied to 1. Therefore,

\[ D_{x_0, y} N_-(0, 0, 0) = i(x_j D_{x_0, y} N_-(0, 0) - D_{x_0, y} N_-(0, 0|x_j)) \]

\[ - 2j_- L_-(0)^{-1} (\partial_{x_0} P_-(0)) x_p + 2\delta_{j, j_-} L(0)^{-1} P_-(0) \]

(4.13)

We use (4.9) to derive

\[ (i(x_j D_{x_0, y} N_-(0, 0) - D_{x_0, y} N_-(0, 0|x_j)) 1, 1) = 2(N_-(0, 0) x_p - n_{-x_0, y}, x_j). \]

We have already shown that \( (n_{-x_0, y}, x_j) = \delta_{j, j_-}[\Omega_+] \), so

\[ (i(x_j D_{x_0, y} N_-(0, 0) - D_{x_0, y} N_-(0, 0|x_j)) 1) = 2(N_-(0, 0) x_p, x_j) + 2\delta_{j, j_-}[\Omega_+]. \]

(4.14)

Let us compute \((j_- L_-(0)^{-1} (\partial_{x_0} P_-(0)) x_p, 1)\). Let \( u(x) = P_-(0) x_p \) and \( v(x) = L_-(0)^{-1} \partial_{x_0} u \). Then

\[ (j_- L_-(0)^{-1} (\partial_{x_0} P_-(0)) x_p, 1) = \int_{\Omega_-} \frac{\partial v(x)}{\partial n_-} dS = \int_{\Omega_-} \Delta v(x) dx \]

\[ = \int_{\Omega_-} \frac{\partial u(x)}{\partial x_0} dx = \int_{\Gamma} x_p \cdot n_{-x_0, y} dS = -\delta_{j, j_-}[\Omega_+]. \]

(4.15)

To compute \((j_- L_-(0)^{-1} P_-(0) 1, 1)\), we note that \( P_-(0) 1 = 1 \). Let \( w(x) \) be a periodic function in \( \Omega_- \) that satisfy the equation \( \Delta w = 1 \), and \( w(x) = 0 \) on \( \Gamma \). Then

\[ (j_- L_-(0)^{-1} P_-(0), 1, 1) = \int_{\Gamma} \frac{\partial w(x)}{\partial n_-} dS = \int_{\Omega_-} \Delta w(x) dx = |\Omega_-|. \]

(4.16)

Equations (4.13)-(4.16) yield

\[ (D_{x_0, y} N_-(0, 0, 0), 1) = 2(N_-(0, 0) x_p, x_j) + 4\delta_{j, j_-}[\Omega_+] + 2\delta_{j, j_-}[\Omega_-]. \]

(4.17)

and, finally, we get (see (4.11))

\[ \omega_{j, j_-}(0) = 2([\Omega_-] \delta_{j, j_-} - (N_-(0, 0)^{-1} n_{-x_0, y}, n_{-x_0, y})). \]

(4.18)

To show that the matrix \( \omega_{j, j_-} \) is positive, we need a variational characterization of the quadratic form \((N_-(0, 0)^{-1} \psi, \psi)\). It is given by the following lemma. The smoothness assumption in Lemma 8 is an overkill, but this is all that we need.
Lemma 8. Let \( \psi(x) \) be a smooth, real valued function defined on \( \Gamma \) that belongs to the space \( \mathcal{M} \) of functions with 0 average. The following variational problem

\[
\left\{ \int_{\Gamma} u(x)\psi(x)dS \rightarrow \max \left[ u(x) \in H^1(T^n \setminus \Omega_+), \int_{\Omega_-} |\nabla u(x)|^2 \leq 1 \right] \right. \tag{4.19}
\]

has a solution, the maximum equals \( \sqrt{(N_-(0, 0)^{-1}\psi, \psi)} \), and any maximizer \( u(x) \) is a harmonic function in \( T^n \setminus \Omega_+ \) that satisfies the boundary condition

\[
\frac{\partial u(x)}{\partial n_-} = a\psi(x) \quad \text{on } \Gamma; \quad |a| = (N_-(0, 0)^{-1}\psi, \psi)^{-1/2}. \tag{4.20}
\]

First, we finish the proof of Lemma 7, and then we will prove Lemma 8. Let \( c = (c_1, \ldots, c_p) \) be a tuple, and let \( \psi_c(x) = c_1 n_{-1} + \cdots + c_p n_{-p} \) be a function on \( \Gamma \). Positivity of the matrix \( (\omega_{j,p}) \) is equivalent (see (4.18)) to the inequality

\[
(N_-(0, 0)^{-1}\psi_c, \psi_c) < |\Omega_-| \quad \text{when } |c| = 1. \tag{4.21}
\]

Take a function \( v_c(x) = x \cdot c \) on \( \Omega_- \). Note that \( v_c(x) \) is not a smooth function in \( T^n \setminus \Omega_+ \), it is not periodic. However, its derivatives are smooth in \( T^n \setminus \Omega_+ \). If \( u(x) \) is an \( H^1(T^n \setminus \Omega_+) \) function, and

\[
\int_{\Omega_-} |\nabla u(x)|^2 \leq 1
\]

then we use harmonicity of \( v_c(x) \) in \( \Omega_- \) and periodicity of \( \nabla v_c(x) \) to obtain

\[
\left| \int_{\Gamma} u(x)\psi_c(x)dS \right| = \left| \int_{\Omega_-} \nabla u \cdot \nabla v_c dx \right| = \left| \int_{\Omega_-} (\nabla u \cdot c) dx \right| \leq |\Omega_-|^{1/2} \left( \int_{\Omega_-} |\nabla u|^2 dx \right)^{1/2} \leq |\Omega_-|^{1/2}. \tag{4.22}
\]

We have used the Cauchy–Schwarz inequality. From Lemma 8 and (4.22), we conclude that \( (N_-(0, 0)^{-1}\psi_c, \psi_c) \leq |\Omega_-| \) (which is not surprising: we have known from the very beginning that the matrix \( (\omega_{j,p}) \) is non-negative because 0 is a point of minimum for \( \omega(k) \)). The equality may hold only when, for a certain function \( u(x) \in H^1(T^n \setminus \Omega_+) \), all inequalities in (4.22) become equalities. The Cauchy–Schwarz inequality becomes an exact equality when \( \nabla u(x) \) is a constant proportional to \( c \). Then \( u(x) \) must be a linear function, but linear functions, except of constants, do not belong to \( H^1(T^n \setminus \Omega_+) \). We conclude that (4.21) holds.

Q.E.D.

Proof of Lemma 8. Though the proof of Lemma 8 is standard, we give it for the sake of completeness. Both the functional and the constraint in the problem (4.19) are invariant under adding a constant to \( u(x) \), so it is sufficient to take the restriction of the problem to the space

\[
\tilde{H}^1(T^n \setminus \Omega_+) = \left\{ u(x) \in H^1(T^n \setminus \Omega_+): \int_{\Omega_-} u(x)dx = 0 \right\}.
\]

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On this space, \( \int |\nabla u|^2 \) is the square of a norm that is equivalent to the \( H^1 \)-norm, and the Sobolev embedding theorem implies that the functional in (4.19) is bounded on 
\( \{ u : \int |\nabla u|^2 dx \leq 1 \} \). Therefore, the problem (4.19) is equivalent to the dual problem

\[
\left\{ \int_{\Omega_-} |\nabla v(x)|^2 dx \rightarrow \min \mid \int_{\Gamma} v(x)\psi(x)dS = 1 \right\},
\]

(4.23)

If \( v(x) \) is a solution of the problem (4.23), then

\[
u(x) = b \left( \int_{\Omega_-} |\nabla v(x)|^2 dx \right)^{1/2} v(x) + c, \quad |b| = 1,
\]

(4.24)
is a solution of the problem (4.19) (here \( b \) and \( c \) are constants), and any solution of the problem (4.19) is of the form (4.24).

Now, let \( v(x) \) be any \( H^1 \)-function in \( \mathbb{T}_n \setminus \Omega_+ \) that meets the constraint in (4.23), and let \( w(x) \) be the harmonic function in \( \mathbb{T}_n \setminus \Omega_+ \) such that

\[
\frac{\partial u(x)}{\partial n_-} = \frac{\psi(x)}{(N_-(0,0)^{-1}\psi, \psi)}, \quad x \in \Gamma; \quad \int_{\Gamma} u(x)dS = 0.
\]

(4.25)

Then

\[
w(x) = \frac{N_-(0,0)^{-1}\psi(x)}{(N_-(0,0)^{-1}\psi, \psi)} \text{ on } \Gamma,
\]

(4.26)

and

\[
\int_{\Gamma} w(x)\psi(x)dx = 1,
\]

so the function \( w(x) \) meets the constraint in (4.23). One has

\[
\int_{\Omega_-} |\nabla w(x)|^2 dx = \int_{\Omega_-} w(x) \frac{\partial w(x)}{\partial n_-}dS = (N_-(0,0)^{-1}\psi, \psi)^{-1}.
\]

(4.27)

Here, we used (4.25) and (4.26).

To show that

\[
\int_{\Omega_-} |\nabla v(x)|^2 dx \geq \int_{\Omega_-} |\nabla w(x)|^2 dx,
\]

(4.28)

we consider a quadratic function of a real variable \( t \)

\[
0 \leq \int_{\Omega_-} |\nabla (v - tw)|^2 dx = \int_{\Omega_-} |\nabla v|^2 dx - 2t \int_{\Omega_-} \nabla v \cdot \nabla w dx + t^2 \int_{\Omega_-} |\nabla w|^2 dx
\]

\[
= \int_{\Omega_-} |\nabla v|^2 dx - 2t \int_{\Gamma} v(x) \frac{\partial w(x)}{\partial n_-}dS + t^2 \int_{\Omega_-} |\nabla w|^2 dx
\]

\[
= \int_{\Omega_-} |\nabla v|^2 dx - 2t (N_-(0,0)^{-1}\psi, \psi)^{-1} + t^2 (N_-(0,0)^{-1}\psi, \psi)^{-1}.
\]

(4.29)
Here, we used (4.25), (4.27), and the fact that \( v(x) \) meets the constraint in (4.23). Therefore,

\[
(N_-(0,0)^{-1} \psi, \psi)^{-1} \int_{\Omega_-} |\nabla v|^2 \, dx \geq (N_-(0,0)^{-1} \psi, \psi)^{-2},
\]

and, finally,

\[
\int_{\Omega_-} |\nabla v|^2 \, dx \geq (N_-(0,0)^{-1} \psi, \psi)^{-1}.
\]

We have shown that \( w(x) \) is a solution of the problem (4.23). The equality in (4.28) holds if \( \nabla v(x) \) is proportional to \( \nabla w(x) \) (see (4.29)). In addition, they satisfy the same constraint (4.23), so any solution of (4.23) equals \( w(x) \), up to an additive constant. Going back to the problem (4.19) (see (4.24)), we derive the statements of Lemma 8.

Q.E.D.

Appendix

In this appendix, we prove that (1.2) holds for a generic domain. To be more precise, we consider the space of all bounded domains in \( \mathbb{R}^d \) with \( C^1 \) boundaries. One easily introduce a topology in this space. Let \( \Omega_0 \) be such a domain. In a tubular neighborhood of its boundary, \( \Gamma_0 \), there exists a \( C^1 \) coordinates \( (y', y_n) = (y_1, \ldots, y_{n-1}, y_n) \) such that \( |y_n| \) is the distance to \( \Gamma_0 \), and \( y_n < 0 \) when a point lies in \( \Omega_0 \). Then, for a sufficiently small neighborhood \( U \) of \( 0 \) in the space \( C^1(\Gamma_0) \), one defines a neighborhood of the domain \( \Omega_0 \) as the set of all domains with the boundaries given by equations \( y_n = f(y') \) where \( y' \in U \).

As in the main text, \( \lambda_j \) is a Dirichlet eigenvalue of the Laplacian in \( \Omega_0 \) and \( u_j \) is the corresponding eigenfunction. We will prove the following proposition.

**Proposition.** For any positive number \( \Lambda \), the set \( \mathcal{D}_\Lambda \) of all \( C^1 \) domains such that all eigenvalues \( \lambda_j \leq \Lambda \) are simple, and (1.2) holds for the corresponding eigenfunctions, is open and dense in the space of all \( C^1 \) domains.

The proposition implies that the set of all domains, for which all Dirichlet eigenvalues are simple and (1.2) holds for all eigenfunctions, is residual. The fact that \( \mathcal{D}_\Lambda \) is open is almost obvious: it follows from the continuity of the Dirichlet spectrum as a function of the domain. The fact that the multiplicity of a Dirichlet eigenvalue can be destroyed by an arbitrarily small perturbation of the domain is well known (e.g., see [A], [M1], [U]). Therefore, we will concentrate on the property (1.2).

Let \( \lambda_j \) be a simple eigenvalue of the Dirichlet Laplacian in \( \Omega_0 \), and let \( u_j(x) \) be the corresponding normalized eigenfunction. For a \( C^1 \) function \( V(x) \) on \( \Gamma_0 \), we define a one-parameter family of domains \( \Omega_\tau \); the boundary \( \Gamma_\tau \) of \( \Omega_\tau \) is given by the equation \( y_n = \tau V(y') \). The eigenvalue \( \lambda_j(\tau) \) of the Dirichlet Laplacian in \( \Omega_\tau \) is simple when \( \tau \) is small enough. It is known (e.g., see [F2]) that the function \( \lambda_j(\tau) \) is differentiable, and

\[
\lambda_j'(0) = - \int_{\Gamma_0} V(x) \left( \frac{\partial u_j(x)}{\partial n} \right)^2 \, dS_x \quad \text{(A1)}
\]
where \( n \) is the outward normal vector to \( \Gamma_0 \).

Let \( u_j(\tau, x) \) be the normalized eigenfunction of the Dirichlet Laplacian in \( \Omega_\tau \), and let

\[
v(x) = \frac{\partial u_j}{\partial \tau}(0, x).
\]

To prove the Proposition, one has to show that if

\[
\int_{\Omega_0} u_j(x) dx = 0
\]

then there exists a function \( V(x) \) such that

\[
\int_{\Omega_0} v(x) dx \neq 0.
\]

Assume that this is not the case, and

\[
\int_{\Omega_0} v(x) dx = 0
\]

for any choice of \( V(x) \). The function \( v(x) \) satisfies the equation

\[
(\Delta + \lambda_j)v(x) + \lambda_j'\partial u_j(0)u_j(x) = 0.
\]

To derive this equation, one differentiates in \( \tau \) the equation for \( u_j(\tau, x) \) and sets \( \tau = 0 \). We will take the variations \( V(x) \) for which \( \lambda_j'(0) = 0 \); that is

\[
\int_{\Gamma_0} V(x) \left( \frac{\partial u_j(x)}{\partial n} \right)^2 dS_\nu = 0.
\]

Then \( v(x) \) satisfies

\[
(\Delta + \lambda_j)v(x) = 0.
\]

It is easy to see that

\[
v(x) = V(x) \frac{\partial u_j(x)}{\partial n}, \quad x \in \Gamma_0.
\]

The equation (A5), with the boundary condition \( v(x) = h(x) \) on \( \Gamma_0 \) is solvable if and only if

\[
\int_{\Gamma_0} h(x) \frac{\partial u_j(x)}{\partial n} dS_\nu = 0,
\]

and it has the unique solution that is orthogonal to \( u_j(x) \) in \( L^2(\Omega_0) \). We denote this solution by \( P(\lambda_j)h(x) \). Let

\[
l(v) = \int_{\Omega_0} v(x) dx.
\]
It follows from the standard elliptic regularity estimates that $IP(\lambda_j)$ is a bounded linear functional on $L^2(\Gamma_\delta)$. If (A3) holds for any function $V(x)$ then this functional vanishes on a linear manifold
\[ \{ V(x) | \partial u_j / \partial n : V(x) \in C^1(\Gamma_\delta) \} \]
The function $\partial u_j / \partial n$ has isolated zeroes, so this linear manifold is dense in $L^2(\Gamma_\delta)$. Therefore, (A3) implies $IP(\lambda_j) = 0$. This means that (A3) holds for any function that satisfies the equation (A5). Integrating both sides of (A5) over $\Omega_\delta$, one derives that
\[ \int_{\Gamma_\delta} \frac{\partial u}{\partial n} \, dS_x = 0 \tag{A8} \]
for any solution $u(x)$ of the equation (A5). Now, (A8) implies that $\lambda_j$ is a Neumann eigenvalue for $\Omega_\delta$, and, moreover, there exists a Neumann eigenfunction $w(x)$ that is constant on $\Gamma_\delta$. Let $w_k(x) = \partial w(x) / \partial x_k$, $k = 1, \ldots, n$. The functions $w_k(x)$ are solutions of the equation (A5), and they equal 0 on $\Gamma_\delta$, so they are Dirichlet eigenfunctions. They are linearly independent. In fact, if $a_1 w_1 + \cdots + a_n w_n = 0$ then the function $w(x)$ is constant along all lines with the direction vector $(a_1, \ldots, a_n)$. On the other hand, $w(x)$ is constant on $\Gamma_\delta$, so $w(x)$ is constant in $\Omega_\delta$. This is impossible because $\lambda_j \neq 0$.

Let us summarize. We assumed that (A3) holds for any function $V(x)$, and, based on this assumption, we have constructed $n$ linearly independent $\lambda_j$-eigenfunctions of the Dirichlet Laplacian in $\Omega_\delta$. However, the domain was taken in such a way that all $\lambda_j$ is a simple eigenvalue of the Dirichlet Laplacian. This contradiction proves that statement of the Proposition.

Q.E.D.

REFERENCES


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