When does a Submodule of $(R[x_1, \ldots, x_n])^o$
 Contain a Positive Element?

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1 Introduction

Let \(S_k = \mathbb{R}[x_1, \ldots, x_k], S_k^+ = \mathbb{R}^+[x_1, \ldots, x_k]\) and \(S_k^{++} = S_k^+ \setminus \{0\}\). Considering Laurent polynomials, similarly write \(R_k = \mathbb{R}[x_k^+, \ldots, x_k] = \mathbb{R}^+[x_k^+, \ldots, x_k] = R_k^{++} = R_k^+ \setminus \{0\}\). When there is no need to make the number of variables explicit we will simply write \(S, S^+, S^{++}, R, R^+, R^{++}\) for these objects.

If \(u = (u(1), \ldots, u(k)) \in \mathbb{Z}^k\), put \(x^u = x_1^{u(1)} \cdots x_k^{u(k)}\) and denote the coefficient of \(x^u\) in \(p \in R\) by \(p_u\). Then \(p = \sum_{u \in \mathbb{Z}^k} p_u x^u\) and the Newton polytope \(N(p)\) of \(p\) is the convex hull of the finite set \(\log(p) = \{u \in \mathbb{Z}^k : p_u \neq 0\}\). For \(v \in \mathbb{R}^k\), let \(\text{in}_v(p)\) be the sum of \(p_u x^u\) over those \(u \in \log(p)\) for which the dot product \(u \cdot v\) is maximal, with the convention that \(\text{in}_v(0) = 0\). For an ideal \(I \subset R\) and \(v \in \mathbb{R}^k\) we have the initial ideal \(\text{in}_v(I) = \langle \text{in}_v(p) : p \in I \rangle \subset R\). It was proved in [ET] that an ideal \(I\) of \(R\) intersects \(R^{++}\) if and only if for every \(v \in \mathbb{R}^k\) and \(a \in (0, \infty)^k\) there exists \(f \in \text{in}_v(I)\) such that \(f(a) > 0\). (It clearly suffices to consider unit \(v\) and \(v = 0\).) We will extend this result to \(R\)-submodules of \(R^n\).

For \(f = (f(1), \ldots, f(n)) \in R^n\) and \(v \in \mathbb{R}^k\), we define \(\text{in}_v(f) \in R^n\) by letting \(\text{in}_v(f)(i) = \text{in}_v(f(i))\). If \(\alpha = (\alpha(1), \ldots, \alpha(n)) \in \mathbb{R}^n\), let

\[
\text{m}_{v, \alpha}(f) = \max \{\max_{1 \leq i \leq n} \{N(f(i)) \cdot v\} + \alpha(i)\}
\]

and define \(\text{in}_{v, \alpha}(f) \in R^n\) by letting the \(i\)-th component

\[
\text{in}_{v, \alpha}(f)(i) = \begin{cases} 
\text{in}_v(f(i)) & \text{if } \max_{1 \leq i \leq n} \{N(f(i)) \cdot v\} + \alpha(i) = \text{m}_{v, \alpha}(f), \\
0 & \text{if } \max_{1 \leq i \leq n} \{N(f(i)) \cdot v\} + \alpha(i) < \text{m}_{v, \alpha}(f).
\end{cases}
\]

Note that if \(f(i) = 0\) then \(\text{in}_{v, \alpha}(f)(i) = 0\). For a module \(M \subset R^n\) we have the initial module \(\text{in}_{v, \alpha}(M) = \langle \text{in}_{v, \alpha}(f) : f \in M \rangle \subset R^n\).

*Key words:* Positive polynomials, vector of polynomials, Gröbner basis

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Let $D_k$ denote the unit sphere in $\mathbb{R}^k$. Let us refer to an element of $D_k$ as a *direction* and say that $v \in D_k$ is *rational* if $tv \in \mathbb{Q}^k$ for some $t \in \mathbb{R}$. For $c \in \mathbb{R}^n$ let us agree to write $c > 0$ if and only if $c \in (0, \infty)^n$.

(1.1) **Theorem** For a submodule $M$ of $R^n$ we have $M \cap (R^+)^n \neq \emptyset$ if and only if the following two conditions hold.

(a) For every $a \in (0, \infty)^k$ there exists $f \in M$ such that $f(a) > 0$.

(b) For every rational $\bar{v} \in D_k$ we have a neighbourhood $U$ of $\bar{v}$ in $D_k$ and a continuous map $v \mapsto \alpha : U \to \mathbb{R}^n$ such that for every $v \in U$ and $a \in (0, \infty)^k$ there exists $f \in \text{in}_{v, a}(M)$ with $f(a) > 0$.

As an immediate corollary we have the analogous result for $S$: An $S$-submodule $M$ of $S^n$ intersects $(S^+)^n$ if and only if (a) and (b) of the theorem hold. (Here, $\text{in}_{v, a}(M)$ may be replaced by the $S$-submodule of $S^n$ generated by $\text{in}_{v, a}(f)$, $f \in M$.)

We will write $\alpha = \alpha_v$ when we wish to make explicit the dependence of $\alpha$ on $v$ in (b).

It is easy to see that (a) and (b) are necessary: Suppose $f \in M \cap (R^+)^n$. For $v \in \mathbb{R}^k$ and $i \in \{1, \ldots, n\}$ put

$$\alpha(i) = \alpha_v(i) = -\max \{N(f(i)) \cdot v\}.$$  

Then $m_{v, a}(f) = 0$ and $\text{in}_{v, a}(f)(i) = \text{in}_v(f(i)) \in R^+$, so that $\text{in}_{v, a}(f)(a) > 0$ for all $a \in (0, \infty)^k$. It is clear from the definition of $\alpha_v$ that $v \mapsto \alpha_v$ is continuous.

Clearly the word 'rational' may be omitted from (1.1).

An example highlighting the importance of the continuity requirement of (b) may be found at the end of section 3. The sufficiency of (a) and (b) is proved in sections 2–5. In sections 2 and 3 we use universal Gröbner bases and an induction on the number $k$ of variables to reduce (1.1) to the following.

(1.2) **Theorem** For a submodule $M$ of $R^n$ we have $M \cap (R^+)^n \neq \emptyset$ if and only if the following two conditions hold.

(a) For every $a \in (0, \infty)^k$ there exists $f \in M$ such that $f(a) > 0$.

(b) For every rational $v \in D_k$ there exists $f \in M$ such that $\text{in}_v(f) \in (R^+)^n$.

Again, the word 'rational' may be omitted and there is an analogous result for $S$. The proof of (1.2) is given in sections 4 and 5. In section 6 we show that it suffices to check (1.2)(b) for finitely many $v$. We then describe a procedure for determining whether $M$ contains an element of $(R^+)^n$ or not, and for finding such an element.

We mention that our interest in the questions addressed in [ET] and the present paper was kindled by our work in ergodic theory [MT].
2 Universal Gröbner bases

Let \( e_1, \ldots, e_n \) be the standard basis of \( S^n \). A monomial of \( S^n \) is an element of the form \( x^u e_i \) for some \( u \in (\mathbb{Z}^+)^k \) and \( i \in \{1, \ldots, n\} \). A term order on the monomials of \( S^n \) is a total order \( \prec \) satisfying the following two conditions:

(i) \( e_i \prec x^u e_i \) for every \( i \in \{1, \ldots, n\} \) and nonzero \( u \in (\mathbb{Z}^+)^k \),
(ii) \( x^u e_i \prec x^{u'} e_i' \) implies \( x^{u+w} e_i \prec x^{u'+w} e_i' \) for all \( i, i' \in \{1, \ldots, n\} \) and \( u, u', w \in (\mathbb{Z}^+)^k \).

Let \( N \) be a submodule of \( S^n \). An element \( f \in S^n \) can be written uniquely as a sum

\[
\sum_{u,i} f_{u,i} x^u e_i,
\]

with coefficients \( f_{u,i} \) in \( \mathbb{R} \). Among the finitely many monomials of \( S^n \) that have nonzero coefficient in this sum, which is maximal according to the term order \( \prec \), is denoted \( \text{in}_\prec(f) \). From \( N \) we obtain the submodule \( \text{in}_\prec(N) = \langle \text{in}_\prec(f) : f \in N \rangle \) of \( S^n \), called the initial module of \( N \) with respect to \( \prec \). Elements \( f_1, \ldots, f_l \in N \) form a Gröbner basis for \( N \) with respect to \( \prec \) if and only if \( \text{in}_\prec(N) = \langle \text{in}_\prec(f_j) : j = 1, \ldots, l \rangle \).

The basic facts we use from the theory of Gröbner bases can all be found in chapters 1 and 3 of [AL].

A universal Gröbner basis of \( N \) is given by elements \( f_1, \ldots, f_l \in N \) that form a Gröbner basis of \( N \) with respect to every term order. The existence of universal bases for submodules \( N \) of \( S^n \) is just like that of \( W, S \) for ideals of \( S \): One verifies that \( N \) has finitely many initial modules by following the proof of (1.2) of [S], using (3.6.4) of [AL] in place of (1.1) of [S].

Fix a submodule \( M \) of \( \hat{R}^n \). Find \( f_1, \ldots, f_l \in M \) that generate \( M \) as an \( \hat{R} \)-module. Let \( \delta = (\delta(1), \ldots, \delta(k)) \in \{-1, 1\}^k \). Pick \( u \in \mathbb{Z}^k \) such that \( x^u f_1, \ldots, x^u f_l \in (\hat{R}[x_1^{\delta(1)}, \ldots, x_k^{\delta(k)}])^n \), and let \( f_{\delta,1}, \ldots, f_{\delta,l(\delta)} \) be a universal Gröbner basis for the \( \hat{R}[x_1^{\delta(1)}, \ldots, x_k^{\delta(k)}] \)-submodule of \( (\hat{R}[x_1^{\delta(1)}, \ldots, x_k^{\delta(k)}])^n \) generated by \( x^u f_1, \ldots, x^u f_l \). List the union of \( \{ f_{\delta,1}, \ldots, f_{\delta,l(\delta)} \} \) over \( \delta \in \{-1, 1\}^k \) as \( g_1, \ldots, g_m \). We call \( g_1, \ldots, g_m \) a super Gröbner basis; we will make use of it throughout the paper.

(2.1) Lemma If \( v \in \mathbb{R}^k \) and \( \alpha \in \mathbb{R}^n \) then \( \text{in}_{\alpha,0}(g_1), \ldots, \text{in}_{\alpha,0}(g_m) \) generate \( \text{in}_{\alpha,0}(M) \).

Proof Let \( \prec_0 \) be an arbitrary term order on the monomials of \( S^n \). Define \( \delta \in \{-1, 1\}^k \) by letting \( \delta(i) = 1 \) if \( v(i) \geq 0 \) and \( \delta(i) = -1 \) if \( v(i) < 0 \). For \( u, u' \in (\mathbb{Z}^+)^k \) and \( i, i' \in \{1, \ldots, n\} \), put \( x_1^{\delta(1)} u^{(1)} \cdots x_k^{\delta(k)} u^{(k)} e_i \prec (x_1^{\delta(1)} u'^{(1)} \cdots x_k^{\delta(k)} u'^{(k)}) e_{i'} \) if

(1) \( \alpha(i) + \sum_{h=1}^k u(h) \delta(h) v(h) < \alpha(i') + \sum_{h=1}^k u'(h) \delta(h) v(h) \), or

(2) \( \alpha(i) + \sum_{h=1}^k u(h) \delta(h) v(h) = \alpha(i') + \sum_{h=1}^k u'(h) \delta(h) v(h) \) and \( x^u e_i \prec_0 x^{u'} e_{i'} \).
This defines a term order \( \prec \) on the monomials of \((\mathbb{R}[x_1^{\delta_1}, \ldots, x_k^{\delta(k)}])^n\). Let \( f \in M \). Considering that \( u \in \mathbb{Z}^k \) involved in the definition of \( f_{\delta_1}, \ldots, f_{\delta_{i(\delta)}} \), find \( w \in \mathbb{Z}^k \) such that \( x^w f \) lies in the submodule \((x^u f_{\delta_1}, \ldots, x^u f_{\delta_{i(\delta)}})\) of \((\mathbb{R}[x_1^{\delta_1}, \ldots, x_k^{\delta(k)}])^n\). Apply the division algorithm [AL] to \( x^w f \) and the subset \( \{f_{\delta_1}, \ldots, f_{\delta_{i(\delta)}}\} \) of \( \{g_1, \ldots, g_m\} \) to find \( p_j \in x^{-w} \mathbb{R}[x_1^{\delta_1}, \ldots, x_k^{\delta(k)}] \) such that \( f = \sum_{j=1}^m p_j g_j \), we have \( p_j = 0 \) if \( g_j \notin \{f_{\delta_1}, \ldots, f_{\delta_{i(\delta)}}\} \), and

\[
\text{in}_<(x^w f) = \max \{\text{in}_<(x^w p_j g_j) : j = 1, \ldots, m\}.
\]

Using (1) of the definition of \( \prec \), it follows that

\[
m_{v, a}(f) = \max \{m_{v, a}(p_j g_j) : j = 1, \ldots, m\}.
\]

Letting \( A = \{1 \leq j \leq m : m_{v, a}(p_j g_j) = m_{v, a}(f)\} \), we have

\[
\text{in}_{v, a}(f) = \sum_{j \in A} \text{in}_{v, a}(p_j g_j) = \sum_{j \in A} \text{in}_{v}(p_j) \text{in}_{v, a}(g_j).
\]

\[\square\]

### 3 Directional positivity

The following inductive step will form the core of our proof of (1.1).

**3.1 Proposition** Suppose (1.1) is valid for fewer than \( k \) variables, and let \( M \) be a submodule of \((R_k)^n\) that satisfies (1.1)(b). Then for every rational \( v \in D_k \) there exists \( f \in M \) such that \( \text{in}_v(f) \in (R^{++})^n \).

For the proof of (3.1), we may employ a suitable change of variables to assume without loss of generality that \( v = -\epsilon_k = (0, \ldots, 0, -1) \). Write \( y = x_k \). For \( f \in R^n \), put \( m(f) = m_{v, a}(f) \) and

\[
\text{in}^0_{v, a}(f)(i) = y^{v(i) - m(f)} \text{in}_{v, a}(f)(i).
\]

Observe that \( \text{in}^0_{v, a}(f)(i) \in R_{k-1} = \mathbb{R}[x_1^{\frac{1}{n}}, \ldots, x_k^{\frac{1}{n}}] \). Let \( \text{in}^0_{v, a}(M) \) be the \( R_{k-1} \)-submodule of \((R_{k-1})^n\) generated by \( \{\text{in}^0_{v, a}(f) : f \in M\} \).

**3.2 Lemma** If \( \text{in}^0_{v, a}(M) \cap (R_{k-1}^{++})^n \neq \emptyset \) then there exists \( f \in M \) such that \( \text{in}_v(f) \in (R^{++})^n \).

**Proof** By assumption there exist \( f_j \in M \) and \( p_j \in R_{k-1} \) such that

\[
g = \sum_j p_j \text{in}^0_{v, a}(f_j) \in (R_{k-1}^{++})^n.
\]

Hence,

\[
\sum_j p_j y^{v(i) - m(f_j)} \text{in}_{v, a}(f_j)(i) \in R_{k-1}^{++}
\]
for \( i = 1, \ldots, n \). As \( m(f_j) \) need not be integral we consider its integral and fractional parts, \( [m(f_j)] \) and \( (m(f_j)) \). Note that either \( y^{-[m(f_j)]} \cdot \text{in}_{w,\alpha}(f_j)(i) \) is homogeneous in \( y \), with \( m(f_j) - [m(f_j)] - \alpha(i) \) as the power of \( y \) common to all its terms, or \( \text{in}_{w,\alpha}(f_j)(i) = 0 \).

Among \( i, j \) with \( \text{in}_{w,\alpha}(f_j)(i) \neq 0 \), the integral quantity \( m(f_j) - [m(f_j)] - \alpha(i) \) is independent of \( j \): To see this, observe that the difference of the quantities corresponding to \( j \) and \( j' \) is an integer and also equals \( (m(f_j)) - (m(f_{j'})) \in (-1, 1) \). So, for each \( i \) we may pick \( j \) with \( \text{in}_{w,\alpha}(f_j)(i) \neq 0 \) and unambiguously write
\[
\beta(i) = m(f_j) - [m(f_j)] - \alpha(i).
\]

When \( \text{in}_{w,\alpha}^0(f_j)(i) \neq 0 \), the lowest \( y \)-degree of the terms of \( y^{-[m(f_j)]} f_j(i) \) equals \( \beta(i) \). On the other hand, in the case \( \text{in}_{w,\alpha}^0(f_j)(i) = 0 \) the \( y \)-degrees of the terms of \( y^{-[m(f_j)]} f_j(i) \) exceed \( \beta(i) \). Since \( p_j \in R_{k-1} \) and \( y = x_k \), it follows that
\[
f = \sum_j p_j y^{-[m(f_j)]} f_j \in M
\]
has \( \text{in}_w(f)(i) = y^{\beta(i)} g(i) \in R^{++} \).

\[\quad\]

**Proof of (3.1)** Without loss of generality we assume that \( v = -e_k = (0, \ldots, 0, -1) \). We continue to write \( y = x_k \). Let \( \alpha = \alpha_v \) be as in (1.1)(b). In view of (3.2) and the assumption that (1.1) is valid for submodules of \( (R_{k-1})^n \), it suffices for the proof of (3.1) to exhibit a continuous function \( w \mapsto \gamma_w : D_{k-1} \to \mathbb{R}^n \) such that for every \( w \in D_{k-1} \) and \( a \in (0, \infty)^{k-1} \) there exists \( g \in \text{in}_{w,\gamma}^0(\text{in}_{w,\alpha}^0(M)) \) with \( g(a) > 0 \).

Let us agree to identify \( w \in D_{k-1} \) with \( (w, 0) \in D_k \). For small \( \epsilon > 0 \), put \( t = \sqrt{1 - \epsilon^2} \), \( \tilde{w} = \epsilon w - t e_k \) and \( \tilde{\alpha} = \alpha_{\tilde{w}} \). Let \( U \) be the neighbourhood of \( v = -e_k \) to which (1.1)(b) applies. Pick \( \epsilon > 0 \) so small that for all \( w \in D_{k-1}, j \in \{1, \ldots, m\} \) and \( i \in \{1, \ldots, n\} \) we have
\[
(1) \quad \tilde{w} \in U,
\]
\[
(2) \quad \text{in}_{\tilde{w}}(g_j(i)) = \text{in}_w(\text{in}_v(g_j(i))),
\]
\[
(3) \quad \text{if } \max\{N(g_j(i)) \cdot v\} + \alpha_v(i) \leq m_{w,\alpha}(g_j) \text{ then } \max\{N(g_j(i)) \cdot \tilde{w}\} + \alpha_{\tilde{w}}(i) < m_{\tilde{w},\tilde{\alpha}}(g_j).
\]

Define
\[
\gamma(i) = \gamma_w(i) = (\alpha_{\tilde{w}}(i) + t \alpha_v(i)) / \epsilon.
\]

Observe that continuity of \( \alpha \) on \( U \) both makes (3) possible and implies the continuity of \( w \mapsto \gamma_w : D_{k-1} \to \mathbb{R}^n \).

Now fix \( w \in D_{k-1} \).

(3.3) **Lemma** For all \( j, i \) we have
\[
(*) \quad \text{in}_{\tilde{w},\tilde{\alpha}}(g_j)(i) = y^{m(\tilde{w}) - \alpha(i)} \text{in}_{w,\gamma}(\text{in}_{w,\alpha}^0(g_j))(i).
\]

5
Proof Observe that (3) of our choice of $\epsilon$ means that we have $\inf_{\gamma}(g_{j})_{i}(i) = 0$ whenever $\inf_{\nu,\alpha}(g_{j})_{i}(i) = 0$, ensuring that the lemma holds whenever $\inf_{\nu,\alpha}(g_{j})_{i}(i) = 0$. Let $i, i'$ be such that $\inf_{\nu,\alpha}(g_{j})_{i}(i), \inf_{\nu,\alpha}(g_{j})_{i'}(i') \neq 0$. From (2) and the definition of $\tilde{w}$ we have

$$\max\{N(g_{j}(i)) \cdot \tilde{w}\} + \tilde{\alpha}(i)$$

$$= \max\{N(y^{m_{j}} - \alpha(i) \inf_{\nu,\alpha}(g_{j})_{i}(i)) \cdot \tilde{w}\} + \tilde{\alpha}(i)$$

$$= \epsilon \max\{N(\inf_{\nu,\alpha}(g_{j})_{i}(i)) \cdot w\} - t(m_{j} - \alpha(i)) - \tilde{\alpha}(i).$$

Using the analogous equalities for $i'$, we see that the equality

$$\max\{N(g_{j}(i)) \cdot \tilde{w}\} + \tilde{\alpha}(i) = \max\{N(g_{j}(i')) \cdot \tilde{w}\} + \tilde{\alpha}(i')$$

holds if and only if

$$\epsilon \max\{N(\inf_{\nu,\alpha}(g_{j})_{i}(i)) \cdot w\} + \tilde{\alpha}(i) + t\alpha(i) = \epsilon \max\{N(\inf_{\nu,\alpha}(g_{j})_{i'}(i')) \cdot w\} + \tilde{\alpha}(i') + t\alpha(i'),$$

which happens if and only if

$$\max\{N(\inf_{\nu,\alpha}(g_{j})_{i}(i)) \cdot w\} + \gamma(i) = \max\{N(\inf_{\nu,\alpha}(g_{j})_{i'}(i')) \cdot w\} + \gamma(i').$$

This means that the left-hand side of (3) is nonzero if and only if the right-hand side is nonzero. Recalling that, by (2), $\inf_{\gamma}(g_{j}(i)) = y^{m_{j}^{(i)}} - \alpha(i) \inf_{\nu,\alpha}(g_{j})_{i}(i)$ whenever $\inf_{\nu,\alpha}(g_{j})_{i}(i)$ is nonzero, the lemma is proved.

Returning to the proof of (3.1), consider $a \in (0, \infty)^{k-1}$. Put $\tilde{a} = (a, 1)$. Since $M$ satisfies (1.1)(b), there exists $f \in \inf_{\gamma}(M)$ with $f(\tilde{a}) > 0$. Use (2.1) to find $p_{j} \in R$ such that

$$f = \sum_{j=1}^{m} p_{j} \inf_{\gamma}(g_{j}).$$

By (3.3),

$$f(i) = \sum_{j} p_{j} y^{m_{j}^{(i)}} \inf_{\nu,\alpha}(g_{j})_{i}(i).$$

Define $q_{j} \in R_{k-1}$ by letting $q_{j}(x_{1}, \ldots, x_{k-1}) = p_{j}(x_{1}, \ldots, x_{k-1}, 1)$ and evaluate the last equation at $\tilde{a}$:

$$\sum_{j} q_{j}(a) \inf_{\nu,\alpha}(g_{j})_{i}(a)(i) = f(i)(\tilde{a}) > 0.$$

Hence, $g = \sum_{j} q_{j} \inf_{\nu,\alpha}(g_{j})_{i}(a)$ is an element of $\inf_{\nu,\alpha}(M)$ with $g(a) > 0$. This completes the proof of (3.1).

We are now in a position to deduce (1.1) from (1.2).
Proof of (1.1) We have already observed that (a) and (b) are necessary. To deduce their sufficiency from (1.2), we verify that (1.1)(b) implies (1.2)(b): When \( k = 1 \), for any \( v \in D_1 = \{-1, 1\} \) and \( f \in R^n \), each entry \( \text{in}_v(f)(i) \) consists of a single term and the existence of \( f \in M \) with \( \text{in}_e(f)(i) \in (R^{++})^n \) is immediate from (1.1)(b). Induction on \( k \), with (3.1) as the inductive step, does the rest. \( \Box \)

We end the section with an example showing that the requirement that \( \alpha \) vary continuously with \( v \) cannot be dropped from (1.1)(b).

(3.4) Example Consider the case \( k = 3 \) and write \( x, y, z \) for the three variables. Let \( p_1 = 1 + x^3 + y^3 - 3xy \) and \( p_2 = 1 - 2x + x^2 + y \). Since \( p_1 \) vanishes when \( x = y = 1 \) and \( \text{in}_{(0,-1)}(p_2) = (1-x)^2 \) vanishes when \( x = 1 \), no multiple of either \( p_1, p_2 \) can belong to \( R^{++} \). Let

\[
f_1 = (z p_1 + 1, z p_1 + 1), \quad f_2 = (z p_2, 1),
\]

and let \( M \) be the submodule of \( R^2 \) generated by \( f_1 \) and \( f_2 \). Put \( w = (0, 0, 1) \). To see that \( M \) does not intersect \( (R^{++})^2 \), suppose \( q_1, q_2 \in R \) are such that \( g = q_1 f_1 + q_2 f_2 \in (R^{++})^2 \). Then \( \text{in}_w(g(2)) \in R^{++} \). Since no multiple of \( p_1 \) belongs to \( R^{++} \), this means that the \( z \)-degree of \( q_2 \) must exceed that of \( q_1 \); that is,

\[
\max\{N(q_2) \cdot w\} > \max\{N(q_1) \cdot w\}.
\]

On the other hand, since \( \text{in}_w(g(1)) \in R^{++} \) and no multiple of \( p_2 \) belongs to \( R^{++} \),

\[
\max\{N(q_2) \cdot w\} \leq \max\{N(q_1) \cdot w\}.
\]

These contradictory inequalities reveal that \( M \cap (R^{++})^2 = \emptyset \).

Observe that \( M \) satisfies (1.1)(a) since \( f_2(a) > 0 \) for all \( a \in (0, \infty)^3 \). Also note that \( \text{in}_v(f_1) \in (R^{++})^2 \) for all \( v \in D_3 \setminus \{w\} \). Define \( v \mapsto \alpha_v : D_3 \to \mathbb{R}^2 \) by putting \( \alpha_w = (0, 1) \) and letting \( \alpha_v = (0, 0) \) for all \( v \in D_3 \setminus \{w\} \). Then \( \text{in}_{w, \alpha}(f_2) = f_2 \), and \( \text{in}_{w, \alpha_v}(f_1) = \text{in}_v(f_1) \in (R^{++})^2 \) for all \( v \in D_3 \setminus \{w\} \). Hence (1.1)(b) is satisfied, except for the fact that \( v \mapsto \alpha_v \) has a discontinuity at \( w \).

4 Irrational directions

In preparation for the proof of (1.2), we next show that positivity in irrational directions follows from that in rational directions. We continue to work with the super Gröbner basis \( g_1, \ldots, g_m \) constructed in section 2.

(4.1) Lemma Let \( \tilde{v} \in D_k \) and \( \epsilon > 0 \). There exists a rational direction \( v \in D_k \) such that \( \|v - \tilde{v}\| < \epsilon \) and \( \text{in}_v(g_j(i)) = \text{in}_{\tilde{v}}(g_j(i)) \) for all \( j \in \{1, \ldots, m\} \) and \( i \in \{1, \ldots, n\} \).

Proof For each \( i, j \) such that \( g_j(i) \neq 0 \), pick \( u_{i,j} \subseteq \log(\text{in}_{\tilde{v}}(g_j(i))) \). The condition \( \text{in}_v(g_j(i)) = \text{in}_{\tilde{v}}(g_j(i)) \) amounts to the requirements that \( u \cdot v = u_{i,j} \cdot v \) for all \( u \subseteq \log(\text{in}_{\tilde{v}}(g_j(i))) \) and
that \( u \cdot v < u_{i,j} \cdot v \) for all \( u \in \log(g_j(i)) \backslash \log(\infty(g_j(i))) \). Treating \( v(1), \ldots, v(k) \) as variables and running through all \( i, j \) with \( g_j(i) \neq 0 \), we obtain a set of homogeneous linear equations and a set of strict linear inequalities in \( v(1), \ldots, v(k) \). These equations and inequalities have integral coefficients. Reduce the equations to echelon form. The reduced equations will have rational coefficients. Since the equations have a nontrivial solution, namely \( v \), the echelon form contains fewer than \( k \) equations, leaving \( l \geq 1 \) of the variables \( v(1), \ldots, v(k) \) free. For the \( l \) free variables choose rational values close to the corresponding entries of \( v \), and use the reduced equations to determine the values of the remaining \( k - l \) variables. The resulting rational direction \( \frac{v}{||v||} \) may be made arbitrarily close to \( v \) by choosing the values of the free variables sufficiently close the the corresponding entries of \( v \), and \( \frac{v}{||v||} \) will satisfy the set of inequalities because \( v \) does. 

\( 4.2 \) Proposition Suppose \( M \) is a submodule of \( \mathbb{R}^n \) and for every rational \( v \in D_k \) we have \( f \in M \) with \( \text{in}_v(f) \in (R^{++})^n \). Then for every \( v \in D_k \) there exists \( g \in M \) with \( \text{in}_v(g) \in (R^{++})^n \).

Proof Let \( \tilde{v} \in D_k \) and use (4.1) to find rational \( v \in D_k \) such that \( \text{in}_v(g_j(i)) = \text{in}(g_j(i)) \) for all \( j, i \) and \( v(i) \tilde{v}(i) > 0 \) whenever \( \tilde{v}(i) \neq 0 \). By assumption there is \( f \in M \) such that, writing \( \beta(i) = -\max\{N(f(i)) \cdot v\} \), we have \( \text{in}_v(f) \in (R^{++})^n \). Put

\[
\tilde{\beta}(i) = -\max\{N(\text{in}_v(f(i))) \cdot \tilde{v}\},
\]

so that \( m_{v,\tilde{\beta}}(\text{in}_v(f)) = 0 \) and \( m_{v,\tilde{\beta}}(\text{in}_v(f)) \in (R^{++})^n \). Let \( \prec_0 \) be an arbitrary term order on the monomials of \( S^n \). Define \( \delta \in \{-1, 1\}^k \) by letting \( \delta(i) = 1 \) if \( v(i) \geq 0 \) and \( \delta(i) = -1 \) if \( v(i) < 0 \). Let \( |v| \) and \( |\tilde{v}| \) denote the elements of \( \mathbb{R}^k \) with \( |v|(i) = |v(i)| \) and \( |\tilde{v}|(i) = |\tilde{v}(i)| \). For \( u, u' \in (\mathbb{Z}^+)^k \) and \( i, i' \in \{1, \ldots, n\} \), put \( (x_1^{\delta(1)})^{u(1)} \cdots (x_k^{\delta(k)})^{u(k)} e_i \prec (x_1^{\delta(1)})^{u'(1)} \cdots (x_k^{\delta(k)})^{u'(k)} e_{i'} \) if

1. \( \beta(i) + u \cdot |v| < \beta(i') + u' \cdot |v| \), or
2. \( \beta(i) + u \cdot |v| = \beta(i') + u' \cdot |v| \) and \( \beta(i) + u \cdot |\tilde{v}| < \beta(i') + u' \cdot |\tilde{v}| \), or
3. \( \beta(i) + u \cdot |v| = \beta(i') + u' \cdot |v| \), \( \beta(i) + u \cdot |\tilde{v}| = \beta(i') + u' \cdot |\tilde{v}| \) and \( x^u e_i \prec_0 x'^{u} e_{i'} \).

This defines a term order on the monomials of \( (\mathbb{R}[x_1^{\delta(1)}, \ldots, x_k^{\delta(k)}])^n \). As in the proof of (2.1), we have \( w \in \mathbb{Z}^k \) and \( p_j \in R \) such that

\[
\text{in}_w(x w f, x w p_j g_j) \in (\mathbb{R}[x_1^{\delta(1)}, \ldots, x_k^{\delta(k)}])^n, \quad f = \sum_{j=1}^m p_j g_j
\]

and

\[
\text{in}_w(x w f) = \max \{\text{in}_w(x w p_j g_j) : j = 1, \ldots, m\}.
\]

Considering (1) of the definition of \( \prec \) and letting

\[
A = \{1 \leq j \leq m : m_{v,\tilde{\beta}}(f) = m_{v,\tilde{\beta}}(p_j g_j)\},
\]

...
we obtain
\[ \text{in}_{\nu,\beta}(f) = \sum_{j \in A} \text{in}_{\nu,\beta}(p_j g_j) \in (R^+)^n. \]

Furthermore, considering (2) of the definition of \( \prec \) and letting
\[ B = \{ j \in A : m_{\bar{\nu},\bar{\beta}}(\text{in}_{\nu,\beta}(p_j g_j)) = 0 \}, \]
we have
\[ (*) \quad \text{in}_{\bar{\nu},\bar{\beta}}(\text{in}_{\nu,\beta}(f)) = \sum_{j \in B} \text{in}_{\nu,\beta}(\text{in}_{\nu,\beta}(p_j g_j)) \in (R^+)^n. \]

Let \( i \in \{1, \ldots, n\} \). Put \( B_i = \{ j \in B : \text{in}_{\nu,\beta}(\text{in}_{\nu,\beta}(p_j g_j))(i) \neq 0 \}, \) so that (*) and the fact that \( \text{in}_{\bar{\nu}}(g_j(i)) = \text{in}_{\nu}(g_j(i)) \) imply
\[ (**) \quad \text{in}_{\bar{\nu}}(\text{in}_{\nu}(f(i))) = \sum_{j \in B_i} \text{in}_{\nu}(p_j) \text{in}_{\nu}(g_j(i)) \in R^+. \]

Consider \( g = \sum_{j \in B} \text{in}_{\nu}(p_j) g_j \in M. \) Since \( B_i = \{ j \in B : N(\text{in}_{\nu}(\text{in}_{\nu}(p_j g_j))(i)) \cdot \bar{\nu} = -\bar{\beta}(i) \}, \) it follows from (**) and the definition of \( g \) that
\[ \text{in}_{\bar{\nu}}(g(i)) = \text{in}_{\bar{\nu},\bar{\beta}}(g(i)) = \sum_{j \in B_i} \text{in}_{\nu}(p_j) g_j(i) \in R^+. \]

The polynomials \( h_j = \text{in}_{\nu}(p_j) \) satisfy \( h_j = \text{in}_{\nu}(h_j). \) Note that
\[ \text{in}_{\bar{\nu}}(h_j g_j(i)) = \text{in}_{\nu}(h_j) \text{in}_{\nu}(g_j(i)) = \text{in}_{\nu}(h_j) \text{in}_{\nu}(\text{in}_{\nu}(g_j(i))) = \text{in}_{\nu}(\text{in}_{\nu}(h_j g_j(i))). \]

Since \( B \) consists of those \( j \in A \) where the maximum \( m_{\bar{\nu},\bar{\beta}}(\text{in}_{\nu,\beta}(p_j g_j)) = m_{\bar{\nu},\bar{\beta}}(h_j \text{in}_{\nu,\beta}(g_j)) \) is attained, we also have
\[ \max_{j \in B} \max_{i \in B} \{ N(h_j g_j(i)) \cdot \bar{\nu} \} = \max_{j \in B} \max_{i \in B} \{ N(\text{in}_{\nu}(h_j g_j(i))) \cdot \bar{\nu} \} = \max_{j \in A} \max_{i} \{ N(\text{in}_{\nu,\beta}(h_j g_j(i))) \cdot \bar{\nu} \} = \max_{i} \{ N(\text{in}_{\nu,\beta}(f(i))) \cdot \bar{\nu} \} = -\bar{\beta}(i). \]

As \( B_i = \{ j \in B : N(\text{in}_{\nu}(\text{in}_{\nu}(p_j g_j))(i)) \cdot \bar{\nu} = -\bar{\beta}(i) \}, \) it then follows from (**) and the definition of \( g \) that
\[ \text{in}_{\bar{\nu}}(g(i)) = \text{in}_{\bar{\nu},\bar{\beta}}(g(i)) = \sum_{j \in B_i} \text{in}_{\nu}(p_j) g_j(i) \in R^+. \]
5  Gluing

We will prove (1.2) by gluing together various elements of $M$ to come up with an element whose components satisfy (it) of the following theorem of Handelman[H].

(5.1) Theorem [H] For $p \in R$ the following are equivalent.

(i) There exists $q \in R^+$ such that $qp \in R^+$.

(ii) $\text{in}_v(p)(a) > 0$ for all $v \in R^k$ and $a \in (0, \infty)^k$.

Handelman’s theorem may be viewed as dealing with principal ideals of $R$; it was also used in proving the result of [ET] for arbitrary ideals. A short self-contained proof of (5.1) may be found in [DT].

We continue to work with a submodule $M \subset R^n$.

(5.2) Lemma Suppose that for every $v \in D_k$ we have $g \in M$ with $\text{in}_v(g) \in (R^+)^n$. Then there exists $f \in M$ such that $\text{in}_v(f) \in (R^+)^n$ for every $v \in D_k$.

Proof For $v \in D_k$, let $f_v \in M$ be such that $\text{in}_v(f_v) \in (R^+)^n$. Note that if $v' \in D_k$ is close enough to $v$ we have $\text{in}_{v'}(f_v) = \text{in}_{v'}(\text{in}_v(f_v)) \in (R^+)^n$. Hence, there exists $\epsilon_v > 0$ such that $\text{in}_{v'}(f_v) \in (R^+)^n$ for all $v'$ in the $\epsilon_v$-ball $B(v, \epsilon_v)$ around $v$. The set of all such balls, $\mathcal{B} = \{B(v, \epsilon_v) : v \in D_k\}$, forms an open cover of the compact set $D_k$ and hence has a Lebesgue number, $2\lambda$.

Take a finite collection of balls of radius $\lambda$ which cover $D_k$, and label their centers $v_1$, $\ldots$, $v_m$. Note that each ball $B(v_j, \lambda)$, $j \in \{1, \ldots, m\}$, is contained in some $B(v_j', \epsilon_v') \in \mathcal{B}$ and let $f_j = f_{v_j'}$, so that $\text{in}_v(f_j) \in (R^+)^n$ for all $v \in B(v_j, \lambda)$. Let $2\kappa$ be a Lebesgue number for the cover $\{B(v_j, \lambda) : j = 1, \ldots, m\}$ of $D_k$. Then for any $v \in D_k$ there exists $j \in \{1, \ldots, m\}$ such that $B(v, \kappa) \subset B(v_j, \lambda)$ and, in particular, $\|v_j - v\| < \lambda - \kappa$.

Let $\delta$ be the infimum of

$$\{v \cdot v_j - v \cdot v_{j'} : v \in D_k, \|v_j - v\| < \lambda - \kappa, \|v_{j'} - v\| \geq \lambda, j, j' \in \{1, \ldots, m\}\}.$$  

Note that $\delta \geq \frac{\kappa}{2} > 0$ since for all $v, w, w' \in D_k$ we have $v \cdot w - v \cdot w' = \frac{1}{2} (\|w' - v\| - \|w - v\|)$. Choose $r$ large enough that $N(f_j(i)) \subset B(0, \frac{\delta + \sqrt{2}}{2})$ for all $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, n\}$.

For $j = 1, \ldots, m$ pick an integral vector $[rv_j]$ such that $\|[rv_j] - rv_j\| \leq \frac{\sqrt{2}}{2}$. Let

$$f = \sum_{j=1}^{m} x^{[rv_j]} f_j.$$
Consider any $v \in D_k$ and $i \in \{1, \ldots, n\}$. Let $j \in \{1, \ldots, m\}$ be such that $\|v - v_j\| < \lambda - \delta$. For all $j' \in \{1, \ldots, m\}$ with $\|v - v_{j'}\| \geq \lambda$ we have
\[
\max \left\{ N \left( x^{v_{j'}} f_{j'}(i) \right) \cdot v \right\} < r v \cdot v_{j'} + \frac{\delta r}{2} < \min \left\{ N \left( x^{v_j} f_j(i) \right) \cdot v \right\}.
\]
Recalling that $n_v(f_{j'}) \in (R^{++})^n$ for all $j' \in \{1, \ldots, m\}$ with $\|v - v_{j'}\| \leq \lambda$, we see that
\[
in_v(f) = n_v \left( \sum_{j: \|v - v_j\| < \lambda} x^{v_j} f_j \right) \in (R^{++})^n.
\]

Let $p \in R$. Put $|p| = \sum_{u \in \log(p)} |p_u| x^u$ and, considering the boundary $\partial N(p)$ of the Newton polyhedron of $p$, let $p_\partial = \sum_{a \partial \log(p)} p_a x^u$ and $p_c = p - p_\partial$. For $v \in \mathbb{R}^k$, write $m_v(p) = \max \{N(p) \cdot v\}$ and $e^v = (e^v(1), \ldots, e^v(k))$. Let $1_\mathbb{R} = (1, \ldots, 1) \in \mathbb{R}^k$.

(5.3) Lemma Let $p \in R$ be such that $n_v(p) \in R^{++}$ for every $v \in D_k$. There exists $d > 0$ such that
\[
p(e^v) \geq e^{m_v(p)} \left( n_v(p)(1) - e^{d t} |p_c(1)| \right)
\]
for all $v \in D_k$ and $t \geq 0$. In particular, there is a compact set $K \subset (0, \infty)^k$ such that $p(a) > 0$ for all $a \in (0, \infty)^k \setminus K$.

Proof Note that $\log(p_c) = \log(p) \setminus \partial N(p)$. Use the compactness of $D_k$ to find $d > 0$ so that, putting $m_v = m_v(p)$, we have
\[
m_v - m_v(p) = \max \{\log(p) \cdot v\} - \max \{\log(p_c) \cdot v\} \geq d
\]
for all $v \in D_k$. For $t \geq 0$ and $v \in D_k$, put $a = e^tv$. Observe that, for $u \in \mathbb{Z}^k$,
\[
a^u = \prod_i a(i)^{u(i)} = \prod_i e^{t u(i) a(i)} = e^{t (v,u)}.
\]
Also using the fact that $p_\partial \in R^{++}$, we have
\[
p(e^tv) = p(a) = p_\partial(a) + p_c(a) \geq n_v(p)(a) + p_c(a) = e^{m_v(p)} n_v(p)(1) + \sum_{u \in \log(p_c)} p_u e^{t (u,v)} \geq e^{m_v(p)} n_v(p)(1) - e^{(m_v - d)t} \sum_{u \in \log(p_c)} |p_u| = e^{m_v(p)} n_v(p)(1) - e^{d t} |p_c(1)|.
\]
Proof of (1.2) Clearly (a) and (b) are necessary. For the converse, use (4.2) and (5.2) to find $f \in M$ such that $\mathfrak{in}(f) \in (R^{++})^n$ for every $v \in D_k$. Applying (5.3) to the entries of $f$, pick $C > 0$ so that we have $f(a) > 0$ whenever $a \in (0, \infty)^k \setminus [C^{-1}, C]^k$. Put $K = [C^{-1}, C]^k$ and $\tilde{K} = [(3C)^{-1}, 3C]^k$. Use (1.1)(a) to find $h_1, \ldots, h_l \in M, a_1, \ldots, a_l \in \tilde{K}$ and $r_1, \ldots, r_l > 0$ such that the open balls $B(a_j, r_j)$ cover $\tilde{K}$ and $h_j > 0$ on $B(a_j, 2r_j)$. For small $\delta > 0$, let $q_j \in R$ be such that $|q_j - 1| < \delta$ on $B(a_j, r_j)$ and $|q_j| < \delta$ on $\tilde{K} \setminus B(a_j, \frac{3}{2}r_j)$. Pick $\delta > 0$ small enough for $g = \sum_{j=1}^l q_j h_j$ to have $g(a) > 0$ for all $a \in \tilde{K}$. Letting

$$q(x_1, \ldots, x_k) = \frac{1}{2k} \sum_{i=1}^k \frac{1}{C}(x_i + x_i^{-1}) \in R,$$

fix $\varepsilon > 0$ small enough to have $\varepsilon f(a) + g(a) > 0$ for all $a \in \tilde{K}$. We will complete the proof by showing that, for sufficiently large $N \in \mathbb{N}$, every entry of $h = \varepsilon q^N f + g \in M$ satisfies (5.1)(ii).

First note that for large $N \in \mathbb{N}$ the Newton polytope of $g$ will be in the interior of that of $\varepsilon q^N f$ and we will have

$$\mathfrak{in}_v(h) = \mathfrak{in}_v(\varepsilon q^N f) = \varepsilon \mathfrak{in}_v(q)^N \mathfrak{in}_v(f) \in (R^{++})^n$$

for all $v \neq 0$. Considering the case $v = 0$, we need to make sure that $h(a) > 0$ for all $a \in (0, \infty)^k$.

Since $0 < q \leq 1$ on $K$, our choice of $\varepsilon$ guarantees that $h(a) > 0$ for all $a \in K$. In fact, $h(a) > 0$ for all $a \in \tilde{K}$ since both $f, g > 0$ on $\tilde{K} \setminus K$. Put

$$\mu = \min \min_i \{m_i(f(i))\},$$

$$\nu = \max \max_i \{m_i(g(i))\}.$$

Use (5.3) to find $\xi > 0$ and $T_0$ so that

(i) $T_0 \geq 2\sqrt{k} \log(2Ck)$,

(ii) $f(i)(e^{tv}) \geq \xi e^{\mu t}$ for all $v \in D_k, t \geq T_0$ and $i \in \{1, \ldots, n\}$.

Since

$$q(e^{tv}) \geq \frac{1}{2Ck} \sum_i (e^{v(i)} + e^{-v(i)}) \geq \frac{1}{2Ck} \sum_i e^{\|v\|_i t} \geq \frac{e^{t\sqrt{\mu}}}{2Ck},$$

for $t \geq T_0$ we have

$$h(i)(e^{tv}) \geq e^{(t\sqrt{\mu} / 2Ck)^N} \xi e^{\mu t} - e^{\mu t} \sum_u \|g(i_u)\|_i$$

$$\geq \xi e^{t\sqrt{\mu} / 2Ck} - e^{\mu t} \|g(i_u)\|_i,$$

for $t \geq T_0$ we have

$$h(i)(e^{tv}) \geq e^{(t\sqrt{\mu} / 2Ck)^N} \xi e^{\mu t} - e^{\mu t} \sum_u \|g(i_u)\|_i$$

$$\geq \xi e^{t\sqrt{\mu} / 2Ck} - e^{\mu t} \|g(i_u)\|_i,$$
where the last inequality results from (i). Provided \( N \) is large enough to satisfy
\[
e^{\frac{T_0}{2} \sqrt{\frac{1}{2} + \mu}} \geq \left| g(i) \right| (\Omega)/(\epsilon \xi),
\]
we will have \( h(i)(e^{\mu}) > 0 \) for \( t \geq T_0 \).

Finally consider \( a = e^{\mu} \) such that \( t \leq T_0 \) and \( a \notin ((3C)^{-1}, 3C)^k \). Since \( \frac{1}{2C}(a(i) + a(i)^{-1}) \geq \frac{3}{2} \) for \( 0 < a(i) \notin ((3C)^{-1}, 3C) \), in this case we have
\[
h(a) \geq \epsilon (3/2)^N f(a) - g(a),
\]
and for each \( a \) the last quantity will be positive for sufficiently large \( N \). By compactness, for large \( N \) we will also have \( h(a) > 0 \) for all \( a = e^{\mu} \) with \( 0 \leq t \leq T_0 \) and \( a \notin ((3C)^{-1}, 3C)^k \). \( \square \)

6 A finite set of directions

In this section we use the super Gröbner basis \( g_1, \ldots, g_m \) to show that it is enough to verify (1.2)(b) for finitely many \( v \in D_k \). We then describe a procedure for checking whether a given module \( M \subset R^n \) contains a positive element and for finding such an element. The procedure will be based on (1.2); it will use recursion on the number of variables, as in the proof of (1.1).

For every polynomial \( p \in R_k \) there exists a finite partition of \( D_k \) such that two directions of the same partition element give you the same initial part of \( p \); in the terminology of polyhedral geometry this partition is the intersection of the normal fan to \( N(p) \) with \( D_k \) (see Chapter 2 in [S]). The next lemma can be considered as a generalization to a module, but first we need some notation.

Let \( v \in D_k, \alpha \in \mathbb{R}^n \) and let \( x^u e_i \) be a monomial of \( R^n \). We introduce the new variables \( t_1, \ldots, t_n \) and define an \( R \)-module homomorphism
\[
\phi : M \rightarrow R[t_1, \ldots, t_n]
\]
by letting \( \phi(x^u e_i) = x^u t_i = x^u t^\alpha \). Then
\[
m_{u, \alpha}(x^u e_i) = u \cdot v + \alpha(i) = (u, e_i) \cdot (v, \alpha)
\]
and therefore we have
\[
(\hat{i}) \quad \phi(\text{in}_{u, \alpha}(f)) = \text{in}_{u, \alpha}(\phi(f))
\]
for every \( f \in M \).

Since \( \phi(g_j) \) is a polynomial there exists a partition of \( D_{k+n} \) such that for any two directions in the same partition element the initial parts are the same. Let \( Q \) be the common refinement of the partitions associated to the polynomials \( \phi(g_1), \ldots, \phi(g_m) \).
Using the map $\rho : D_k \times \mathbb{R}^n \to D_{k+n} : (v, \alpha) \mapsto \frac{[v, \alpha]}{\|v, \alpha\|}$, we can consider $D_k \times \mathbb{R}^n$ as a subset of $D_{k+n}$. We also have the scaled projection $\pi : \text{Im}(\rho) \subseteq D_{k+n} \to D_k$ with

$$\pi((v_1, \ldots, v(k + n))) = (v(1), \ldots, v(k))/\|(v(1), \ldots, v(k))\|.$$ 

For each $Q \in \mathcal{Q}$ we consider the two-element partition $\{\pi(Q), D_k \setminus \pi(Q)\}$ and define

$$\mathcal{P} = \bigvee_{Q \in \mathcal{Q}} \{\pi(Q), D_k \setminus \pi(Q)\}.$$

The following lemma clarifies the connection between the finite partitions $\mathcal{P}$, $\mathcal{Q}$ and the initial modules $\text{in}_{v, \alpha}(M)$.

(6.1) Lemma For a submodule $M$ of $\mathbb{R}^n$ there are finitely many initial modules $\text{in}_{v, \alpha}(M)$. In fact, we have $\text{in}_{v_1, \alpha_1}(g_j) = \text{in}_{v_2, \alpha_2}(g_j)$ and $\text{in}_{v_1, \alpha_1}(M) = \text{in}_{v_2, \alpha_2}(M)$ whenever $\rho((v_1, \alpha_1))$ and $\rho((v_2, \alpha_2))$ belong to the same element of $\mathcal{Q}$. In addition, if $v_1, v_2 \in D_k$ belong to the same element of $\mathcal{P}$ and $\alpha_1 \in \mathbb{R}^n$, then there exists $\alpha_2 \in \mathbb{R}^n$ such that $\text{in}_{v_1, \alpha_1}(g_j) = \text{in}_{v_2, \alpha_2}(g_j)$ and $\text{in}_{v_1, \alpha_1}(M) = \text{in}_{v_2, \alpha_2}(M)$.

Proof Assume that $\rho((v_1, \alpha_1))$ and $\rho((v_2, \alpha_2))$ lie in the same element of $\mathcal{Q}$. Since $\mathcal{Q}$ is defined as the refinement of the partitions associated to $\phi(g_j)$ we have

$$\text{in}_{(v_1, \alpha_1)}(\phi(g_j)) = \text{in}_{(v_2, \alpha_2)}(\phi(g_j)),$$

and it follows from (†) that $\text{in}_{v_1, \alpha_1}(g_j) = \text{in}_{v_2, \alpha_2}(g_j)$ for $j = 1 \ldots m$. We obtain $\text{in}_{v_1, \alpha_1}(M) = \text{in}_{v_2, \alpha_2}(M)$ by (2.1).

For the final assertion, suppose $v_1$ and $v_2$ belong to the same element $P$ of $\mathcal{P}$ and $\alpha_1 \in \mathbb{R}^n$. Let $Q$ be the element of $\mathcal{Q}$ such that $\rho((v_1, \alpha_1)) \in Q$. Since $v_1$ belongs to both $P$ and $\pi(Q)$, the set $P$ is contained in $\pi(Q)$. Hence, for $v_2 \in P$ there exists $\alpha_2 \in \mathbb{R}^n$ with $\rho((v_2, \alpha_2)) \in Q$. Since $\text{in}_{v_1, \alpha_1}(M) = \text{in}_{v_1, t \alpha_1}(M)$ for any $t > 0$, we then have $\text{in}_{v_1, \alpha_1}(M) = \text{in}_{v_2, \alpha_2}(M)$ by the first part of the lemma.

We can now construct an element $M$ that is positive for all directions in $P$.

(6.2) Lemma Let $P \in \mathcal{P}$ and assume that after a change of coordinates $P$ is an open subset of $D_k \cap (e_1, \ldots, e_d)^\perp$. Let $v \in P$ and $\alpha \in \mathbb{R}^n$. There exist $b_i \in \mathbb{Z}^k$ and $c_j \in \mathbb{Z}^k$ such that $x_i x_j \text{in}_{v, \alpha}(g_j)(i) \in R_d$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. If the module $\text{in}_{v, \alpha}(M)$ contains a positive element, then there are polynomials $p_j \in x^{i_j} R_d$ such that

(i) $\sum p_j \text{in}_{v, \alpha}(g_j) \in (R_k^{++})^n$, and

(ii) $f_P = \sum p_j g_j$ has $\text{in}_v(f_P) \in (R_k^{++})^n$ for every $v \in P$.

Proof Let $h_j = \text{in}_{v, \alpha}(g_j)$, and let $Q$ be the element of $\mathcal{Q}$ to which $\rho((v, \alpha))$ belongs. Then $h_j = \text{in}_{v', \alpha'}(g_j)$ whenever $\rho((v', \alpha')) \in Q$. Suppose $i \in \{1, \ldots, n\}$ is such that $h_j(i) \neq 0$. 

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Then \( h_j(i) = \text{in}_{v'}(h_j(i)) \) for every \( v' \in P \). Since \( P \) is assumed to be an open subset of \( D_k \cap \{e_1, \ldots, e_d\}^k \) this shows that \( x^{a_j} h_j(i) \in R_d \) for some \( a_{ij} \in \{0\}^d \times \mathbb{Z}^{k-d} \). Letting \( F = \{(i, j) : \text{in}_{v, a}(g_j)(i) \neq 0\} \), this defines \( a_{ij} \) for all \((i, j) \in F \). Note that for \( (i, j) \in F \) we have
\[
m_{v', a'}(g_j) = -a_{ij} \cdot v' + a(i).
\]

Considering a sequence
\[
(*) \quad (i_0, j_0), (i_1, j_0), (i_1, j_1), \ldots, (i_l, j_{l-1}), (i_l, j_l), (i_0, j_l)
\]
in \( F \), and writing \( i_{l+1} = i_0 \), we find that
\[
0 = \sum_{s=0}^{l} m_{v', a'}(g_{j_s}) - m_{v', a'}(g_{j_s}) = \sum_{s=0}^{l} a_{i_{s+1}, j_s} \cdot v' - a_{i_{s}, j_s} \cdot v'
\]
for every \( v' \in P \). By the assumption on \( P \) we get
\[
(**) \quad \sum_{s=0}^{l} a_{i_{s+1}, j_s} = a_{i_{s}, j_s} = 0
\]
for every allowed sequence \( (*) \) in \( F \).

We now extend \( a_{ij} \) and \( ** \) to all pairs \((i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}\). Assume \( a_{ij} \) is already defined on a set \( E \supseteq F \) and \( ** \) is valid on \( E \). Pick \((i, j) \notin E \). If there exists a sequence
\[
(i_1, j), (i_1, j_1), \ldots, (i_l, j_{l-1}), (i_l, j_l), (i_0, j_l) \in E,
\]
we put \( i_{l+1} = i \) and define
\[
a_{ij} = a_{i_{l+1}, j} - \sum_{s=1}^{l} a_{i_{s+1}, j_s} - a_{i_{s}, j_s}.
\]
One easily verifies that \( ** \) then holds for every allowed sequence \( (*) \) in \( E \cup \{(i, j)\} \).

If there is no sequence
\[
(i_1, j_0), (i_1, j_1), \ldots, (i_l, j_{l-1}), (i_l, j_l), (i_0, j_l)
\] in \( E \), we can take \( a_{ij} \) to be any element of \( \{0\}^d \times \mathbb{Z}^{k-d} \) and have \( ** \) hold for all sequences \( (*) \) in \( E \cup \{(i, j)\} \).

Having thus extended \( a_{ij} \) to all pairs \((i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}\), we define
\[
b_i = a_{i1}
\]
and
\[
c_j = a_{ij} - a_{i1},
\]
which is independent of \( i \) by \( ** \) (using the allowed sequence \((i, j), (i', j), (i', 1), (i, 1)\)). Then \( a_{ij} = b_i + c_j \) and the first part of the lemma follows.
Multiplying every \( i \)-th coordinate of every element of \( M \) with \( x^k_i \) we get a conjugated module, and multiplying \( g_j \) with \( x^k_j \) we get a different set of generators. So, we can assume that \( b_i = c_j = 0 \). This means that \( \text{in}_{\nu, \alpha}(g_j) \in R^+_d \) for every \( j \). If \( \text{in}_{\nu, \alpha}(M) \) contains a positive element, we can choose \( p_j \in R_d \) such that

\[
g = \sum p_j \text{in}_{\nu, \alpha}(g_j) \in (R^+_d)^n.
\]

Define \( f = \sum p_j g_j \) and fix \( \nu' \in P \) and \( \alpha' \) with \( \rho((\nu', \alpha')) \in Q \). As in (3.2), we have \( \text{in}_{\nu', \alpha'}(f) = \sum p_j \text{in}_{\nu', \alpha'}(g_j) = g \in (R^+_d)^n \). Going back to the original module and the original generators we see that \( p_j \in x^\nu R_d \) and \( \text{in}_{\nu', \alpha'}(f) \in (R^+_k)^n \) only. \( \square \)

We now describe a procedure for deciding whether \( M \) contains a positive element and for finding such an element.

(6.3) Procedure 1. Construct a super Gröbner basis \( g_1, \ldots, g_m \in M \).

2. Using the Newton polytopes of \( \phi(g_j) \) calculate the partition \( Q \) of \( D_{k+n} \) and project its sets to define \( P \).

3. Pick for every \( P \in P \) a rational direction \( v_P \in P \) and, for every \( Q \in Q \) with \( P \subset \pi(Q) \), pick a vector \( \alpha_{PQ} \in \mathbb{R}^n \) with \( \rho((v_P, \alpha_{PQ})) \in Q \).

4. Fix a partition element \( P \in P \).
   - Make a coordinate change in the variables \( x_1, \ldots, x_k \) using a matrix \( A \in \text{Gl}(k, \mathbb{Z}) \) such that after the change \( v_P = e_k \).
   - For every \( Q \in Q \) with \( P \subset \pi(Q) \), consider the module
     \[
     \text{in}_{\nu, \alpha_{PQ}}^0(M) = \langle \text{in}_{\nu, \alpha_{PQ}}^0(g_1), \ldots, \text{in}_{\nu, \alpha_{PQ}}^0(g_m) \rangle \subset R_{k-1}^n
     \]
     and determine whether this \( R_{k-1} \) module contains a positive element \( h_P \).
   - If there is a positive element \( h_P \) in one of the above modules, use (6.2) to construct an element \( f_P \in M \) with \( \text{in}_v(f_P) \in (R^+_k)^n \) for every \( v \in P \).
   - If there is no positive element in any of the above modules, then \( M \) does not contain a positive element either.

5. Having completed the last step for every \( P \in P \), glue the vectors \( f_P \) together to get an element \( f \in M \) with \( \text{in}_v(f) \in (R^+_k)^n \) for every \( v \in D_k \) (see (5.2)).

6. Use (5.3) to find a compact set \( K \subset (0, \infty)^k \) such that \( f(a) > 0 \) for \( a \notin K \).

7. Check condition (1.2)(a) for \( a \in K \). If the condition fails for some \( a \in K \), the module \( M \) does not contain a positive element. If the condition holds for every \( a \in K \) find, as in the proof of (1.2), an element \( h \in M \) which satisfies (5.1)(ii) in every coordinate.
8. Let $q = \prod_i \left( \sum_{n \in \mathbb{N}} (h(i)) x^n \right)$, and find $l \in \mathbb{N}$ such that $q'(h) \in (\mathbb{R}_k^{++})^n$.

The above procedure might be called an algorithm except for two questions. The first is whether (1.2)(a) can be checked algorithmically; as we have seen above, it is sufficient to have an algorithm for checking this condition on a compact set $K$. In particular, when $k = 1$ we have polynomials and we require an algorithm for checking a compact set for zeros. The existence of $l$ as in step 8 is a consequence of the proof of Handelman’s theorem (see [DT]); the second question is whether there is a computable bound on $l$.

References


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