Classification on the Average of Random Walks

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Abstract. We introduce a new method for studying large scale properties of random walks. The new concepts of transience and recurrence on the average are compared with the ones introduced in [1] and with the usual ones; their relationships are analyzed and various examples are provided.

Keywords: Random walk, generating function, limit on the average, summability methods.

Mathematics Subject Classification: 82B41, 60G50

1. Introduction

Random walks on graphs provide a mathematical model in many scientific areas, from finance (financial modelling), to physics (magnetization properties of metals, evolution of gases, etc.) and biology (neural networks, disease spreading, etc.). In particular graphs describe the microscopical structure of solids, ranging from very regular structures like crystals or ferromagnetic metals which are viewed as Euclidean lattices, to the irregular structure of glasses, polymers or biological objects.

Geometrical and physical properties of these discrete structures are linked by random walks (especially the simple random walk), which usually describe the diffusion of a particle in these more or less regular media.

An interesting feature of random walks on graphs is their large time scale asymptotics which is deeply connected with the concept of recurrent or transient random walk. This classification was first introduced by Pólya for simple random walks on lattices (see [2]) to distinguish between random walks which return to the starting point with probability one (these are recurrent), and those whose return probability is less than one (which are transient).

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We observe that in a vertex-transitive graph (such as the lattice $\mathbb{Z}^d$) the return probabilities of the simple random walk do not depend on the starting vertex; although in the case of any irreducible random walk they may differ from vertex to vertex, nevertheless being strictly less than one in one vertex is equivalent to being strictly less than one in any vertex. The distinction between recurrent and transient random walks is known as the type-problem (for the type-problem for random walks on infinite graphs, see [3]).

It has been recently observed that even though the type of a random walk describes local properties of the physical model, average values of return probabilities over all starting sites play a key role in the comprehension of the macroscopical behaviour of the model itself (like spontaneous breaking of continuous symmetries [4], critical exponents of the spherical model [5], or harmonic vibrational spectra [6]).

These observations lead to the definition of a new type-problem: the type-problem on the average (see [1]). The definition introduced by Burioni, Cassi and Vezzani is the following: given the family of the generating functions of the $n$-step return probabilities of a random walk, $\{F(x,x|z)\}_{x \in X}$, and a reference vertex $o \in X$ (where $(X,E(X))$ is the graph to which the random walk is adapted), the random walk is recurrent on the average if

$$\lim_{z \to 1^{-}} \lim_{n \to \infty} \frac{\sum_{x \in B(o,n)} F(x,x|z)}{|B(o,n)|} = 1,$$

and transient on the average if the value of the double limit is less than 1 ($B(o,n)$ is the closed ball of center $o$ and radius $n$, $| \cdot |$ denotes cardinality).

The “average” mentioned in the name given to this new type-problem is here a repeated average over balls with fixed center and increasing radii (of course existence of the limit of these averages is implicitly required). This procedure is a particular case of the following: given a sequence $\{\lambda_n\}_n$ of probability measures on the set $X$, for each $n$ we consider the average of $F$ with respect to $\lambda_n$ (that is the expected value of $F$ with respect to $\lambda_n$) and then we take the limit of these averages when $n$ goes to infinity. Note that in definition (1) one has to evaluate a further limit (namely the one for $z$ going to 1) and $\lambda_n(x) = \chi_{B(o,n)}(x)/|B(o,n)|$.

From a mathematical point of view the definition of this “limit on the average” leads to some problems, like the existence of the limit, the possibility of exchanging the order of the two limits and the dependence on the reference vertex $o$.

We provide an example of random walk which has no classification on the average (the simple random walk on a bi-homogeneous tree), thus the classification on the average is not complete, while the classical one in recurrent and transient random walks is complete (we will refer to the usual classification as the “local” one, in contrast with the one “on the average”).

We then propose a new classification on the average which is complete and is in many cases an extension of the one given in [1]. For this new definition we analyze its independence on the reference vertex and provide a sufficient condition which is weaker than the one produced in [1]. We make comparisons between the former and the new definitions of classification on the average and with the local one; we study when these definitions agree and we give examples of random walks which behave differently according to different classifications (that is, which are transient with respect to one of these classifications...
Another question which naturally arises is what can be said when averages are taken over general sets (not necessarily balls), that is when \( \{ \lambda_n \}_n \) is defined as \( \{ \chi_{B_n}/|B_n| \}_n \) where \( \{ B_n \}_n \) is an increasing family of subsets. Moreover \( \{ \lambda_n \}_n \) could be a general family of probability measures (for instance, for some reasons one would like to give to some subgraphs a greater weight than the one given to other subgraphs). We deal with these more general averaging procedure and prove results which generalize the particular cases.

We briefly outline the content of the paper. In Section 2 we define the limit on the average, we recall the distinction between thermodynamically transience and recurrence on the average (TOA\(_t\) and ROA\(_t\)) as defined in [1]. Moreover we give the definitions of the generating functions \( F(x,x|z) \) and \( G(x,x|z) \) and we state the well known flow criterion which characterize transient networks.

In Section 3 we show that the simple random walk on the bihomogeneous tree has no thermodynamical classification (Examples 3.1 and 3.11) and we introduce our classification on the average of random walks (TOA and ROA). The rest of the section is devoted to the study of averages over balls: we prove that under certain conditions the classification is independent of the centre of the balls (Proposition 3.3). We compare the classical concepts of recurrence and transience with the corresponding on the average and the thermodynamical ones (Theorem 3.5 and Examples 3.7 and 3.8, see also the table at the end of this section). The analogous of the flow criterion on the average is stated (Theorem 3.10).

In Section 4 we connect the behaviour of the random walk on the subgraph to the behaviour of the random walk on the whole graph. Corollary 4.4 and Theorem 4.6 deal with the classification on the average, Theorem 4.10 with the thermodynamical one.

Sections 5 and 6 are devoted respectively to averages over families of finite sets and general averages. The two appendices present technical results for averages of general functions and families of power series.

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<tr>
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<tr>
<td>Recurrent</td>
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<td>impossible (Th. 6.2)</td>
<td>Ex. 5.4</td>
<td>( \mathbb{Z}^2 )</td>
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<tr>
<td>Transient</td>
<td>( \mathbb{Z}^3 )</td>
<td>impossible (Th. 6.2)</td>
<td>Ex. 3.7</td>
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2. Basic definitions

We start giving the general definition of a large scale average depending on a sequence of probability measures on an at most countable set \( X \) (we will usually think of \( X \) as the vertex set of an infinite, connected and locally finite graph).
Definition 2.1. Let \( \lambda = \{ \lambda_n \}_{n \in \mathbb{N}} \) be a sequence of probability measures on \( X \); we call limit on the \( \lambda \)-average (or, if there is no ambiguity, limit on the average) the linear map
\[
L_\lambda(f) := \lim_{n \to +\infty} \sum_{x \in X} f(x)\lambda_n(x).
\]
We call \( D(L_\lambda) \) the domain of \( L_\lambda \), that is
\[
D(L_\lambda) := \left\{ f : X \to \mathbb{C} : \sum_{x \in X} |f(x)|\lambda_n(x) < +\infty, \forall n \in \mathbb{N}, \quad \text{and} \quad \exists \lim_{n \to +\infty} \sum_{x \in X} f(x)\lambda_n(x) \in \mathbb{C} \right\}.
\]
If \( A \subseteq X \) is such that \( \chi_A \in D(L_\lambda) \), then \( A \) is called \( L_\lambda \)-measurable (or briefly measurable) and with a slight abuse of notation, we write \( L_\lambda(A) \) instead of \( L_\lambda(\chi_A) \) (and we call it the \( L_\lambda \)-measure of \( A \) or simply the measure of \( A \)).

If \( \mathcal{F} = \{ B_n \}_{n \in \mathbb{N}} \) is an increasing family of finite subsets whose union is \( X \), then we call limit on the average with respect to \( \mathcal{F} \) (we denote it by \( L_\mathcal{F} \)) the limit on the \( \lambda \)-average where \( \lambda_n(x) = \chi_{B_n}(x)/|B_n| \).

When \( X \) is a metric space (in our case a locally finite, non-oriented graph with its natural distance) and \( o \in X \), if not otherwise stated, we will refer to the limit on the average as the limit on the \( \lambda \)-average, where \( \lambda_n(x) = \chi_{B(o,n)}(x)/|B(o,n)| \), and we will write \( L_o \) instead of \( L_\lambda \).

The limits on the average are particular cases of summability methods (see for instance [7] Paragraph 4.10); note that if \( \lim_{n \to \infty} \lambda_n(x) = 0 \) for any \( x \in X \) (i.e. every finite subset of \( X \) is measurable and its measure is zero) then the limit on the average is called regular. An explanation for this terminology is given by the Toeplitz limit theorem (see [7] Theorem 4.10-1).

We want now to point out some remarks on the Definition 2.1.

Remark 2.2. Given a limit on the average on \( X \), the set of measurable subsets is not, in general, a \( \sigma \)-algebra (nor an algebra; see Proposition 5.2). Anyway it is easy to show that \( A \mapsto L_\lambda(A) \) satisfies a sort of completeness property: more precisely if \( S \) is a measurable set such that \( L_\lambda(S) = 0 \) and \( S' \subseteq S \) then \( S' \) is measurable too and \( L_\lambda(S') = 0 \). Moreover if \( A \) is measurable and its measure is 0 then for every bounded complex function \( f : X \to \mathbb{C} \), we have that \( \chi_A f \in D(L_\lambda) \) and \( L_\lambda(\chi_A f) = 0 \).

Remark 2.3. We defined \( L_\lambda \) on complex valued functions mostly for technical reasons (integrals in the complex field will be needed), nevertheless we are interested in real valued functions. In particular functions taking possibly the values \( \pm \infty \) should be admitted (take for instance \( f \) equal to the Green function of a random walk, which we will define in a moment). To this aim let us consider a function \( f : X \to \mathbb{R} \cup \{ \pm \infty \} \), such that, for all \( n \in \mathbb{N} \), at least one of the following conditions holds:

\[
\begin{cases}
\sum_{x \in X : f(x) > 0} f(x)\lambda_n(x) < +\infty \\
\sum_{x \in X : f(x) < 0} f(x)\lambda_n(x) > -\infty.
\end{cases}
\]
For any such function we introduce the upper limit on the λ-average and lower limit on the λ-average as
\[
\sup L_\lambda(f) := \limsup_{n \to +\infty} \sum_{x \in X} f(x) \lambda_n(x),
\]
\[
\inf L_\lambda(f) := \liminf_{n \to +\infty} \sum_{x \in X} f(x) \lambda_n(x).
\]
We easily note that if \( f \) is any real valued function satisfying the above condition then \( f \in \mathcal{D}(L_\lambda) \) if and only if \( \inf L_\lambda(f) = \sup L_\lambda(f) \in \mathbb{R} \); in this case \( L_\lambda(f) = \inf L_\lambda(f) = \sup L_\lambda(f) \).

**Remark 2.4.** Since any bounded function (hence any characteristic function of a subset of \( X \)) satisfies equation (2), we have that every subset is \( \inf L_\lambda \)-measurable and \( \sup L_\lambda \)-measurable (note that these “measures” are not even finitely additive, although if they are defined on \( \mathcal{P}(X) \)); moreover \( A \subseteq X \) is \( L_\lambda \)-measurable if and only if the \( \inf \)-measure and the \( \sup \)-measure agree on \( A \). Also note that the following assertions are equivalent for every subset \( A \subseteq X \):
1. \( A \) is \( L_\lambda \)-measurable and its measure is 1,
2. \( \inf L_\lambda(A) = 1 \),
3. \( A^c \) is \( L_\lambda \)-measurable and its measure is 0,
4. \( \sup L_\lambda(A^c) = 0 \).

Now we define the functions related with random walks, of which we will consider averages through the main part of this paper.

Associated to a given random walk \((X,P)\) there are usually two generating functions, the Green function \(G(x,y|z)\) and the generating function of the first time return probabilities \(F(x,y|z)\). We denote by \(p^{(n)}(x,y)\) the \(n\)-step transition probabilities from \(x\) to \(y\) \((n \geq 0)\) and by \(f^{(n)}(x,y)\) the probability that the random walk starting from \(x\) hits \(y\) for the first time after \(n\) steps \((n \geq 1)\). Then \(G(x,y|z) = \sum_{n \geq 0} p^{(n)}(x,y)z^n\) and \(F(x,y|z) = \sum_{n \geq 1} f^{(n)}(x,y)z^n\), where \(x,y \in X, z \in \mathbb{C}\). When \(z = 1\) we write \(G(x,y)\) and \(F(x,y)\) (further details can be found in [3]).

An irreducible random walk \((X,P)\) is recurrent if \(F(x,x) = 1\) for some \(x \in X\) (equivalently for all \(x\)) and transient if \(F(x,x) < 1\) for some \(x \in X\) (equivalently for all \(x\)).

We recall here the well known flow-criterion which allows to distinguish between transient and recurrent networks. Given a reversible random walk \((X,P)\) (that is a random walk such that there exists a measure \(m\) on \(X\), \(m(x) \neq 0\) for every \(x\) and \(m(x)p(x,y) = m(y)p(y,x)\)), we give to \((X,E(X))\) the structure of an electric network in the following way. We endow any edge with an orientation \(e = (e^-,e^+)\) and with a resistance \(r(e) = 1/m(e^-)p(e^-,e^+)\). For instance in the case of the simple random walk \(r(e) = 1\) for every edge \(e\).

A flow \(u\) from a vertex \(x\) to infinity with input \(i_0\) is a function defined on \(E(X)\) such that
\[
\begin{align*}
\sum_{e: e^- = x} u(e) &= \sum_{e: e^+ = x} u(e) + i_0, \\
\sum_{e: e^- = y} u(e) &= \sum_{e: e^+ = y} u(e), \quad \forall y \neq x
\end{align*}
\]
(these equalities express balance in any vertex of the graph, the first for \(x\), the second
for any other vertex). The energy of $u$ is defined as $<u,u> := \sum_{e \in E(X)} u^2(x)v(e)$. The existence of finite energy flows is related with transience by the following theorem.

**Theorem 2.5.** Let $(X, P)$ be a reversible random walk. The following are equivalent:

(a) the random walk is (locally) transient;

(b) there exists $x \in X$ (equivalently for all $x \in X$) such that it is possible to find a finite energy flow with non-zero input, from $x$ to infinity;

(c) there exists $x \in X$ (equivalently for all $x \in X$) such that $\text{cap}(x) > 0$.

Here $\text{cap}(x)$ is the capacity of the set $\{x\}$; we refer to Woess [3] for the definition.

We restate the definition of the type-problem according to [1] (and we call it “thermodinamical” to distinguish it from the definition which will be given later).

**Definition 2.6.** Let $(X, P)$ a random walk, and $o \in X$ a fixed vertex. Suppose that $F(\cdot, z) \in \mathcal{D}(L_0)$, for all $z \in \mathbb{C}$, $|z| < 1$. The random walk is said thermodynamically transient on the average with respect to $o$ (briefly $\text{TOTA}_t$) if

$$\lim_{z \to 1^{-}} L_0(F(\cdot, z)) < 1,$$  \hspace{1cm} (3)

thermodinamically recurrent on the average with respect to $o$ ($\text{ROA}_t$) if the limit is equal to $1$.

3. The classification on the average (over balls)

From now on, if not otherwise stated, we will assume that $(X, E(X))$ is a connected (infinite), locally finite, non oriented graph, that $o$ is a fixed vertex of $X$, and that $(X, P)$ is a random walk adapted to the graph $(X, E(X))$. Moreover, for the sake of simplicity, $\text{TOTA}_t$ and $\text{ROA}_t$ will be tacitly understood to be with respect to $o$.

We start considering some natural questions about the classification on the average: is any random walk either $\text{TOTA}_t$ or $\text{ROA}_t$ (that is, is the classification on the average complete)? Does the classification depend on the choice of $o$? Can we reverse the order of the two limits in equation (3)?

Regarding the first question, it is not difficult to find examples of random walks with no thermodinanical classification.

**Example 3.1.** Let us consider the class of bi-homogeneous trees (this coincides with the class of trees which are radial with respect to every point, see [8] Proposition 2.9). Despite its property of symmetry, the simple random walk on a bi-homogeneous tree $\mathbb{T}_{n,m}$ (with $n \neq m$) is neither $\text{ROA}_t$ nor $\text{TOTA}_t$ (for the proof, see Example 3.11).

It would be desirable that the classification on the average would not depend on the choice of the reference vertex $o$. It has been shown in [1] Section 4 that if

$$\lim_{n \to +\infty} \frac{\partial B(o, n)}{|B(o, n)|} = 0$$ \hspace{1cm} (4)

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(where $\partial B(o, n) := \{x \in B(o, n) : \exists y \not\in B(o, n), (x, y) \in E(X)\}$) for some $o$, then the limit on the average is independent of the choice of $o$. This condition is not satisfied, for instance, by any homogeneous tree of degree greater than 2, or by any “fast growing” graph.

As for the last question, that is whether the limit in equation (3) coincides with

$$L_o(F(\cdot, \cdot)) := L_o(F(\cdot, \cdot|1)),$$

in general the answer is no. Anyway, exploiting the fact that $F$ is a power series with non negative coefficients one can show that at least when $\sum_{n \geq 1} k_n$ converges, where $k_n = \sup_{x \in X} f^{(n)}(x, x)$, then existence of the limit in (3) implies existence of (5) and these limits coincide (see Proposition 3.5 (iii) and (iv)).

In order to overcome these difficulties, we introduce a new classification on the average.

**Definition 3.2.** Let $(X, P)$ a random walk, and $\{\lambda_n\}_n$ a sequence of probabilities measures on $X$, the random walk is called transient on the average with respect to $\lambda$ ($\lambda$-TOA) if

$$\inf L_\lambda(F) := \liminf_{n \to \infty} \sum_{x \in X} F(x, x) \lambda_n(x) < 1,$$

recurrent on the average with respect to $\lambda$ ($\lambda$-ROA) if the limit is equal to 1.

With this definition, since the limit always exists, any random walk is classifiable on the average (that is, it is either $\lambda$-TOA or $\lambda$-ROA). Of course, if $L_\lambda(F)$ exists, then $\inf L_\lambda(F) = L_\lambda(F)$.

In this section, if not otherwise stated, $\lambda_n = \chi_{B(o, n)}/|B(o, n)|$ (we consider classification with $\inf L_o$) and we write TOA and ROA instead of $\lambda$-TOA and $\lambda$-ROA.

Now we exhibit a condition implying that this new classification (which in the rest of this paper we denote by “the classification on the average”, in contrast with the “thermodinamical” one defined by (3)) does not depend on the fixed vertex $o$. As in the thermodinamical case, this condition is a topological one for the underlying graph.

**Proposition 3.3.** Let $(X, E(X))$ be such that there exists $x \in X$ satisfying

$$\sup_{n \in \mathbb{N}} \frac{|S(x, n + 1)|}{|B(x, n)|} < +\infty,$$

then the classification on the average of any random walk is independent of the choice of $o$.

**Proof.** We note that eq. (6) holds for one $x$ if and only if it holds for any vertex of $X$. It is easy to show that (6) is equivalent, in the case of the $\inf L_o$ classification, to the requests of Proposition 6.1 (i).

This condition is weaker than the one for the thermodinamical classification, (4); indeed observe that (6) is satisfied by any graph with bounded geometry. On the other hand, bounded geometry is not necessary, as is shown by the following example.
Example 3.4. Given a strictly increasing sequence of natural numbers \( \{s_j\}_j \), such that \( s_0 \geq 1 \), and a vertex \( x_0 \), the tree \( T \) is constructed as follows (see Figure 1, where \( s_j = j \)). Each element on the sphere \( S(x_0, m) \) has exactly one neighbour on the sphere \( S(x_0, m + 1) \) if \( m \neq s_j \) for any \( j \in \mathbb{N} \) and exactly \( j \) neighbours if \( m = s_j \). If we choose \( s_{j+1} \geq s_j + j + 1 \) then \( T \) satisfies eq. (6) and has not bounded geometry.

![Figure 1](image)

Now we start comparing the two classifications on the average and the local one.

Proposition 3.5. Let \( (X, P) \) be a random walk.

(i) If there exists \( A \subseteq X \) measurable, such that \( L_0(A) = 1 \) and \( \lim_{x \to \infty} F(x, x) = \alpha \) then 
\( L_0(F(\cdot, \cdot)) \) exists, and it is less than 1 (respectively equal to 1) if and only if \( \alpha < 1 \) (respectively \( \alpha = 1 \)), and the random walk is TOA (respectively ROA);

(ii) if \( (X, P) \) is (locally) recurrent then \( L_0(F(\cdot, \cdot)) \) exists, is equal to 1 and the random walk is ROA;

(iii) if \( (X, P) \) is ROA then \( L_0(F(\cdot, \cdot)) \) exists, is equal to 1 and the random walk is ROA;

(iv) if the series \( F(x, x) \) is totally convergent (with respect to \( x \in X \)) and \( (X, P) \) is TOA, then \( L_0(F(\cdot, \cdot)) \) exists, it is less than 1 and the random walk is TOA;

(v) \( (X, P) \) is ROA \( \iff \) for every \( \varepsilon > 0 \) the set \( \{x : F(x, x) \geq 1 - \varepsilon\} \) is measurable with measure 1;

(vi) \( (X, P) \) is ROA \( \iff \) there exists \( A \subseteq X \) measurable, such that \( L_0(A) = 1 \) and 
\( \lim_{x \to \infty} F(x, x) = 1; \)

(vii) \( (X, P) \) is TOA \( \iff \) there exists \( A \subseteq X \) such that \( \sup L_0(A) > 0 \) and \( \sup_A F(x, x) < 1. \)

The proof is a particular case of the proof of Proposition 6.2.

Proposition 3.5(iv) states that being the series \( F(x, x) \) totally convergent guarantees that the classification on the average and the thermodinamical one agree (if the last one is admissible). Obviously the function \( F(x, x) \) needs not to be totally convergent even in the case of simple random walks (see Examples 3.7 and 5.4).

Under certain conditions, the series \( F(x, x) \) is indeed totally convergent.
Proposition 3.6. Let $(X, P)$ be a random walk. If one of the following conditions hold then the series $F(x, x)$ is totally convergent.

(i) There exists a subset $\Gamma$ of $\text{AUT}(X)$ (the automorphism group of the graph) and a finite subset $X_0 \subset X$ with the property that for any $y \in X$ there exist $x \in X_0$ and $\gamma \in \Gamma$ such that $\gamma(x) = y$ and $P$ is $\Gamma$-invariant.

(ii) The radius of convergence of the Green function $G(x, x | z)$ (which is independent of $x$) is $r > 1$.

(iii) $(X, P)$ is reversible (with reversibility measure $\nu$ and total conductance $a(x, y) := \nu(x)p(x, y)$) and it satisfies the strong isoperimetric inequality that is

$$\sup_{A \subseteq X} \frac{\nu(A)}{s(A)} < +\infty$$

where the supremum is taken over finite subsets $A$ and $s(A) := \sum_{x \in A, y \in A^c} a(x, y)$.

Proof. We just outline the main points.

(i) If $y = \gamma(x)$ and $P$ is invariant under the action of $\gamma$ then $f^{(n)}(x, x) = f^{(n)}(y, y)$. By hypotheses $k_n := \sup_{x \in X} f^{(n)}(x, x) = \max_{x \in X_0} f^{(n)}(x, x) \geq c_n := \sum_{x \in X_0} f^{(n)}(x, x)$. Hence the radius of convergence of the generating function of the sequence $\{k_n\}$ is greater or equal than the radius of convergence of the generating function of the sequence $\{c_n\}$ and this one is the minimum of the radius of $F(x, x | z)$ where $x \in X_0$.

(ii) It follows immediately by $f^{(n)}(x, x) \leq p^{(n)}(x, x) \leq 1/r^n$ which holds for every $x \in X$ and every $n \in \mathbb{N}$.

(iii) See Woess [3] Chapter 2 Theorems 10.3 and 10.9 and apply (ii). \qed

For instance if $\Gamma$ is a subgroup then property (i) says that $X$ has a finite number of orbits with respect to $\Gamma$. Examples are random walks adapted to Cayley graphs or the simple random walk on quasi transitive graphs.

As for condition (ii), an example is given by a locally finite tree with minimum degree 2 and with finite upper bound to the length of its unbranched geodesics.

We observe that even if $(X, P)$ is both thermodynamically classifiable and classifiable on the average, the two classifications may not agree, as is shown by the following example.

Example 3.7. Let $X := \bigcup_{n \in \mathbb{N}} \{n\} \times \mathbb{Z}_{n+1}$ where $\mathbb{Z}_p$ is the set of the equivalence classes of natural numbers $\mod(p)$ (for any $p \in \mathbb{N}, p > 0$) and let us denote, as usual, $(n, p)$ any element of $X$. For any $n, m \in \mathbb{N}, p, q \in \mathbb{Z}_{n+1}, q \in \mathbb{Z}_{m+1}$ we define the (non oriented) edge set by $(n, p) \sim (m, q)$ if and only if one of the following holds

1) $p = 0_{m+1}$ and $q = 0_{m+1}$ and $|m - n| = 1$,

2) $m = n$ and $p - q = \pm 1$

(where $p - q$ is the usual operation in $\mathbb{Z}_{n+1}$).

![Figure 2](image-url)
If \( \{ p_n \} \) is a \( (0,1) \)-valued sequence such that \( p_n^m \uparrow 1 \) and \( \alpha \in \mathbb{R}, \alpha < 1/3 \), then we define the (adapted) transition probabilities as follows

\[
p((0,0),(1,0)) := p((1,1),(1,0)) = 1
\]

\[
p((1,0),(0,0)) := (1 - p_1)(1 - 2\alpha),
\]

\[
p((1,0),(2,0)) := (1 - p_1)\alpha,
\]

\[
p((1,0),(1,1)) := p_1 + (1 - p_1)\alpha
\]

\[
p((n, p), (m, q)) :=
\begin{cases}
1 & \text{if } n = 0, m = 1, p = 0, q = 0; \\
(1 - p_n)(1 - 2\alpha) & \text{if } m = n + 1, p = 0, q = 0; \\
(1 - p_n)\alpha & \text{if } m = n - 1, p = 0, q = 0; \\
p_n & \text{if } m = n, p = q - 1; \\
1 - p_n & \text{if } m = n, p = q + 1, p \neq 0; \\
(1 - p_n)\alpha & \text{if } m = n, p = 0, q = -1; \\
0 & \text{otherwise;}
\end{cases}
\]

By using standard stopping time arguments we easily see that this random walk is locally transient. If we denote by \( C_n := \{(n, p) : p \in \mathbb{Z}_{n+1}\} \) for every \( n \in \mathbb{N} \), hence for any \( x \in C_n, m \in \mathbb{N} \) we have that \( f^{(n)}(x, x) \geq p_n^m \) and \( f^{(m)}(x, x) \leq 1 - f^{(n)}(x, x) \). We immediately note that \( \lim_{x \to \infty} f^{(m)}(x, x) = 0 \) for any \( m \in \mathbb{N} \) and if \( z \in (0,1) \) by Bounded Convergence Theorem (using \( z^m \geq f^{(m)}(x, x)z^m \)) we derive \( \lim_{x \to \infty} F(x, x|z) = 0 \). Whence for any regular convex limit \( \lambda \) we obtain \( L_\lambda(F(\cdot, \cdot|z)) = 0 \) which implies thermodinamical \( L_\lambda \)-transience. On the other hand \( F(x, x|1) \geq f^{(m)}(x, x) \) for any \( x \in X, m \in \mathbb{N} \), hence if \( x \in \bigcup_{m \geq n} C_m \) we have that \( F(x, x|1) \geq \inf_{m \geq n} p_m^m = p_n^m \) which implies \( \lim_{x \to \infty} F(x, x|1) = 1 \) and, for any regular limit on the average \( \lambda \), \( L_\lambda(F(\cdot, \cdot|1)) = 1 \) (which is \( L_\lambda \)-recurrence). Since the classification on the average and the thermodinamical one are different this provides an example of a random walk for which \( F(x, x) \) is not totally convergent (Proposition 3.5(iv)).

Let us now make some comparisons between the local classification and the classification on the average of a random walk. The previous example shows also that while local recurrence imply recurrence on the average, local transience does not imply transience on the average. Indeed 3.7 is a random walk which is locally transient, TOA, and ROA. There are also examples of locally transient, ROA, and ROA random walks, as is shown by the following.

**Example 3.8.** Given the sequence of natural numbers \( \{ s_j = \sum_{i=1}^{j} \beta^i \}_{j \geq 1} \), where \( \beta \geq 2 \) is an integer number, \( s_0 = 0 \) and \( \alpha \) is a vertex, the construction of the tree \( T \) is similar to the one in Example 3.4. Each element on the sphere \( S(\alpha, m) \) has exactly one neighbour on the sphere \( S(\alpha, m + 1) \) if \( m \neq s_j \) for any \( j \geq 0 \) and exactly \( \alpha \) neighbours if \( m = s_j \) \( (\alpha \in \mathbb{N}) \): Figure 3 represents the case \( \alpha = 3, \beta = 2 \). An application of Nash-Williams criterion proves that \( T \) is locally transient if and only if \( \alpha > \beta \) (see for instance [9] Proposition 2.4).

It is easy to prove that the set \( A \) obtained by removing from \( X \) the balls of radius \( k \) centered in the elements of \( S(\alpha, s_k) \), for all \( k \in \mathbb{N} \) has \( L_\alpha \)-measure equal to 1. Moreover on \( A \), for every fixed \( n \), as \( x \) tends to infinity \( f^{(n)}(x, x) \) is definitively equal to \( f^{(n)}_Z(0,0) \) (the
first time return probabilities of the simple random walk on $\mathbb{Z}$). Hence by Proposition 3.5(i) the graph is ROA$_t$ (thus ROA) with respect to any reference vertex.

![Figure 3](image)

It is known that (local) transience is equivalently expressed by one of the following conditions: (i) $G(x, x) = +\infty$ for some (i.e. for every) $x \in X$; (ii) $F(x, x) = 1$ for some (i.e. for every) $x \in X$. In the average case we can only claim a partial result.

**Proposition 3.9.** Let $(X, P)$ be a random walk. Then:
(i) if the random walk is ROA$_t$ then $\lim_{z \to 1^-} \inf L_0(G(\cdot, \cdot|z)) = +\infty$;
(ii) if the random walk is ROA then $L_0(G(\cdot, \cdot)) = +\infty$.

For the proof we refer to the general case, see Proposition 6.5. Observe that in Proposition 3.9 (i) existence of $L_0(G(\cdot, \cdot|z))$ is not guaranteed and then we had to consider $\inf L_0$ instead. Also notice that reversed implications are not true, see for instance Example 5.3 (according to [1] this is an example of a mixed TOA$_t$ graph).

As we recalled in Section 2, an instrument which can be used to (locally) classify a reversible random walk is the Theorem of Flow. A similar result can be stated also for the classification on the average.

**Theorem 3.10.** Let $(X, P)$ be a reversible random walk, with reversibility measure $m$ satisfying $\inf m(x) > 0$, $\sup m(x) < +\infty$ (in particular this condition is satisfied by the simple random walk on a graph with bounded geometry). For the associated network $N$

- $N$ is TOA;
- there exists $A \subseteq X$ such that $\sup L_0(A) > 0$, there is a finite energy flow $u^x$ from $x$ to $\infty$ with non-zero input for every $x \in A$ and $\sup_{x \in A} < u^x, u^x > < +\infty$;
- there exists $A \subseteq X$ such that $\sup L_0(A) > 0$ and $\inf L_0 A \cap (x) > 0$. 

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For the proof see Theorem 6.3.

As an application we classify bi-homogeneous trees and a whole family of inhomogeneous trees.

**Example 3.11.** Consider the bi-homogeneous tree $T_{m,n}$ and a couple of vertices $x_n$ and $x_m$, the first with degree $n$ and the second with degree $m$ (see Figure 4 for the case $m = 3$ and $n = 2$). We can construct two finite energy flows $u^n$ and $u^m$ with fixed input $i_0$, respectively from $x_n$ to infinity and from $x_m$ to infinity. But then we can obtain a finite energy flux from any vertex $x$ to infinity (with input $i_0$) by translating $u^n$ or $u^m$ (depending on the degree of $x$). Thus we can construct a family of fluxes with bounded energy and this proves that the simple random walk on $T_{m,n}$ is TOA. The proof, which is based on the ideas of Theorem 4.6, can be repeated for any $\lambda$. Moreover, since $L_0(F(\cdot, \cdot))$ does not exist for any reference vertex $o$, and $F(\cdot, \cdot)$ is totally convergent by Proposition 3.6(i) (with $X_0 = \{x_n, x_m\}$ and $\Gamma = \text{AUT}(X)$) then by Proposition 3.5(iii) and (iv) the simple random walk on $T_{m,n}$ cannot be thermodynamically classifiable.

Analogously one can show that a tree $T'_{k,n}$ whose vertices have degree $2$ or $k$ ($k \geq 3$) and such that the distance between ramifications is $n$ ($n \geq 2$) is TOA (while in the former case we had essentially only two fluxes, here we have at most $\lceil n/2 \rceil + 1$ fluxes).

Now consider an inhomogeneous tree $T''_{k,n}$ whose vertices have degree $2$ or $k$ ($k \geq 3$) and such that the distance between ramifications does not exceed $n$ ($n \geq 2$): see Figure 5 for the case $k = 3$ and $n = 2$. Here the family of finite energy fluxes with fixed input, constructed from any vertex to infinity, is in general infinite, but the supremum of the energy is bounded by the supremum of the energy of the fluxes on $T'_{k,n}$. Indeed $T''_{k,n}$ can be mapped into $T''_{k,n}$ by deleting vertices in order to (if necessary) decrease the distance between ramifications. By this procedure fluxes on the first tree are mapped into fluxes (with equal or less energy) on the second one.

As we have seen, the family of $L_0$-measurable sets plays an important role in the classification on the average of random walks, but one has to be careful when dealing with such sets, since they are not an algebra. We state the following proposition, as for the proof, see Proposition 5.2.
Proposition 3.12. Let $o \in X$. The class of $L_o$-measurable subsets is not an algebra; in particular there exist two measurable subsets of $X$, $A$ and $B$, such that $A \cap B$ is not measurable.

4. Subgraphs and graphs

In this section we study information which can be inferred from the knowledge of the behaviour of random walks on subgraphs.

Since we have to average on a subgraph, the first thing to do is to rescale the weights. We start in a general setting, where $X$ is an at most countable set, and we average a general function $f$.

Definition 4.1. Let $\{\lambda_n\}_n$ be a sequence of probability measures on $X$ and $S \subseteq X$ such that $\lambda_n(S) > 0$, for all $n \in \mathbb{N}$. Then the limit on the average $L^S_\lambda$ defined by $\lambda^S_n := \lambda_n\chi_S/\lambda_n(S)$ for every $n \in \mathbb{N}$ and $x \in S$ is called rescaled limit on the average.

As usual, $L^S_0$ will be the limit in the case of the average over balls. With this definition, we link the rescaled average on $S$ and the average on $X$.

Proposition 4.2. Let $S \subseteq X$ be an $L^\lambda$-measurable subset with positive $L^\lambda$-measure. Then:

(1) $f|_S \in \mathcal{D}(L^S_\lambda) \iff \chi_S \cdot f \in \mathcal{D}(L_\lambda),$

(2) if $f|_S \in \mathcal{D}(L^S_\lambda)$ then

$$L_\lambda(\chi_S \cdot f) = L^S_\lambda(f|_S) \cdot L_\lambda(S).$$

(7)

In the case of the lower limit, we have

$$\inf L_\lambda(\chi_S \cdot f) = \inf L^S_\lambda(f|_S) \cdot L_\lambda(S).$$

(8)

Proof. By definition,

$$\sum_{x \in X} f(x)\chi_S(x)\lambda_n(x) = \left( \sum_{x \in X} f(x)\lambda^S_n(x) \right) \cdot \lambda_n(S),$$

hence

$$\exists L_\lambda(\chi_S \cdot f) \iff \exists L^S_\lambda(f|_S),$$

since we assume that $\lim_{n \to +\infty} \lambda_n(S)$ exists and is equal to $L_\lambda(S)$; equation (8) is derived analogously.

From now on, we consider the averaging process over balls of the generating function $F$ related to a random walk $(X, P)$. There are two different ways of looking at the behaviour of the random walk on a subgraph $S \subseteq X$. The first one is to consider $S$ as a subset of the graph, that is with the same transition probabilities of $X$ and hence with $F(x, x)$
restricted to the sites in $S$. The second approach is to view $S$ as an independent graph, that is with possibly different generating functions $F(x, x)$.

We start with the first point of view. Proposition 4.2 implies that sets of measure zero have no weight in the averaging procedure (think of $S$ such that $L_\lambda(S) = 1$: then $L_\lambda^S(F|_S) = L_\lambda(F)$). In particular, if we can find two subsets such that one of the two grows strictly slower than the other one, then the first one can be neglected in the averaging process.

**Remark 4.3.** Let $X = X_1 \cup X_2$, where $X_1 \cap X_2$ is finite. Suppose that $|B(o, n) \cap X_1| \leq f(n), \ |B(o, n) \cap X_2| \geq g(n)$ for all $n \in \mathbb{N}$, where $f$ and $g$ are two functions such that $f(n)/g(n)$ tends to zero as $n$ goes to infinity. Then $L_\alpha(X_1) = 0$ (hence the classification of any random walk on $X$ depends only on the restriction of the generating function $F$ on $X_2$).

Moreover, the following corollary links classification on a subgraph with classification on the whole graph.

**Corollary 4.4.** Let $(X, P)$ be a random walk and let $S$ be a subgraph of $X$ such that $L_\alpha(S) > 0$. If the restriction of the random walk on $S$ is TOA with respect to the rescaled limit on the average, then $(X, P)$ is TOA.

Another result which links the knowledge of $L_\lambda(F(\cdot, \cdot))$ on subsets with the knowledge of $L_\lambda(F(\cdot, \cdot))$ on the whole graph is the following (note that this result holds for any $\lambda$ and any function $f$ in place of $F$).

**Proposition 4.5.** Let $\overline{X} := X \cup \{\infty\}$ be the Alexandroff compactification of $X$ with the discrete topology and let $\{A_i\}_{i=1}^n \subset \mathcal{P}(X)$ be a partition of $X$ such that $A_i$ is $L_\alpha$-measurable and for every $i$ such that if $L_\alpha(A_i) > 0$ then there exists $\lim_{x \to \infty} F|_{A_i}(x, x) =: \alpha_i$.

(i) $L_\alpha(F(\cdot, \cdot))$ exists and is equal to $\sum_{i=1}^n L_\alpha(A_i)\alpha_i$ (where if $L_\alpha(A_i) = 0$ then $\alpha_i$ can be any real number if $L_\alpha(A_i) = 0$).

(ii) Moreover, if $\sum_{i \in \mathbb{N}} F(x, x)\chi_{A_i}$ is uniformly convergent (to $F$) then $L_\alpha(F(\cdot, \cdot))$ is equal to $\sum_{i=1}^\infty L_\alpha(A_i)\alpha_i$ (where if $L_\alpha(A_i) = 0$ $\alpha_i$ can be any real number if $L_\alpha(A_i) = 0$).

**Proof.**

(i) It is easily proved by induction on $n$ and using Theorem A.1 and Proposition 4.2.

(ii) If $L_\alpha(A_i) > 0$ then $|A_i| = +\infty$ and then $\infty$ is an accumulation point of $A_i$ in $\overline{X}$. Since $A_i \cup \{\infty\}$ with the induced topology from $\overline{X}$ is homeomorphic to the Alexandroff compactification of $A_i$ (with the induced topology from $X$), then it is possible to apply Theorem A.1 to $F|_{A_i}$ obtaining $F|_{A_i} \in \mathcal{D}(L_\alpha)$. By Proposition 4.2 we have that $\chi_{A_i}F \in \mathcal{D}(A_i)$ and then by (i) $L_\alpha(\sum_{i=1}^\infty \chi_{A_i}F) = \sum_{i=1}^\infty L_\alpha(A_i)\alpha_i$. Using Proposition A.4 we have the conclusion.

We remark that even though sets of measure zero have no influence on the resulting limit on the average of the function $F(\cdot, \cdot)$, their presence may change the return probabilities and hence the function $F(\cdot, \cdot)$ that we average. This is the main difficulty in the second
approach: the fact that $F(\cdot, \cdot)$ on the whole graph and $F(\cdot, \cdot)$ on the subgraph considered as an independent set may be very different.

Anyway, under certain regularity conditions we can gain information on the whole graph from the knowledge of what happens on its subgraph (regarded as an independent graph). With the next Theorem we give a sufficient condition for the simple random walk on a general graph to be TOA when one of its subgraph is locally transient (that is the simple random on it walk is locally transient). The proof is an application of Theorem 3.10.

**Theorem 4.6.** Let $(A, E(A))$ be a subgraph of $X$ such that $\sup L_o(A) > 0$. Suppose that there exists $x_0 \in A$ such that for every vertex $y \in A$ there exists an injective map $\gamma_y : A \to A$ such that (i) $\gamma_y(x_0) = y$ and (ii) for any $w, z \in A$, $(w, z) \in E(A)$ implies $(\gamma_y(w), \gamma_y(z)) \in E(A)$. If the simple random walk on $(A, E(A))$ is transient then the simple random walk on $(X, E(X))$ is TOA.

The proof will be given in the general case, see Theorem 6.4.

We observe that the condition on $A$ in the previous statement is a requirement of “self-similarity” of $A$. This self-similarity is of course more easily checked on graphs with a group structure, like Cayley graphs.

**Corollary 4.7.** Let $(G, E(G))$ be a Cayley graph and $A \subseteq G$ such that (i) the group identity $e \in A$, (ii) for any $x, y \in A$ we have that $xy \in A$ and (iii) the simple random walk on $(A, E(A))$ is transient. If $(X, E(X))$ is a locally finite graph which contains $(A, E(A))$ as a subgraph and $\sup L_o(A) > 0$ then the simple random walk on $(X, E(X))$ is TOA.

We observe that Theorem 4.6 and Corollary 4.7 hold for any $\lambda$ (indeed once the hypothesis $\sup L_\lambda(A) > 0$ is satisfied, $\lambda$ plays no role in the proof). Moreover, in Example 5.3 we use the Corollary in an explicit case.

The last result of this section deals with knowledge of random walks on subgraphs and the thermodinamical limit on the average. Before stating it we need a technical lemma and a definition.

**Lemma 4.8.** Let $(X, E(X))$ a graph with bounded geometry and $C$ a measurable subset of $X$ such that $L_o(C) = 0$. If $X_n := \{x \in X : d(x, C) \leq n\}$ for every $n \in \mathbb{N}$, then $X_n$ is measurable and $L_o(X_n) = 0$ for every $n \in \mathbb{N}$.

**Proof.** We note that for every $n, r \in \mathbb{N}$, $X_n \cap B(o, r) \subseteq \bigcup_{x \in C \cap B(o, n+r)} B(x, n)$; by hypotheses, for every $m, r \in \mathbb{N}$ we have that

$$\sup_{x \in X} |B(x, m)| = \max_{x \in X} |B(x, m)| \geq |B(o, r + m)|/|B(o, r)|,$$

then there exists $M$ such that

$$\frac{|X_n \cap B(o, r)|}{|B(o, r)|} \leq M \frac{|C \cap B(o, r + n)|}{|B(o, r)|} \leq M^2 \frac{|C \cap B(o, r + n)|}{|B(o, r + n)|} \xrightarrow{r \to +\infty} 0.$$

Our purpose is now to consider a subgraph $(A, E(A))$ as an independent graph, nevertheless the random walk we study on it should be closely related to the random walk
\((X, P)\) on \((X, E(X))\) (think for instance of the simple random walk). Namely, whenever two vertices \(x, y\) are in \(A\) but not on the boundary of \(A\) (that is they have no neighbours outside \(A\)), the 1-step transition probability \(p(x, y)\) should be the same for the random walk on \(A\) and the random walk on \(X\). This is the sense of the following definition.

**Definition 4.9.** Let \((A, E(A))\) be a subgraph on \((X, E(X))\). Let \((X, P)\) be a random walk on \((X, E(X))\), a random walk \((A, P_A)\) is called induced random walk if for every \(x \in A \setminus \partial A\) and every \(y \in A\) we have that \(p(x, y) = p_A(x, y)\).

We note that in general the induced random walk is not uniquely determined. Moreover if \(n \in \mathbb{N}\) and \(x \in A\) are chosen such that \(d(x, \partial A) \geq n\) then

\[
f^{(n)}(x, x) = f_A^{(n)}(x, x), \quad p^{(n)}(x, x) = p_A^{(n)}(x, x).
\]

In the next proposition we deal with a graph which is partitioned in two subgraphs with known properties. We require that a subset \(A\) is convex w.r. to a vertex \(o \in A\), that is that for every \(x \in A\) at least one geodesic path from \(o\) to \(x\) lies in \(A\) (hence we are sure that \(d_A(o, x) = d_X(o, x)\) and we denote this distances by simply \(d\)).

**Theorem 4.10.** Let \((X, E(X))\) be an infinite graph with bounded geometry, and let \((A, E(A))\) and \((B, E(B))\) be two subgraphs such that \(o = A \cap B, X = A \cup B\) and \(A, B\) are both convex w.r. to \(o\). Moreover suppose that \(L_o(A) > 0\) and \(L_o(\partial A) = 0\). Let \(P\) be a stochastic matrix representing a random walk on \(X\) (adapted to \((X, E(X))\)) and let us consider two induced random walks (represented by \(P_A\) and \(P_B\)) on the subgraphs \((A, E(A))\) and \((B, E(B))\). Under the previous hypotheses we have that

(i) any two of the following assertion imply the remaining one

(i.a) \((X, P)\) is \(L_o\)-thermodinamically classifiable;

(i.b) \((A, P_A)\) is \(L_o\)-thermodinamically classifiable;

(i.c) \((A^c, P_B)\) is \(L_o\)-thermodinamically classifiable;

(ii) if two of the assertions in (i) hold then \((A, P_A)\) TOA\(t\) implies \((X, P)\) TOA\(t\);

(iii) if two of the assertions in (i) hold and \(L_o(A) < 1\) then \((X, P)\) is ROA\(t\) if and only if \((A, P_A)\) and \((B, P_B)\) are both ROA\(t\).

**Proof.** (i) It follows easily by Proposition 4.2 and by the equation (10) below.

(ii) Let \(F\) and \(F_A\) the generating function of the hitting probabilities associated to \(P\) and \(P_A\) respectively. By the hypotheses \(F \in D(L_o)\) and \(F_A \in D(L_o^A)\). By equation (9) and Lemma 4.8 we can apply Proposition B.3 to \(F_A\) and \(F_{|A}\) obtaining that \(F_{|A} \in D(L_o^A)\) and \(L_o^A(F_{|A}) = L_o^A(F_A)\).

From Proposition 4.2 we have that \(\chi_A F \in D(L_o)\), moreover

\[
L_o^A(F_A) = L_o^A(F_{|A}) = \frac{L_o(\chi_A F)}{L_o(A)}.
\]

We note that

\[
L_o(F) = L_o(\chi_A F) + L_o(\chi_B F) - L_o(\chi_{A \cap B} F) \leq L_o^A(F_A) L_o(A) + (1 - L_o(A)), \quad (10)
\]

hence if \(\lim_{z \to 1^-} L_o^A(F_A) < 1\) we obviously have \(\lim_{z \to 1^-} L_o(F) < 1\).
(iii) The only if part is a consequence of (ii). Let us define $p_s$ such that

$$p_s(x, y) = \begin{cases} p_A(x, y) & \text{if } x, y \in A, x \neq o \\ p_B(x, y) & \text{if } x, y \in B, x \neq o \\ (1/2)p_A(o, y) & \text{if } x = o, y \in A \\ (1/2)p_B(x, o) & \text{if } x \in B, y = o \\ 0 & \text{otherwise} \end{cases}$$

which is clearly a stochastic matrix satisfying $p_s(x, y) = p(x, y)$ for every $x \notin \partial A \cup \partial B$. Since $\deg(\cdot)$ is bounded, we have that $L_o(\partial A) = 0$ if and only if $L_o(\partial B) = 0$; moreover if we have $x \in X$ and $n \in \mathbb{N}$ such that $n \leq d(x, \partial A \cup \partial B)$ then $f_s^{(n)}(x, x) = f^{(n)}(x, x)$. Whence, using again Proposition B.3 (with $C := \partial A \cup \partial B$ and $X_n$ defined as in Lemma 4.8) and Proposition 4.2, we show that

$$L_o(F) = L_o(F_s) = L_o(\chi_A F_s) + L_o(\chi_B F_s)$$
$$= L_o^A(F_A) L_o(A) + L_o^B(F_B) L_o(B) \xrightarrow{z \to +\infty} L_o(A) + L_o(B) = 1,$$

whence the proof is complete.

The previous theorem is different from those in [1] since here a subgraph $A$ is regarded as an independent graph with an induced random walk. In [1], one is supposed to study the generating function $F_1$ of $(X, P_1)$ to classify the random walk; in our approach one can study independently two (hopefully) simpler random walk $P_2$ and $P_3$ (on $A$ and $A^c$ respectively) and then the classification of the main random walk can be inferred.

5. Averages over increasing sequences of subsets

In this section we deal with the classification on the average for a random walk when the average is taken over a family $\mathcal{F}$ of subsets of $X$. More precisely, we consider $\mathcal{F} = \{B_n\}_n$ an increasing sequence of finite subsets of $X$ such that $\bigcup_n B_n = X$ (we call $\mathcal{F}$ an increasing covering family or ICF), and denote by $L_\mathcal{F}$ the corresponding limit on the average.

Clearly, two families of subsets $\mathcal{F}_1 = \{B_n\}_n$ and $\mathcal{F}_2 = \{C_n\}_n$ may give the same classification on the average of a random walk. The following proposition provides a sufficient condition for this to happen.

**Proposition 5.1.** Given two increasing covering families $\mathcal{F}_1 = \{B_n\}_n$ and $\mathcal{F}_2 = \{C_n\}_n$ in $X$, such that

(i) there exist a divergent sequence $\{i_n\}_n$ of natural numbers and $K > 0$ such that $B_{i_n} \supseteq C_n$ and $|B_{i_n}|/|C_n| \leq K$, for every $n$;

(ii) there exist a divergent sequence $\{j_n\}_n$ of natural numbers and $K' > 0$ such that $C_{j_n} \supseteq B_n$ and $|C_{j_n}|/|B_n| \leq K'$, for every $n$;

then $L_{\mathcal{F}_1}$ and $L_{\mathcal{F}_2}$ induce the same classification on the average for any random walk.

**Proof.** These conditions are equivalent to the ones in Proposition 6.1 (i).
We already observed that the family of $L_0$-measurable sets is not an algebra. We now prove it for the more general case of $L_\mathcal{F}$-measurable sets (where $\mathcal{F}$ is an ICF).

**Proposition 5.2.** Let $\mathcal{F} = \{B_n\}_n$ be an ICF of $X$. Then the class of $L_\mathcal{F}$-measurable subsets is not an algebra; in particular there exist $A, B$ $L_\mathcal{F}$-measurable subsets of $X$ such that $A \cap B$ is not $L_\mathcal{F}$-measurable.

**Proof.** We first note that $A \subseteq X$ is measurable if and only if $A^c$ is measurable. Let us define, for every $n \in \mathbb{N}$, $m_n := |B_n|$; let $\{A_k, C_k\}_{k \in \mathbb{N}}$ be a family of subsets of $X$ such that for every $k \in \mathbb{N}$, $\{A_k, C_k\}$ is a partition of $S_k = B_k \setminus B_{k-1}$ with the following two properties

$$|A_k| - |C_k| \in [-1, 1], \quad \forall k \in \mathbb{N},$$

$$|\bigcup_{i=0}^k A_i| - |\bigcup_{i=0}^k C_i| \in [-1, 1], \quad \forall k \in \mathbb{N}.$$ 

Let us define for every $k \in \mathbb{N}$, $a_k := |\bigcup_{i=0}^k A_i|$, $c_k := |\bigcup_{i=0}^k C_i|$; obviously $a_k + c_k = m_k$ and since $X$ is infinite, we can choose an increasing sequence of natural numbers $\{k_n\}$ such that

$$m_{k_{n+1}} / m_{k_n} \geq 4. \quad (11)$$

It is easy to note that by our hypotheses, for every $n, i \in \mathbb{N}$,

$$\frac{1}{2} - \frac{1}{m_n} \leq \frac{a_n}{m_n} \leq \frac{1}{2} + \frac{1}{m_n},$$

$$\frac{1}{2} - \frac{1}{m_n} \leq \frac{c_n}{m_n} \leq \frac{1}{2} + \frac{1}{m_n},$$

$$\frac{m_i - 2}{m_n + 2} \leq \frac{a_i}{a_n} \leq \frac{m_i + 2}{m_n - 2}. \quad (12)$$

We finally define the two sets

$$A := \bigcup_{i=0}^\infty A_i,$$

$$B := \bigcup_{i=0}^\infty \left( \bigcup_{j=k_{2i+1}}^{k_{2i+2}} A_j \cup \bigcup_{j=k_{2i+1}+1}^{k_{2i+2}} C_j \right);$$

by equation (12) (since $|A \cap B_n| = a_n$) we have that $A$ is measurable and $L_\mathcal{F}(A) = 1/2$; similarly $||B \cap B_n| - c_n| \leq 1 + ||\{i \in \mathbb{N} : k_i < n\}|$. Since by equation (11)

$$\lim_{n \to +\infty} ||\{i \in \mathbb{N} : k_i < n\}||/m_n = 0$$

(observe that if $||\{i \in \mathbb{N} : k_i < n\}| = j$ then $m_n \geq 4^j$), then by equation (12) we obtain that $B$ is also measurable and $L_\mathcal{F}(B) = 1/2$. Moreover

$$A \cap B = \bigcup_{i=0}^\infty \bigcup_{j=k_{2i+1}}^{k_{2i+2}} A_j,$$

Finally, we define

$$X := \bigcup_{i=0}^\infty X_i,$$

and we have that $X$ is not measurable. This contradicts the assumption, and hence the proposition.
hence if $n$ is odd

$$\frac{|A \cap B \cap B_{k_n}|}{|B_{k_n}|} \geq \frac{|A \cap B \cap S_{k_n}|}{|B_{k_n}|} \geq \frac{a_{k_n}}{m_{k_n}} \left(1 - \frac{a_{k_{n-1}}}{a_{k_n}}\right) \xrightarrow{n \to +\infty} \frac{1}{4};$$

similarly when $n$ is even (using equations (11) and (12))

$$\frac{|A \cap B \cap B_{k_n}|}{|B_{k_n}|} = \frac{|A \cap B \cap B_{k_{n-1}}|}{|B_{k_n}|} \leq \frac{a_{k_{n-1}}}{m_{k_n}} = \frac{a_{k_{n-1}}}{m_{k_{n-1}}} \frac{m_{k_{n-1}}}{m_{k_n}}.$$

Since $\liminf_{n \to +\infty} \frac{a_{k_{n-1}}}{m_{k_{n-1}}} m_{k_n} \leq 1/8$, we have that

$$\inf_{\mathcal{F}} L\mathcal{F}(A \cap B) \leq 1/8 < 1/4 \leq \sup_{\mathcal{F}} L\mathcal{F}(A \cap B)$$

which implies that $A \cap B$ is not measurable.

We give two examples of averages over subsets which are not balls. The first one shows that with two different ICFs the classification of a random walk can be different; it is also an example of a random walk which is TOA even if $L_0(G(\cdot, \cdot)) = +\infty$ (recall Proposition 3.9). The second one is an example of classification on the average with an ICF which appears natural and with respect to which the random walk is ROA and TOA$_t$.

**Example 5.3.** Let $X$ be the graph obtained from $Z^3$ by deleting all horizontal edges joining vertices with positive height (compare with [1] where this graph is an example of mixed TOA$_t$): we call $X_+$ the set of vertices with (strictly) positive height and $X_- = X_+^c$. The graph $X$ is a locally transient graph; using Theorem 2.5 one can construct a finite energy flow $u$ defined on $E(Z^3)$ from the origin $o$ to $\infty$ with input 1. By Corollary 4.7 we have that the simple random walk is TOA (more generally it is TOA with respect to any $\lambda$ such that $\sup_{\mathcal{F}} L\mathcal{F}(X_-) > 0$). Observe that $L\mathcal{F}(G(\cdot, \cdot)) = +\infty$ if we restrict to $X_+$, while $L\mathcal{F}(G(\cdot, \cdot)) < +\infty$ if we restrict to $X_-$. The same graph is ROA$_t$ (thus ROA) if we choose a different $\lambda$ given by the following ICF: $\mathcal{F} = \{B_n = (B(o, 2^n) \cap X_+) \cup (B(o, n) \cap X_-)\}_{n}$. In this case $X_-$ has $L\mathcal{F}$-measure zero and by Lemma 4.8, for every $n$, $f^{(n)}(x, x) = f_{Z^3}^{(n)}$ outside a set of $L\mathcal{F}$-measure zero. Thus by Theorem A.1 every $f^{(n)}$ has $L\mathcal{F}$ limit equal to $f_{Z^3}^{(n)}$ and the graph is ROA$_t$ (Theorem B.1(a.1)).
The following example was suggested by D. Cassi, R. Burioni and A. Vezzani.

**Example 5.4.** Let $X$ be the graph obtained by attaching at each vertex $i$ of $\mathbb{N}$ a cube lattice $C_i$ of side $n_i$ by one of its corners. Suppose that $n_i$ diverges. Then $X$ is locally recurrent (by an application of Theorem 2.5), hence it is also ROA with respect to any $\lambda$.

Consider the following ICF: $\mathcal{F} = \{\bigcup_{i=1}^{n} C_i\}$. The simple random walk on $X$ is TOA$_t$ with respect to the limit on the average $L_{\mathcal{F}}$. Indeed for each $k \in \mathbb{N}$, the set $X_k$ obtained removing from $X$ all the vertices at distance $k$ from the surface of the cubes has $L_{\mathcal{F}}$-measure equal to 1 and if $x \in X_k$, $f^{(k)}(x, x) = f^{(k)}_{x^2}$. The thesis is an easy consequence of Theorem A.1 and Theorem B.1(a.1). Note that since the classification on the average and the thermodinamical one are different this also provides an example of a random walk for which $F(x, x)$ is not totally convergent (Proposition 6.2(iv)).

6. The general case

In this section we consider a general sequence $\lambda = \{\lambda_n\}_n$ of probability measures on $X$. We already stated many results for the average over balls which hold also for a general $\lambda$: Proposition 3.5, Proposition 3.9, Theorem 3.10, Proposition 4.2, Corollary 4.4, Theorem 4.6 and Corollary 4.7.

We now prove a result, which we already used in the previous sections: it is a comparison between classifications with two different limits on the average (see Propositions 3.3 and 5.1).

**Proposition 6.1.** Let $\lambda = \{\lambda_n\}_n$, $\eta = \{\eta_n\}_n$ two sequences of probability measures on $X$. Let us consider the following assertions:

(i) there exist two divergent sequences $\{i_n\}_n$ and $\{j_n\}_n$ of natural numbers, and two positive constants $C, K$ such that $C \lambda_{i_n}(x) \geq \eta_n(x)$ and $K \eta_{j_n}(x) \geq \lambda_n(x)$ for every $n$ and $x$;

(ii) for every $A \subseteq X$, $A \in \mathcal{D}(L_{\lambda})$, $L_{\lambda}(A) = 1$ if and only if $A \in \mathcal{D}(L_{\eta})$, $L_{\eta}(A) = 1$;

(iii) every random walk $(X, P)$ is ROA (respectively TOA) with respect to $\lambda$ if and only if it is ROA (respectively TOA) with respect to $\eta$.

Then the following chain of implications holds: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).
Proof. (i) ⇒ (ii) It is easy (recall also Remark 2.4).
   (ii) ⇒ (iii) It is true because of the equivalence between (ii) and (iii) in Proposition A.2.

We state and prove for the general case some of the results quoted in Sections 3 and 4.

**Proposition 6.2.** Let \( (X, P) \) be a random walk.

(i) If there exists \( A \subseteq X \) measurable, such that \( L_\lambda(A) = 1 \) and \( \lim_{x \to +\infty} F(x, x) = \alpha \) then \( L_\lambda(F(\cdot, \cdot)) \) exists, and it is less than 1 (respectively equal to 1) if and only if \( \alpha < 1 \) (respectively \( \alpha = 1 \)), and the random walk is \( \lambda \)-TOA (respectively \( \lambda \)-ROA);

(ii) if \( (X, P) \) is (locally) recurrent then \( L_\lambda(F(\cdot, \cdot)) \) exists, is equal to 1 and the random walk is \( \lambda \)-ROA;

(iii) if \( (X, P) \) is \( \lambda \)-ROA, then \( L_\lambda(F(\cdot, \cdot)) \) exists, is equal to 1 and the random walk is \( \lambda \)-ROA;

(iv) if the series \( F(x, x) \) is totally convergent (with respect to \( x \in X \)) and \( (X, P) \) is \( \lambda \)-TOA, then \( L_\lambda(F(\cdot, \cdot)) \) exists, it is less than 1 and the random walk is \( \lambda \)-TOA;

(v) \( (X, P) \) is \( \lambda \)-ROA \iff for every \( \varepsilon > 0 \) the set \( \{ x : F(x, x) \geq 1 - \varepsilon \} \) is measurable with measure 1;

(vi) if there exists \( A \subseteq X \) measurable, such that \( L_\lambda(A) = 1 \) and \( \lim_{x \to +\infty} F(x, x) = 1 \) then \( (X, P) \) is \( \lambda \)-ROA. Moreover, if \( \lambda \) is regular the converse holds;

(vii) \( (X, P) \) is \( \lambda \)-TOA \iff there exists \( A \subseteq X \) such that \( \sup_{A} L_\lambda(A) > 0 \) and \( \sup_{A} F(x, x) < 1 \).

Proof.

(i) It is an easy consequence of Proposition 4.5.

(ii) It follows trivially from (i).

(iii) It is a consequence of Remark A.5 since \( z \mapsto F(x, x|z) \) is a non decreasing function on \([0, 1]\) bounded from above by 1.

(iv) It follows by Theorem B.1(d).

(v) and (vii) From the relation \( 0 \leq \inf_{A} L_\lambda(F(1)) \leq 1 \) we have that \( (X, P) \) is \( \lambda \)-TOA if and only if it is not \( \lambda \)-ROA; Proposition A.2 yields the conclusion.

(vi) It is a consequence of Proposition A.2.

**Theorem 6.3.** Let \( (X, P) \) be a reversible random walk, with reversibility measure \( m \) satisfying \( \inf_{x \in A} m(x) > 0 \), \( \sup_{x \in A} m(x) < +\infty \) (in particular this condition is satisfied by the simple random walk on a graph with bounded geometry). For the associated network \( N \) TFAE:

(a) \( N \) is \( \lambda \)-TOA;

(b) there exists \( A \subseteq X \) such that \( \sup_{A} L_\lambda(A) > 0 \), there is a finite energy flow \( u^* \) from \( x \) to \( \infty \) with non-zero input for every \( x \in A \) and \( \sup_{x \in A} u^* > \infty \);

(c) there exists \( A \subseteq X \) such that \( \sup_{A} L_\lambda(A) > 0 \) and \( \inf_{x \in A} \text{cap}(x) > 0 \).

Proof. First note that the interesting case is \( (X, P) \) (locally) transient. We therefore restrict
to this particular case.

(a) ⇒ (b) Recall that \( u^x = -\frac{i_0}{m(x)} \nabla G(\cdot, x) \) is a finite energy flow from \( x \) to \( \infty \) with input \( i_0 \) and energy

\[
< u^x, u^x > = \frac{i_0^2}{m(x)} G(x, x),
\]

(where \( \nabla \) denotes the difference operator, see [3]). But by Proposition 6.2(vii) the network is \( \lambda \)-TOA if and only if there exist \( \alpha < 1, A \subseteq X \) such that \( \inf L_A(A) > 0 \) and \( F(x, x) < \alpha < 1 \) for every \( x \in A \). Since \( G(x, x) = 1/(1 - F(x, x)) \) this is equivalent to \( \sup_{x \in A} G(x, x) < +\infty \). By eq. (13) and our hypotheses on the reversibility measure, this implies (b).

(b) ⇒ (c) This is an obvious consequence of \( \text{cap}(x) \geq 1 / < u^x, u^x > \) (see for instance Woess [3]).

(c) ⇒ (a) This follows from \( G(x, x) \leq m(x) / \text{cap}(x) \) for every \( x \in A \), and from our hypotheses on \( m \).

\[ \square \]

Theorem 6.4. Let \( (A, E(A)) \) be a subgraph of \( X \) such that \( \sup L_A(A) > 0 \). Suppose that there exists \( x_0 \in A \) such that for every vertex \( y \in A \) there exists an injective map \( \gamma_y : A \to A \) such that (i) \( \gamma_y(x_0) = y \) and (ii) for any \( w, z \in A, (w, z) \in E(A) \) implies \( (\gamma_y(w), \gamma_y(z)) \in E(A) \). If the simple random walk on \( (A, E(A)) \) is transient then the simple random walk on \( (X, E(X)) \) is \( \lambda \)-TOA.

Proof. Let us consider \( (A, E(A)) \) with the edge orientation induced by \( X \). Let \( u \) be a flow on \( (A, E(A)) \) with finite energy starting from \( x_0 \) to \( \infty \) with input \( 1 \). Given any simple random walk, the conductance is the characteristic function of the edges, hence for any \( y \in A \) it is easy to show that the following equations

\[
u_y(a, b) := \begin{cases} 
\varepsilon_y u(\gamma_y^{-1}(a), \gamma_y^{-1}(b)) & \text{if } (a, b) \in E(A); \\
0 & \text{if } (a, b) \notin E(A);
\end{cases} \quad \forall (a, b) \in E(X)
\]

(where \( \varepsilon_y(a, b) \) is equal to \( +1 \) or \( -1 \) according to \( (\gamma_y(a), \gamma_y(b))^+ = (x, y)^+ \) or not) define a finite energy flow \( u_y \) on \( (X, E(X)) \) starting from \( y \) to \( \infty \) with input one. Apply now Theorem 6.3.

\[ \square \]

Proposition 6.5. Let \( (X, P) \) be an irreducible random walk, adapted to a (connected) infinite graph \( (X, E(X)) \) and \( \{\lambda_n\}_n \) a sequence of probability measures on \( X \). If \( F(\cdot, \cdot | z) \in D(L_\lambda) \) for every \( z \in [0, 1) \) and \( \lim_{z \to 1^-} L_\lambda(F(\cdot, \cdot | z)) = 1 \) (respectively \( L_\lambda(F(\cdot, \cdot)) = 1 \)) then \( \lim_{z \to 1^-} \inf L_\lambda(G(\cdot, \cdot | z)) = +\infty \) (respectively \( \inf L_\lambda(G(\cdot, \cdot)) = +\infty \)).

Proof. From equation \( G(x, x | z) = 1/(1 - F(x, x | z)) \) (see Woess [3]) we have that for all \( x \in X \) and all \( z \in \mathbb{R}, |z| < 1 \),

\[ G(x, x | z) = \varphi(F(x, x | z)), \]

where \( \varphi(t) := 1/(1 - t) \) is a convex function. By Jensen’s inequality

\[
\varphi \left( \sum_{x \in X} F(x, x | z)\lambda_n(x) \right) \leq \sum_{x \in X} \varphi(F(x, x | z))\lambda_n(x) = \sum_{x \in X} G(x, x | z)\lambda_n(x).
\]

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If we take the limit of both sides of the previous equation, taking into account the continuity of $\varphi$,
\[
\varphi(L_{\lambda}(F(z))) = \lim_{n \to +\infty} \varphi \left( \sum_{x \in X} F(x, x|z)\lambda_n(x) \right) \\
\leq \liminf_{n \to +\infty} \sum_{x \in X} G(x, x|z)\lambda_n(x) =: \inf L_{\lambda}(G(z));
\]
hence
\[
\lim_{z \to 1-} \inf L_{\lambda}(G(z)) \geq \lim_{z \to 1-} \varphi(L_{\lambda}(F(z))) = \lim_{z \to 1-} \varphi(L_{\lambda}(F(z))) = +\infty.
\]
The case $L_{\lambda}(F(\cdot, \cdot)) = 1$ is completely analogous (please note that could happen that $\sum_{x \in X} G(x, x)\lambda_n(x) = +\infty$ for some $n \in \mathbb{N}$).

**Appendix A: limits on the average of general functions**

In this appendix we consider a very general setting: if not otherwise stated $X$ is a countable set, $\lambda$ is a sequence of probability measures on $X$ and $f$ is a real-valued function defined on $X$. We look for sufficient conditions on $f$ such that $L_{\lambda}(f)$ exists and we study what can be said about its value.

First, we observe that if $\lambda$ is regular, then values taken by $f$ on finite subsets (which have $\lambda$-measure zero) do not influence neither the existence nor the value of $L_{\lambda}(f)$. In some sense only values $f(x)$ for $x$ tending to $\infty$ will matter, more precisely we must study the Alexandroff compactification of $X$ (see for instance [10]). Let us call the compactification set $\overline{X}$ and let $\infty$ be the point at infinity.

**Theorem A.1.** Let $\lambda$ be regular, then for any real valued function $f$,
\[
\liminf_{x \to \infty} f(x) \leq \inf L_{\lambda}(f) \leq \sup L_{\lambda}(f) \leq \limsup_{x \to \infty} f(x).
\]
In particular if there exists $\lim_{x \to \infty} f = \alpha$ then $f \in \mathcal{D}(L_{\lambda})$ and $L_{\lambda}(f) = \alpha$. Moreover, if $f$ is bounded and $A \subseteq X$ such that $L_{\lambda}(A) = 1$, then
\[
\liminf_{x \to \infty} f(x) \leq \inf L_{\lambda}(f) \leq \sup L_{\lambda}(f) \leq \limsup_{x \to \infty} f(x).
\]

**Proof.** We easily observe that, in this case,
\[
\liminf_{x \to \infty} f(x) \equiv \sup_{S \subseteq X: |S| < +\infty, x \not\in S} \inf_{x \to \infty} f(x), \quad \limsup_{x \to \infty} f(x) \equiv \inf_{S \subseteq X: |S| < +\infty} \sup_{x \not\in S} f(x).
\]

If $\liminf_{x \to \infty} f(x) = -\infty$ there is nothing to prove; if $\liminf_{x \to \infty} f(x) = \alpha > -\infty$ then for every $\varepsilon > 0$ there exists a finite subset $S$ such that for every $x \not\in S$, $f(x) > \alpha - \varepsilon$. Hence
\[
\sum_{x \in X} f(x)\lambda_n(x) = \sum_{x \in S} f(x)\lambda_n(x) + \sum_{x \not\in S} f(x)\lambda_n(x) \geq
\]

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\[
\geq \sum_{x \in S} f(x)\lambda_n(x) + (1 - \lambda_n(S))(\alpha - \varepsilon)^n \underset{n \to \infty}{\to} \alpha - \varepsilon.
\]

This means that \(\liminf_{x \to \infty} f(x) \leq \inf_L f_L(f)\); by changing \(f\) into \(-f\) we obtain the other inequality. The last assertion is an easy consequence of the following facts:

\[
f \in D(L_\lambda) \iff \inf_L f_L(f) = \sup_L f_L(f)
\]

\[
\exists \lim_{x \to \infty} f(x) \iff \liminf_{x \to \infty} f(x) = \limsup_{x \to \infty} f(x).
\]

The case involving the set \(A\) is analogous.

We note that the previous result means that, for \(\lambda\) regular, if \(f\) converges ("almost surely") to infinity then \(f \in D(L_\lambda)\) and \(L_\lambda(f)\) does not depend on the choice of \(\lambda\) (only the notion of "almost surely" does).

Moreover, only the topological (discrete) structure of \(X\) is involved; if \(X\) has a graph structure then the topology is obviously independent of the choice of the edge set (the topology is always the discrete one). The link between the two structures is more evident in the connected, locally finite case. In this case (and only in this case) every ball is a finite subset of \(X\), hence \(A \subset X\) is compact if and only if it is bounded. This condition means that for every \(x \in X\), the set \(\{B(x, n)\}_n\) is a base for the set of the neighbours of \(x\).

If we restrict to bounded functions, we get a stronger result.

**Proposition A.2.** Let \(\{\lambda_n\}_n\) be a sequence of probability measures on \(X\), and \(f : X \to \mathbb{R}\) be a bounded function \((N \leq f(x) \leq M, \text{ for every } x \in X)\). Then TFAE:

1. \(f \in D(L_\lambda)\) and \(L_\lambda(f) = M\) (resp. \(L_\lambda(f) = N\));
2. \(\inf_{x \in \lambda} L_\lambda(f) = M\) (resp. \(\sup_{x \in \lambda} L_\lambda(f) = N\));
3. \(\forall \varepsilon > 0, L_\lambda(\{x : f(x) > M - \varepsilon\}) = 1\) (respectively \(L_\lambda(\{x : f(x) < N + \varepsilon\}) = 1\));
4. \(\forall \varepsilon > 0, L_\lambda(\{x : f(x) \leq M - \varepsilon\}) = 0\) (respectively \(L_\lambda(\{x : f(x) \geq N + \varepsilon\}) = 0\)).

Moreover if \(\lambda\) is regular then

5. there exists a measurable set \(A\) with measure 1 such that \(\lim_{x \to +\infty} f(x) = M\) (respectively \(\lim_{x \to +\infty} f(x) = N\))

is equivalent to each of the previous ones.

**Proof.** We consider the case involving the superior limit \(M\) (the other one is completely analogous). Let us define \(F_\varepsilon^+ := \{x : f(x) > M - \varepsilon\}\) and \(F_\varepsilon^- := X \setminus F_\varepsilon^+\).

(i) \(\iff\) (ii) and (iii) \(\iff\) (iv) are trivial.

(i) \(\Rightarrow\) (iii). For every \(n \in \mathbb{N}, \varepsilon > 0\),

\[
\sum_{x \in X} f(x)\lambda_n(x) \leq M\lambda_n(F_\varepsilon^+) + (M - \varepsilon)\lambda_n(F_\varepsilon^-)
\]
which is
\[
\lambda_n(F^-_\varepsilon) \leq \frac{1}{\varepsilon} \left( M - \sum_{x \in X} \lambda_n(x) f(x) \right),
\]
then
\[
0 \leq \sup L_\lambda(F^-_\varepsilon) \leq \frac{1}{\varepsilon} (M - \inf L_\lambda(f)) \leq \frac{1}{\varepsilon} (M - L_\lambda(f)) = 0.
\]

(iii) \(\Rightarrow\) (i). Since \(f = f_{\chi_{F^+}} + f_{\chi_{F^-}}\), by Remark 2.2, \(\chi_{F^+} f \in \mathcal{D}(L_\lambda)\) and \(L_\lambda(f_{\chi_{F^+}}) = 0\) for every \(\varepsilon > 0\). Whence for every \(\varepsilon > 0\)
\[
M \geq \sup L_\lambda(f) \geq \inf L_\lambda(f) + \inf L_\lambda(f_{\chi_{F^+}}) \geq (M - \varepsilon) L_\lambda(F^+_{\varepsilon}) = M - \varepsilon
\]
which easily implies (i).

(v) \(\Rightarrow\) (iii). Let \(n \in \mathbb{N}\) and \(A_n := \{x \in X : f(x) > M - 1/n\}\), then \(|A \setminus A_n| < +\infty\)
hence
\[
\inf L_\lambda(A_n) = \inf L_\lambda(1 - \chi_{A \setminus A_n} - \chi_{A \setminus A_n}) = 1 - L_\lambda(A^c \setminus A_n) - L_\lambda(A \setminus A_n) = 1.
\]

(i) \(\Rightarrow\) (v). Let \(\{B_n\}_n\) be any increasing sequence of finite subsets of \(X\) such that
\[
\cup_{n \in \mathbb{N}} B_n = X
\]
(that is \(\{B_n\}_n\) is a basis for the set of neighbours of \(\infty\)). Let, for any \(n \in \mathbb{N}\),
\(A_n\) defined as in the previous point (it is non-empty since \(L_\lambda(f) = M\)). Let us construct
recursively two increasing sequences \(\{m_i\}_i\) and \(\{n_i\}_i\) with values in \(\mathbb{N}\) satisfying
\[
\begin{align*}
\lambda_m(A_i) &> 1 - 1/i, \quad \forall m \geq m_i; \\
\lambda_m(A_i \cap B_{n_i}) &> 1 - 1/i, \quad \forall m : m_i \leq m < m_{i+1}.
\end{align*}
\]
This is possible since \(\lim_{m \to +\infty} \lambda_m(A_i) = 1\) for any \(i \in \mathbb{N}\) and since (using Monotone
Convergence Theorem) \(\lim_{n \to +\infty} \lambda_m(A_i \cap B_n) = \lambda_m(A_i) > 1 - 1/i\) and the set \(\{m : m_i \leq m < m_{i+1}\}\) is finite. We prove now that \(A := \cup_{i=1}^\infty (A_i \cap B_{n_i})\) satisfies the two conditions
in (v).
By regularity we have that \(L_\lambda(A \setminus B_{n_i}) = 1\) hence \(A \setminus B_{n_i} \neq \emptyset\) and \(\infty\) is an accumulation
point for \(A\). Moreover if \(x \in A \setminus B_n\) we have that \(x \in A_j\) for some \(j > i\) and hence
\(f(x) > M - 1/j > M - 1/i\); this proves that \(\lim_{x \to +\infty} f(x) = M\).
If \(m\) satisfies \(m_i \leq m < m_{i+1}\) then
\[
\lambda_m(A) \geq \lambda_m(A_i \cap B_{n_i}) > 1 - 1/i
\]
whence \(\lim_{m \to +\infty} \lambda_m(A) = 1\).

In the hypotheses of the previous theorem, if \(\{\lambda_n\}_n\) is regular and if for some \(\varepsilon > 0\),
\[\{|x : f(x) > M - \varepsilon|\} < +\infty\] (resp. \[\{|x : f(x) < N + \varepsilon|\} < +\infty\]) then \(L_\lambda\{x : f(x) = M\} = 1\) (resp. \(L_\lambda\{x : f(x) = N\} = 1\)).

We show now that \(\mathcal{D}(L_\lambda)\) is closed under uniform convergence and that \(L_\lambda\) is continuous. In order to be able to prove Proposition A.4 we need the generalization of a well
known theorem which we report here for completeness.
Lemma A.3. Let \((W, \tau)\) and \((Y, \sigma)\) two topological spaces and \(g : E \times Y \to Z\) where \(E \subset W\) and \((Z, d)\) is a complete metric space. If \(x_0 \in X, y_0 \in Y\) are accumulation points of \(E\) and \(Y\) respectively, and such that \(g(x, y) \xrightarrow{y \to y_0} h(x)\) uniformly with respect to \(x \in E\) and for all \(y \in Y \setminus \{y_0\}\) there exists \(\lim_{x \to x_0} g(x, y) =: \varphi(y)\) then there exist \(\lim_{y \to y_0} \varphi(y) =: \alpha\) and \(\lim_{x \to x_0} h(x) =: \beta\) and \(\alpha = \beta\).

The proof is similar to that of Theorem 7.11 of [11] and we omit it.

Proposition A.4. Let \(\{\lambda_n\}_n\) be a sequence of probability measures on \(X\) and \((\Gamma, \geq)\) be a directed, partially ordered set. If \(\{f_\gamma\}_\Gamma \subseteq D(L_\Lambda)\) is a net with the property that \(\lim_{\Gamma} f_\gamma =: f\) holds uniformly with respect to \(x \in X\), then \(f \in D(L_\Lambda)\) and

\[
L_\Lambda(f) = \lim_{\Gamma} L_\Lambda(f_\gamma).
\]

Proof. It is trivial to note that Lemma A.3 implies that Theorem 7.11 of [11] holds considering a net of functions instead of a sequence.

Since \(f_\gamma - f \to 0\) in \(l^\infty(X)\) and \(f_\gamma \in D(L_\Lambda)\) for every \(\gamma \in \Gamma\), then it is easy to show that \(\sum_{x \in X} \lambda_n(x) f(x)\) exists for every \(n \in \mathbb{N}\) and

\[
g_\gamma(n) := \sum_{x \in X} \lambda_n(x) f_\gamma(x) \xrightarrow{\Gamma} \sum_{x \in X} \lambda_n(x) f(x) =: \varphi(n)
\]

uniformly with respect to \(n \in \mathbb{N}\) (since \(\sum_{x \in X} \lambda_n(x) = 1\)).

Since \(\lim_{n \to +\infty} g_\gamma(n) =: L_\Lambda(f_\gamma)\), using the previous Lemma we have that \(\lim_{\Gamma} L_\Lambda(f_\gamma)\) and \(\lim_{n \to +\infty} \varphi(n) =: L_\Lambda(f)\) both exist and they are equal.

The previous result leads to an alternative proof Theorem A.1: it is enough to observe that if \(o \in X\) is fixed and \(f\) satisfies \(\lim_{x \to \infty} f(x) = \alpha\) then \(f_n(x) := (f - \alpha) \cdot \chi_{B(o, n)} + \alpha\) is a sequence of functions uniformly convergent to \(f\), such that \(f_n \in D(L_\Lambda)\) and \(L_\Lambda(f_n) = \alpha\).

Furthermore, from a functional point of view, \(D(L_\Lambda)\) is a closed linear subspace of the linear generalized metric (generalized normed) space \(\mathbb{C}^X\) with the supremum distance, which turns out to be complete; it is easy to note, hence, that \(L_\Lambda\) is a linear continuous map from \(D(L_\Lambda)\) to \(\mathbb{C}\).

Remark A.5. The averaging procedure is particularly simple for some classes of functions: namely, if we have a non negative function \(f(x, z)\) such that for every \(x \in X\) \(z \mapsto f(x, z)\) is real and not decreasing (when \(z \in (\varepsilon, r)\)) then

\[
\lim_{z \to r^-} \inf_{z \in (\varepsilon, r)} \inf_{z \in (\varepsilon, r)} L_\Lambda(f(\cdot, z)) = \sup_{z \in (\varepsilon, r)} \inf_{z \in (\varepsilon, r)} L_\Lambda(f(\cdot, z)) \leq \inf_{z \in (\varepsilon, r)} L_\Lambda(f(\cdot, z)),
\]

moreover if \(\lim_{z \to r^-} \inf_{z \in (\varepsilon, r)} L_\Lambda(f(\cdot, z)) = \sup_{x \in X} f(x, r) < +\infty\) then \(f(\cdot, r) \in D(L_\Lambda)\) and \(L_\Lambda(f(\cdot, r)) = \sup_{x \in X} f(x, r) < +\infty\).
Appendix B: limits on the average of power series

In this appendix we study the limit on the average of families of power series defined on an at most countable set $X$ and for general sequences of probability measures on $X$. In particular we search for conditions on $\sum_{n=0}^{\infty} a_n(x)z^n$, to belong to $\mathcal{D}(L_\lambda)$ for every fixed $z$ in the common domain of convergence; moreover, provided that $L_\lambda(\sum_{n=0}^{\infty} a_n(x)z^n)$ exists we ask when

$$\lim_{z \to r-} L_\lambda \left( \sum_{n=0}^{\infty} a_n(x)z^n \right) = L_\lambda \left( \sum_{n=0}^{\infty} a_n(x)r^n \right),$$

where $z \in \mathbb{R}$ and $B(0,r)$ is a common domain of convergence.

**Theorem B.1.** Let $\{\lambda_n\}_n$ be a sequence of probability measures on $X$ and a family of power series $\sum_{n=0}^{\infty} a_n(x)z^n$ such that $a_n(x) \geq 0$ for every $n \in \mathbb{N}, x \in X$. Suppose that the series $\sum_{n=0}^{\infty} k_n z^n$, where $k_n := \sup_{x \in X} a_n(x)$, has a positive radius of convergence $r'$. If $r \in (0, r')$ then the following results hold:

a) if $\{a_n(\cdot)\}_n \subset \mathcal{D}(L_\lambda)$ then
   a.1) $\sum_{n=0}^{\infty} a_n(\cdot)z^n \in \mathcal{D}(L_\lambda)$ and $L_\lambda(\sum_{n=0}^{\infty} a_n(\cdot)z^n) = \sum_{n=0}^{\infty} L_\lambda(a_n)z^n$, for every $z \in \mathbb{C}$ such that $\sum_{n=0}^{\infty} k_n|z|^n < +\infty$;
   a.2) $\lim_{z \to r-} L_\lambda(\sum_{n=0}^{\infty} a_n(\cdot)z^n) = \sum_{n=0}^{\infty} L_\lambda(a_n)r^n \leq \inf L_\lambda(\sum_{n=0}^{\infty} a_n(\cdot)r^n)$;
   a.3) if $\sum_{n=0}^{\infty} k_n r^n < +\infty$ then $\lim_{z \to r-} L_\lambda(\sum_{n=0}^{\infty} a_n(\cdot)z^n) = L_\lambda(\sum_{n=0}^{\infty} a_n(\cdot)r^n)$;

b) if $\sum_{n=0}^{\infty} a_n(\cdot)z^n \in \mathcal{D}(L_\lambda)$ for all $z \in \mathbb{C}, z \in \Gamma$ where $\Gamma$ is a circuit with $\sup_{z \in \Gamma} |z| = r_1 < r$, and $d(\Gamma, 0) = r_2 > 0$ then $\{a_n\}_n \subset \mathcal{D}(L_\lambda)$ and (a.1) holds;

c) if $\sum_{n=0}^{\infty} k_n r^n < +\infty$ then $\lim_{z \to r-} \sum_{n=0}^{\infty} a_n(x)z^n = \sum_{n=0}^{\infty} a_n(x)r^n$ holds uniformly with respect to $x \in X$;

d) if $\sum_{n=0}^{\infty} k_n r^n < +\infty$, $f(x, z) := \sum_{n=0}^{\infty} a_n(x)z^n$ and there exists a complex sequence $\{z_n\}_n$ such that $|z_n| < r, z_n \to r$ and $f(\cdot, z_n) \in \mathcal{D}(L_\lambda)$ for every $n$, then $L_\lambda(f(\cdot, z_n))$ tends to $L_\lambda(f(\cdot, r))$.

**Proof.** a.1) Since $0 \leq a_n(x) \leq k_n$ for every $n \in \mathbb{N}, x \in X$ then the radius of convergence $r_x$ of $\sum_{n=0}^{\infty} a_n(x)z^n$ must satisfy $r_x \geq r'$ for all $x \in X$. Moreover for any $z \in \mathbb{C}$ such that $\sum_{n=0}^{\infty} k_n |z|^n < +\infty$ then

$$\sum_{x \in X} \sum_{i \in \mathbb{N}} |a_i(x)z^i\lambda_n(x)| \leq \sum_{x \in X} \lambda_n(x) \sum_{i \in \mathbb{N}} k_i |z|^i < +\infty,$$

whence, using Tonelli-Fubini’s Theorem, we have

$$\sum_{x \in X} \left( \sum_{i \in \mathbb{N}} a_i(x)z^i \right) \lambda_n(x) = \sum_{i \in \mathbb{N}} \left( \sum_{x \in X} a_i(x)\lambda_n(x) \right) z^i.$$
Applying Bounded Convergence Theorem to the sequence of complex functions $g_n(i) := \sum_{x \in X} \lambda_n(x) a_i(x) z^j$ (bounded by the summable function $g(i) := k_i |z|^j$) we have that

$$\lim_{n \to +\infty} \sum_{i \in \mathbb{N}} \left( \sum_{x \in X} a_i(x) z^j \lambda_n(x) \right) = \sum_{i \in \mathbb{N}} \left( \lim_{n \to +\infty} \sum_{x \in X} a_i(x) z^j \lambda_n(x) \right)$$

which is

$$L_\lambda \left( \sum_{i=0}^{\infty} a_i(z) z^j \right) = \sum_{i=0}^{\infty} L_\lambda(a_i) z^j.$$ 

a.2) Moreover $L_\lambda(a_n) \geq 0$, hence by (a.1) and Monotone Convergence Theorem

$$\lim_{z \to r^-} L_\lambda \left( \sum_{n \in \mathbb{N}} a_n(z) z^j \right) = \lim_{z \to r^-} \sum_{n \in \mathbb{N}} L_\lambda(a_n) z^j = \sum_{n \in \mathbb{N}} L_\lambda(a_n) r^j$$

where the last sum could possibly be infinite. The inequality in (a.2) is a trivial consequence of the monotonicity of both $f \mapsto \inf L_\lambda(f)$ and of the power series $z \mapsto \sum_{n=0}^{\infty} a_n(z) z^j$ (as a function of the real variable $z$).

a.3) If $\sum_{n \in \mathbb{N}} k_n r^n < +\infty$ then by (a.1),

$$\sum_{n \in \mathbb{N}} L_\lambda(a_n) r^n = L_\lambda \left( \sum_{n \in \mathbb{N}} a_n(z) r^n \right)$$

hence, by (a.2),

$$\lim_{z \to r^-} L_\lambda \left( \sum_{n \in \mathbb{N}} a_n(z) z^j \right) = \sum_{n \in \mathbb{N}} L_\lambda(a_n) r^n = L_\lambda \left( \sum_{n \in \mathbb{N}} a_n(z) r^n \right).$$

b) Using Cauchy integral formula we have that

$$a_k(x) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\sum_{j=0}^{\infty} a_j(x) z^j}{z^{k+1}} dz$$

where the curve $\Gamma$ is parametrized by $\gamma(t), t \in [0, T]$, $\gamma(t)$ is in $\mathcal{C}^1$ and $\sup_{t \in [0, T]} |\gamma(t)| = r_1, \inf_{t \in [0, T]} |\gamma(t)| = r_2 > 0$. Since $z \mapsto \sum_{j=0}^{\infty} k_j z^j$ is continuous on $\gamma([0, T])$ (which is a compact subset of $\mathbb{C}$) then it is bounded and Fubini’s Theorem holds

$$\sum_{x \in X} a_k(x) \lambda_n(x) = \frac{1}{2\pi i} \int_{\Gamma} \sum_{x \in X} \left( \frac{\sum_{j=0}^{\infty} a_j(x) z^j \lambda_n(x)}{z^{k+1}} \right) dz.$$ 

Now

$$\int_{\Gamma} \sum_{x \in X} \left( \frac{\sum_{j=0}^{\infty} a_j(x) z^j \lambda_n(x)}{z^{k+1}} \right) dz = \int_{0}^{T} \sum_{x \in X} \left( \frac{\sum_{j=0}^{\infty} a_j(x) \lambda_n(x) \gamma(t)^j}{\gamma(t)^k} \right) \gamma'(t) dt,$$

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and (since $\sup_{t \in [0, T]} |\gamma'(t)| = C$)
\[
\left| \frac{1}{2\pi i} \sum_{x \in X} \frac{\sum_{j=0}^{\infty} a_j(x) \lambda_n(x) \gamma(t)^j}{\gamma(t)^k} \right| \gamma'(t) \leq \frac{1}{2\pi r_{1,k+1}} C \sum_{x \in X} \left( \sum_{j=0}^{\infty} a_j(x) r_{1,j}^j \right) \lambda_n(x) \leq \frac{1}{2\pi r_{1,k+1}} C \sum_{x \in X} \left( \sum_{j=0}^{\infty} k_j r_{1,j}^j \right) \lambda_n(x) \leq M.
\]

Using Bounded Convergence Theorem (note that the integrand is bounded by $\frac{C'}{r_{2,k+1}} \sum_{j=0}^{\infty} k_j r_{1,j}^j$ uniformly with respect to $n \in \mathbb{N}$, $x \in X$)

\[
L_\Lambda(a_k) := \lim_{n \to +\infty} \sum_{x \in X} a_k(x) \lambda_n(x) = \\
= \frac{1}{2\pi i} \int_0^T \lim_{n \to +\infty} \sum_{x \in X} \frac{\left( \sum_{j=0}^{\infty} a_j(x) \lambda_n(x) \gamma(t)^j \right)}{\gamma(t)^k} \gamma'(t) dt = \\
= \frac{1}{2\pi i} \int_{\Gamma} L_\Lambda \left( \sum_{j=0}^{\infty} a_j(\cdot) z^j \right) \frac{dz}{z^{k+1}}
\]

\(c)\) is straightforward.

\(d)\) The proof is quite simple and we just outline it (taking for simplicity of notation, \(z\) as a real variable instead of a discrete one): \(f(x, r) - f(x, z) = \sum_{n=0}^{\infty} h_n(x, z)\) where \(h_n(x, z) := a_n(x)(r^n - z^n)\); by the estimate \(|h_n(x, z)| \leq k_n |r^n - z^n| \leq 2k_n r^n\) it is possible to apply the Bounded Convergence theorem to the sum \(\sum_{n=0}^{\infty} k_n |r^n - z^n|\) obtaining that \(\lim_{|z| < r} \sum_{n=0}^{\infty} |h_n(x, z)| = 0\) holds uniformly with respect to \(x \in X\). Proposition A.4 implies the conclusion.

The meaning of the previous theorem is that for a family of power series satisfying the hypotheses of the theorem, the following assertions are equivalent:

(i) every coefficient is in the domain of \(L_\Lambda\);

(ii) there exists a circuit \(\Gamma\) as in Theorem B.1(b), such that for all \(z \in \Gamma\), \(\sum_{n=0}^{+\infty} a_n(x) z^n\)

is in the domain of \(L_\Lambda\);

(iii) for every \(z \in \mathbb{C}\), \(|z| < r', \sum_{n=0}^{+\infty} a_n(x) z^n\) is in the domain of \(L_\Lambda\).

By means of the previous theorem we can state and prove a result which we call identity principle on the average for power series.

**Proposition B.2.** If \(w_j(x, z) := \sum_{j=0}^{\infty} a_j^{(j)}(x) z^j\), \(j = 1, 2\), is a couple of families of power series on a graph \((X, E(X))\), such that \(a_i^{(j)}(x) \geq 0\) for every \(i \in \mathbb{N}, x \in X, j = 1, 2\).

Suppose that the series \(\sum_{i=0}^{\infty} k_i^{(j)} z^i\), where \(k_i^{(j)} := \sup_{x \in X} a_i^{(j)}(x)\), has a positive radius of convergence \(r_j\), \(j = 1, 2\). Let \(\{\lambda_n(x)\}_n\) be a sequence of probability measures such that \(w_j(\cdot, z) \in \mathcal{D}(L_\Lambda)\) \(j = 1, 2\), for all \(z \in B(0, \min(r_1, r_2))\), then TFAE
(i) there exists a subset $E \subseteq B(0, \min(r_1, r_2))$ with an accumulation point $x_0$ which belongs
to the domain $B(0, \min(r_1, r_2))$ such that

$$L_\lambda(w_1(\cdot, z)) = L_\lambda(w_2(\cdot, z)), \quad \forall z \in E;$$

(ii) for every $n \in \mathbb{N}$ and for every $x \in X$ we have that $L_\lambda(a_n^{(1)}) = L_\lambda(a_n^{(2)}).

Proof. By Theorem B.1(a.1) we have that

$$L_\lambda(w_j(\cdot, z)) = \sum_{i=0}^{\infty} L_\lambda(a_i^{(j)}) z^i, \quad \forall z \in B(o, r_j), \ j = 1, 2, \quad (14)$$

then (i) $\implies$ (ii) is trivial.

(ii) $\implies$ (i) It is a consequence of (14) and Theorems 8.1.2 and 8.1.3 of [12].

We can state a similar result which takes also into account the presence of zero-measure sets.

**Proposition B.3.** Let $\lambda$ and $w_i(x, z) := \sum_{j=0}^{\infty} a_j^{(i)}(x) z^j, \ i = 1, 2$ satisfy the hypotheses of the previous theorem. Suppose that for every $n \in \mathbb{N}$ there exists a subset $X_n \subset X$ such that $L_\lambda(X_n) = 0$ and $a_n^{(1)}(x) = a_n^{(2)}(x)$ for every $x \notin X_n$. If $z \in \mathbb{C}, \ |z| \in (0, \min(r_1, r_2))$ then

$$w_1(\cdot, z) \in \mathcal{D}(L_\lambda) \iff w_2(\cdot, z) \in \mathcal{D}(L_\lambda) \quad (15)$$

and

$$L_\lambda(w_1(\cdot, z)) = L_\lambda(w_2(\cdot, z)). \quad (16)$$

Proof. Let us first consider a general family of power series, namely $w(x, z) := \sum_{i=0}^{\infty} a_i(x) z^i$ such that the series $\sum_{i=0}^{\infty} k_i z^i$ has a positive radius of convergence $r (k_i := \sup_{x \in X} |a_i(x)|)$. Then for every $n \in \mathbb{N}$ and $z \in \mathbb{C}$ such that $|z| < r$, \n
$$|a_i(x) z^i \lambda_n(x)| \leq k_i |z|^i \lambda_n(x);$$

then by Fubini’s Theorem (applied to the function $(x, i) \mapsto a_i(x) z^i \lambda_n(x)$)

$$\sum_{x \in X} \sum_{i=0}^{\infty} a_i(x) z^i \lambda_n(x) = \sum_{i=0}^{\infty} \sum_{x \in X} a_i(x) z^i \lambda_n(x) + \sum_{i=0}^{\infty} \sum_{x \in X} a_i(x) z^i \lambda_n(x).$$

Now

$$\left| \sum_{x \in X} a_i(x) z^i \lambda_n(x) \right| \leq \sum_{x \in X} \sup_{x \in X} \left| a_i(x) \right| |z|^i \lambda_n(x) \leq k_i |z|^i \lambda_n(X_i) \leq k_i |z|^i,$$

implies that $\sum_{x \in X} a_i(x) z^i \lambda_n(x)$ converges (uniformly with respect to $|z| \leq r - \varepsilon$) to 0 if $n$ diverges to infinity (since $\lim_{n \to +\infty} \lambda_n(X_i) = 0$) and it is bounded (uniformly) by the
summable function $k_i z^i$, then using Bounded Convergence Theorem (applied to the series) we obtain

$$\lim_{n \to +\infty} \sum_{i=0}^{\infty} \sum_{x \in X_i} a_i(x) z^i \lambda_n(x) = 0$$

(uniformly with respect to $|z| \leq r - \varepsilon$).

Hence if $|z| < r$ it is obvious that if $w = w_j$ (where $j \in \{1, 2\}$),

$$w_j(\cdot, z) \in \mathcal{D}(L_\lambda) \iff \exists \lim_{n \to +\infty} \sum_{i=0}^{\infty} \sum_{x \in X_i} a_i^{(j)}(x) z^i \lambda_n(x)$$

$$L_\lambda(w_j(\cdot, z)) = \lim_{n \to +\infty} \sum_{i=0}^{\infty} \sum_{x \in X_i} a_i^{(j)}(x) z^i \lambda_n(x),$$

but by our hypotheses $a_i^{(1)}(x) = a_i^{(2)}(x)$ for every $i \in \mathbb{N}$ and for every $x \in X_i^\ast$, whence the existence and the value of the last limit does not depend on $j$. \qed

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**Bibliography**


