Genericity of Rigidity and Multiple Recurrence
for Infinite Measure Preserving
and Nonsingular Transformations

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Abstract

We show that the generic transformation is rigid and hence multiply recurrent for both the group of infinite measure preserving transformations \( M \) and the group of nonsingular transformations \( G \).

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1 Introduction

Let \((X, \mathcal{B}, \mu)\) be a \(\sigma\)-finite nonatomic Lebesgue measure space. A **nonsingular automorphism** or **transformation** is an invertible map \(T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)\) such that is \(T, T^{-1}\) are measurable and \(\mu(A) = 0\) if and only if \(\mu(T^{-1}(A)) = 0\). \(T\) is **ergodic** if for all \(A \in \mathcal{B}\) with \(T^{-1}(A) = A\), we have \(\mu(A)\mu(A^c) = 0\). \(T\) is **conservative** if for every \(A\) with \(\mu(A) > 0\) there is an integer \(n > 0\) such that \(\mu(A \cap T^{-n}(A)) > 0\). \(T\) is said to have **infinite ergodic index** if for all \(k > 0\), the \(k\)-fold Cartesian products \(T \times \ldots \times T\) are ergodic. A nonsingular automorphism \(T\) is **measure preserving** if \(\mu(T^{-1}(A)) = \mu(A)\); it is finite measure preserving if \(X\) is of finite measure and infinite measure preserving if \(X\) is of infinite measure.

If \(T\) on \((X, \mathcal{B}, \mu)\) is a nonsingular automorphism we will denote by \(\omega_i\) the Jacobian or Radon-Nikodym derivative of \(T^{-i}\):

\[
\mu(T^{-i} A) = \int_A \omega_i d\mu \quad \forall A \in \mathcal{B}.
\]

When \(\mu\) or \(T^{-i}\) needs to be emphasized we write \(\omega_{\mu, T^{-i}}\). One has the cocycle relation

\[
\omega_i(x)\omega_j(T^{-i} x) = \omega_{i+j}(x) \quad a.e.
\]

Let \(d > 0\) be an integer. A nonsingular automorphism \(T\) on \((X, \mathcal{B}, \mu)\) is said to be **\(d\)-recurrent** if for all sets of positive measure \(A\), there exists an integer \(n > 0\) with \(\mu(A \cap T^n A \cap \ldots \cap T^{nd} A) > 0\). As is well known, Furstenberg [Fu] has shown that every finite measure preserving transformation is multiply recurrent. It is clear that multiple recurrence implies conservativity, and that there exist infinite measure preserving ergodic automorphisms (on atomic spaces) that are not conservative. However, Eigen, Hajian and Halverson in [EHH] showed that there exist ergodic infinite measure preserving transformations that are \(d\) recurrent but not \(d+1\) recurrent, for every \(d > 1\); these transformations do not have infinite ergodic index. More recently, Aaronson and Nakada [AN] have shown that if \(S\) is an infinite measure preserving Markov shift, then \(S\) is \(d\)-recurrent if and only if the Cartesian product of \(d\) copies of \(S\) is conservative; and in particular an infinite ergodic index Markov shift is multiply recurrent. They also construct an infinite odometer with some additional properties that is not \(2\)-recurrent. However, Adams, Friedman and Silva [AFS] have shown that there exists a rank one infinite measure preserving transformation \(T\) that has infinite ergodic index but that is not \(2\)-recurrent; for this transformation \(T \times T^2\) is not conservative.
Our aim in this article is to show that multiple recurrence is generic for the weak and uniform topologies in both the group of nonsingular transformations and the group of infinite measure preserving transformations. In fact we show something stronger. We define a nonsingular transformation $T$ to be **power multiply recurrent** if for all finite sequences of integers $\{k_1, \ldots, k_r\}$, $T^{k_1} \times \ldots \times T^{k_r}$ is multiply recurrent. We show that power multiple recurrence is generic.

## 2 The weak and uniform topologies

Let $G(X, \mu)$ denote the group of all nonsingular automorphisms on $(X, \mu)$ and $M(X, \nu)$ the group of all measure preserving automorphisms on $(X, \nu)$ where $\nu(X) = \infty$; in this paper we will not be concerned with the group of finite measure preserving automorphisms. We consider two well-known topologies on these groups.

Given a nonsingular automorphism $T$ define the operator $U_T$ on $L^1(X, \mu)$ by

$$U_T(f) = f \circ T^{-1} \omega_i(x).$$

(We should note here that sometimes one lets $\omega_i(x)$ denote the Radon-Nikodym derivative of $T^i$ and not of $T^{-i}$ as we are doing here and defines $U_T$ using $T$ rather than $T^{-1}$; the reason for our choice here is so that, for example, $U_T$ preserves composition.)

Clearly, $U_T$ is a positive isometry on $L^1(X, \mu)$. The **coarse topology** or **weak topology** on $G(X, \mu)$ is defined by net convergence so that $T_n \to T$ if and only if $\|U_{T_n}(f) - U_T(f)\|_1 \to 0$ for all $f \in L^1(X, \mu)$. We will use the characterization of this topology in the lemma below. For a proof and further properties of this topology we refer to A. Ionescu Tulcea [I], Choksi-Kakutani [CK] and Hamachi-Osikawa [HO]. We note that while in some works this topology is referred to as the coarse topology, we will call it the weak topology as this agrees with what is commonly called the weak topology on the group of finite measure preserving automorphisms.

**Lemma 2.1.** $G(X, \mu)$ is a complete metrizable group under the weak topology. The topology is the same for any other finite or $\sigma$-finite measure $\nu$ equivalent to $\mu$. Furthermore, if $\{T_n\}_{n=1}^{\infty}$, $T$ are nonsingular automorphisms, $T_n \to T$ weakly as $n \to \infty$ if and only if

$$\mu(T_n(A) \triangle T(A)) \to 0 \quad \text{as} \quad n \to \infty$$
and

$$\omega_{T_n} \to \omega_T \text{ as } n \to \infty \text{ in } L^1.$$

The uniform topology on $G(X, \mu)$ is defined by the metric

$$d(T, S) = \mu\{x : T(x) \neq S(x)\}.$$

This topology is complete and finer than the weak topology. It can also be shown that it depends only on the measure class of the measure $\mu$. For properties of this topology we refer the reader to Friedman [F] and Choksi-Kakutani [CK]; this last paper may also be consulted for a brief survey on the weak and uniform topologies.

The phrase the generic transformation $T$ satisfies property $P$ will mean that the set of transformations in $G(X, \mu)$ that satisfy property $P$ contains a dense $G_{\delta}$, in the appropriate topology.

3 Multiple Recurrence in the Weak Topology

Theorem. The generic T is $d$-recurrent for any $d$ in both $M(\mathbb{R}, \nu)$ and $G(X, \mu)$.

Proof. In the case of $M(\mathbb{R}, \nu)$ the proof is a more or less standard reduction to the case of finite measure and present it for completeness. We say that $T$ is rigid iff $T^{k_i} \to E$ for some sequence $k_i \to \infty$, where we use the weak operator convergence, and $E$ is the identity. It is clear that if $T$ is rigid, then $T$ is multiply recurrent.

By cyclic permutation of rank $k$ we mean any transformation $T$ which is equal to $E$ on $\mathbb{R} \setminus [-2^k, 2^k)$ and a couple of sets $I_k, T I_k, ..., T^{2^{k+1} - 1} I_k$ is exactly a partition of $[-2^k, 2^k)$ on half-open intervals of equal length, where $I_k = [0, 1/2^k)$ and $T^{2^{k+1}} = E$. Let $O_k$ be a set of all such permutations of rank $k$.

We will prove a little more, i.e., that the generic transformation is rigid. Denote $C_n = \bigcup_{k > n} \bigcup_{S \in O_k} B_{\delta_k}(S)$;

$$C(\delta_1, \delta_2, \ldots) = \bigcap_{n=1} \bigcap_{C_n},$$

where $B_{\delta}(S)$ is the open ball of radius $\delta$, centered at $S$. For any $n$, $C_n$ is dense, because all cyclic permutations are dense (see, for example, [S71]). This gives that $C(\delta_1, \delta_2, \ldots)$ is a dense $G_{\delta}$ set for any sequence of positive $\delta_k$. 

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Fix $\varepsilon_k \searrow 0$. We choose $\delta_k$ such that

$$T \in B_{\delta_k}(S) \Rightarrow T^{2k+1} \in B_{\varepsilon_k}(S^{2k+1}) = B_{\varepsilon_k}(E).$$

Obviously, if $T \in C(\delta_1, \delta_2, \ldots)$, then $T$ is rigid.

The case $G(X, \mu)$ needs for more careful consideration. Fix $T \in G(X, \mu)$.

Let $T^k = E$, i.e. $U_T^{k} = U_T^k \rightarrow E$. Using the isometry of $U_T$, we have

$$\forall m U_T^{mk} = U_T^{mk} \rightarrow E.$$ 

Finally, by Lemma 2.1,

$$U_{T_n} \rightarrow U_{T_0} \Leftrightarrow \omega_n \rightarrow \omega_0 \text{ in } L_1 &$$

$$\mu(T_n A \triangle T_0 A) \rightarrow 0 \text{ for any measurable set } A,$$

and thus

$$\forall A \forall m \mu(T^{mk} A \triangle A) \rightarrow 0;$$

Therefore $T$ is multiply recurrent.

In order to prove the genericity of rigid transformations, we need for a new set $O_n \subset G$. Let $T \in O_n$ iff the sequence $\{T^k(x)\}_{0 \leq k < \infty}$ is periodic with strict period $n$ for a.e. $x$. The set $\bigcup_{n \geq k} O_n$ is dense for any integer $k$ (see [I]). It follows that $C_n$ is dense and open for any $n$ and any sequence of positive $\delta_k$, where

$$C_n = \bigcup_{k>n} \bigcup_{S \in O_k} B_{\delta_k}(S).$$

It is clear that

$$U_{T_n} \rightarrow U_S \Rightarrow \exists k U_T^{k} \rightarrow U_S^{k}.$$ 

If $S \in O_k$, then $U_S^k = E$. It remains for $C = \bigcap_{n=1} C_n$ to repeat the end of the proof for $M(\mathbb{R}, \nu)$, where $\delta_k$ are sufficiently small.

**Theorem.** The generic $T$ is power multiply recurrent in both $M(\mathbb{R}, \nu)$ and $G(X, \mu)$.

**Proof.** It is enough to prove that if $T$ is rigid, then $T$ is power multiply recurrent. But the property $T^{kn} \rightarrow E$ for $T$ gives the same for $T^{k_1} \times T^{k_2} \times \ldots \times T^{k_k}$ in both $M(\mathbb{R}, \nu)$ and $G(X, \mu)$. (We use here that for any isometry $U$ on $L_1$, $U^{kn} \rightarrow E \Rightarrow U^{kl_n} \times U^{ml_n} \rightarrow E$ for any $k, m \in \mathbb{Z}$). Therefore, consequently, $T^{k_1} \times T^{k_2} \times \ldots \times T^{kn}$ is rigid too, and is multiply recurrent.
It is not difficult to show that the set of rigid transformations, say \( C_r \), is exactly a dense \( G_\delta \)-set in both \( M(\mathbb{R}, \nu) \) and \( G(X, \mu) \). Indeed, it is

\[
\bigcap_{n=1}^{\infty} \bigcap_{l=1}^{n} \bigcup_{k \geq l} \{ T : T^k \in B_{1/n}(E) \}.
\]

The density of \( C_r \) follows from \( C \subseteq C_r \) (see proof of Theorem 1), or directly if we add the obvious note that if \( T \in O_n \) for some \( n \), then \( T \) is rigid, i.e. \( T \in C_r \).

4 Multiple Recurrence in the Uniform Topology

**Theorem.** The generic \( T \) is power multiply recurrent in both \( M(\mathbb{R}, \nu) \) and \( G(X, \mu) \) with respect to the uniform topology.

**Proof.** We will use the same way. Let \( T \) be a rigid transformation. Rewrite this condition in the following form:

\[
\forall A(\mu(A) < \infty) \mu \{ x \in A : T^n x \neq x \} \to 0 \text{ as } n_i \to \infty.
\]

This gives an analogous statement for \( T^{k_1} \times T^{k_2} \times \cdots \times T^{k_m} \) too. Therefore \( T \) is power multiply recurrent.

Consider

\[
C_r = \bigcap_{n=1}^{\infty} \bigcap_{l=1}^{n} \bigcup_{k \geq l} \{ T : T^k \in B_{1/n}(E) \}.
\]

We use here that the uniform topology in \( M(\mathbb{R}, \nu) \), as it is independent of the measure class, it may be defined by the metric

\[
d(T, S) = \sum_{n=1}^{\infty} \frac{\nu \{ x \in E_n : T x \neq S x \}}{2^n n},
\]

where \( E_n = [-n, n] \). It remains to prove that \( C_r \) is dense, because it is clear that \( \{ T : T^k \in B_{1/n}(E) \} \) is open. Note that the set of all periodic transformations is dense in \( G(X, \mu) \) (see [4,5]). This is also well known for \( M(\mathbb{R}, \nu) \), but, in principle, it is possible to have from the proof of Prop.2 and 3 in [CK]. Finally, every periodic \( T \) is rigid with respect to the uniform topology, or, equivalently, belongs to \( C_r \). □
5 The Case of $\mathbb{Z}^n$ Actions

All theorems mentioned above admit a natural extension to the case of more general group actions, but the analog of $T^k$ for actions is not so natural. Therefore we will work only with a weaker version of the power multiply recurrence. More precisely, we say that a $G$-action $T$ has infinite multiply recurrence index iff a $G$-actions which consist of elements $T_g \times \ldots \times T_g$ ($k$ times) ($g \in G$) are multiply recurrent for any positive $k$.

We announce here the following theorems which will be published separately in a fuller form.

**Theorem.** The generic $\mathbb{Z}^n$-action $T$ is $d$-recurrent for any $d$ and $n$ in both $M_{\mathbb{Z}^n}(\mathbb{R}, \nu)$ and $G_{\mathbb{Z}^n}(X, \mu)$ with respect to the weak and uniform topologies.

**Theorem.** The generic $\mathbb{Z}^n$-action $T$ has infinite multiply recurrent index for any $n$ in both $M_{\mathbb{Z}^n}(\mathbb{R}, \nu)$ and $G_{\mathbb{Z}^n}(X, \mu)$ with respect to the weak and uniform topologies.

References


