On Asymmetry of the Future and the Past for Limit Self-Joinings

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ON ASYMMETRY OF THE FUTURE AND THE PAST FOR LIMIT SELF-JOININGS

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Abstract. Let $\Delta_T$ be an off-diagonal joining of a transformation $T$. We construct a non-typical transformation having asymmetry between limit sets of $\Delta_T^\pm$ for positive and negative powers of $T$. It follows from a correspondence between subpolymorphisms and positive operators, and from structure of limit polynomial operators. We apply this technique to find all polynomial operators of degree 1 from a weak closure (in the space of positive operators on $L_2$) of powers for Chacon’s automorphism and for its generalizations.

Introduction

In [1], D. Rudolph introduced the notion of joinings. This notion turned out to be very fruitful (see [2]-[6]). It is known that every automorphism or endomorphism can be characterized both as a measure on a graph of this map, i.e. a joining or, more general, a polymorphism [7], and as an operator on $L_2$. This gives a way to study structure of joinings by operator methods, and automorphisms by joinings.

It is not difficult to show that limit sets for positive and negative powers of a transformation $T$ are equal in the space of all automorphisms if $T$ is rigid. If $T$ is not rigid, then these sets are empty. Moreover, for rigid or mixing transformations (see Section 1) these limit sets also are equal in the space of all linear operators on $L_2$.

Nevertheless we prove (Theorem 2.4) that there exist rank-one transformations $T$ with different limit sets of off-diagonal joinings for positive and negative powers of $T$ or, in terms of operators, the sets of limit operators are different.

We apply a new approach, via limit polynomials. This approach gave recently a possibility to solve [8],[9] old Rokhlin’s problem (see [10]-[13]) and to answer some well-known questions, for example, the paper [8] contains an answer to Katok’s question.

In Section 1 we introduce subpolymorphisms and a natural homeomorphism between the space of such measures and a subspace of positive operators on $L_2$. Note that, for any transformation $T$, the space of its self-joinings lies in the space of subpolynomials.

Theorem 2.1 describes all possible linear polynomials in $T^{(k)}$ for powers of transformations $T^{(k)}$, where $T^{(k)}$ are constructed in Section 1. In particular, this
implies that only one linear polynomial \(1/2E+1/2T\) is a limit of powers for classical Chacon’s transformation \(T\) (see [3],[14],[15]).

In Section 3 we show that the future and the past for limit self-joinings cannot be completely different for any automorphism. Moreover, their intersection contains an abelian semigroup that is trivial (i.e. \(\{\mu \times \mu\}\)) only for mixing transformations.

1. Basic definitions and notations

Let \(T\) be a transformation defined on a non-atomic standard Borel probability space \((X,\mathcal{F},\mu)\). A transformation and a unitary operator on \(L_2(\mu) : Tf(x) = f(Tx)\) are often called automorphisms and denoted by the same symbol \(T\). It is clear that \(T\) is contained in the space of positive operators on \(L_2(\mu) : L^+ = \{U : Uf \in L_2 \text{ if } f \in L_2\} \triangleleft Ug \geq 0 \text{ if } g \geq 0\} \) equipped with the weak operator convergence. Everywhere below the identity automorphism will be denoted by \(E\). The group of all automorphisms \(\mathcal{A}(\mu)\) of \((X,\mathcal{F},\mu)\) becomes a completely metrizable topological group when endowed with the weak convergence of transformations \((T^n \rightarrow T\) iff for any measurable \(\mu(T_n(A) \Delta T(A)) \rightarrow 0\) as \(n \rightarrow \infty\). Note that this topology is a restriction of the weak operator topology in \(L^+\) to the non-closed \(\mathcal{A}(\mu)\). Denote by \(C(T)\) the commutant of \(T\), i.e. the set \(\{S \in \mathcal{A}(\mu) : ST = TS\}\).

Let \(\nu\) be a finite Borel measure on \(X \times X\) with marginal measures, say \(\pi_1\nu\) and \(\pi_2\nu\), such that

\[
(1.1) \quad \pi_i\nu(A) \leq \mu(A) \text{ for any } \mu\text{-measurable set } A.
\]

Obviously, marginal measures are \(\mu\)-absolutely continuous and \(\nu = \nu(X \times X) \leq 1\).

The set of all such measures, say \(M(\mu)\), is a convex compact metrizable space with respect to the topology determined by

\[
\mu_n \rightarrow \mu_0 \Leftrightarrow \mu_n(A \times B) \rightarrow \mu_0(A \times B)
\]

for any \(\mu\text{-measurable sets } A\) and \(B\). Now we shall give the following definition.

**Definition 1.1.** Each measure from \(M(\mu)\) is called a subpolymorphism.

Fix \(T \in \mathcal{A}(\mu)\). The set \(J_{c}(T,T)\) of \(c\)-self-joinings, i.e. \(T \times T\)-invariant elements \(\nu\) of \(M(\mu)\), with \(||\nu|| = c\), is a closed subspace of \(M(\mu)\) for any \(c \in [0,1]\). If \(T\) is ergodic, then each \(\nu\) from \(J_{c}(T,T)\) has \(c\mu\) as marginal measures. This gives that \(J(T,T) = J_{1}(T,T)\) is exactly the set of well-known self-joinings. Let \(S \in C(T)\). As usual, by an off-diagonal joining \(\Delta_S\) we mean a measure from \(J(T,T)\) completely defined by \(\Delta_S(A \times B) = \mu(S^{-1}A \cap B)\), where \(A, B\) are any \(\mu\text{-measurable sets}.\)

Denote by \(L_{j}^{+}(T)\) and \(L_{j}^{-}(T)\) limit sets of \(\Delta_{T^n}\) in \(M(\mu)\) for all positive \(n\) and negative \(n\) respectively. It is clear that \(L_{j}^{-}(T) \subseteq J(T,T)\). If \(T\) is mixing, then \(L_{j}^{-}(T)\) consist of only one point \(\mu \times \mu\). It is well known that \(T\) is weakly mixing iff \(L_{j}^{-}(T)\) contain at least \(\mu \times \mu\). We say \(T\) is rigid if \(T^n \rightarrow E\) for some sequence \(n_k \rightarrow \infty\).

**Proposition 1.2.** \(L_{j}^{-}(T) = L_{j}^{+}(T)\) for rigid \(T\).

**Proposition 1.3.** \(L_{j}^{+}(T^n) = L_{j}^{-}(T) = \sigma L_{j}^{+}(T),\)

where \(\sigma\) is a flip map (i.e. \(\sigma(x,y) = (y,x)\)), and \(T^n\) denotes the operator adjoint to \(S\).

**Corollary 1.4.** \(L_{j}^{-}(T) = L_{j}^{+}(T)\) iff \(L_{j}^{+}(T)\) is invariant with respect to \(\sigma\).

We leave simple proofs of statements above to the reader. It seems to be well known that the set of rigid transformations is a dense \(G_{\delta}\)-set in \(\mathcal{A}(\mu)\) (see [16]
about close results in different subclasses of the set of non-singular transformations). It turns out that $LJ_\pm(T) = LJ_\pm(T)$ for a typical transformation $T$.

1.1. **Construction of $T_{(k)}$.** We consider the following “generalized” Chacon’s automorphisms. For each $k \geq 3$, let $T_{(k)}$ be a rank-one transformation, where each column $C_{n+1}$ is obtained by cutting $C_n$ into $k$ subcolumns, say $C_n(i)$, of equal width, placing a spacer only on the subcolumn $C_n(k - 1)$, and then stacking the subcolumn $C_n(i + 1)$ on top of $C_n(i)$ for $1 \leq i < k$. It is clear that $T_{(k)}$ is exactly Chacon’s automorphism. For the column $C_n$, let $h_n$ be its height and let $d_n$ be a measure of its one level, where $n \geq 1$.

1.2. **Correspondence between positive operators and $M(\mu)$.** Consider

$$\mathcal{L}^+_\mu = \{ U \in \mathcal{L}^+ : \int_A U(1) d\mu \leq \mu(A) \}$$

& $\int_A U^*(1) d\mu \leq \mu(A)$ for any $\mu$-measurable set $A$

with a restriction of the weak operator topology to this set. Obviously,

$$\mathcal{L}^+_\mu = \{ U \in \mathcal{L}^+ : U(1) \leq 1 \}$$

& $U^*(1) \leq 1$ for a.e. $x$ with respect to $\mu$.

**Proposition 1.5.** 

**Natural correspondence given by**

$$(1.2) \quad \langle U_\nu f, g \rangle = \int f \otimes \tilde{g} dv,$$

for any $f, g \in L_2(\mu)$, defines a linear homeomorphism, say $\phi$, between topological spaces $M(\mu)$ and $\mathcal{L}^+_\mu$.

Note that in (1.2) $f$ and $U_\nu f$ are from different spaces $L_2(\mu)$, but we naturally identify these spaces.

**Proof.** Indeed, the right part of (1.2) is a linear bounded functional for every $f \in L_2(\mu)$ because

$$(1.3) \quad \int f \otimes \tilde{g} dv \leq ||f||_\nu ||g||_\nu = ||f||_{\pi_1 \nu} ||g||_{\pi_2 \nu} \leq ||f||_\mu ||g||_\mu,$$

where $||h||_\nu$ means a norm of $h \in L_2(\nu)$. Thus $U_\nu f \in L_2(\mu)$ for each $f \in L_2(\mu)$. Remain properties of the map $\phi$ are obvious.

Some properties of operators $U_\nu$ were considered in [17] due to Vershik for polymorphisms $\nu$, i.e., for elements of $M(\mu)$ with exact equality in (1.1) for each $A$.

**Remark 1.6.** Clearly, $\phi(\Delta_T) = T$, where $T \in \text{Aut}(\mu)$. Also, $U_\nu$ commutes with $T$ if and only if $\nu \in J_{[\nu]}(T, T)$.

Note that

$$\int f \otimes \tilde{g} dv = \int_x \tilde{g}(y) d\pi_2 \nu \int_x f(x) dv_\mu(x) = \int_x \tilde{g}(y) \rho_2(y) d\mu(y) \int_x f(x) dv(y)(x),$$

where $\rho_2(y)$ is a density of $\pi_2 \nu$ with respect to $\mu$, and $\nu(y)(x)$ is a canonical system of conditional measures corresponding to $\nu$. Hence, changing $g$ to $U_\nu f$ in (1.2) and (1.3), we get
Corollary 1.7. \( \mathcal{L}^+_\mu \) is a compact convex metrizable space. For any subpolynomial \( \nu \in M(\mu) \), \( \|U_\nu\| \leq 1 \), and

\[
U_\nu(f) = \rho_2(y) \int_X f(x) d\nu(y)(x).
\]

Remark 1.8. The space \( \mathcal{L}^+_\mu \) is a semigroup, and closed with respect to taking parts of operators, i.e. if \( 0 \leq V \leq U \) and \( U \in \mathcal{L}^+_\mu \), then \( V \in \mathcal{L}^+_\mu \).

Remark 1.9. It is readily seen that \( \phi(LJ_+ (T)) \) and \( \phi(LJ_- (T)) \) are \( T \)-invariant closed semigroups.

2. LIMIT POLYNOMIALS

The following theorem completely determines the simplest limit polynomial.

**Theorem 2.1.** Let \( \mathcal{P}_1[x] \) be the set of polynomials of degree at most 1. Then

\[
\phi(LJ_+ (T_{(i)}) \cap \mathcal{P}_1[T_{(i)}]) = \left\{ \frac{1}{k-1} E + \frac{k-2}{k-1} T_{(i)} \right\}.
\]

**Proof.** In order to prove Theorem 2.1 we need for some definitions and a technical lemma. For some clarity we restrict our attention on the case \( k = 4 \). The proof for \( k \neq 4 \) is analogous.

Let \( m_i \to +\infty \) as \( i \to +\infty \). Fix \( i \) and choose \( n \) such that \( h_n \leq m_i < h_{n+1} \). Consider the \( (n+1)^{\text{st}} \) column. Number its levels by \( 1, 2, ..., 4h_n + 1 \) from the base consequence. There exists \( i_0 \) such that \( (3h_n - m_i)^{\text{th}} \) level belongs to \( C_n(i_0) \). Let \( p_i \) be a number of higher levels in \( C_n(i_0) \). Denote by \( B_i(j) \) the set of the top \( p_i \) levels of \( C_n(j) \) and \( A_i(j) = C_n(j) \setminus B_i(j) \) (\( A_i(j) \) or \( B_i(j) \) can be empty).

Define \( O_i = \{ x \in C_{n+1} : T^r x \in C_{n+1} \text{ for } r = 1, 2, ..., n+1 \} \). It is clear that on each level a measure of \( x \) from \( O_i \) is \( 2/3h_{n+1} \).

Define operators

\[
Q_i = \chi_{A_i(i_0)} T^{m_i}_{(4)}, \quad R_i = \chi_{B_i(i_0)} T^{m_i}_{(4)},
\]

\[
Q_i' = \chi_{O_i(Q_i)}, \quad R_i' = \chi_{O_i(Q_i)}.
\]

Note that \( Q_i, Q_i', R_i, R_i' \) are in \( \mathcal{L}^+_\mu \) because they less than \( T^{m_i}_{(4)} \).

**Lemma 2.2.**

\[
T^{m_i}_{(4)} - P_i \to 0 \text{ as } i \to +\infty,
\]

where

1. \( P_i = (4/3T^{m_i}_{(4)} + 8/3E)Q_i + (1/3T^{m_i}_{(4)} + 2E + 5/3T_{(4)})R_i \) if \( h_n \leq m_i < 2h_n \).
2. \( P_i = (1/3T^{m_i}_{(4)} + 2T^{m_i}_{(4)} + 5/3E)Q_i + (2/3T^{m_i}_{(4)} + 8/3E + 2/3T_{(4)})R_i \) if \( 2h_n \leq m_i < 3h_n \).
3. \( P_i = (2/3T^{m_i}_{(4)} + 8/3T^{m_i}_{(4)} + 2/3E)3/2Q_i + (4/3T^{m_i}_{(4)} + 8/3E)3/2R_i \) if \( 3h_n \leq m_i < h_{n+1} \).

**Proof.** It is enough to show (2.1) on pairs of functions from some dense set in \( L_2(\mu) \).

Therefore we can assume that \( f \) and \( g \) are constant, say \( f_i(j) \) and \( g_i(j) \), on each \( j^{\text{th}} \) level of \( C_n \) for sufficiently large \( n \).

Obviously, if \( \mu(D_i) \to 0 \), then \( \chi_{D_i} T^{m_i}_{(4)} \to 0 \). Thus

\[
(2.2) \quad T^{m_i}_{(4)} = \sum_j \chi_{A_i(j)} T^{m_i}_{(4)} - \sum_j \chi_{B_i(j)} T^{m_i}_{(4)} \to 0.
\]
Next we will calculate the connection between components of each sum in (2.2), using that $T_{(4)}^{m_i}f$ has a "regular" structure on sets $A_i(j)$ and $B_i(j)$ and $g$ is independent of $j$. Consider the $(n+1)^{\text{st}}$ column.

1. If $m_i < 2h_n$, then $i_0 = 2$. Clearly,

$$
\langle \chi_{A_i(1)} T_{(4)}^{m_i} f, g \rangle = d_{n+1} \sum_{l=1}^{h_n - p_i} f_n(l + p_i) \tilde{g}_n(l) = \langle Q_i f, g \rangle,
$$

$$
\langle \chi_{B_i(1)} T_{(4)}^{m_i} f, g \rangle = d_{n+1} \sum_{l=h_n - p_i + 1}^{h_n} f_n(l + p_i - h_n - 1) \tilde{g}_n(l) = \langle T_{(4)} f, g \rangle.
$$

Here and next values of $f$ and $g$ can be written wrong on bases and tops of $A_i(j)$ and $B_i(j)$, but this fact does not essential for convergence of such operators. It is clear that

$$
\chi_{A_i(1)} T_{(4)}^{m_i} - Q_i \to 0,
$$

$$
\chi_{B_i(1)} T_{(4)}^{m_i} - T_{(4)} R_i \to 0.
$$

Analogously,

$$
\chi_{A_i(3)} T_{(4)}^{m_i} - T_{(4)}^{-1} Q_i \to 0.
$$

By construction of $T_{(4)}$, the function $f(T_{(4)} x)$ has two values on each level from $B_i(3), A_i(4), B_i(4)$. The first one is exactly at $x$ from $O_i$, and the second one is elsewhere when one point from $\{x, T_{(4)} x, \ldots, T_{(4)}^{m_i} x\}$ is outside of $C_{n+1}$. Thus

$$
\langle \chi_{B_i(3)} T_{(4)}^{m_i} f, g \rangle = \langle \chi_{B_i(3) \cap O_i} T_{(4)}^{m_i} f, g \rangle + \langle \chi_{B_i(3) \setminus O_i} T_{(4)}^{m_i} f, g \rangle =
$$

$$
\frac{2}{3} d_{n+1} \sum_{l=h_n - p_i + 1}^{h_n} f_n(l + p_i - h_n - 1) \tilde{g}_n(l) +
$$

$$
\frac{1}{3} d_{n+1} \sum_{l=h_n - p_i + 3}^{h_n} f_n(l + p_i - h_n - 2) \tilde{g}_n(l) = \frac{2}{3} \langle R_i f, g \rangle + \frac{1}{3} \langle T_{(4)}^{-1} R_i f, g \rangle.
$$

Hence

$$
\chi_{B_i(3)} T_{(4)}^{m_i} - \left( \frac{2}{3} E + \frac{1}{3} T_{(4)}^{-1} \right) R_i \to 0.
$$

In the same way, we get

$$
\chi_{A_i(4)} T_{(4)}^{m_i} - \left( \frac{2}{3} E + \frac{1}{3} T_{(4)}^{-1} \right) Q_i \to 0,
$$

$$
\chi_{B_i(4)} T_{(4)}^{m_i} - \left( \frac{2}{3} T_{(4)} + \frac{1}{3} E \right) R_i \to 0.
$$

This completes the calculation in the case 1.

2. Here $i_0 = 1$. As above, we have

$$
\chi_{A_i(2)} T_{(4)}^{m_i} - T_{(4)}^{-1} Q_i \to 0,
$$

$$
\chi_{B_i(2)} T_{(4)}^{m_i} - \left( \frac{2}{3} E + \frac{1}{3} T_{(4)}^{-1} \right) R_i \to 0,
$$

$$
\chi_{A_i(3)} T_{(4)}^{m_i} - \left( \frac{2}{3} T_{(4)}^{-1} + \frac{1}{3} T_{(4)}^{-2} \right) Q_i \to 0,
$$

$$
\chi_{B_i(3)} T_{(4)}^{m_i} - \left( \frac{2}{3} E + \frac{1}{3} T_{(4)}^{-1} \right) R_i \to 0,
$$

$$
\chi_{A_i(4)} T_{(4)}^{m_i} - \left( \frac{2}{3} E + \frac{1}{3} T_{(4)}^{-1} \right) Q_i \to 0,
$$

$$
\chi_{B_i(4)} T_{(4)}^{m_i} - \left( \frac{2}{3} T_{(4)} + \frac{1}{3} E \right) R_i \to 0.
3. In this case, \( i_0 = 4 \), and \( f(T_{(4)}^n x) \) has two values on each level, except for from \( A_i(1) \). Using
\[
\langle Q'_i f, g \rangle = \frac{2}{3} d_{n+1} \sum_{l=1}^{h_n - p_i} f_o (l + p_i) \tilde{g}_o (l),
\]
calculate
\[
\langle \chi_{A_i(1)} T_{(4)}^m i f, g \rangle = \frac{2}{3} d_{n+1} \sum_{l=1}^{h_n - p_i} f_o (l + p_i - 1) \tilde{g}_o (l) = \frac{3}{2} \langle T_{(4)} f, g \rangle.
\]
\[
\langle \chi_{A_i(1)} T_{(4)}^m i f, g \rangle = \langle \chi_{A_i(1)} \cap \mathcal{O}_i T_{(4)}^m i f, g \rangle + \langle \chi_{A_i(1)} \setminus \mathcal{O}_i T_{(4)}^m i f, g \rangle = \langle Q'_i f, g \rangle + \frac{1}{3} d_{n+1} \sum_{l=1}^{h_n - p_i} f_o (l + p_i - 1) \tilde{g}_o (l) = \langle T_{(4)} f, g \rangle + \frac{1}{2} \langle T_{(4)}^2 f, g \rangle.
\]

For \( j = 2, 3 \), we get
\[
\langle \chi_{A_i(j)} T_{(4)}^m i f, g \rangle = \frac{2}{3} d_{n+1} \sum_{l=1}^{h_n - p_i} f_o (l + p_i) \tilde{g}_o (l) + \frac{1}{3} d_{n+1} \sum_{l=1}^{h_n - p_i} f_o (l + p_i - 2) \tilde{g}_o (l) = \langle T_{(4)} f, g \rangle + \frac{1}{2} \langle T_{(4)}^2 f, g \rangle.
\]
As before, for any \( j \)
\[
\langle \chi_{B_i(j)} T_{(4)}^m i f, g \rangle = \frac{2}{3} d_{n+1} \sum_{l=1}^{h_n} f_o (l + p_i - h_n - 1) \tilde{g}_o (l) + \frac{1}{3} d_{n+1} \sum_{l=1}^{h_n} f_o (l + p_i - h_n - 2) \tilde{g}_o (l) = \langle R'_i f, g \rangle + \frac{1}{2} \langle T_{(4)}^2 f, g \rangle.
\]

\[ \square \]

1. Fix \( P = a E + b T_{(4)} \). Let \( T_{(4)}^m \rightarrow P \) for some \( m_i \rightarrow +\infty \). It is clear that \( a, b \geq 0 \), because \( P \in \mathcal{L}_+^+ \). Choose a subsequence of \( m_i \) (if necessary) such that
\[
(2.4) \quad Q_{i}^{(x_i)} \rightarrow Q_i,
\]
where \( (x_i) \) means nothing or prime for each \( i \), and \( (x_i) \) is completely defined by \( m_i \) as in Lemma 2.2. Obviously, \( Q, R \in \mathcal{L}_+^+ \). By construction of \( T_{(4)} \),
\[
\| T_{(4)} \chi_{D_i} - \chi_{D_i} \|_\mu^2 \leq 2d_{n+1} \rightarrow 0,
\]
where \( D_i \) is either \( A_i(i_0) \) or \( A_i(i_0) \cap O_i \). Thus
\[
\| (T_{(4)} Q_{(e_i)} - Q_{(e_i)} T_{(4)}) f \|_\mu^2 \leq \| T_{(4)} \chi_{D_i} - \chi_{D_i} \|_\mu \| T_{(4)}^2 f \|_\mu \rightarrow 0,
\]
for any \( f \in L_2 (\mu) \). Therefore \( Q \) commutes with \( T_{(4)} \). Analogously, we have that \( R \) commutes with \( T_{(4)} \). Denote by \( \mathcal{P}_1^+ \) the subset of \( \mathcal{P}_1 \) with non-negative coefficients.
Next we will show that if \( P = \sum_{i=1}^d S_i \), where \( S_i \in \mathcal{L}_+^+ \), and \( S_i \) commute with \( T_{(4)} \), then \( S_i \in \mathcal{P}_1[T_{(4)}] \). Indeed, measures \( \phi^{-1} S_i \) are \( T_{(4)} \times T_{(4)}^{-1} \) invariant and absolutely continuous with respect to the subpolynomial \( \phi^{-1} P = a\Delta_E + b\Delta_{T_{(4)}} \).

The transformation \( T_{(4)} \times T_{(4)} \) is ergodic for measures \( \Delta_E \) and \( \Delta_{T_{(4)}} \). Hence every \( T_{(4)} \times T_{(4)}^{-1} \)-invariant part of the measure \( \Delta_E \) \( (\Delta_{T_{(4)}}) \) is \( c\Delta_E \) \( (c\Delta_{T_{(4)}}) \) for some \( c > 0 \).

This gives \( \phi^{-1} S_i = a_i\Delta_E + b_i\Delta_{T_{(4)}} \) for some \( a_i, b_i \geq 0 \).

From Lemma 2.2 and (2.4) we have that

\[
(2.5) \quad P = UQ + VR,
\]

where \((U, V)\) is at least one of the following pairs \((4/3T_{(4)}^{-1} + 8/3E, 1/3T_{(4)}^{-1} + 2E + 5/3T_{(4)}), (1/3T_{(4)}^{-2} + 2T_{(4)}^{-1} + 5/3E, 2/3T_{(4)}^{-1} + 8/3E + 2/3T_{(4)}), (T_{(4)}^{-2} + 4T_{(4)}^{-1} + E, 2T_{(4)} + 4E)\). In any case \( P \) contains \( c_1 Q \) and \( c_2 R \) as parts. Thus \( Q, R \in \mathcal{P}_1[T_{(4)}] \). Obviously,

\[
(2.6) \quad 1 = \langle T_{(4)}\mathbf{1}, \mathbf{1} \rangle \Rightarrow \langle \mathbf{1}, \mathbf{1} \rangle.
\]

This implies that \( P \neq 0 \). Therefore (2.5) is possible only when \( P = (4/3T_{(4)}^{-1} + 8/3E)cT_{(4)} \). It remains to mention, using (2.6), that \( c = 1/4 \).

2. The proof that \( 1/3E + 2/3T_{(4)} \in \phi(LJ_+(T_{(4)})) \) is almost obvious. Namely, consider \( m_i = h_n \). We obtain \( i_n = 2, p_i = 0, R_i = 0 \). Thus (2.3) gives

\[
\langle Q; f, g \rangle = d_n + \left| \sum_{l=1}^{b_n} f_n(l)\tilde{g}_n(l) \right| = \frac{d_n}{4} \left| \sum_{l=1}^{b_n} f_n(l)\tilde{g}_n(l) \right| = \frac{1}{4}\langle f, g \rangle.
\]

This implies that

\[
\langle P; f, g \rangle = \langle \left( \frac{1}{3}T_{(4)}^{-1} + \frac{2}{3}E \right) f, g \rangle.
\]

Thus Theorem 2.1 follows from Lemma 2.2 and Remark 1.9. \( \square \)

Remark 2.3. By the same argument as in [3], it is not difficult to show that \( T_{(k)} \)

have minimal self-joinings. Then \( Q, R \) can be written in the following form

\[
\alpha \int + \sum_j a_j T_{(k)}^j,
\]

where \( f \) is an orthogonal projection onto the space of constants, and \( 0 \leq \alpha, 0 \leq a_i \).

This gives that the first part of Theorem 2.1 also follows directly from (2.5).

Our main result is the following.

Theorem 2.4. \( LJ_+(T_{(k)}) \neq LJ_-(T_{(k)}) \) for \( k > 3 \).

Proof. Indeed, it is clear that

\[
\phi(LJ_+(T_{(k)})) = \{ U^*_\nu : \nu \in LJ_+(T) \}.
\]

Therefore, using Proposition 1.3, Remark 1.9, and Theorem 2.1, we have

\[
\phi(LJ_-(T_{(k)})) \cap \mathcal{P}_1[T_{(k)}] = \left\{ \frac{k-2}{k-1}E + \frac{1}{k-1}T_{(k)} \right\},
\]

and Theorem 2.4 is proved. \( \square \)
3. Closing remarks

**Proposition 3.1.** For any $T \in \text{Aut}(\mu)$

$$LJ_+(T) \cap LJ_-(T) \neq \emptyset.$$ 

This is an immediate consequence of the next proposition.

**Proposition 3.2.**

\[ \phi(LJ_+(T)) \cap \phi(LJ_-(T)) \supseteq \{ T_+T_-, T_+ \in \phi(LJ_+(T)) \}, \]

and

\[ \{ T_+T_-, T_+ \in \phi(LJ_+(T)) \} = \{ f \} \Leftrightarrow \text{This is mixing}, \]

where $f$ is defined as in Remark 2.3.

**Proof.** Fix $T_+ \in \phi(LJ_+(T))$, and $n_i, k_i \to +\infty$ such that $T^{k_i} \to T_+$, $T^{-n_i} \to T_-$. Consider also a dense set of functions from $L_2(\mu)$, say $f_i$. For each $\epsilon > 0$ and $m \in \mathbb{N}$, choose $i$ such that

$$|\langle T^{-n_i} f_i, T^*_+ f_+ \rangle - \langle T_+ f_+, T^*_+ f_+ \rangle| < \epsilon,$$

for all $l_1, l_2 \leq m$. Finally choose $j = j(i)$ such that $k_j(i) - n_i > m$ and

$$|\langle T^{k_j(i)} T^{-n_i} f_i, f_+ \rangle - \langle T_+ T^{-n_i} f_i, f_+ \rangle| < \epsilon,$$

for all $l_1, l_2 \leq m$. Thus

$$|\langle T^{k_j(i)} T^{-n_i} f_i, f_+ \rangle - \langle T_+ f_i, f_+ \rangle| < 2\epsilon.$$

Therefore $T^{k_j(i)-n_i} \to T_+ T_-$, where $k_j(i) - n_i \to +\infty$.

Operators $T_+$ and $T_-$ belong to the von Neumann algebra generated by $T$. Thus $T_+ T_- = T_- T_+$. This implies that arguing as above, we see that $T^{-n_j(i)+k_i} \to T_+ T_-$, where $-n_j(i) + k_i \to -\infty$.

The second part of the proof is more or less standard. Indeed, if $T$ is not mixing, then there exists $T_+$ from $\phi(LJ_+(T)) \setminus \{ f \}$. Thus $T_- = T^*_+ \in \phi(LJ_-(T)) \setminus \{ f \}$.

Next for $S^* S = f$, where $S \in \phi(J(T, T))$, we have

$$\int f d\mu = 0 \Rightarrow \langle S f, S f \rangle = \langle S^* S f, f \rangle = \langle f, f \rangle = 0.$$ 

This means that $S f = 0$, and finally $S = f$. Therefore the operator $T_+ T_- = T_- T_+ = T^*_+ T^*_+$ is not $f$. \hfill \Box

**Remark 3.3.** Obviously, in (3.1) we have exact equality if $T$ is rigid or mixing. However, taking into account Remark 2.3, the operator $1/2E + 1/2T$ can not be represented as $T_+ T_-$ for Chacon's transformation $T$. This yields that, in general, the left part of (3.1) is different from the right part.

**REFERENCES**

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