The Supersymmetry Representation
for Correlation Functions of Disordered Systems

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Abstract

Two disordered models are considered: spherical model and non-interacting electron gas on lattice. The supersymmetry representation for correlation functions of these models is obtained. Using this representation the configurational averaging can be performed before thermodynamical one and the problem of calculation of configurationally averaged correlation functions of disordered spherical model is reduced to the calculation of correlation functions for Fermi-Bose interacting regular system.
I. Introduction

It is known that the calculation of configurationally averaged free energy for disordered systems is not an easy problem. In 1975 Edwards and Anderson proposed a new method (the so-called replica method) for calculation of free energy of disordered systems [1]. In this method the disordered system is replaced by systems ("replicas") which are identical to the original one. The limit $n \to 0$ gives initial model. Using replica method the configurational averaging of free energy can be performed before thermodynamical averaging. The partition function of a regular model is obtained after the averaging and standard methods developed for regular systems can be used. However, the replica method has some difficulties. For example, the procedure of analytical continuation of non-zero integer $n$ to $n = 0$ is not simple, the Hamiltonian obtained by the replica method is very complicated and cannot be solved exactly.

The second method, where configurational averaging can be performed at the very beginning of calculations is the supersymmetry method [2]. This method is applicable to the description of the motion of non-interacting particles in a random potential and is based on the use of functional integrals over both commuting and anticommuting variables. After configurational averaging the problem is reduced to the supersymmetry field theory. The supersymmetry method can be used for a essentially narrower class of problems than replica method but it is free of the problem of analytical continuation.

In the present paper the correlation functions of disordered spherical model and non-interacting electrons on lattice are considered. The main problem in calculation of configurational averaged correlation functions of disordered systems is factor $1/Z$ ($Z$ is partition function) in statistical operator. The idea of present paper is to rewrite $1/Z$ as a partition function $Z' = S e^{-\beta H'}$ of some new system with the Hamiltonian $H'$ and then to perform configurational averaging. In the present paper this idea is realized for spherical model and non-interacting electrons on lattice and as result the supersymmetry representation for correlation functions is obtained. Using this representation the configurational averaging can be performed before thermodynamical one and the problem of calculation of configurationally averaged correlation functions of disordered systems is reduced to the calculation of correlation functions for interacting Fermi-Bose regular systems.

II. The spherical model

Note, that spherical model is one of the most studied models of statistical mechanics (see, for example, [4] and references therein). The disordered spherical model was investigated by Pastur [5]. Nevertheless the supersymmetry representation for correlation functions of this model was not discussed in literature.

First of all we consider spherical model with 2-dimensional classical spins. In this case the supersymmetry representation for correlation functions can be performed in the simplest way. The generalization for $2n$-dimensional spins can be easily performed. The supersymmetry representation in the case of $2n + 1$-dimensional classical spins faces some difficulties but can be performed as well.
2.1. The supersymmetry representation for correlation functions

The spherical model Hamiltonian [3] of two-dimensional classical spins reads

\[ H = - \sum_{i,j} J_{ij} S_i S_j + \mu \sum_i S_i^2, \quad (2.1) \]

where \( i, j \) are the site numbers of lattice, \( S_i = (S_i^x, S_i^y) \) is 2-dimensional vector, \( J_{ij} \) is the random exchange interaction of \( i \) and \( j \) spins, \( \mu \) is determined from spherical condition

\[ \frac{1}{N} \sum_i S_i^2 = 1. \quad (2.2) \]

Let us introduce a complex variables

\[ \varphi_j = S_j^x + iS_j^y, \quad \varphi_j^* = S_j^x - iS_j^y. \quad (2.3) \]

Then the Hamiltonian and the spherical condition read

\[ H = - \sum_{i,j} J_{ij} \varphi_i \varphi_j + \mu \sum_i \varphi_i^* \varphi_i, \quad (2.4) \]

\[ \frac{1}{N} \sum_{i,j} < \varphi_i \varphi_j >= 1. \quad (2.5) \]

Now let us consider the spin correlation functions or the mean value of function of spin variables

\[ < A > = \frac{1}{Z} \int (d\varphi^* d\varphi) A(\varphi^*, \varphi) \exp \left( - \sum_{i,j} I_{ij} \varphi_i^* \varphi_j \right), \quad (2.6) \]

where

\[ (d\varphi^* d\varphi) = \prod_i dS_i^x dS_i^y, \]
\[ A(\varphi^*, \varphi) = A(\varphi_1^*, \varphi_1, \ldots, \varphi_N^*, \varphi_N) = A(S_1^x, S_1^y, \ldots, S_N^x, S_N^y), \]
\[ I_{ij} = (\mu \delta_{ij} - J_{ij})/T. \]

The partition function \( Z \) is

\[ Z = \int (d\varphi^* d\varphi) \exp \left( - \sum_{i,j} I_{ij} \varphi_i^* \varphi_j \right) = \frac{\pi^N}{\det \| I_{ij} \|}. \quad (2.7) \]

We will be interested in the configurationally averaged mean value \( \overline{< A >} \), where

\[ \overline{< A >} = \int \prod_{i<j} d\Delta J_{ij} \, P(\ldots \Delta J_{ij} \ldots) \, < A >, \quad (2.8) \]
\[ \Delta J_{ij} = J_{ij} - \overline{J_{ij}}, \]
\( P(\ldots \Delta J_{ij} \ldots) \) is the distribution function of exchange integral. We assume that
\[
P(\ldots \Delta J_{ij} \ldots) = \prod_{i<j} \frac{1}{\sqrt{2\pi D_{ij}}} \exp \left( -\frac{1}{2} \frac{\Delta J_{ij}^2}{D_{ij}} \right),
\]  
(2.9)

\( D_{ij} = \Delta J_{ij}^2 \) is quadratic fluctuation of exchange integral.

Note, that the calculation of configurationally averaged quantity \( \langle A \rangle \) using (2.6) is difficult because of \( 1/Z \). The main idea is to write \( 1/Z \) as a partition function of a some new system and then to perform configurational averaging.

Using Gaussian Grassman integral we obtain
\[
\frac{1}{Z} = \frac{1}{\pi^N} \det \|I_{ij}\| = \int d\eta_1^* d\eta_2 \ldots d\eta_N^* d\eta_N \exp \left( -\sum_{ij} I_{ij}\eta_i^* \eta_j \right) = Z'.
\]  
(2.10)

Note, that \( Z' \) can be treated as partition function of spherical model with anticommuting variables.

Then \( \langle A \rangle \) reads
\[
\langle A \rangle = \frac{1}{\pi^N} \int (d\varphi^* d\varphi)(d\eta_1^* d\eta_2 \ldots d\eta_N^* d\eta_N) A(\varphi^*, \varphi) \exp \left( -\sum_{ij} I_{ij}\phi_i^* \phi_j \right),
\]  
(2.11)

where we introduce the supervectors
\[
\phi_i = \left( \varphi_i \eta_i \right), \quad \phi_i^+ = \left( \varphi_i^* \eta_i^* \right).
\]

The functional
\[
F = -\sum_{ij} I_{ij}\phi_i^+ \phi_i
\]
is invariant under linear transformation
\[
\begin{pmatrix} \varphi_i \\ \eta_i \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix} \begin{pmatrix} \varphi_i \\ \eta_i \end{pmatrix},
\]  
(2.12)

where \( \varepsilon \) is Grassman variable.

This transformation can be treated as a rotation in the space of supervectors. That is why the functional has supersymmetry under the group of transformations mixing commuting and anticommuting variables.

Note that important result of supersymmetry is
\[
\langle \varphi_i^* \varphi_j \rangle = -\langle \eta_i^* \eta_j \rangle.
\]  
(2.13)

It can be easily proved. Consider (2.11) in the case \( A = 1 \). Then taking the derivative of (2.11) with respect to \( I_{ij} \) we obtain (2.13).
Using (2.11) the configurationally averaged correlation function $\langle A \rangle$ can be written in the form

\[
\langle A \rangle = \frac{1}{\pi N} \int (d\varphi^* d\varphi) (d\eta^* d\eta) A(\varphi^*, \varphi) \times \exp \left( -\sum_{i,j} T_{ij} \phi_i^+ \phi_j \right) \exp \left( \frac{1}{T} \sum_{i,j} \Delta J_{ij} \phi_i^+ \phi_j \right).
\]

(2.14)

For distribution function (2.9) one can perform configurational averaging exactly

\[
\exp \left( \frac{1}{T} \sum_{i,j} \Delta J_{ij} \phi_i^+ \phi_j \right) = \exp \left( \frac{1}{4T^2} \sum_{i,j} D_{ij} (\phi_i^+ \phi_j + \phi_j^+ \phi_i)^2 \right)
\]

(2.15)

and for $\langle A \rangle$ we have

\[
\langle A \rangle = \frac{1}{\pi N} \int (d\varphi^* d\varphi) (d\eta^* d\eta) A(\varphi^*, \varphi) \exp \left( F(\varphi, \eta) \right),
\]

(2.16)

where

\[
F(\varphi, \eta) = F_0(\varphi, \eta) + U(\varphi, \eta),
\]

\[
F_0(\varphi, \eta) = -\sum_{i,j} T_{ij} \phi_i^+ \phi_j,
\]

\[
U(\varphi, \eta) = \frac{1}{4T^2} \sum_{i,j} D_{ij} (\phi_i^+ \phi_j + \phi_j^+ \phi_i)^2,
\]

\[
\Delta J_{ij} = (\mu \delta_{ij} - J_{ij}) / T = (\mu \delta_{ij} - J(R_i - R_j)) / T, \quad D_{ij} = D(R_i - R_j).
\]

Because $\phi_i^+ \phi_j$ is invariant under linear transformation (2.12) the full functional $F = F_0 + U$ is invariant under this transformation too. Thus configurational averaging does not break the supersymmetry. Note also that configurational averaging over the fluctuations of exchange integral with Gaussian distribution function leads to $\phi^I$ interaction.

It is convenient to introduce new variables

\[
\varphi_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \varphi_j e^{-i k \cdot R_j},
\]

(2.17)

\[
\eta_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \eta_j e^{-i k \cdot R_j}.
\]

Then

\[
(d\varphi^* d\varphi) = \prod_k (d\varphi_k^* d\varphi_k),
\]

\[
(d\eta^* d\eta) = \prod_k (d\eta_k^* d\eta_k),
\]

and

\[
\langle A \rangle = \prod_k \langle A_k \rangle,
\]

where

\[
A_k = A(\varphi_k, \eta_k).
\]

(2.18)
\[ F_0 = - \sum_k I(k)(\phi_k^+ \phi_k), \]  
\[ U = \frac{1}{2T^2} \frac{1}{N} \sum_{k_1,k_2} \sum_{k_1',k_2'} (D(k_1 + k_1') + D(k_1 - k_2')) \times \]
\[ \times \delta(k_1 + k_1' - k_2 - k_2')(\phi_{k_1}^+ \phi_{k_2}) (\phi_{k_1'}^+ \phi_{k_2'}), \]
where
\[ I(k) = (\mu - J(k))/T, \]
\[ J(k) = \sum_j J(R_j) e^{-ikR_j}, \]
\[ D(k) = \sum_j D(R_j) e^{-ikR_j}. \]

Because in the limit \( N \to \infty \) one has \( \frac{1}{N} \sum_i < \varphi_i^* \varphi_i > = \frac{1}{N} \sum_j < \varphi_j^* \varphi_j > \) the spherical condition now can be written as
\[ \frac{1}{N} \sum_k < \varphi_k^* \varphi_k > = 1. \]  
(2.20)

Note, that
\[ < \varphi_{k=0}^* \varphi_{k=0} > = N | < \varphi_i > |^2 = N | \sigma | ^2, \]  
(2.21)

here we take into account that in the limit \( N \to \infty \) \( \frac{1}{\sqrt{N}} \varphi_{k=0} = \frac{1}{N} \sum_i \varphi_i \) tends to configurationally averaged mean value \( < \varphi_i > = \sigma \). Finally spherical condition reads
\[ \sigma^2 + \frac{1}{N} \sum_{k \neq 0} < \varphi_k^* \varphi_k > = 1. \]  
(2.22)

### 2.2. Gaussian approximation

Unfortunately, the calculation of functional integral (2.15) with full functional cannot be done exactly. Consider the simplest approximation that we called Gaussian approximation. We carry out this approximation for the case \( T > T_c \), where \( T_c \) is the critical temperature. Using the Wick’s theorem the product of two commuting or anticommuting variables in (2.18) are replaced by mean values
\[ \varphi_{k_1}^* \varphi_{k_2} \rightarrow \delta(k_1 - k_2) \frac{< \varphi_{k_1}^* \varphi_{k_2} >}{}, \]  
(2.23)
\[ \eta_{k_1}^* \eta_{k_2} \rightarrow \delta(k_1 - k_2) \frac{< \eta_{k_1}^* \eta_{k_2} >}{}, \]
where taking into account (2.13) we have \( < \varphi_{k}^* \varphi_{k} > = -< \eta_{k}^* \eta_{k} > \).

After suggested approximation for \( U \) we obtain
\[ U = \frac{1}{T} \sum_k A(k)(\phi_k^+ \phi_k), \]  
(2.24)
where
\[
A(k) = \frac{1}{2T} \frac{1}{N} \sum_{q \neq 0} (D(k + q) + D(0)) \langle \varphi_q^* \varphi_q \rangle.
\]

In the frames of Gaussian approximation the full functional takes the form of \( F_0 \) with renormalized exchange integral and \( \mu \)
\[
F = -\sum_k \frac{1}{T} (\tilde{\mu} - \tilde{J}(k)) \left( \phi_k^+ \phi_k \right),
\]

where
\[
\tilde{J}(k) = J(k) + \frac{1}{2N} \sum_{q \neq 0} D(k + q) \langle \varphi_q^* \varphi_q \rangle,
\]
\[
\tilde{\mu} = \mu - \frac{D(0)}{2T}.
\]

On the basis of this approximation we can easily calculate the correlation function \( \langle \varphi_k^* \varphi_k \rangle \). The result is the following
\[
\langle \varphi_k^* \varphi_k \rangle = \frac{T}{\tilde{\mu} - \tilde{J}(k)}. \tag{2.26}
\]

Then for \( \mu \) we obtain the equation
\[
\frac{1}{N} \sum_{k \neq 0} \frac{T}{\tilde{\mu} - \tilde{J}(k)} = 1. \tag{2.27}
\]

Substituting (2.26) into (2.25) we obtain the equation for \( \tilde{J}(k) \)
\[
\tilde{J}(k) = J(k) + \frac{1}{2N} \sum_{q \neq 0} D(k + q) \frac{T}{\tilde{\mu} - \tilde{J}(q)} \tag{2.28}
\]

The solution of this equation can be presented in the form of a series with respect to \( D \).

In zero-order approximation
\[
\tilde{J}(k) = J(k). \tag{2.29}
\]

In first-order approximation (linear approximation with respect to \( D \)) \( \tilde{J}(k) \) is determined by (2.28) where \( \tilde{J}(k) \) is replaced by \( J(k) \) in the second term.

The critical temperature can be obtained from the pole of the correlation function (2.26) at \( k = 0 \). Then
\[
\tilde{\mu}_c = \tilde{J}(0). \tag{2.30}
\]

Substituting (2.30) into (2.27) for the critical temperature we obtain
\[
\frac{1}{T_c} = \frac{1}{N} \sum_{k \neq 0} \frac{1}{J(0) - J(k)}. \tag{2.31}
\]
The asymptotic behaviour of $\tilde{\mu}$ at $T \to \infty$ is

$$\tilde{\mu} = T,$$  \hfill (2.32)

and for the correlation function in the limit of great temperatures we obtain

$$\langle \varphi_k \varphi_k \rangle \to 1 \text{ while } T \to \infty.$$  

Then in the site representation

$$\langle \varphi_i \varphi_j \rangle \to \delta_{ij} \text{ while } T \to \infty.$$  

Let us consider for example the case of nearest-neighbors interaction. In linear with respect to $D$ approximation for renormalized exchange integral we obtain

$$J(k) = 2J \sum_{\alpha=1}^{d} \cos k_\alpha a,$$  \hfill (2.33)

where

$$J = \mathcal{J} \left(1 + \frac{\Delta J^2}{J^2} \frac{1}{4d} \left( \frac{\mu}{2\mathcal{J}} f\left( \frac{\mu}{2\mathcal{J}} \right) - 1 \right) \right),$$

$$f(x) = \frac{1}{N} \sum_{k \neq 0} \frac{1}{x - \sum_{\alpha=1}^{d} \cos k_\alpha a} = \frac{1}{(2\pi)^d} \int_0^{2\pi} dy_1 \cdots \int_0^{2\pi} dy_d \frac{1}{x - \sum_{\alpha=1}^{d} \cos y_\alpha},$$

$d$ is lattice dimension, $a$ is lattice constant.

At critical temperature $\mu = \mu_c = 2\mathcal{J}d$ and then $f(\mu_c/2\mathcal{J}) = f(d) = 2\mathcal{J}/T_c^0$, where $T_c^0$ is the critical temperature of ideal spherical model with exchange integral $\mathcal{J}$. Using (2.31) and (2.33) for renormalized critical temperature we obtain

$$T_c = T_c^0 \left(1 + \frac{\Delta J^2}{J^2} \left( \frac{\mathcal{J}}{T_c^0} - \frac{1}{2d} \right) \right).$$  \hfill (2.34)

It is interesting to note that $\mathcal{J}/T_c^0 > 1/2d$ and thus the fluctuations of exchange integral in the case of nearest-neighbors interaction lead to increase of critical temperature.
2.3. The generalization of supersymmetry representation for arbitrary dimension of spin

The generalization of supersymmetry representation for correlation functions in the case of even dimension of spins faces no problems. The supersymmetry representation in the case of odd dimension of spins is more complicated but also can be performed. Let us consider the spherical model with one dimensional spins. In this case $1/Z = \sqrt{\det I_{ij}} / \pi^N$ and for mean value we have

$$< A > = \frac{1}{\pi^{N/2}} \sqrt{\det I_{ij}} \int (dS^x) \exp \left( -\sum_{i,j} I_{ij} S^x_i S^x_j \right) A(S^x_1, \ldots, S^x_N).$$ \hspace{1cm} (2.35)

The main problem is that (2.35) contains $\sqrt{\det I_{ij}}$. Note, that $\sqrt{\det I_{ij}}$ where $I_{ij}$ is symmetric matrix can not be written as Gaussian integral over Grassman variables. This problem can be solved if we rewrite (2.35) in the following form

$$< A > =$$ \hspace{1cm} (2.36)

$$= \frac{1}{\pi^N} \det I_{ij} \sqrt{\det I_{ij}} \int (dS^x) \exp \left( -\sum_{i,j} I_{ij} S^x_i S^x_j \right) A(S^x_1, \ldots, S^x_N) =$$ \hspace{1cm} (2.36)

$$= \frac{1}{\pi^N} \det I_{ij} \int (dS^x dS^y) \exp \left( -\sum_{i,j} I_{ij} (S^x_i S^x_j + S^y_i S^y_j) \right) A(S^x_1, \ldots, S^x_N).$$

Equation for $\mu$ is

$$\frac{1}{N} \sum_{i=1}^{N} < (S^x_i)^2 > = 1.$$ \hspace{1cm} (2.37)

Taking into account that $< (S^x_i)^2 > = < (S^y_i)^2 >$ the equation (2.37) can be written as

$$\frac{1}{2N} \sum_{i=1}^{N} < S^2_i > = 1.$$ \hspace{1cm} (2.38)

On the basis of (2.36) and (2.38) the calculation of correlation functions of spherical model with one dimensional spins is reduced to the calculation of correlation functions of spherical model with two dimensional spins. Therefore the method described in sections 1 and 2 can be used.

III. Non-interacting electron gas on a lattice

In [2] the supersymmetry representation for Green function of electron in a random potential was obtained. In this section we propose the new supersymmetry method for calculation of configurationally averaged correlation functions of non-interacting electrons on lattice.
3.1. The Hamiltonian and the main definition

Consider non-interacting electron gas which is described by the Hamiltonian

\[ H_f = \sum_{i,\sigma} (\epsilon_i - \mu) a_{i\sigma}^+ a_{i\sigma} + \sum_{i,j} V_{ij} a_{i\sigma}^+ a_{j\sigma}, \]  

(3.1)

where \( a_{i\sigma}^+, a_{i\sigma} \) are creation and annihilation operators of fermions with spin \( \sigma \) on the lattice site \( i \), \( \epsilon_i \) is the energy of fermion located on site \( i \), \( V_{ij} \) is the hopping amplitude between sites \( i \) and \( j \), \( \mu \) is a chemical potential.

The electron correlation functions are defined in standard way

\[ \langle A \rangle = \frac{1}{Z_f} \text{Sp}_f A e^{-H_f/T}, \]  

(3.2)

where \( Z_f \) is partition function of fermions.

In the case when \( \epsilon_i \) and \( V_{ij} \) take random values we will be interested in the configurationally averaged correlation functions \( \overline{\langle A \rangle} \), where configurational averaging contains the averaging over \( \epsilon_i \) and \( V_{ij} \)

\[ \overline{\langle A \rangle} = \int (d\epsilon dV) P(\epsilon, V) \langle A \rangle, \]  

(3.3)

\( P(\epsilon, V) \) is distribution function of \( \epsilon_i \) and \( V_{ij} \).

The performing of configurational averaging using (3.2) is difficult because of factor \( 1/Z_f \). The aim of next section is to solve this problem.

3.2. The relation between fermion and boson partition functions

In this section we show that partition functions of non-interacting fermions \( Z_f \) and non-interacting bosons \( Z_b \) are connected by the following formula

\[ \frac{1}{Z_f(\mu)} = Z_b(\mu \pm i\pi T). \]  

(3.4)

For this purpose first of all let us consider the system described by the Hamiltonian of simple fermion harmonic oscillator

\[ H_f = (\epsilon - \mu) a^+ a. \]  

(3.5)

The partition function of this system is

\[ Z_f(\mu) = 1 + e^{-(\epsilon - \mu)/T}. \]  

(3.6)

Now consider the boson harmonic oscillator

\[ H_b = (\epsilon - \mu) b^+ b, \]  

(3.7)
with partition function
\[
Z_b(\mu) = \frac{1}{1 - e^{-(\mu - \nu)/T}}.
\]  
(3.8)

From (3.8) and (3.6) we see that
\[
Z_b(\mu \pm i\pi T) = \frac{1}{1 - e^{\pm i\pi} e^{-(\mu - \nu)/T}} = \frac{1}{Z_f(\mu)}
\]
and thus (3.4) is true.

It is interesting to note that correlation function of fermion and boson subsystem are connected in the following way
\[
\langle a^+ a \rangle_\mu = -\langle b^+ b \rangle_{\mu \pm i\pi T}.
\]  
(3.9)

The formula (3.4) is also true for the case of non-interacting electrons described by the Hamiltonian (3.1) and non-interacting bosons described by the same Hamiltonian where \(a_{i\sigma}, a_{i\sigma}^+\) are replaced by boson operators \(b_{i\sigma}, b_{i\sigma}^+\). This can be proved using the procedure of diagonalization. After diagonalization (3.1) takes the form
\[
H_f = \sum_{n,\sigma} (E_n - \mu) A_{n\sigma}^+ A_{n\sigma}.
\]  
(3.10)

It is obvious that the Hamiltonian of bosons satisfying (3.4) is
\[
H_b = \sum_{n,\sigma} (E_n - \mu) B_{n\sigma}^+ B_{n\sigma}.
\]  
(3.11)

After returning to initial operators the Hamiltonian of bosons (3.11) takes the form
\[
H_b = \sum_{i,\sigma} \epsilon_i - \mu \sum_{i\sigma} V_{ij} b_{i\sigma}^+ b_{j\sigma}.
\]  
(3.12)

Thus the partition function of fermions described by the Hamiltonian (3.1) and partition function of bosons described by (3.12) are connected via formula (3.4).

The correlation functions of fermion and boson subsystems satisfy the following relation
\[
\langle a_{i\sigma}^+ a_{j\sigma} \rangle_{\mu} = -\langle b_{i\sigma}^+ b_{j\sigma} \rangle_{\mu \pm i\pi T}.
\]  
(3.13)

This also can be proved using the procedure of diagonalization.

3.3. The supersymmetry representation for correlation functions

Using (3.4) for correlation functions of fermions we obtain
\[
\langle A \rangle = Z_b(\mu \pm i\pi T) S p_f A e^{-H_f/T} =
\]  
(3.14)

\[
= S p_b e^{-H_b/T \mp i\pi N_b} S p_f A e^{-H_f/T} = S p A e^{-H/T \pm i\pi N_b},
\]
where full Hamiltonian is
\[ H = H_f + H_b, \]
and operator of boson occupation number is
\[ N_b = \sum_{i,\sigma} b_{i,\sigma}^+ b_{i,\sigma}. \]

Note, that full Hamiltonian is supersymmetrical one. It is obvious in the case of system described by simple Hamiltonian (3.5). Then
\[ H = (\epsilon - \mu)(a^+ a + b^+ b). \]  
(3.16)

In the general case full Hamiltonian
\[ H = \sum_{i,\sigma}(\epsilon - \mu)(a_{i,\sigma}^+ a_{i,\sigma} + b_{i,\sigma}^+ b_{i,\sigma}) + \sum_{i,j} V_{i,j}(a_{i,\sigma}^+ a_{j,\sigma} + b_{i,\sigma}^+ b_{j,\sigma}) \]
(3.17)
is also supersymmetrical one that can be seen after diagonalization.

The Hamiltonian (3.17) can be written in the form of supersymmetric quantum mechanics
\[ H = Q^+ Q^- + Q^- Q^+. \]
(3.18)
The generators of supersymmetry read
\[ Q^+ = \sum_{\sigma} \sum_{i,j} T_{i,j} a_{i,\sigma}^+ b_{j,\sigma}, \]
(3.19)
\[ Q^- = \sum_{\sigma} \sum_{i,j} T_{i,j}^* b_{j,\sigma}^+ a_{i,\sigma}, \]
(3.20)
where Hermitian Matrix \( T \) satisfies the equation
\[ (T^2)_{ij} = (\epsilon - \mu)\delta_{i,j} + V_{ij}. \]
(3.21)

Therefore the Hamiltonian (3.17) is supersymmetrical Hamiltonian and that is why the representation for correlation functions (3.14) can be called the supersymmetry representation.

### 3.4. The configurational averaging

The supersymmetry representation is convenient for performing the configurational averaging. Using (3.14) the configurational averaging can be performed before the thermodynamical averaging
\[ \langle \overline{A} \rangle = \text{Sp} A e^{\pm i \pi N_b - \tilde{H}/T} \]
(3.22)
where the result of configurational averaging can be presented in the following form
\[ e^{-\tilde{H}/T} \]
(3.23)
The effective Hamiltonian $\tilde{H}$ does not contain random parameters. Therefore $\tilde{H}$ can be considered as the Hamiltonian of regular system. Note, that, generally speaking, the effective Hamiltonian includes the interaction between bosons and fermions. Thus the disorder is reduced to Fermi-Bose interaction. In fact, our problem now is to calculate the correlation functions of Fermi-Bose interacting regular systems.

Unfortunately the effective Hamiltonian can not be calculated exactly. Using well known approximation in the case of high temperatures

$$e^{-\tilde{H}/T} \approx e^{-\Pi/T + (\Delta H)^2/2T^2}$$

we obtain

$$\tilde{H} = \Pi - \frac{1}{2T} (\Delta H)^2,$$

(3.24)

where $\Pi$ is the Hamiltonian (3.17) with mean value of random parameters $\bar{\epsilon}_i$ and $\bar{V}_{ij}$. $(\Delta H)^2$ contains Fermi-Bose interaction. For example, in the case when only $\epsilon_i$ takes random value we have

$$\langle (\Delta H)^2 \rangle = \sum_{\sigma'} \sum_{i,j} \bar{\epsilon}_i \bar{\epsilon}_j \left( a_{i\sigma} a_{i\sigma'} + b_{i\sigma} b_{i\sigma'} \right) \left( a^+_{j\sigma} a_{j\sigma'} + b^+_{j\sigma} b_{j\sigma'} \right),$$

(3.25)

where $\bar{\epsilon}_i \bar{\epsilon}_j = D(R_i - R_j)$.

Note, that even within the simplest approximation the effective Hamiltonian (3.24) is not simple and can not be solved exactly. But it is the Hamiltonian of regular system and therefore the method developed for regular systems can be used.

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**References**