Coulomb gas representation
of the SU(2) WZW correlators at higher genera

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Abstract
We extend the analysis of [1] to the case with insertion points. The result allows to express the correlation functions of the SU(2) WZW conformal field theory on Riemann surfaces of genus > 1 by finite dimensional integrals.

1. Introduction

The present paper completes the work [1] where we have computed the scalar product of the SU(2) Chern-Simons states on a Riemann surface of genus > 1 in the absence of Wilson lines. Here, we treat the case with Wilson lines \(C_l\), in representations of spin \(j_l\), cutting the Riemann surface at marked points. Let us briefly list the basic notations, definitions and relations, referring to [1] for more extensive introduction. Below:

\(\Sigma\) denotes the Riemann surface (of genus \(\gamma > 1\)) with distinct marked points \(\xi_l\), \(l = 1, \ldots, L\).

\(A^{01}\) stands for the space of smooth \(sl(2, \mathbb{C})\)-valued 0,1-forms \(A^{01}\) on \(\Sigma\).

\(G^C\) is the group of complex gauge transformations given by smooth maps \(h : A^{01} \rightarrow SL(2, \mathbb{C})\) which act by \(A^{01} \rightarrow h A^{01} \equiv h A^{01} h^{-1} + h \partial h^{-1}\) on \(A^{01}\).

\(S(h, A^{10} + A^{01})\) denotes the WZW model action in the presence of the gauge field.

\(k = 1, 2, \ldots\) is the level of the theory.

\(V_j\) stands for the space of spin \(j\) representation with \(g \in SL(2, \mathbb{C})\) acting on it by linear automorphism \(g_j\).
\( \Psi : A^{01} \rightarrow \otimes \mathbb{V}_j \) is a CS state if it is holomorphic and if \( \Psi = \Psi \) where
\[
(\Psi)(A^{01}) = e^{-k S(h, A^{01})} \otimes \mathcal{h}(\xi_i)_{\otimes \mathbb{V}_j} \Psi(h^{-1} A^{01}). \tag{1.1}
\]

\( W_k((\xi_i), (j_i)) \) denotes the (finite-dimensional) space of CS states.

The scalar product of the CS states is formally given by the integral
\[
\| \Psi \|^2 = \int |\Psi(A^{01})|^2 \otimes \mathbb{V}_j e^{i \frac{i}{\tau} \mathcal{g}(\Psi, A^{01})} \mathcal{h}^0 \mathcal{A}^0 \mathcal{A} \mathcal{D} \mathcal{A} . \tag{1.2}
\]

The aim of the paper is to compute the functional integral (1.2) over the \( su(2) \) gauge fields \( A = A^{10} + A^{01} \) with \( A^{10} = -(A^{01})^\dagger \) by reducing it to an explicit finite-dimensional integral. Such a reduction allows to express the correlation functions in the external gauge field of the WZW model of conformal field theory,
\[
\Gamma((\xi_i), (j_i), A) = \int \otimes \mathcal{g}(\xi_i)_{\otimes \mathbb{V}_j} e^{-k S(A)} \mathcal{D} \mathcal{g} , \tag{1.3}
\]
by finite-dimensional integrals. According to [2],
\[
\Gamma((\xi_i), (j_i), A) = \sum_{r, r'} H^{r r'} \Psi_r(A^{01}) \otimes \overline{\Psi}_{r'}(A^{01}) e^{i \frac{i}{\tau} \mathcal{g}(\Psi, A^{01})} \mathcal{h}^0 \mathcal{A}^0 \mathcal{A} \mathcal{D} \mathcal{A} \mathcal{d} \mathcal{A} . \tag{1.4}
\]
for any basis \( (\Psi_r) \) of \( W_k((\xi_i), (j_i)) \) with matrix \( (H^{r r'}) \) inverting the matrix \( ((\Psi_r, \Psi_{r'})) \) of scalar products of \( \Psi_r \)'s. Hence
\[
\Gamma((\xi_i), (j_i), A) = \sum_{r, r'} \Psi_r(A^{01}) \otimes \overline{\Psi}_{r'}(A^{01}) e^{i \frac{i}{\tau} \mathcal{g}(\Psi, A^{01})} \mathcal{h}^0 \mathcal{A}^0 \mathcal{A} \mathcal{D} \mathcal{A} \mathcal{d} \mathcal{A} \cdot \int \mathcal{Z}^1 \mathcal{Z}^1 \exp[- \sum_{s, s'} z_s^s(\Psi_s, \Psi_{s'}) z_s^{s'}] \prod_l d^2 z_l \int \exp[- \sum_{s, s'} z_s^s(\Psi_s, \Psi_{s'}) z_s^{s'}] \prod_l d^2 z_l \tag{1.5}
\]
into which one should substitute the finite-dimensional integral expressions for the scalar products \( (\Psi_s, \Psi_{s'}) \).

The functional integral in Eq. (1.2) is calculated by a change of variables \( A^{01} \rightarrow (h, n) \),
\[
A^{01} = h^{-1} A^{01}(n) , \tag{1.6}
\]
where \( h \in G^C \) and \( n \) parametrizes a slice \( \{ A^{01}(n) \} \) (of complex dimension \( 3(\gamma - 1) \equiv N \) in \( A^{01} \) which cuts a generic orbit of \( G^C \) a finite number, say \( \nu \), of times. Upon the change of variables, Eq. (1.2) becomes
\[
\| \Psi \|^2 = \frac{1}{\nu} \int |\Psi(A^{01}(n))|^2 \otimes \mathbb{V}_j e^{-k S(h h^\dagger, A^{01}(n))} \mathcal{h}^0 \mathcal{A}^0 \mathcal{A} \mathcal{D} \mathcal{A} \mathcal{d} \mathcal{A} \cdot \det(D^1_{\mathcal{h}^0} D_n) \det(\Omega(1, n))^{-1} \left| \det \left( \int \mathcal{g} \mathcal{h}^{\beta}(n) \mathcal{A}^0(\mathcal{h}^\dagger) \mathcal{A} \mathcal{D} \mathcal{A} \mathcal{d} \mathcal{A} \right) \right|^2 \mathcal{d} \mathcal{h}^0 \prod_{\mathcal{a}} d^2 n_{\mathcal{a}} . \tag{1.7}
\]
where \( | \cdot |_{\otimes \mathbb{V}_j} \) is the norm induced from the scalar product of spaces \( \mathbb{V}_j, D_n \equiv \partial + [A^{01}(n), \cdot] \) acts on \( sl(2, \mathbb{C}) \)-valued functions and the matrix \( \Omega(1, n) = (i \int \mathcal{g} \mathcal{h}^{\beta}(n) \mathcal{A}^0(\mathcal{h}^\dagger) \mathcal{A} \mathcal{D} \mathcal{A} \mathcal{d} \mathcal{A} \right) \).
annihilated by the dual of $D_n$. The passage from Eq. (1.2) to (1.7) has been discussed in details in the beginning of Sec. 6 of [1].

We shall use the same slice $\{ A^{01}(n) \}$ as in [1]. Let us briefly recall its construction. It is based on representing the gauge fields $A^{01}$ in the trivial bundle $\Sigma \times \mathbb{C}$ by gauge fields $B^{01}$ in the bundle $L_0^{-1} \oplus L_0$ where $L_0$ is a fixed holomorphic line bundle over $\Sigma$ of degree $g - 1$ s.t. $L_0^2 \neq K$, where $K$ is the canonical bundle (of 1,0-covectors) on $\Sigma$. We shall provide $L_0$ with a hermitian structure. $\Sigma \times \mathbb{C}$ and $L_0^{-1} \oplus L_0$ are equivalent as smooth rank two bundles with trivial determinant and hermitian structures. Choosing the equivalence $U : L_0^{-1} \oplus L_0 \to \Sigma \times \mathbb{C}$, we shall put

$$B^{01} = U^{-1} A^{01} U + U^{-1} \partial U \quad (1.8)$$

$B^{01}$ is a 0,1-form with values in $\text{End}(L_0^{-1} \oplus L_0)$. We shall consider triangular forms

$$B^{01}_{x, b} = \begin{pmatrix} -a_x & b \\ 0 & a_x \end{pmatrix}, \quad (1.9)$$

where $b$ is a 0,1-form with the values in $L_0^{-2}$ (i.e. $b \in \wedge^{01}(L_0^{-2})$) and

$$a_x = \pi (\int_{x_0}^{x} \omega)(\text{Im}\tau)^{-1}\omega \quad (1.10)$$

in the shorthand notation using the vector $\omega \equiv (\omega^i)$ of holomorphic 1,0-forms and the period matrix $\tau \equiv (\tau^j)$ corresponding to a fixed marking of $\Sigma$. $x_0$ is a fixed point on $\Sigma$ and $x$ is the point in the covering space $\tilde{\Sigma}$ corresponding to the path from $x_0$ to $x$ used in the integral in (1.10). The slice $\{ A^{01}(n) \}$ of $A^{01}$ is obtained by choosing (in a piece-wise holomorphic way) the gauge fields $A^{01}_{x, b}$ corresponding to $B^{01}_{x, b}$ with one $x$ for each $x \in \Sigma$ and one $b$ in each complex ray in $\wedge^{01}(L_0^{-2})/(\partial - 2a_x)(\mathcal{C}^\infty(L_0^{-2}))$. The latter space may be identified with the cohomology space $H^1(L_x^{-2})$ where $L_x$ denotes the line bundle $L_0$ with the holomorphic structure given by the operator $\partial_{x_0} \equiv \partial + a_x$. $\dim(H^1(L_x^{-2})) = N(\equiv 3(\gamma - 1))$, the number equal to the dimension of the slice (the one dimension subtracted by considering projectivized $H^1(L_x^{-2})$ is added by changing $x \in \Sigma$).

2. CS states with insertions

We shall represent the CS states $\Psi$ by functions $\psi$ of $x \in \tilde{\Sigma}$ and of $b \in \wedge^{01}(L_0^{-2})$, with values in $\otimes_i V_{j_i}$. To this end, it will be convenient to realize each $V_{j_i}$ as the space of polynomials $P(u_i)$ of degree $\leq 2j_i$ of variable $u_i$ taking values in the fiber $(L_0^{-2})_{k_i}$ of $L_0^{-2}$. It will be done so that the action of the element $U(\begin{pmatrix} a & b \\ c & d \end{pmatrix})^{-1}U^{-1}$ of $SL(2, \mathbb{C})$, where

$$(c \ c \ c \ d)$$

is a linear endomorphism with determinant one of $(L_0^{-1} \oplus L_0)_{k_i}$ is given by

$$P(u_i) \longmapsto (cu_i + d)^{2j_i} P((au_i + b)/(cu_i + d)). \quad (2.1)$$
The scalar product in \( V_j \) is then given by
\[
|P|^2 = \frac{2^{j+1}}{\pi} \int |P(u_i)|^2 \left(1 + |u_i|^2\right)^{-\frac{2j-2}{2}} \, d^2 u_i ,
\]
where \( |u_i|^2 \) is the hermitian square of \( u_i \). In such a polynomial realization of \( \otimes_i V_j \), \( \psi(x, b) \) will be defined by the relation:
\[
\psi(x, b)(u) = \exp\left[ \frac{ib}{2\pi} \int \text{tr} \, A_{0}^{10} \wedge A_{x, b}^{01} \right] \, \Psi(A_{x, b}^{01})(u - v(b)) ,
\]
where \( u = (u_i) \) and \( v(b) = ((v_i)(\xi_i)) \) with
\[
v(b)(\xi) = (\partial^{-1}_{\xi_i}) b(\xi) .
\]

In the latter formula, \( \partial^{-1}_{\xi_i} \) stands for the orthogonal projection (in the \( L^2 \) scalar product) on the image of the operator \( \partial_{\xi_i} \equiv \partial - 2\partial x : C^{\infty}(L_0^{-2}) \rightarrow \wedge^0(L_0^{-2}) \) followed by the inverse of \( \partial_{\xi_i} \) mapping onto its image. Note that, for \( v \in C^{\infty}(L_0^{-2}) \),
\[
v(b + \partial^{-1}_{\xi_i} v) = v(b) + v .
\]

Generalizing the arguments of Sect. 3 of [1] to the present case, one obtains, as consequences of the gauge invariance of the CS states, the following relations
\[
\begin{align*}
\psi(x, b + \partial^{-1}_{\xi_i} v)(u) &= \psi(x, b)(u) , \\
\psi(x, \lambda b)(\lambda u) &= \lambda^{k(\gamma - 1) + J} \psi(x, b)(u) , \\
\psi(px, c_p^2 b)(u) &= \mu(p, x)^k \nu(c_p)^k \prod_l c_p(\xi_i)^{2j_l} \psi(x, b)(c_p(\xi_i)^{-2} u_l) \end{align*}
\]
for \( v \in C^{\infty}(L_0^{-2}) \), \( \lambda \in \mathbb{C} \), \( J \equiv \sum_l j_l \), \( p \in \Pi_1(\Sigma, x_0) \) and \( c_p, \mu(p, x) \) and \( \nu(c_p) \) as in Sect. 3 of [1]. In particular, writing
\[
\psi(x, b)(u) = \sum_{q \equiv (0)} \psi_q(x, b) \prod_l q_l^{q_l} ,
\]
with \( q_l = 0, 1, \ldots, 2j_l \), we infer that \( \psi_q(x, \cdot) \) is a homogeneous polynomial on \( H^1(L_0^{-2}) \) of degree \( k(\gamma - 1) + J - Q \) where \( Q \equiv \sum_l q_l \), with values in \( \otimes_i (L_0^{2j_l})_{\xi_l} \). Moreover,
\[
\psi_q(px, c_p^2 b)(u) = \mu(p, x)^k \nu(c_p)^k \prod_l c_p(\xi_i)^{2(j_l - q_l)} \psi_q(x, b) .
\]

For the interpretation of the last relation, one should remark that the function \( \Phi \) on \( \Sigma \) transforming under the action of \( \Pi_1(\Sigma, x_0) \) as
\[
\Phi(px) = \mu(p, x)^k \nu(c_p)^k \prod_l c_p(\xi_i)^{2(j_l - q_l)} \Phi(px) ,
\]
(2.9)
defines a holomorphic section of the bundle \( L_0^2 K^k (2 \sum_i (j_i - q_i) \xi_i + 2(k(2 - \gamma) - J + Q)x_0) \) 
\((MN(D) \equiv M \otimes N \otimes O(D)\) for line bundles \(M, N\) and a divisor \(D\). In the notation of Sect. 3 and 4 of [1], \(\psi_q\) is a holomorphic section of the bundle

\[
\varpi^* (L^k K^k (2 \sum_i (j_i - q_i) \xi_i + 2(k(1 - \gamma) - J + Q)x_0)) \ Hf(W_0)^{(1 - \gamma) - J + Q}
\]

\[
\cong \varpi^* (L^k K^k (2 \sum_i (j_i - q_i) \xi_i)) \ Hf(W_0')^{(1 - \gamma) - J + Q}
\]

with values in \(\otimes_i (L_0^{2q_i})_{\xi_i}\).

3. Scalar product formula

With the use of the slice of \(A^{01}\) described above, the functional integral (1.7) reduces to the integral over fields \(U^{-1} h U = (\begin{smallmatrix} e^{i/2} & w e^{i/2} \cr 0 & e^{-i/2} \end{smallmatrix})\) (\(\varphi\) is a real function on \(\Sigma\), \(w\) is a section of \(L_0^{-2}\), over the projective space \(\mathbb{P}H^1(L_X^{-2})\) and over \(x\) belonging to a fundamental domain in \(\Sigma\). The functional integration will be performed essentially as in [1]. In the first step, one rewrites the expression under the integral in Eq. (1.7) in an explicit way (Sect. 6.1 to 6.3 and Sect. 7 of [1]). The only place where the arguments of [1] have to be modified is the treatment of the term

\[
| \otimes_i h(\xi_i)^{-1} \Psi (A^{01}(n)) |^2 \varepsilon_{ij}
\]

appearing in the place of \(|\Psi (A^{01}(n))|^2\) treated in Sect. 6.1 therein. The modification replaces \(|\Psi (x, b)|^2\) at the end of Eq. (6.17) of [1] by

\[
\prod_i \frac{2^{j_i + 1}}{\pi} \int |\psi(x, b)(e^{i\psi(x)}(u_1 + w(\xi_i)) + v(b_i))|^2 \prod_i \varepsilon^{2j_i \psi(x)} d^2 u_i \equiv (1 + |w|[F])^{2j_i + 2},
\]

where \(u(b_i)\) is given by Eq. (2.4). Altogether, the explicit form of Eq. (1.7) is

\[
||\Psi||^2 = \text{const.} \left( \frac{\det(-\Delta_{L_0^2})}{\text{area} \cdot \text{det}(\text{Im} \tau)} \right) e^{-\frac{1}{2} \rho} \int e^{-2\pi k (f(x_0, \xi)) \rho \cdot \text{Im} \tau (f(x_0, \xi))}
\]

\[
\cdot \det \left( \int_{\Sigma} \kappa_{\Sigma} \wedge \kappa_{\Sigma}^p \right) \cdot \det \left( \int_{\Sigma} \kappa_{\Sigma}^n \wedge \kappa_{\Sigma}^{p+1} \right) \cdot \det \left( \int_{\Sigma} \kappa_{\Sigma}^n \wedge \kappa_{\Sigma}^{p+2} \right)
\]

\[
\cdot \left| \sum_{j=1}^g (-1)^j \int_{\Sigma} f_{\Sigma}^{\kappa_{\Sigma}^j \wedge b, \omega_j} \omega_j (x) \right|^2 \left( \int_{\Sigma} \psi(x, b)(e^{i\psi(x)}(u_1 + w(\xi_i)) + v(b_i)) \right)^2
\]

\[
\cdot \prod_i \varepsilon^{2j_i \psi(x)} d^2 u_i \equiv (1 + |w|[F])^{2j_i + 2}
\]

\[
\cdot \left( \int_{\Sigma} \kappa_{\Sigma}^n \wedge \kappa_{\Sigma}^{p+3} \wedge \ldots \wedge \kappa_{\Sigma}^{p+3} \right) D w D \varphi,
\]

where \((\kappa_{\Sigma}^n)^{-1}\) is a basis of \(H^0(L_X^{-2})\), \((\eta_{\Sigma}^n)^{N_{\Sigma}}\) is a basis of \(H^0(L_X^{2k})\), \(\kappa_{\Sigma}^n = \int_{\Sigma} \kappa_{\Sigma}^n \wedge b\) provide homogeneous coordinates on \(\mathbb{P}H^1(L_X^{-2})\), \(\langle \cdot, \cdot \rangle\) denotes the hermitian structures.
on powers of the bundle $L_x$ induced by the fixed hermitian metric on $L_0$ (of curvature $F_0$) and vol is the Riemannian volume on $\Sigma$. We shall shift above the field $w$ by $-e^{-\varphi}v(b)$, see Eq. (2.4). Note that
\[ b - \partial_{L_x^0} v(b) = i z_x^0 (H_0^{-1})_{\alpha \beta} \eta_x^\beta j \]  \hspace{1cm} (3.4)
where $(H_0)^{\alpha \beta} = \frac{i}{2} \int_x <\eta_x^\alpha, \wedge \eta_x^\beta>$ (the right hand side of Eq. (3.4) gives the component of $b$ orthogonal to $\partial_{L_x^0} (C^\infty(L_x^{-2}))$). The shift of $w$ removes $v(b)$ from the argument of $\psi(x, b)$ and replaces $e^{-\varphi}b + (\partial_{L_x^0} + \partial_{\varphi})w$ by
\[ d \equiv i e^{-\varphi} z_x^0 (H_0^{-1})_{\alpha \beta} \eta_x^\beta j + (\partial_{L_x^0} + \partial_{\varphi})w . \]  \hspace{1cm} (3.5)
It will be convenient to decompose
\[ \varphi = a + \bar{\varphi} , \quad D\varphi = \text{area}^{1/2} da D\bar{\varphi} , \]  \hspace{1cm} (3.6)
where $a \in \mathbb{R}$ is the constant contribution to $\varphi$ and $\bar{\varphi}$ is orthogonal to the constant mode. We shall also multiply the right hand side of Eq. (3.3) by $1 = \int_0^{2\pi} d\theta / 2\pi$ and replace the variables $z_x^0$ by $e^{-i\theta} z_x^0$. Finally, we shall perform the change of variables
\[ (\theta, a, z_x^0, \ldots, z_x^N, w) \rightarrow (z_x^0 \equiv e^{-a-i\theta} z_x^0, \ldots, z_x^N \equiv e^{-a-i\theta} z_x^N, w) \]  \hspace{1cm} (3.7)
The corresponding Jacobian is easily calculated to be
\[ e^{2\pi N} |z_x^0|^{-2} \det (H_0) \det \left( (\partial_{L_x^0} + \partial_{\varphi})^{1}(\partial_{L_x^0} + \partial_{\varphi})^{-1} \right) = e^{2\pi N} |z_x^0|^{-2} \det (H_0) \det \left( (\partial_{L_x^0} \partial_{L_x^0}^{1} + \partial_{\varphi} \partial_{\varphi}^{1})^{-1} \right) e^{2\pi / \pi} \int \bar{\varphi} R \]  \hspace{1cm} (3.8)
where $(H_0)^{\alpha \beta} = \frac{i}{2} \int_x <\eta_x^\alpha, \wedge \eta_x^\beta>$ and $R$ denotes the metric curvature of $\Sigma$. In the last line we have used the chiral anomaly to extract the explicit dependence of the determinants on $\bar{\varphi}$. Implementing all the above transformations in Eq. (3.3), we obtain
\[ \|\Psi\|^2 = \text{const.} \left( \frac{\det'((-\Delta))}{\text{area}^{1/2} \det (i m r^{-1})} \right) \int e^{-2\pi k / \int x} A_0^0 \wedge A_0^1 \int e^{-2\pi k (f r_0) / \int x} M^{\alpha \beta}_{\gamma \delta} \]  \hspace{1cm} (3.9)
where we have denoted:
\[ M^{\alpha \beta}_{\gamma \delta} \equiv \int x \kappa_x^\alpha \omega_i^j \wedge \eta_x^\beta j , \quad \psi(x, \zeta_x) \equiv \psi(x, i \zeta_x^0 (H_0^{-1})_{\alpha \beta} \eta_x^\beta j) . \]  \hspace{1cm} (3.10)
Observe that the zero mode $a$ of $\varphi$ has been completely absorbed into $\zeta_X$. The change of variables (3.7) is easy to invert. The relation

$$e^\varphi d = i \zeta_X^\alpha (H_0^{-1})_{\alpha \beta} \eta_X^{\beta \dagger} + \partial_{x^2} v$$

(3.11)

where $v \equiv e^\varphi w$ implies that

$$v = \partial_{x^2}^{-1} e^\varphi d \quad \text{and} \quad \zeta_X^\alpha = \int_{\Sigma} \eta_X^{\alpha} \land e^\varphi d.$$  

(3.12)

The $d$ integral is now straightforward. Since, by (3.12),

$$\frac{\xi}{\xi_d(y)} = \left( \frac{\partial}{\partial \xi} \eta_X^{\alpha}(y) + \frac{\xi}{\xi_0(\xi)} \partial_{x^2}^{-1}(\xi, y) \right) e^{\varphi(y)},$$

(3.13)

performing the $d$ integration in (3.9) gives:

$$||\Psi||^2 = \text{const. } \left( \frac{\text{det}(-\Delta)}{\text{area}^{1/2} \cdot \text{det}(A_0)} \right) e^{-\frac{it}{2\pi} \int_{\Sigma} \text{tr} A_0^0 \land A_0^1} \int e^{-2\pi k f(x)} \frac{1}{\text{Im}^{1/2}} (f_{x_0} \omega)$$

$$\cdot \text{det} \left( \frac{\xi}{\xi_d(y)} \text{vol} \right)^{-1} \text{det}(\partial_{x^2} \partial_{x_2}) \text{ exp} \left[ \int_{\Sigma} \left( \frac{\partial}{\partial \xi} \eta_X^{\alpha}(y) \right) e^{\varphi(y) \left( \eta_X^{\beta \dagger}(y) \partial_{x^2} \eta_X^{\alpha} + \int_{\Sigma} \partial_{x^2}^{-1}(y, \xi') \frac{\xi}{\xi_0(\xi')} \right) \right]_{\zeta=0}$$

$$\cdot \left| \sum_{j=1}^g (-1)^j \text{det} \left( \zeta_X^\alpha \left( H_0^{-1} \right)_{\alpha \beta} M r_{ij}^{\beta \dagger}, \omega_j(x) \right) \right|^2 \left( \int_{\Sigma} |\psi(x, \zeta_X)(e^{\varphi(\xi)} u_j + v(\xi))|^2$$

$$\cdot \prod_l \frac{e^{-2j} \varphi(\xi(\xi)) d^2 u_l}{(1 + |u_l|^2)^{2j+2}} \right) e^{(k+2) f(x)} \frac{1}{\text{Im}^{1/2}} \int_{\Sigma} \varphi(\xi, \xi_0(\xi_0) - 2R) D\varphi.$$  

(3.14)

The $u_l$-integral may be written in the following form:

$$\int_{\Sigma} |\psi(x, \zeta_X)(e^{\varphi(\xi)} u_l + v(\xi))|^2 \prod_l \frac{e^{-2j} \varphi(\xi(\xi)) d^2 u_l}{(1 + |u_l|^2)^{2j+2}}$$

$$= \sum_p \left| \tilde{\psi}_p(x, \zeta_X)(v(\xi)) \right|^2 \prod_l e^{-2(\xi - p_0) \tilde{\varphi}(\xi)}$$

(3.15)

where $\tilde{\psi}_p(x, \zeta_X)(v(\xi))$ is a homogeneous function of $\xi_X^\alpha$ and $v(\xi)$ of degree $k(\gamma - 1) + P$ with $P \equiv \sum_i p_i$. Inserting this decomposition into Eq. (3.14) and denoting $(k+1)(\gamma - 1)$ by $M$, we obtain

$$||\Psi||^2 = \text{const. } \left( \frac{\text{det}(-\Delta)}{\text{area}^{1/2} \cdot \text{det}(A_0)} \right) e^{-\frac{it}{2\pi} \int_{\Sigma} \text{tr} A_0^0 \land A_0^1} \int e^{-2\pi k f(x)} \frac{1}{\text{Im}^{1/2}} (f_{x_0} \omega)$$

$$\cdot \text{det} \left( \frac{\xi}{\xi_d(y)} \text{vol} \right)^{-1} \text{det}(\partial_{x^2} \partial_{x_2}) \text{ exp} \left[ \int_{\Sigma} \left( \frac{\partial}{\partial \xi} \eta_X^{\alpha}(y) \right) e^{\varphi(y) \left( \eta_X^{\beta \dagger}(y) \partial_{x^2} \eta_X^{\alpha} + \int_{\Sigma} \partial_{x^2}^{-1}(y, \xi') \frac{\xi}{\xi_0(\xi')} \right) \right]_{\zeta=0}$$

$$\cdot \left| \sum_{j=1}^g (-1)^j \text{det} \left( \zeta_X^\alpha \left( H_0^{-1} \right)_{\alpha \beta} M r_{ij}^{\beta \dagger}, \omega_j(x) \right) \right|^2 \left| \tilde{\psi}_p(x, \zeta_X)(v(\xi)) \right|^2$$

(3.16)
\[ \prod_l e^{-2(j_l-p_l)} \bar{\varphi} (\xi_l) e^{i(k+2) \bar{\varphi}(\partial \bar{\varphi} - 2F_0)} e^{i \frac{1}{2\pi} \int_x \bar{\varphi} R} D\bar{\varphi}. \] (3.16)

We are still left with the Gaussian integration over \( \bar{\varphi} \). It is of the form

\[
\int \exp \left[ \sum_{m=1}^{M+J-P} 2 \bar{\varphi}(y_m) - \sum_l 2(j_l-p_l)\bar{\varphi}(\xi_l) + \frac{i(k+2)}{2\pi} \int \bar{\varphi}(\partial \bar{\varphi} - 2F_0) + \frac{i}{2\pi} \int_x \bar{\varphi} R \right] D\bar{\varphi} \equiv \int \exp \left[ - \frac{i}{\epsilon} \int \bar{\varphi} \sigma + \frac{i(k+2)}{2\pi} \int \bar{\varphi} \sigma \right] D\bar{\varphi}
\]

\[
= \text{const.} \ \text{det}'(-\Delta)^{-1/2} e^{\frac{i}{\epsilon(\xi_2-x)} \int_x \bar{\varphi} \sigma(x,y)} G(x,y) \sigma(y),
\] (3.17)

where \( \sigma \equiv \frac{i(k+2)}{2\pi} F_0 + \frac{i}{\epsilon} R - 2i \sum_l (j_l-p_l)\delta(\xi_l) + 2i \sum_m \delta y_m \) and \( G(x,y) \) is a Green function of the Laplacian \( \Delta \) on \( \Sigma \). Note that \( \int_x \sigma = 0 \). For convenience, we shall choose \( G(\cdot,\cdot) \) so that \( \int_x G(x,\cdot)((k+2)F_0+R/2) = 0 \). Then the right hand side of Eq. (3.17) becomes

\[
\text{const.} \ \text{det}'(-\Delta)^{-1/2} \exp \left[ - \frac{i}{\epsilon(\xi_2-x)} \sum_{m_1,m_2} G(y_{m_1}, y_{m_2}) \right.
\]

\[
- \sum_{m_l,j_l} (j_l-p_l)G(y_{m_l}, \xi_l) + \sum_{i_l,i_2} \sum_{j_l,j_2} (j_l-p_l)(j_2-p_l)G(\xi_l, \xi_2) \right].
\] (3.18)

Note the divergent contribution from coinciding points due to the short distance singularity \( G(x,y) \approx \frac{1}{\epsilon} \ln \text{dist}(x,y) \). These divergences may be regularized by Wick ordering, i.e. by splitting the coinciding points at distance \( \epsilon \), extracting the most divergent factor \( e^{-\frac{i}{\epsilon(\xi_2-x)} \sum_l \frac{1}{2} (j_l+1)G_x} \) appearing in the term with \( p = 0 \) and taking the limit \( \epsilon \to 0 \). The less singular terms with \( p \neq 0 \) will be annihilated by this multiplicative renormalization. As the result, one obtains the following scalar product formula:

\[
\|\Psi\|^2 = \text{const.} \ \frac{1}{\text{det}(\text{Im} \tau)} \left( \frac{\text{det}'(-\Delta)}{\text{area}} \right)^{1/2} e^{-\frac{i}{\epsilon} \int \bar{\varphi} \sigma} \text{tr} A_0^{\text{vol}} \prod_{l_1 \neq l_2} \int_x \bar{\varphi}(\xi_1) G(\xi_1, \xi_2)
\]

\[
\times \prod_l \int_x \bar{\varphi}(\xi_l) G(\xi_l,\xi_2) e^{-\frac{i}{\epsilon} \int \bar{\varphi} \sigma} \text{tr} A_0^{\text{vol}} \prod_{l_1 \neq l_2} \int_x \bar{\varphi}(\xi_1) G(\xi_1, \xi_2)
\]

\[
\times \prod_{m=1}^{M+J} \left( \delta \frac{\partial G_x}{\partial y} (y_m) + \int_\Sigma \frac{\delta}{\delta y} (G_x - \delta^{-1} y_m) \right) \left( \delta \frac{\partial G_x}{\partial x} (y_m) + \int_\Sigma \frac{\delta}{\delta x} (G_x + \delta^{-1} y_m) \right)
\]

\[
\times \left( \prod_{j=1}^l (-1)^j \text{det} \left( \frac{\partial G_x}{\partial y} (H_0^{-1})_{\alpha\beta} \left( \delta \frac{\partial G_x}{\partial y} \right)_{\alpha\beta} \right) \right)
\]

\[
\times \prod_{m_1 \neq m_2} \int_x \bar{\varphi}(y_{m_1}, y_{m_2}) G(y_{m_1}, y_{m_2}) e^{-\frac{i}{\epsilon} \int \bar{\varphi} \sigma} G(y_{m_1}, y_{m_2}) e^{-\frac{i}{\epsilon} \int \bar{\varphi} \sigma} G(y_{m_1}, y_{m_2})
\] (3.19)

where \( G(y, y) := \lim_{y' \to y} (G(y, y') - \frac{1}{\epsilon} \ln \text{dist}(y, y')) \).

Eq. (3.19) generalizes the integral formula (9.5) of \cite{1} for the scalar product of CS states to the case with insertions of the Wilson lines. It reduces to the latter expression if no lines are present. The integral in (3.19) is over the modular parameter \( x \) running through a fundamental domain of \( \Sigma \) and over the positions \( y_m \in \Sigma \) of \( (k+1)(\gamma-1) + \sum_l j_l \equiv M + J \) screening charges. As in the case with no insertions, we expect the integrals to
converge only if $\psi$ describes a (globally defined) CS state. The resulting scalar product should define the projectively flat Knizhnik-Zamolodchikov-Bernard connection [3][4] on the holomorphic bundle whose fibers are formed by the spaces of CS states.

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References


