Index formulas for geometric Dirac operators in Riemannian Foliations

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INDEX FORMULAS FOR GEOMETRIC DIRAC OPERATORS IN RIEMANNIAN FOLIATIONS

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ABSTRACT. With regards to certain Riemannian foliations we consider Kasparov pairings of leafwise and transverse Dirac operators. Relative to a pairing with a transversal class we commence by establishing an index formula for foliations with leaves of non-positive sectional curvature. The underlying ideas are then developed in a more general setting leading to pairings of images under the Baum-Connes map in geometric K-theory with transversal classes. Several ideas implicit in the work of Connes and Hilsum-Skandalis are formulated in the context of Riemannian foliations. From these we establish the notion of a dual pairing in K-homology and a theorem of Grothendieck-Riemann-Roch type.

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0. Introduction

Index theory was born in the mid 1960's with the celebrated theorem of Atiyah and Singer which responded to a question raised by Gelfand a few years earlier. Elliptic operators on compact manifolds without boundary define Fredholm operators which have an integer-valued index. Gelfand observed that this index depended on topological properties of the operator and the Atiyah-Singer theorem provides an explicit formula for the index in terms of these properties. Since then, generalizations of this result have been sought for more general classes of operators.

One line of investigation, begun by Atiyah [1], concerns differential operators which are only partially elliptic but that are elliptic in the directions transverse to the action of a compact group. Also, the operator is assumed to be invariant under this action. Atiyah showed that the index in this case, while no longer an integer, could be viewed as a distribution. To obtain a numerical-valued index one had to introduce an auxiliary function. Moreover, he was able to exhibit specific properties of the index distribution in special cases.

A related situation was investigated by Connes in [11] [10] [12], where he considered differential operators that are transversally elliptic relative to a foliation of a compact manifold. In Connes' approach, the index appears as a complex trace which in some cases is again integer-valued but to understand how this is possible we need to view the index as a K-homology class with the numerical index resulting from a pairing.

An elliptic operator on a compact manifold $M$ yields a class in the K-homology of $C(M)$ [15] (see also [6] in the context of $spin^c$-structures) and pairing this class with the K-theory class of a vector bundle on $M$ yields an integer, the index of the operator with coefficients in the vector bundle. For foliations, Connes defined a $C^*$-algebra consisting of operators that are 'smoothing along the leaves of the foliation'. He could then show that a transversally elliptic operator defined a class in the K-homology of the foliation algebra which can be paired with classes in the K-cohomology of the foliation algebra to yield an integer. This now makes it possible to seek an explicit index formula for transversally elliptic operators (cf. [31] [33]).

The challenges in obtaining a concrete formula are formidable since we must translate properties of the foliation and the transversally elliptic operator into homological objects that can be paired with cohomological objects obtained from the K-theory of the foliation algebra. In this paper we consider a Riemannian foliation $(M, \mathcal{F})$ on a compact Riemannian manifold $M$ and combine the geometric features of the foliation with the relevant operator algebra techniques. Instrumental here is the Kasparov KK-theoretic method which has been previously under-exploited for the case of Riemannian foliations. The geometric Dirac operators of the title are the longitudinal and transverse Dirac operators relative to $(M, \mathcal{F})$ which have been considered by Connes and his group, Baum and Connes [3] [4] [5], Hilsum and Skandalis [22] Vergne [31], as well as a number of other authors.

On taking the K-homology classes of such operators, we aim to establish an index formula which arises via a Kasparov pairing of a longitudinal class with a transversal class. In §3 we establish such a formula for a special class of foliations, where the K-theory class is taken to be the index of a 'test' longitudinally elliptic operator. This allows us to bypass an explicit knowledge of the
K–theory of the foliation algebra. Although the question of how much of the K–theory we are able to pair with depends on the outcome of the Baum-Connes conjecture, we do obtain an explicit index formula for all geometrical classes. Such a formula may be viewed, in a sense, as a ‘transverse index formula’ and as such, it is more concrete than viewing the transverse index in its distributional form. The main point being that the Atiyah-Singer Index Theorem can be implemented to capture the ingredients of the longitudinal and transverse geometry of the foliation.

The ‘test’ case of § 3 motivates our considerations at the more general level of pairings of KK–classes of transversally elliptic operators with images under the Baum-Connes map. The explicit details of this construction are outlined in § 4. In § 5 we draw upon the more general setting of Hilsum and Skandalis [22] to describe the geometric topological K-theory $K^*_\text{top}(M,F)$ in a more geometrical context applicable to the foliation $(M,F)$. This leads to the notion of a dual pairing in K–homology in terms of the symbol of a transversally elliptic operator. The underlying principles are shown to be suitably exemplified in terms of the discussion and results of § 3 and § 4. Furthermore, we obtain a transversal Grothendieck-Riemann-Roch theorem for K–oriented maps (§ 5). Similar ideas relating to the nature of the KK–pairings which we describe here in the foliation context, have been considered in [14] [17] [18] and [20].

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1. Foliations: The leafwise and transversal Dirac operators

Let $M$ be a compact oriented $n$-dimensional Riemannian manifold and let $F$ be an oriented Riemannian foliation on $M$ of $\dim F = p$ even and $\operatorname{codim} F = n - p = q$. Denoting by $TF$ the tangent bundle along the leaves and by $Q$ the normal bundle to $F$, we have the exact sequence

$$0 \longrightarrow TF \longrightarrow TM \longrightarrow Q \longrightarrow 0. \quad (1.1)$$

For a Riemannian foliation the metric $g_M$ on $M$ is said to be bundle-like, giving an identification of $(TF)^\bot \cong Q$ and $g_M = g_L + g_Q$, where the transverse metric $g_Q$ satisfies the condition of holonomy invariance (for references to Riemannian foliations see e.g. [25] [27]). We shall occasionally use the letters $L$ and $T$ as subscripts to denote objects ‘leafwise’ (or ‘longitudinal’) and ‘transverse’ relative to $F$ respectively.

In the following sections we make the assumption that both $Q$ and $TF$ are endowed with a spin$^c$-structure. Then the above identification shows immediately that $TM$ also inherits this structure where we have elected to take the Clifford structure on $T^*M$ as

$$c(T^*M) \cong c(T^*F) \otimes c(Q^*). \quad (1.2)$$

Thus if $S_M$ is taken to denote the Clifford bundle associated to the spin$^c$-structure of $TM$, we consider the Dirac operator

$$D_M : C^\infty_c(S_M) \longrightarrow C^\infty_c(S_M) \quad (1.3)$$

whose symbol $\sigma_M$ is prescribed below. Such an operator defines a KK-class $[D_M] \in KK(C(M),\mathbb{C})$ where $C(M)$ denotes the C*-algebra of $M$ [6] [7].

As for the C*-algebras relative to $F$, in the first instance we take $C^*(M/F) = C^*(G_c)$ to denote the C*-algebra of the fundamental groupoid $G_c$ of $F$ as described by Baum and Connes in [5]. The other case is to take $C^*(M/F)$ as the C*-algebra [10], over the graph $G$ or holonomy groupoid of $F$. 

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In both cases we have the maximal and reduced representations of these algebras taken wherever these are appropriate. In order to simplify matters, we shall often denote both of these groupoids by the same symbol $G$ and make the distinction whenever necessary. Since $F$ is a Riemannian foliation on a compact manifold, it follows from [32] that on the groupoid $G$ (dim $G = m + p$) we have the fibrations

$$G \xrightarrow{r,s} M$$

where $r$ and $s$ are respectively the range and source maps which satisfy $r([m, \gamma, m']) = m$ and $s([m, \gamma, m']) = m'$ for $[m, \gamma, m'] \in G$, where $m'$ lies on a leaf $L_m$ through $m \in M$ and in the case of the holonomy groupoid (fundamental groupoid) $\gamma$ is a holonomy (homotopy) class of paths joining $m$ to $m'$. Thus the fibres in (1.4) are the holonomy coverings (universal coverings) of the leaves of $F$.

In terms of a Haefliger cocycle $\{U_i, g_{ij}, \gamma_{ij}\}$ with $g_i : U_i \rightarrow T_i$ Riemannian submersions and $T_{i,j} = g_i(U_{i,j}) \subset T_i$ (see [19]), the transition isometries $\gamma_{ij} : T_{j,i} \cong T_{i,j}$, satisfying $g_i = \gamma_{ij} \circ g_j$ on $U_{i,j}$; must preserve (by our assumptions) a spin$^c$-structure on the tangent bundle of the transversal manifold $T = \coprod_i T_i$ (the $T_i$ may of course be viewed as transversal discs in the foliation charts $U_i$).

The data may be summarized in the following diagram:

$$\begin{array}{cccc}
\cdots & \longrightarrow & \coprod_{i,j} U_{i,j} & \longrightarrow & \prod_i U_i & \longrightarrow & M \\
\downarrow{(\beta_i, U_i)} & & \downarrow{(\beta_i)} & & \downarrow{g_0} & \\
\cdots & \longrightarrow & \coprod_{i,j} T_{i,j} & \longrightarrow & \prod_i T_i & \longrightarrow & M/F \\
\end{array}$$

where the top row

$$\mathcal{N}(\mathcal{U}) : \cdots \longrightarrow \coprod_{i,j} U_{i,j} \xrightarrow{\simeq} \prod_i U_i$$

is the nerve of the covering $\mathcal{U}$ whose realization is homotopy equivalent to $M$ and

$$\mathcal{T}(F) : \cdots \longrightarrow \coprod_{i,j} T_{i,j} \xrightarrow{\simeq} \prod_i T_i$$

is a semi-simplicial resolution of $M/F$ whose realization is homotopy equivalent to $BGd$.

In terms of (1.5), (1.7), the discrete holonomy groupoid $Gd$ is given by

$$Gd : \coprod_{i,j} T_{i,j} \xrightarrow{\simeq} \prod_i T_i$$

with $s : T_{j,i} \subset T_j$ and $r : T_{j,i} \xrightarrow{\gamma_{ij}} T_{i,j} \subset T_i$. Note also that $BGd \simeq BG$. This yields the commutative diagram (up to homotopy)

$$\begin{array}{ccc}
|\mathcal{N}(\mathcal{U})| & \xrightarrow{\simeq} & M \\
\downarrow{\tilde{g}_0} & & \downarrow{g_0} \\
BGd & \xrightarrow{b_F} & M/F
\end{array}$$

A smooth mapping (morphism) $f : M/F \rightarrow M'/F'$ is defined in [13] [22] in terms of its graph or, equivalently, by a cocycle from $G$ to $G'$. For our purposes it will be more convenient to view a morphism as a smooth map of diagrams (1.5) which induces a map of discrete groupoids (1.8) $\tilde{f} : Gd \rightarrow G'd$. This also produces in an obvious way a mapping $f : T(F) \rightarrow T(F')$ in
and thus an induced map $f: B\mathcal{G}_d \to B\mathcal{G}_d'$ in (1.9). Intuitively, a morphism is a leaf preserving smooth map $f: M \to M'$ which is compatible with holonomy. The vertical maps in (1.5) and (1.9) show, for instance, that the canonical projection $g_0: M \to M/F$ is a smooth map. Henceforth, such mappings $M/F \to M'/F'$, characterized as above, are assumed to be smooth.

Since the foliation $\mathcal{F}$ is Riemannian, we are assured the existence of a transverse invariant measure, so that the longitudinal Dirac operator $D_L$ is a self-adjoint leafwise elliptic operator $[28]$. Relative to each of the abovementioned foliation algebras, $D_L$ defines by $[13]$ a class

$$[D_L] \in KK(C(M), C^*(M/\mathcal{F})).$$  

(1.10)

Let $S_Q$ denote the Clifford module over $Q$ associated to its $spin^c$-structure. Here we assume that the principal $spin^c$-frame bundle of $Q$ carries a foliated structure compatible with that of the SO-frame bundle $F_{SO}(Q)$. The geometric Dirac operator which we consider in the transversal directions is the invariant transverse Dirac operator

$$D_T: C^\infty_c(S_Q) \to C^\infty_c(S_Q)$$

whose principal symbol $\sigma_T$ regarded as a symbol map

$$\sigma_T: Q \to \text{End}(S_Q)$$

given by Clifford multiplication, is holonomy invariant as a consequence of $\mathcal{F}$ being Riemannian and the above assumption. On the holonomy groupoid $\mathcal{G}$ this condition amounts to $r^*(\sigma_T) = s^*(\sigma_T)$. It follows that the procedure described in $[11]$ $[22]$ can be applied to construct an element of $KK(C^*(M/\mathcal{F}), \mathbb{C})$ relative to $C^*(M/\mathcal{F})$ taken to be the maximal $C^*$-algebra of the foliation. We shall denote the corresponding class by $[D_T]$.

We shall denote by $\hat{\otimes}$ tensor products over the $C^*$-algebras in question as well as for admissible objects $x, y$ their Kasparov pairing $x \hat{\otimes} y$ (following the notation used in e.g. [7]). Relative to the above Clifford module structures, we take the cup-product pairing of symbols as described in $[13]$ $[22]

$$\sigma_M = J^{1/2} \sigma_L \otimes 1_T + (1 - J)^{1/2} 1_L \otimes \sigma_T$$

(1.12)

where $J$ is a continuous function with values in $[0, 1]$. Then taking $D_M$ with the cup-product symbol $\sigma_M$ as above and observing (1.2), we infer:

1. The transverse symbol of $D_M$ is given by

$$\sigma_M \mid_T = 1_L \hat{\otimes} \sigma_T$$

and

2. $[\sigma_L \hat{\otimes} 1_T, \sigma_M] \geq 0$.

Then it follows from $[22]$ Theorem A.11, on reducing the case of a triple foliation to a single one in the obvious way, that (up to operator-homotopy equivalence)

$$[D_M] = [D_L] \hat{\otimes} [D_T].$$

(1.13)

In § 4 we shall state a more general result (Proposition 4.1) from which (1.13) may be seen to be a special case. The pairing in (1.13) will eventually lend itself to establishing an index formula under some extra hypotheses.
2. Construction of the dual Dirac class

To proceed, we consider a special situation which will motivate the more general considerations in the later sections. We assume in this section and §3 that the leaves of $\mathcal{F}$ are complete Riemannian manifolds of non-positive sectional curvature. This condition is reflected on the fundamental groupoid $\mathcal{G} = \mathcal{G}_\pi$ as follows. The leafwise metric $g_L$ at each point lifts to give a Riemannian metric with those same properties on $r^{\perp 1}(m)$ noting that $s : r^{\perp 1}(m) \to L_m$ is onto and $r^{\perp 1}(m)$ is the universal cover of $L_m$. In this way, $r^{\perp 1}(m)$ itself becomes a complete simply connected Riemannian manifold of non positive sectional curvature. Rescaling if necessary, the assumptions imply that for points $a, b \in r^{\perp 1}(m)$ with geodesic distance $d(a, b) < 1$ with respect to $r^*_L g_L$, there is a unique tangent vector $X_{(a, b)} \in T_a(r^{\perp 1}(m))$ with $\|X_{(a, b)}\| < 1$ and $exp_a X_{(a, b)} = b$. In other words, we can find a unique geodesic between $a$ and $b$ with $\|d(a, b)\| < 1$ [21].

For a small compact ball $B(z_0, r_0)$ centered at $z_0 \in r^{\perp 1}(m)$ of radius $r_0$, we define a $C^\infty$ function $\theta$ which satisfies

$$
\theta(z) = \begin{cases} 
0 & \text{for } z \in B(z_0, r_0/2) \\
1 & \text{for } z \in B(z_0, r_0).
\end{cases}
$$

(2.1)

If $X(z_0, z)$ is taken to be the unit tangent vector to the unique geodesic joining $z_0$ to $z$, then we shall write $X(z_0, z) = \theta(z)X(z_0, z)$. Note that as a consequence of the curvature condition on the leaves, the function $z \mapsto ||X(z_0, z) - X(z_1, z)||$ converges to 0 as $z \to \infty$ with convergence uniform on compact subsets. This technical point assures the continuity of the Hilbert fields and corresponding endomorphisms (see e.g. [16]).

Let $S_L$ denote the Clifford bundle associated to the $spin^c$-structure of $T\mathcal{F}$ and let $S = r^*S_L$. We shall describe how to obtain a $C^*$-module over $C(M)$ provided by a continuous field $(H_m)_{m \in M}$ of Hilbert spaces on $M$. Setting $S_m = S |_{r^{-1}(m)}$, we let

$$
H_m = L^2(r^{\perp 1}(m)) \otimes S_m.
$$

(2.2)

Recall from [10] the representation

$$
\Pi_m : C^\infty_c(\mathcal{G}) \longrightarrow L^2(r^{\perp 1}(m))
$$

(2.3)

where

$$
\Pi_m(\phi)f(m, m'') = \int_{r^{-1}(m)} \phi(m, m')f(m', m'') \, dm''.
$$

From the completion of $C^\infty_c(\mathcal{G})$ in its regular representation, we obtain the representation

$$
\Pi_m : C^*(\mathcal{G}_\pi) \longrightarrow L^2(r^{\perp 1}(m))
$$

(2.4)

which by our assumptions on $T\mathcal{F}$, uniquely defines a representation

$$
\Pi_m^\pm : C^*(M/\mathcal{F}) \longrightarrow H_m^\pm = L^2(r^{\perp 1}(m)) \otimes S_m^\pm
$$

(2.5)

such that for all $\zeta \in C^*(M/\mathcal{F})$, we have

$$
\Pi_m^\pm(\zeta)(\xi_1 \otimes \xi_2) = \Pi_m(\zeta)\xi_1 \otimes \xi_2.
$$

This then leads to a $\star$-homomorphism

$$
\Pi^\pm : C^*(M/\mathcal{F}) \longrightarrow \mathcal{L}(H_m^\pm).
$$

(2.6)
We proceed along lines similar to those in [10] [16]. For all \( \xi \in \mathcal{C}_c^\infty (\mathcal{G}, S) \), we set

\[
\xi_m = \xi |_{\nu^{-1}(m)} \varepsilon_z
\]

(2.7)

where for \( m \in M \), \( z \in r^{-1}(m) \), \( \varepsilon_z \) is a continuous section of the Clifford bundle satisfying \( \varepsilon_z \varepsilon_z^* = -1 \) and \( \varepsilon_z = \varepsilon_z^* \). In this way we obtain sections which permit us to endow \( H_m \) with a unique structure of a continuous field of Hilbert spaces on \( M \).

Let \( \mathcal{H} \) denote the \((\mathbb{Z}_2,\text{graded}) \) \( C(M) \)-module of continuous sections of \((H_m)_{m \in M} \) which is a Hilbert module for the scalar product

\[
\langle \xi, \eta \rangle_{C(M)} (m) = \langle \xi_m, \eta_m \rangle
\]

\[
= \int_{r^{-1}(m)} \langle \xi(z), \eta(z) \rangle_{s_m} \, dz .
\]

For each \( m \in M \), let

\[
\Delta_m : H^+_m \to H^-_m
\]

be defined as follows: for each \( \xi \in H_m \), define \( \Delta_m \) by

\[
(\Delta_m \xi)(z) = c(X_m(z)) \xi(z),
\]

(2.9)

where for \( z \in r^{-1}(m) \), \( X_m(z) \) is the vector in \( T_m \mathcal{F} \) satisfying

\[
\text{exp}(X_m(z)) = s(z)
\]

for which \( X_m(z) = \theta(z)X_m(z) \) and \( c \) denotes Clifford multiplication.

For \( \zeta \in \mathcal{C}_c^\infty (\mathcal{G}, \Omega^1_+ \mathcal{F}) \) a smooth \( \frac{1}{2} \)-density and for each \( m \in M \), \( \Pi^\pm_m(\zeta) \) is a smoothing operator in \( H^\pm_m \). Observing that \( z \mapsto \|X(z)\|^2 - 1 \) defines a proper map \( r^{-1}(m) \to \mathbb{R} \), consider the operators

\[
\Pi^+_m(\zeta)(\Delta_m^* \Delta_m - 1)
\]

\[
\Pi^-_m(\zeta)(\Delta_m^* \Delta_m - 1)
\]

\[
\Pi^+_m(\zeta)\Delta_m - \Delta_m^* \Pi^-_m(\zeta) .
\]

Given \( \epsilon > 0 \) and \( C > 0 \), there exists by assumption, a compact subset \( K \subset r^{-1}(m) \), such that for \( z_1, z_2 \in r^{-1}(m) - K, d(z_1, z_2) \leq C \) we then have

\[
\|X_m(z_1) - X_m(z_2)\| \leq \epsilon .
\]

Modulo compact operators, \( \Delta_m \) intertwines the representation \( \Pi^\pm_m \) of \( C^*(M/\mathcal{F}) \) and the above operators are in \( K(\mathcal{H}) \). Thus we obtain the Kasparov bimodule \( (\mathcal{H}, \Delta) \) defining the dual Dirac class

\[
[\hat{D}] \in KK(C^*(M/\mathcal{F}), C(M))
\]

(2.10)

(‘l'élément de Kasparov-Misčenko’ in the terminology of [16]).

Let the identity elements in \( KK(C(M), C(M)) \) and \( KK(C^*(M/\mathcal{F}), C^*(M/\mathcal{F})) \) be denoted by \( 1_M \) and \( 1\mathcal{F} \) respectively. Taking \( [D_L] \in KK(C(M), C^*(M/\mathcal{F})) \), it is asserted in [5] that

\[
[D_L] \otimes [\hat{D}] = 1_M
\]

(2.11)

although a proof does not seem to have been given explicitly in our context (cf [16]). On the other hand, it seems to be known that \( [\hat{D}] \otimes [D_L] \neq 1\mathcal{F} \) in general [29], but

\[
[D_L] \otimes [\hat{D}] = \nu
\]

(2.12)

is an idempotent in \( KK(C^*(M/\mathcal{F}), C^*(M/\mathcal{F})) \) by (2.11).
3. Pairing with the dual Dirac class

The assumptions on $F$ made in § 2 carry over to this section. Let us write

$$\alpha = [D_L] = [(\mathcal{E}, F)] \in KK(C(M), C^*(M/F))$$

where $(\mathcal{E}, F)$ is the associated Kasparov bimodule (see [13]). Here, $\mathcal{E}$ is the ($\mathbb{Z}_2$-graded) Hilbert $C^*(M/F)$-module defined as the completion of $C_\infty^*(G, \Omega^1)$ with respect to the inner product

$$\langle \xi, \eta \rangle(\gamma) = \int_{\gamma_1 \gamma_2 = \gamma} \xi(\gamma_1) \eta(\gamma_2)$$

for $\gamma_1, \gamma_2, \gamma \in \mathcal{G}$. Likewise, we write

$$\beta = [\hat{D}] = [(\mathcal{H}, \Delta)] \in KK(C^*(M/F), C(M)).$$

In view of the assumptions of the previous section, we now proceed to outline our proof of the assertion in [5]:

**Proposition 3.1.** (cf [16])

$$\alpha \otimes \beta = [D_L] \otimes [\hat{D}] = 1_M$$

**Proof.** In the usual way, let us take the pairing over $C^*(M/F)$ of the $\mathbb{Z}_2$-graded Hilbert modules

$$\mathcal{E} \otimes \mathcal{H} = \mathcal{N}.$$

We can write the operator $F$ corresponding to $\alpha$ above, as $F = g(D_L)$ where $g$ is a smooth real-valued function on $\mathbb{R}$ satisfying $\lim_{x \to -\infty} g(x) = 1$ and $\lim_{x \to +\infty} g(x) = -1$. Then it is a straightforward matter to check that

$$\mathcal{E} \otimes \mathcal{H} \cong \{S_m \otimes S_m\}_{m \in M}$$

is an isomorphism of $(C(M), C(M))$-bimodules.

For $m \in M$ and $z \in r_{+1}(m)$ a path along a leaf with endpoint $m$, the bimodule action on both sides is given by

$$((\xi_m \otimes \eta_m)f)(z) = \xi_m(z) \otimes \eta_m(z)f(m) \quad (3.1)$$

$$(f(\xi_m \otimes \eta_m))(z) = f(z)\xi_m(z) \otimes \eta_m(z) \quad (3.2)$$

where $f \in C(M)$ (by abuse of notation, we have retained $f$ when pull-backs are understood). Let $z_m$ be a canonical origin in $r_{+1}(m)$ given by the trivial path. For $z \in r_{+1}(m)$ and $\lambda \in [0, 1]$, we take $w_\lambda(z, m)$ to denote the unique point of the geodesic on the covering of the leaf containing the path from $m$ to $s(z)$ satisfying

$$(1 - \lambda)d(w_\lambda(z, m), z_m) = \lambda d(w_\lambda(z, m), z).$$

Now, a homotopy of the bimodule action is obtained by replacing $f(z)$ by $f(w_\lambda(z, m))$ above (thus for $\lambda = 1$, we have $f(w_\lambda(z, m)) = f(m)$).

Consider now the pseudodifferential operator of order 0 on $S_m \otimes S_m$, defined by

$$G = \begin{bmatrix} 0 & G_{+1,m} \\ G_{-1,m} & 0 \end{bmatrix} = 1 \otimes \Delta_m + (1 - 1 \otimes \Delta_m^2)^{1/2} F \otimes 1. \quad (3.3)$$
For all \( \lambda \), it can be checked that \( G \) is a \( \Delta \)-connection on \( \mathcal{E} \) (see [13][16]). For \( \mathcal{E} \otimes \mathcal{H} \) we vary the \( C^* \)-algebra actions via \( \lambda \) to obtain the continuous family of bimodules \( (\mathcal{E} \otimes \mathcal{H})_\lambda \) and hence a continuous family

\[
(\alpha \otimes \beta)_\lambda \in KK(C(M), C(M)).
\]

At \( \lambda = 0 \), we consider the pointwise action of \( G_{+,m} \) on \( L_m \) where \( T \mathcal{F} |_{L_m} \) is locally an \( \mathbb{R}^p \)-bundle. In which case, the \( O(p) \)-invariant operator \((D_L)_m \) yields a class \([(D_L)_m] \in K^0(C_0(L_m))\) and \( \Delta_m \) gives the Bott element \([(\Delta_m)] \in K(c_0(L_m))\). We have for the ordinary index

\[
\text{Ind } G_{+, m} = \text{Ind}((D_L(1 + D_L^{-1/2})_m \# \Delta_m) = 1
\]

and this together with the \( O(p) \)-invariance allows us to infer from [13]

\[
(\alpha \otimes \beta)_0 = (\alpha \otimes \beta)_1 = 1_M. \quad \Box
\]

In the following we shall denote by \( \mathbf{1} \) the class \([M \times \mathbb{R}] \in \text{KK}(\mathbb{C}, C(M))\) corresponding to the trivial line bundle. For a leafwise elliptic pseudodifferential operator \( D_L \) defining an element \([D_L] \in KK(C(M), C^*(M/\mathcal{F}))\), we define its index class by

\[
[\text{Ind } D_L] = \mathbf{1} \otimes [D_L].
\]

The following Proposition appears to be known in a broader generality but does not seem to have been recorded explicitly in the foliation literature:

**Proposition 3.2.** Let \([D_L] \in KK(C(M), C^*(M/\mathcal{F}))\) be the class of a leafwise elliptic pseudodifferential operator \( D_L \) on \( M \). Then we have

i) \([D_L] \otimes \nu = [D_L]\)

ii) \(\text{Ind } D_L] \otimes \nu = [\text{Ind } D_L].\)

In other words, \( \nu \) pairs as the identity on these classes.

**Proof.** Let \( \sigma_L \in K^1(T^*\mathcal{F}) \) be the principal symbol of \( D_L \). The assignment \([\sigma_L] \to [D_L]\) defines the symbol map [7]

\[
\Sigma : K^1(T^*\mathcal{F}) \to KK(C(M), C^*(M/\mathcal{F}))
\]

for which we have \( \Sigma([\sigma_L]) = \alpha \) and \( \Sigma([\sigma_L]) = [D_L] \). We also have the isomorphism

\[
\ell : K^0(M) \xrightarrow{\cong} KK(C(M), C(M))
\]

following which the diagram below is commutative

\[
\begin{array}{ccc}
K^0(M) \times K^1(T^*\mathcal{F}) & \xrightarrow{\ell \times \Sigma} & K^1(T^*\mathcal{F}) \\
\downarrow{\ell \times \Sigma} & & \downarrow{\Sigma} \\
KK(C(M), C(M)) \times KK(C(M), C^*(M/\mathcal{F})) & \longrightarrow & KK(C(M), C^*(M/\mathcal{F})).
\end{array}
\]

Following the leafwise Thom isomorphism [26]

\[
\Phi : \xrightarrow{\cong} K^0(M)
\]

(3.7)
we have \( \Phi([\sigma_L]) = 1_M \). Let us set \( \zeta = \Phi([\sigma_L]) \in K^0(M) \). From the commutativity of the diagram
\[
\begin{array}{ccc}
K^0(M) \times K^1(T^*F) & \longrightarrow & K^1(T^*F) \\
\downarrow_{1_M \times \Phi} & & \downarrow_{\Phi} \\
K^0(M) \times K^0(M) & \longrightarrow & K^0(M)
\end{array}
\]
we deduce that
\[
\zeta \hat{\otimes} [\sigma_L] = [\sigma_L] \in K^1(T^*F) .
\] (3.8)
Also, it follows that
\[
[D_L] = \Sigma([\sigma_L]) \\
= \Sigma(\zeta \hat{\otimes} [\sigma_L]) \\
= \zeta \otimes \Sigma([\sigma_L]) \\
= \zeta \otimes \alpha .
\]

In view of Proposition 3.1, computing the product
\[
[D_L] \hat{\otimes} \nu = [D_L] \hat{\otimes} (\beta \otimes \alpha) \\
= \zeta \otimes \alpha \hat{\otimes} (\beta \otimes \alpha) \\
= \zeta \hat{\otimes} 1_M \otimes \alpha \\
= [D_L]
\]
establishes i). Pairing both sides with the class \( 1 \in KK(\mathbb{C}, C(M)) \) then establishes ii). \( \square \)

**Remark 3.3.** Returning to Proposition 3.1, we recall that we employed the homotopy \((\alpha \hat{\otimes} \beta)_\lambda\) rather than \(\alpha_\lambda \hat{\otimes} \beta_\lambda\). For a family of groupoids \((\mathcal{G}_\lambda, r_\lambda, s_\lambda)\) as discussed in [10] relative to some special cases, the tangential groupoid construction yields \(\alpha_0 \hat{\otimes} \beta_0 = \alpha_1 \hat{\otimes} \beta_1 = 1_M\), but there it was not shown that \(\beta \hat{\otimes} \alpha\) is \(1_F\). As was the case in [10], consider a family of leafwise elliptic pseudodifferential operators (of order 0)
\[
(D_L)_\lambda : L^2(\mathcal{G}_\lambda, r_\lambda^* S_L) \longrightarrow L^2(\mathcal{G}_\lambda, r_\lambda^* S_L) .
\]
Setting \(\beta_\lambda = [D_L]_\lambda \hat{\otimes} \beta_\lambda\), one might then proceed to think of a modified homotopy of the sort
\[
\beta_\lambda \hat{\otimes} \alpha_\lambda = [D_L]_\lambda .
\] (3.9)
The problem would then be to show that for \(\lambda \in [0, 1]\), (3.9) is operator-homotopic to zero. For the particular situations described in [10] and in [30], where the leaf type is a rank 1 locally symmetric space of non-compact type, it does indeed turn out to be the case that \(\beta \hat{\otimes} \alpha = 1_F\) and Connes’ conjecture is true (see e.g. [26] Appendix A4 by Hurder for a discussion).

The operator \(D_L\) in Proposition 3.2 may be regarded as the “test” longitudinally elliptic operator mentioned in §0. Recall from the proof of Proposition 3.2 that we have
\[
[D_L] = \zeta \hat{\otimes} [D_L] = \zeta \hat{\otimes} \alpha
\] (3.10)
where we recall that \(\zeta = \Phi([\sigma_L]) \in KK(C(M), C(M))\). Pairing both sides with \(\beta\), it follows from Proposition 3.1 that
\[
\zeta = [D_L] \hat{\otimes} \beta .
\] (3.11)
Note that in $KK(C(M), C^*(M/F))$ we then have

$$\zeta \hat{\otimes} \alpha = [D_L] \hat{\otimes} \nu$$

(3.12)

(cf. the above remark).

At this stage we now wish to reintroduce the pairing

$$[D_M] = [D_L] \hat{\otimes} [D_T]$$

in (1.13), where we had considered $[D_L] \in KK(C(M), C^*(M/F))$ and $[D_T] \in KK(C^*(M/F), C)$ relative to the maximal algebra $C^*(M/F)$ over the holonomy groupoid. In view of having defined our dual Dirac class $[D] \in KK(C^*(M/F), C(\mathcal{M}))$ relative to $\mathcal{G}_\tau$, we consider a `shifting' class

$$\varphi \in KK(C^*(M/F), C^*(M/F))$$

between the two foliation algebras. For instance, taking $[D_L] \in KK(C(M), C^*(M/F))$, we can define $\varphi$ via the pairing $\varphi = \beta \hat{\otimes} [D_L]$. Alternatively, we can define $\varphi$ as the class in

$$KK(C^*(M/F), C^*(M/F))$$

determined via the canonical map $\mathcal{G}_\tau(\mathcal{F}) \to \mathcal{G}(\mathcal{F})$. Thus when the `test' class $[D_L]$ was taken to be in $KK(C(M), C^*(M/F))$, the pairing $[D_L] \hat{\otimes} \varphi$ gives then the class of $D_L$ in $KK(C(M), C^*(M/F))$.

In proceeding to our next result, we let $E \in K^0(M)$ be the complex vector bundle obtained from $\zeta$ in (3.11) by collapsing $C(M)$ in the first factor. To see this explicitly, we again pair with the class of the trivial line bundle

$$1 \hat{\otimes} \zeta = 1 \hat{\otimes} [D_L] \hat{\otimes} \beta$$

which in $KK(C, C(M))$, gives

$$[E] = [\text{Ind } D_L] \hat{\otimes} \beta.$$ 

(3.14)

**Theorem 3.4.** In $KK(C, C) \cong \mathbb{Z}$, we have

$$[\text{Ind } D_L] \hat{\otimes} [D_T] = \langle \text{ch}^*([\sigma_L]) \cup \text{ch}^*([\sigma_L]^{-1}) \cup \text{Td}(T\mathcal{F}) \cup \text{Td}(Q), [M] \rangle.$$ 

(3.15)

**Proof.** Commencing from (1.13) we apply $\beta \in KK(C^*(M/F), C(M))$ to both sides:

$$\beta \hat{\otimes} [D_M] = \beta \hat{\otimes} [D_L] \hat{\otimes} [D_T]$$

and then apply the shift class (3.13) defined by $\varphi = \beta \hat{\otimes} [D_L]$, to obtain

$$\beta \hat{\otimes} D_M = \varphi \hat{\otimes} [D_T].$$

Now pairing $[D_L] \in KK(C(M), C^*(M/F))$ with both sides, we obtain

$$[D_L] \hat{\otimes} \beta \hat{\otimes} [D_M] = [D_L] \hat{\otimes} \varphi \hat{\otimes} [D_T]$$

and so

$$\zeta \hat{\otimes} [D_M] = [D_L] \hat{\otimes} \varphi \hat{\otimes} [D_T] = [D_L] \hat{\otimes} [D_T]$$

where the shifting by $\varphi$ gives $[D_L] \in KK(C(M), C^*(M/F))$ and the last pairing is taken over $C^*(M/F)$. The left-hand-side now defines an element $[D_E] \in KK(C(M), C)$ where $D_E$ denotes
the Dirac operator $D_M$ twisted by the complex vector bundle $E$ given by (3.14). On pairing $[D_E]$ with the class $1 = [M \times \mathbb{R}]$ in $KK(\mathbb{C}, C(M))$, we obtain $\text{Ind } D_E$ and so

$$\text{Ind } D_E = 1 \hat{\otimes} [D_L] \hat{\otimes} [D_T] = [\text{Ind } D_L] \hat{\otimes} [D_T].$$

The Index Theorem gives

$$\text{Ind } D_E = (\text{ch}^*(E) \cup \text{Td}(M), [M]).$$

In order to calculate $\text{ch}^*(E) = \text{ch}^*(1 \otimes \zeta)$, we observe from (3.8) that

$$1 \otimes \zeta = \Phi([\sigma_L] \otimes [\sigma_L]^{11})$$

$$= \Phi([\sigma_L]) \cdot \Phi([\sigma_L]^{11})$$

in $K^0(M)$. From this we deduce that

$$\text{ch}^*(E) = \text{ch}^*([\sigma_L]) \cup \text{ch}^*([\sigma_L]^{11})$$

and since $TM = T\mathcal{F} \oplus Q$, the multiplicative property of the Todd class completes the formula. □

**Remark 3.5.**

1. The above formula would also apply to an arbitrary element $\tau \in KK(\mathbb{C}, C^*(M/\mathcal{F}))$ leading us to define $[E] = \tau \otimes 1$. However, $\text{ch}^*(E)$ would not be known explicitly in general.

2. Although we commenced with the assumption that $p = \dim \mathcal{F}$ is even, the preceding results apply equally well to the case of $p$ odd where $\mathcal{F}$ can be replaced by $\mathcal{F} \times I$ (as confirmed in e.g. [12]).

In § 4 and § 5 to follow, we will consider other $KK$-classes which can be represented geometrically. It would be desirable to obtain formulas as explicit as that of (3.15). Elements such as $\text{Ind}[D_L]$ can be seen to be obtained as images under the Baum-Connes map with respect to suitable data in the geometric $K$-theory of the foliation. In this more general setting we shall see how such images can be paired with transversal classes in $KK(C^*(M/\mathcal{F}), \mathbb{C})$ so as to produce similar index-theoretic results.

4. **Pairings with the transverse class and the Baum-Connes map**

In what follows, we turn to a more general situation where we need only assume that $Q$ is furnished with a spin$^c$-structure and the curvature condition on the leaves is relaxed. The results here apply when $C^*(M/\mathcal{F})$ is taken as the maximal $C^*$-algebra over the holonomy groupoid $\mathcal{G}$. References for this section are [3] [4] [10] and [12].

From the construction in § 1 it follows that the normal bundle of $(M, \mathcal{F})$ is defined naturally over all objects in diagram (1.9). In particular, there is a (simplicial) vector bundle $\tilde{Q}$ on $BG$ (associated to the universal $\mathcal{G}$-bundle $E\mathcal{G}$), corresponding to $Q$ on $M$. The **restricted geometric $K$-cohomology** is defined by

$$K^0_{res}(M, \mathcal{F}) := K_0(B\tilde{Q}, S\tilde{Q})$$

(4.1)

where $B$ and $S$ here denote the unit ball and sphere bundles respectively of $\tilde{Q}$ over $BG$. The definition of $K$-homology is that with compact supports. Elements in $K^0_{res}(M, \mathcal{F})$ are called **geometric cycles** and are realized by triples $(N, E, g)$ where

1. $N$ is a compact smooth manifold;
2. $E$ is a complex vector bundle on $N$;
3. $g : N \to M/\mathcal{F}$ is a $K$-oriented map.
Giving such a map $g$ is equivalent to defining a $G$-bundle on $N$ and hence a classifying map $g : N \to B\tilde{G}$ related by $g = b_G \circ g$. Recall then that $K$-oriented implies that $TN \oplus g^*\tilde{Q}$ is endowed with a $\text{spin}^c$-structure. This is equivalent to the existence of a $\text{spin}^c$-structure on $TN$ under our assumption that $Q$ has a foliated $\text{spin}^c$-structure.

Let $\mathcal{B}$ and $\mathcal{S}$ denote the unit ball and sphere bundles respectively of $g^*\tilde{Q}$ on $N$. The total space of $\mathcal{B}$ is a $\text{spin}^c$-manifold with boundary, so that by the isomorphism of Poincaré duality (denoted PD), a class $\xi \in K_0(\mathcal{B}, \mathcal{S})$ is assigned to the pull-back of $E$ to $\mathcal{B}$. We may then write

$$[N, E, g] = g_*(\xi) \in K_0(B\tilde{Q}, S\tilde{Q}) = K^0_{\text{res}}(M, \mathcal{F}).$$

Now we recall how the Chern character is defined in geometric $K$-theory. Let $H^0_c(B\tilde{G}, \mathbb{Q})$ be the even dimensional singular homology of $(B\tilde{Q}, S\tilde{Q})$ over $\mathbb{Q}$. Then the Chern character of $K^0(M, \mathcal{F})$ is defined as the following composition

$$ch_* : K^0_{\text{res}}(M, \mathcal{F}) \xrightarrow{\pi^*} H^0_c(B\tilde{G}, \mathbb{Q}) \xrightarrow{\Phi} H^*_c(B\tilde{G}, \mathbb{Q}),$$

where $\Phi$ denotes the Thom isomorphism. For the homology Chern character of $y = [N, E, g] \in K^0_{\text{res}}(M, \mathcal{F})$ we have the formula

$$ch_*(y) = ch_*(g_*(\xi))$$

$$= g_*(ch^*(E) \cup Td(TN \oplus g^*\tilde{Q}) \cap [N])$$

$$= Td(\tilde{Q}) \cap g_*(ch^*(E) \cup Td(N) \cap [N])$$

in $H^*_c(B\tilde{G}, \mathbb{Q})$ (cf [12]). This formula may also be expressed as

$$Td^{11}(\tilde{Q}) \cap ch_*(y) = g_*(ch^*(E) \cup Td(N) \cap [N]).$$

**Proposition 4.1.** If $N$ is a compact manifold and $f : N \to M/\mathcal{F}$ is a $K$-oriented submersion, then

$$[D_N] = f! \otimes [D_T]$$

(4.4)

where $f! \in KK(C(N), C^*(M/\mathcal{F}))$ and $D_N$ is the Dirac operator on $N$.

**Proof.** We recall how $f!$ is defined. The map $f$ is a $G$-valued cocycle $(O_i, f_i)$ over $N$, where $\{O_i\}$ is an open covering of $N$ and $f_{ij} : O_i \cap O_j \to G$ satisfies the usual condition $f_{ij}(x) \circ f_{jk}(x) = f_{ik}(x)$, for all $x \in O_i \cap O_j \cap O_k$. The map $f$ also induces a foliation on $N$ since $f$ is a submersion. Then $f!$ is defined via the pairing of $p_N! \in KK(C(N), C^*(N/f^*\mathcal{F}))$ and an element

$$e_f \in KK(C^*(N/f^*\mathcal{F}), C^*(M/\mathcal{F}))$$

where $p_N : N \to N/f^*\mathcal{F}$ is the natural projection and $f^*\mathcal{F}$ denotes the pull-back foliation by $f$. The element $e_f$ is described below.

As in [13] §4, let us denote by $G_f$ the glueing together of the open sets

$$\tilde{O}_i = \{(x, \gamma) \in O_i \times G, f_i(x) = r(\gamma)\}$$

by the map $\tilde{O}_i \to \tilde{O}_j$ defined by $(x, \gamma) \to (x, f_{ij}(x) \circ \gamma)$. Let $E_f$ be the Hilbert $C^*(M/\mathcal{F})$-module defined as the completion of $C^\infty_c(G_f, \Omega^{1/2})$ with respect to its $C^\infty_c(G, \Omega^{1/2})$-valued inner product

$$\langle \xi, \eta \rangle(\gamma) = \int_{\alpha \circ \gamma = \beta} \xi(\alpha)\eta(\beta).$$
The right-module structure is given by

\[(\xi h)(\gamma) = \int_{\alpha \circ \gamma = \beta} \xi(\alpha) h(\beta)\]

where \(\xi \in C_c^\infty(\mathcal{G}_f, \Omega^{1/2})\) and \(h \in C_c^\infty(\mathcal{G}, \Omega^{1/2})\). The left-module \(C_c^\infty(\mathcal{G}_N, \Omega^{1/2})\) action can be defined in a similar way, where \(\mathcal{G}_N\) is the graph of the pull back foliation on \(N\). Then we can define \(\epsilon_f\) via the Kasparov triple \((\mathcal{E}_f, \phi_1, 0)\) where \(\phi_1\) is the representation of \(C^*(N/f^*\mathcal{F})\) on \(\mathcal{E}_f\) defined by the left-module structure.

Taking pairings over \(C^*(M/\mathcal{F})\), we now proceed to establish the identity of bimodules

\[\mathcal{E}_f \otimes L^2(S_Q) \cong L^2(S_{Q_N})\]  \hspace{1cm} (4.5)

where \(S_{Q_N}\) is the normal bundle of the foliation on \(N\) and we have taken the metric on \(S_{Q_N}\) as the pull-back under \(f\) of the metric on \(S_Q\). We choose open coverings \(\{\Lambda_i\}\) of \(N\) and \(\{\Lambda_i'\}\) of \(M\) such that \(\Lambda_i = T_i \times I_i\) and \(\Lambda_i' = T_i \times I_i'\) are regular foliation charts where the \(T_i\) are transversals to the respective foliations and the \(I_i\) and \(I_i'\) are the corresponding plaques. Recall from e.g. [10] [14], the restriction \(C^*(M/\mathcal{F})|_I\) is isomorphic to the algebra of smoothing operators along the leaves. Let \(K(I_i')\) be the completion of smoothing operators on \(I_i'\). Since locally \(\mathcal{G}_f\) can be identified with the product \(T_i \times I_i \times I_i'\), a careful calculation involving integration over the the \(I_i\) coordinates, shows that

\[\mathcal{E}_f|_{\Lambda_i} \cong C_0(T_i) \hat{\otimes} L^2(I_i) \hat{\otimes} K(I_i')\]

We also have

\[L^2(S_Q)|_{\Lambda_i} \cong L^2(I_i') \hat{\otimes} L^2(I_i, S_Q)\]

With these identities, we now proceed to establish the isomorphism

\[C_0(T_i) \hat{\otimes} L^2(I_i) \hat{\otimes} K(I_i') \hat{\otimes} L^2(I_i') \hat{\otimes} L^2(T_i, S_Q) \cong L^2(I_i) \hat{\otimes} L^2(T_i, S_Q)\]  \hspace{1cm} (4.6)

where the second product on the left hand side of the identity is taken over \(C_0(T_i) \hat{\otimes} K(I_i')\) and we have used the identity \(K(I_i') \hat{\otimes} L^2(I_i') \cong \mathbb{C}\). Now this latter identity holds by virtue of the relationship

\[x \hat{\otimes} (u \hat{\otimes} v)y = x \hat{\otimes} (v, y)u = x(u \hat{\otimes} v) \hat{\otimes} y\]

where \(x \in C_0(I_i)\), \(y \in L^2(I_i')\) and \(u \hat{\otimes} v\) is a rank 1 operator in \(K(I_i')\) for \(u, v \in L^2(I_i')\). Since

\[L^2(S_{Q_N})|_{\Lambda_i} \cong L^2(I_i) \hat{\otimes} L^2(T_i, S_Q)\]

and any element in \(\mathcal{E}_f \hat{\otimes} L^2(S_Q)\) can be approximated by linear combinations of elements in

\[\bigcup_i \mathcal{E}_f|_{\Lambda_i} \hat{\otimes} L^2(S_Q)|_{\Lambda_i'}\]

the isomorphisms of (4.6) can be glued together to yield a global isometry \(\mathcal{E}_f \hat{\otimes} L^2(S_Q) \rightarrow L^2(S_{Q_N})\).

Let \(D_{T_N}\) be the transverse Dirac operator acting on \(S_{Q_N}\) with the pull back metric and let \([D_{T_N}]\) be the corresponding K-homology class. It is now easy to show that

\[\epsilon_f \hat{\otimes} [D_T] = [D_{T_N}]\]

It is enough to check the connection condition since \(\epsilon_f\) is represented by the 0-operator. Using a local argument this can be easily done by reducing matters to the case whereby \(N = T \times I\) and
\( M = T \times I' \), with leaves \( \{ t \} \times I \) and \( \{ t \} \times I' \). Then to establish our proposition it is enough to show that

\[ p_N! \hat{\otimes} [D_{T_N}] = [D_N]. \]

Recall that \( p_N! \) is represented by the longitudinal Dirac operator \( D_L \) on \( f^*F \) [13]. As before, the above pairing can be achieved by reducing matters to the local case. \( \square \)

Note that with the assumptions of §1, the pairing in (1.13) is reduced to a special case of the above proposition when \( M = N \) and \( g_0 : M \to M/F \) is the canonical map.

Recall from [3] [4] that the Baum–Connes map is a natural map

\[ \mu : K^0_{\text{res}}(M, F) \longrightarrow K_0(C^*(M/F)) \cong KK(C, C^*(M/F)) \]

(4.7)
given by the assignment of the geometric cycle \( y = [N, E, f] \in K^0_{\text{res}}(M, F) \) to its ‘Index class’ \( \text{Ind } y \). Specifically, \( \mu(y) \) is defined by the relationship \( f_! [E] = [E] \hat{\otimes} f! \).

With this in mind, we now proceed to establish:

**Theorem 4.2.** Let \( N \) be a compact manifold and \( f : N \to M/F \) a \( K \)-oriented mapping. Then for the geometric cycle \( y = [N, E, f] \in K^0_{\text{res}}(M, F) \), we have

\[ \mu(y) \hat{\otimes} [D_T] = \langle \text{ch}_*(y), \text{Td}^{11}(\tilde{Q}) \rangle = \text{Ind}(D_{N,E}) . \]

(4.8)

**Proof.** We follow along the lines of [13] Proposition 4.13. For such a triple \( y = [N, E, f] \), we can always take a factorization \( f = p \circ e \circ j \) where the maps are indicated in the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{e} & M/F \times \mathbb{R}^N \\
\uparrow & & \downarrow p \\
N & \xrightarrow{j} & M/F .
\end{array}
\]

(4.9)

Here \( X \) is a smooth manifold, \( j \) is a \( K \)-oriented immersion, \( e \) is an étale map and \( p \) is the projection.

We now make the following formal calculation implicit in which is the use of the composition rule \( (f \circ g)! = f! \circ g! : \)

\[
\mu([N, E, f]) \hat{\otimes} [D_T] = f_! [E] \hat{\otimes} [D_T] \\
= (p_! \circ e_! \circ j_!) [E] \hat{\otimes} [D_T] \\
= (p_! \circ e_!) (j_! [E]) \hat{\otimes} [D_T] \\
= (j_! [E]) \hat{\otimes} (p_! \circ e_!) \hat{\otimes} [D_T] \\
= (j_! [E]) \hat{\otimes} [D_X] .
\]

But the last statement equals \( \text{Ind } D_{N,E} \) where \( D_{N,E} \) is the Dirac operator on \( N \) twisted by \( E \). This in turn gives us:

\[ \text{Ind } D_{N,E} = \langle \text{ch}^*(E) \cup \text{Td}(N), [N] \rangle = \langle \text{ch}_*(y), \text{Td}^{11}(\tilde{Q}) \rangle \]

where we have taken account of (4.3) and so have established the result. \( \square \)

Theorem 4.2 can be applied to obtain some partial affirmative answers to some questions raised by A. Connes ([10] p.588, [12] p.2):
Corollary 4.3.
\[ \langle \mathrm{ch}_*(y), \mathrm{Td}^{11}((\tilde{Q})) \rangle \in \mathbb{Z}. \]

Returning to the context of the proof of Theorem 4.2, suppose \( M = N \) and \( X = D^q \) the \( q \)-disc \((q = \text{codim} \mathcal{F} \text{ even})\). With regards to (4.9) consider a transverse (orientation preserving) embedding \( \nu : D^q \rightarrow M \) at a fixed point \( m \in M \) and let \( \lambda_q \) denote the generator of \( KK(C, D^q) \cong \mathbb{Z} \). The element \( \nu! \lambda_q = \lambda_q \otimes \nu! \in KK(C, C^*(M/\mathcal{F})) \) is independent of \( m \) and \( \nu \) and defines the transverse orientation class \([M/\mathcal{F}]^* \in KK(C, C^*(M/\mathcal{F}))\). This embedding of \( D^q \) induces a geometric cycle \( x \in K^0_\text{res}(M, \mathcal{F}) \) satisfying \( \mu(x) = \nu! \lambda_q = [M/\mathcal{F}]^* \). The usual index formula for the Dirac operator on \( S^q \) applied to (4.3), yields \( \langle \mathrm{ch}_*(x), \mathrm{Td}^{11}(Q) \rangle = 1 \). Thus we obtain:

Corollary 4.4.
\[ [M/\mathcal{F}]^* \otimes [D_T] = 1. \]

We note that Theorem 4.2 may also be useful in computing the distributional index of \( D_T \) when \( \mathcal{F} \) arises from a Lie group \( G \) acting locally-freely on \( M \) and \( \mu \) is an isomorphism (\( \mu \) is conjectured to be an isomorphism if the holonomy of \( \mathcal{F} \) is torsion-free [3][4]).

Example 4.5. It is instructive to consider the situation in which the foliation on \( M \) is given by a fibration \( F \rightarrow M \rightrightarrows B \) where \( F \) and \( B \) are compact manifolds. The map \( \pi \) is \( K \)-oriented if \( T(\pi) = \ker(d\pi) \) is spin\(^c\) and then the spin\(^c\)-structures on \( B \) and \( M \) are in one-to-one correspondence. Assuming this to be the case, \( \pi! = [\tilde{D}] \in KK(C(M), C(B)) \) is a class representing a family of Dirac operators along the fibers. The families index theorem [2] gives the formula for \( \text{Ind}(\tilde{D}_E) = \pi!(E) \in KK(C, C(B)) \) :
\[ \text{ch}^*(\text{Ind}(\tilde{D}_E)) = \pi_*(\text{ch}^*(E) \cup \text{Td}(T(\pi))) \]
where \( \pi_* \) is integration along the fibers. Since \( TM \cong T(\pi) \oplus \pi^*TB \), this is equivalent to the Grothendieck-Riemann-Roch formula (cf §)
\[ \text{ch}^*(\pi!(E)) \cup \text{Td}(B) = \pi_*(\text{ch}^*(E) \cup \text{Td}(M)). \]

Thus for the twisted Dirac operator \( D_{M,E} \), we obtain
\[ \text{Ind}(D_{M,E}) = \langle \text{ch}^*(E) \cup \text{Td}(M), [M] \rangle = \langle \pi_*(\text{ch}^*(E) \cup \text{Td}(T(\pi))) \cup \text{Td}(B), [B] \rangle = \langle \text{ch}^*(\pi!(E)) \cup \text{Td}(B), [B] \rangle = \text{Ind}(\tilde{D}_E) \otimes [D_B] \]
which leads to the formula
\[ \text{Ind}(D_{M,E}) = \text{Ind}(\tilde{D}_E) \otimes [D_B] = \text{Ind}(D_B, \text{Ind}(\tilde{D}_E)). \quad (4.10) \]

In view of the fact that \( C^*(M/\mathcal{F}) \) is Morita equivalent to \( C(B) \) and \( B\mathcal{G} \cong B \), (4.10) falls into the context of Theorems above formula to Riemannian foliations.

In the case where the fibration is simply taken to be the product \( M = B \times F \) and \( D_M = (D_B, D_F) \) is the sharp product of the Dirac operators along \( B \) and \( F \), then (4.10) reduces to
\[ \text{Ind} D_M = \text{Ind} D_B \cdot \text{Ind} D_F. \]

The case of a foliation on a generalized flat bundle will be exemplified in § 5.
Let $R(G)$ denote the representation ring of a compact Lie group $G$ and $C^*(G)$ its $C^*$-algebra. Then for $x \in R(G) = KK(C_\ast, C^*(G)) \subseteq KK(C_\ast, C^*(M/F))$, we have

$$x \otimes [D_f] = \langle \chi_* (\mu^{-1}(x)), Td^{-1}(h) \rangle.$$ 

These happen to be topological invariants of $F$ (note the choice of $x$ does not depend on $M$). We briefly recall the known facts, for $G$ compact, that the reduced algebra $C^*_r(G) \cong C^*(G)$ and $\mu$ is not an isomorphism in general.

Let us now assume that $F$ arises as the foliation by the orbits of $G$ acting locally-freely on $M$. We have $C^*(M/F) = C(M) \times G$ and there is a natural map $\phi : K^*_G(Q) \to K^*(C^*(M/F))$ defined as follows. When $* = 0$, every element $\sigma \in K^*_G(Q)$ can be represented as a transverse symbol which can be extended to a symbol $\tilde{\sigma}$ of a pseudodifferential operator (noting that $\tilde{\sigma}$ is equivariant). We define $\phi(\sigma)$ as the K-homology class of the transverse pseudodifferential operator of order $0$ with symbol $\tilde{\sigma}$. The map $\phi$ can be defined similarly for the odd case (see $[6]$). It is then natural to ask if every K-homology class over $M/F$ can be realized as a transversally elliptic pseudodifferential operator on $M$.

Let us now proceed to:

**Theorem 4.6.** If $q = \text{codim } F$ is even, then the map

$$\phi : K^*_G(Q) \to K^*(C^*(M/F))$$

is an isomorphism.

**Proof.** The main ingredients will be the Mayer-Vietoris sequence and equivariant Poincaré duality. We shall consider a 'good' cover of $M$ in the following sense and proceed inductively. A good cover of $M$ will mean a finite open cover $\{U_i\}_{i \in I}$ of $M$, such that:

1. There is a slice $S_i$ of the action that is considered to be a smooth submanifold with metric induced by the foliated structure on $M$, where taking $G_x = \{g : gx = x\}$, $x \in S_i$, we have $U_i \cong G \times_{G_x} S_i$.
2. Each $S_i$ is a manifold with boundary $S_i - S_i$.
3. Each $S_i$ is geodesically convex and the exponential map is a diffeomorphism onto its image via $\exp^1_{S_i} S_i - S_i$.

Such a good cover always exists by virtue of the Slice Theorem $[8]$. Since the action is locally free, $G_x$ is a finite group.

We now proceed by induction on the cardinality of the good cover. When $I = \{1\}$, then $C(U_1) \times G$ is Morita equivalent to $C(S_1) \times G$ and $K^*_G(Q | U_1)$ is naturally equivalent to $K^*_{G_1}(TS_1)$. Hence the result follows from equivariant Poincaré duality (see e.g., $[24]$).

Suppose $\phi$ is an isomorphism for any manifold having a good cover by at most $k$ open sets. Consider now $M$ with such a good cover $\{U_1, \ldots, U_{k+1}\}$ by $k+1$ open sets. Now $(U_1 \cup \cdots \cup U_k) \cap U_{k+1}$ has a good cover $\{U_1 \cap U_{k+1}, \ldots, U_k \cap U_{k+1}\}$ of $k$ open sets. For the sake of clarity in presentation, we now define the following terms:

$$A = K^*_G(Q | (U_{k+1} \cap (U_1 \cup \cdots \cup U_k)))$$
$$B = K^*_G(Q | U_{k+1}) \oplus K^*_G(Q | (U_1 \cup \cdots \cup U_k))$$
$$C = K^*_G(Q | (U_1 \cup \cdots \cup U_{k+1}))$$
$$A' = K^*_G(C(U_{k+1} \cap (U_1 \cup \cdots \cup U_k)) \times G)$$
$$B' = K^*_G(C(U_{k+1}) \times G) \oplus K^*_G(C(U_1 \cup \cdots \cup U_k) \times G)$$
$$C' = K^*_G(C(U_1 \cup \cdots \cup U_{k+1}) \times G).$$
Then it follows that $\phi$ is an isomorphism for $U_1 \cup \cdots \cup U_{k+1}$ by an application of the Five Lemma:

$$
\cdots \rightarrow A \rightarrow B \rightarrow C \rightarrow \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
\cdots \rightarrow A' \rightarrow B' \rightarrow C' \rightarrow \cdots
$$

The verification that this diagram commutes is a routine exercise and is omitted. Granting this, the steps of the induction argument are now complete. □

Remarks 4.7.

1. The above theorem should be useful in establishing an index formula for the distributional index of transversally elliptic operators in view of the Chern map for $K^0_G(Q)$ using cyclic homology [9]. But there remains the question of calculating the cyclic cohomology of $C^\infty(M) \times G$ in which the Chern character of a transversally elliptic operator resides. It is shown in [33] that the continuous cyclic cohomology of $C^\infty(M) \times G$ can be computed combinatorially.

2. The K-theory of $C(M) \times G$ is equivalent to the equivariant K-theory of of $C(M)$ [24], but this is not the case for K-homology as the theorem above indicates.

3. For an arbitrary $G$-action on $M$, the K-homology of $C(M) \times G$ is not generated by classes of transversally elliptic pseudodifferential operators. For example, for the case that $M$ is a homogeneous space, the K-homology element represented by every transversally elliptic pseudodifferential operator may be trivial although the the equivariant K-homology itself may not be trivial.

5. The Chern character and the dual pairing in geometric K-theory

We now continue with the basic assumptions of § 4, recalling that $Q$ is endowed with a spin$^c$-structure and commence by considering a larger class of geometric cycles by assuming a foliated structure on the test manifold $N$ [22]. Specifically, we consider those foliations $(N, \mathcal{F}_N)$ which are proper in the sense that the map $(r, s) : G_N \rightarrow N \times N$ is a proper map. We recall from [3] [22] that the geometric topological K-theory $K^*_\text{top}(M, \mathcal{F})$ is then generated by elements $(N, \mathcal{F}_N, x, g)$ where $(N, \mathcal{F}_N)$ is a proper foliation, $x \in K_*(C^*(N/\mathcal{F}_N))$ and $g : N/\mathcal{F}_N \rightarrow M/\mathcal{F}$ is a K-oriented map. Equivalence of elements is given by

$$
(N, \mathcal{F}_N, x, g \circ h) \sim (N', \mathcal{F}_{N'}, h_!(x), g)
$$

where $h : N/\mathcal{F}_N \rightarrow N'/\mathcal{F}'_N$ is K-oriented. In other words, we have

$$
K^*_\text{top}(M, \mathcal{F}) := \lim K_*(C^*(N/\mathcal{F}_N))
$$

with the relations given by (5.1) and the limit is over triples $(N, \mathcal{F}_N, g)$. Following the construction outlined at the beginning of § 4, there is a surjection

$$
K^*_\text{top}(M, \mathcal{F}) \twoheadrightarrow K^*_\text{res}(M, \mathcal{F})
$$

and the Baum-Connes map (4.7)

$$
\mu : K^*_\text{top}(M, \mathcal{F}) \rightarrow K_*(C^*(M/\mathcal{F})) = KK^*(\mathcal{C}, C^*(M/\mathcal{F}))
$$
is then given in terms of $K^*_{\text{top}}(M, \mathcal{F})$ by
\[
\mu([N, \mathcal{F}_N, x, g]) = g(x) = x \otimes g!
\] (5.2)
where $g! \in KK^*(\mathcal{F}_N, \mathcal{F})$ is the element constructed in [22] (see also [12]). If $\mathcal{F}_N = 0$ is the trivial foliation then the resulting triples $(N, x, g) = (N, 0, x, g)$ lie in $K^*_\text{res}(M, \mathcal{F})$ and thus we recover the situation in § 4.

For a K-oriented map $f : M/\mathcal{F} \to M'/\mathcal{F}'$ we shall use the notation $f! = \hat{\otimes} f!$ and $f! = f! \hat{\otimes}$ where $f! \in KK(C^*(M/\mathcal{F}), C^*(M'/\mathcal{F}'))$. The homomorphism
\[
f_i^! : K^*_{\text{top}}(M, \mathcal{F}) \to K^*_{\text{top}}(M', \mathcal{F}')
\] is given by $f_i^!(N, \mathcal{F}_N, x, g) = (N, \mathcal{F}_N, x, f \circ g)$. It follows that the diagram below
\[
\begin{array}{ccc}
K^*_{\text{top}}(M, \mathcal{F}) & \xrightarrow{f_i^!} & K^*_{\text{top}}(M', \mathcal{F}') \\
\downarrow \mu & & \downarrow \mu \\
K_*(C^*(M/\mathcal{F})) & \xrightarrow{f_i^!} & K_*(C^*(M'/\mathcal{F}'))
\end{array}
\] (5.3)
is commutative. Henceforth, we often suppress ‘top’ and the foliation $\mathcal{F}_N$ in order to simplify the notation.

On $K^*(M, \mathcal{F})$ we have as before a Chern character via
\[
\text{ch}_*: K^*(M, \mathcal{F}) \xrightarrow{\text{ch}} H^*_q(B\mathcal{G}, \mathbb{Q}) \xrightarrow{\Phi} H_{q-1}(B\mathcal{G}, \mathbb{Q}).
\]
Just as in (4.3) we can infer that
\[
\text{ch}_*(y) = \text{Td}(\bar{Q}) \cap g_*(\text{ch}^*(E) \cup \text{Td}(N) \cap [N])
\] (5.4)
in $H_*(B\mathcal{G}, \mathbb{Q})$.

Consider now two K-oriented maps $g : N' \to M/\mathcal{F}$ and $h : N \to N'$. For $x \in K^0(N)$ let $x' = h_!(x)$; we consider the equivalence relation
\[
(N, x, g \circ h) \sim (N', x', g).
\]
To ensure that $\text{ch}_*(y)$ on $K^*(M, \mathcal{F})$ is well-defined, the following formula characterizing ‘$\sim$’
\[
h_*(\text{ch}^*(E) \cup \text{Td}(N) \cap [N]) = \text{ch}^*(h_!(E)) \cup \text{Td}(N') \cap [N']
\] (5.5)
is applied to yield:
\[
\text{ch}_*([N', h_!(x); g]) = \text{Td}(\bar{Q}) \cap g_*(\text{ch}^*(h_!(x)) \cup \text{Td}(N') \cap [N'])
\]
\[
= \text{Td}(\bar{Q}) \cap g_*(\text{ch}^*(x) \cup \text{Td}(N) \cap [N])
\]
\[
= \text{Td}(\bar{Q}) \cap (g \circ h)_*(\text{ch}^*(x) \cup \text{Td}(N) \cap [N])
\]
\[
= \text{ch}_*([N, x, g \circ h]).
\]

Recall from [22] §6 that a $\mathcal{G}(\mathcal{F})$-equivariant vector bundle $V$ on $(M, \mathcal{F})$ determines an element $[V] \in KK(C^*(M/\mathcal{F})) := KK(C^*(M/\mathcal{F}), C^*(M/\mathcal{F}))$ by multiplicity, as well as a corresponding vector bundle $\tilde{V}$ on $B\mathcal{G}$. From the K-oriented projection $p : M/\mathcal{F} \to pt$ we obtain the class $p! \in KK_*(C^*(M/\mathcal{F}), C^*(M/\mathcal{F}))$. Since $\mathcal{F}$ is Riemannian, the abstract twisted element $pV = [V] \hat{\otimes} p!$ of [22], can be identified with the transverse Dirac operator $D_T$ twisted by $V$, thus obtaining a class $pV = [D_{T, V}]$ in $KK(C^*(M/\mathcal{F}), \mathbb{C})$. For Riemannian foliations, Theorem 6.4 of [22] can thus be rephrased as follows, with the second equation in ii) below providing the link with Theorem 4.2:
Theorem 5.1. Let $V \rightarrow M$ be a $\mathcal{G}(\mathcal{F})$-equivariant complex vector bundle with corresponding class $[V] \in KK(C^*(M/\mathcal{F}), C^*(M/\mathcal{F}))$. Then there exists a class $p_V = [D_T, V] \in KK(C^*(M/\mathcal{F}), \mathbb{C})$ such that:

i) for $y = [N, \mathcal{F}_N, x, g]$ in $K^0(M, \mathcal{F})$, we have

$$\mu(y) \otimes [D_T, V] = (p \circ g)_!(x \otimes [g^*V])$$

$$= \mu([N, \mathcal{F}_N, x \otimes [g^*V], g]) \otimes p!$$

$$= \mu([N, \mathcal{F}_N, x \otimes [g^*V], g]) \otimes [D_T];$$

ii) for $y = [N, 0, x, g]$ where $x \in K^0(N)$, we have

$$\mu(y) \otimes [D_T, V] = \langle ch^*(y) \cup ch^*(g^*\tilde{V}) \cup Td(N), [N] \rangle$$

$$= \langle ch^*(y), ch^*(\tilde{V}) \cup Td^{\perp, 1}(\tilde{Q}) \rangle .$$

We note that the last expression in ii) is a consequence of (5.4).

Remark 5.2. The $\mathcal{G}(\mathcal{F})$-equivariant vector bundles in [22] are very closely related to the foliated vector bundles in [23]. In fact, every $\mathcal{G}(\mathcal{F})$-equivariant bundle determines a foliated bundle in an obvious way (since covariant differentiation along the leaves is determined by path-lifting). On the other hand, parallel transport in a foliated bundle along paths within a leaf determines an action of the fundamental groupoid $\mathcal{G}_x(\mathcal{F})$ on $V$ which may not necessarily descend to $\mathcal{G}(\mathcal{F})$ via the canonical map $\mathcal{G}_x(\mathcal{F}) \rightarrow \mathcal{G}(\mathcal{F})$. Note that the foliated bundle $Q \rightarrow M$ is always $\mathcal{G}(\mathcal{F})$-equivariant.

Example 5.3. Let $X$ and $F$ be closed Riemannian manifolds with $\pi_1(X) = \Gamma$. Relative to a holonomy representation $h : \Gamma \rightarrow \text{Isom}(F)$ we consider the action of $\gamma \in \Gamma$ on $\tilde{X} \times F$:

$$\gamma(\tilde{x}, f) = (\tilde{x} \cdot \gamma_{\perp, 1}, h(\gamma)f).$$

Let $M$ be the quotient of $\tilde{X} \times F$ by this action. We then obtain a generalized flat bundle

$$M = \tilde{X} \times_\Gamma F \xrightarrow{\pi} X$$

and a foliation $\mathcal{F}$ on $M$ transverse to the fibers of $\pi$ where the leaves of $\mathcal{F}$ are coverings of $X$. In this case the leaf space $M/\mathcal{F}$ is the quotient $F/h(\Gamma)$.

Consider the diagram of maps

$$\begin{array}{ccc}
M = \tilde{X} \times_\Gamma F & \longrightarrow & B(\Gamma \times F) = E_\Gamma \times_\Gamma F \\
\downarrow & & \downarrow \\
X & \longrightarrow & B\Gamma
\end{array}$$

where $\Gamma \times F \rightarrow F$ is viewed as a groupoid for which the range and source maps respectively are given by $r(\gamma, x) = h(\gamma)(x)$ and $s(\gamma, x) = x$ and the homotopy quotient $F_\Gamma = E_\Gamma \times_\Gamma F$ is homotopy equivalent to $B\mathcal{G}$. Furthermore, we have $C^*(M/\mathcal{F}) \cong C(F) \rtimes \Gamma$ via Morita equivalence. For any $\Gamma$-equivariant complex vector bundle $V_0 \rightarrow F$, we have $\mathcal{G}(\mathcal{F})$-equivariant vector bundles

$$\begin{array}{ccc}
V = \tilde{X} \times_\Gamma V_0 & \longrightarrow & M \\
\downarrow & & \downarrow \\
\tilde{V} = E_\Gamma \times_\Gamma V_0 & \longrightarrow & F_\Gamma
\end{array}$$
(see e.g. [12]). As before, we assume compatible \(\text{spin}^c\)-structures and K-orientations on \(X\), \(F\) and \(\pi\).

Consider now the K-oriented inclusion \(j : F \to M\) and the corresponding map \(j : F \to F_\Gamma\). By [12] Lemma 6.5, \(j! \in KK(C(F), C(F) \times \Gamma)\) is induced by the canonical inclusion \(I : C(F) \to C(F) \times \Gamma\). The \textit{canonical class} \(\{F\}\) of the transversal \(F\) in \(F_\Gamma\) is given by

\[
\{F\} = \mu([F, 1, j]) = 1 \hat{\otimes} j! = \hat{j}(1) = I_*(1)
\]

(\(1\) here denotes the class \([F \times \mathbb{R}]\)). For the triple \(y = [F, V_0, j] \in K^0(M, \mathcal{F})\), we have

\[
\mu(y) = 1 \hat{\otimes} [V_0] \hat{\otimes} j!
\]

\[
= 1 \hat{\otimes} [j^* \hat{V}] \hat{\otimes} j!
\]

\[
= 1 \hat{\otimes} j! \hat{\otimes} [\hat{V}]
\]

\[
= \{F\} \hat{\otimes} [\hat{V}].
\]

Then by Theorem 5.1 we obtain

\[
\mu(y) \hat{\otimes} [D_T] = (\{F\} \hat{\otimes} [\hat{V}]) \hat{\otimes} [D_T]
\]

\[
= \{F\} \hat{\otimes} [D_{T,V}]
\]

\[
= \langle \text{ch}_*(y), \text{Td}^1(\hat{Q}) \rangle.
\]

But the latter expression equals

\[
\langle \text{ch}^*(V_0) \cup \text{Td}(F), [F] \rangle = \text{Ind}(D_{F, V_0}).
\]

(5.6)

Since \(V_0 = j^* \hat{V}\) and \(TF = j^* (\hat{Q})\), we have

\[
\text{ch}_*(y) = \text{Td}(\hat{Q}) \cap j_* (\text{ch}^*(V_0) \cup \text{Td}(F) \cap [F])
\]

\[
= \text{Td}(\hat{Q} \otimes \mathcal{C}) \cup \text{ch}^*(\hat{V}) \cap j_* [F]
\]

so that \(\text{ch}_*(y)\) depends only on \(j_* [F] \in H_*(F_\Gamma, \mathbb{Q})\). Combining this last relationship with (5.6), we arrive at

\[
\{F\} \hat{\otimes} [D_{T,V}] = \text{Ind}(D_{F, V_0}) =
\]

(5.7)

In particular, if we have \(j_* [F] = 0\), then \(0 = \text{ch}_*(y) \in H_*(F_\Gamma, \mathbb{Q})\) and so \(y \in K^*(M, \mathcal{F})\) is a \textit{torsion class} since the Chern character \(\text{ch}_*\) is a rational isomorphism. Consequently, we have the following (cf [12] Theorem 6.6) : if \(j_* [F] = 0\), i.e. \(F\) is homologous to zero in \(F_\Gamma\), then \(\mu(y) \in KK(C(C(F), C(F)) \times \Gamma)\) is a torsion class.

In order to see how the classes \([D_T]\) and \([D_F]\) are related in this case, we proceed as follows. The inclusion \(j : F \to M\) induces a K-oriented \(\acute{e}\)tal submersion \(j : F \to M / \mathcal{F}\). Recalling that \([V_0] \in KK(C(F), C(F))\) (see Theorem 5.1), we deduce from Proposition

\[
[V_0] \hat{\otimes} j! [D_T] = [V_0] \hat{\otimes} j! \hat{\otimes} [D_T]
\]

\[
= [j^* \hat{V}] \hat{\otimes} j! \hat{\otimes} [D_T]
\]

\[
= j! \hat{\otimes} [\hat{V}] \hat{\otimes} [D_T]
\]

\[
= j! \hat{\otimes} [D_{T,V}]
\]

\[
= j^! ([D_{T,V}]).
\]
In other words $[V_0] \otimes j^! [D_T] = j^! [D_{T,V}]$ is the class of the Dirac operator on $F$ twisted by $V_0$. Thus for the generalized flat bundle case we have established the following relationship

$$[D_{F,V_0}] = [V_0] \otimes j^! [D_T] = j^! [D_{T,V}] \in KK(C(F), \mathbb{C})$$

(5.8)

which completes the example.

Returning to the general situation, we now arrive at a formula of Grothendieck-Riemann-Roch type:

**Theorem 5.4.** Let $f : M/F \to M'/F'$ be $K$-oriented. If the map

$$\mu^! : K^*(M', F') \to KK(C, C^*(M'/F')) = K_* (C^*(M'/F'))$$

is an isomorphism, then we have $f^!_i = f_i \circ \mu$ and

$$Td^{11} (\bar{Q}') \cap ch_* (f_i^! (y)) = f_*(Td^{11} (\bar{Q}) \cap ch_* (y)).$$

(5.9)

**Proof.** From the K-oriented map $f : M/F \to M'/F'$ we obtain the diagram

\[
\begin{array}{ccc}
H_*(BG', \mathbb{Q}) & \xrightarrow{ch_*} & K^*(M', F') \\
\uparrow f_* & & \uparrow f_* \\
H_*(BG, \mathbb{Q}) & \xrightarrow{ch_*} & K^*(M, F') \\
\end{array}
\]

(5.10)

whose right square is commutative by (5.3). From (5.4) it follows that

$$f_*(Td^{11} (\bar{Q}) \cap ch_* (y)) = Td^{11} (\bar{Q}') \cap ch_* (f^!_i (y)).$$

(5.11)

In fact, for $y = [N, x, g]$, the left member equals

$$f_*(g_* (ch^*(x) \cup Td(N) \cap [N]))$$

and the right member equals

$$Td^{11} (\bar{Q}') \cap ch_* (f^!_i (y)) = (f \circ g)_* (ch^*(x) \cup Td(N) \cap [N]) = f_*(g_* (ch^*(x) \cup Td(N) \cap [N])).$$

By assumption, we have $f^!_i = f_i \circ \mu$ and thus

$$Td^{11} (\bar{Q}') \cap ch_* (f^!_i (y)) = f_*(Td^{11} (\bar{Q}) \cap ch_* (y)).$$

\[
\square
\]

We note that Theorem 5.4 essentially implies Theorem 5.1 for $f = p : M/F \to pt$. The index formula in Theorem 5.1 can be interpreted as a formula on $BG$, but it can also be interpreted as a formula on the ‘test manifold’ $N$, where $(N, 0, x, g)$ represents $y \in K^*(M, F)$. This fact is the motivation for the following dual construction to $\mu$ which is essentially given by the symbol of a transversally elliptic operator.
Thus we introduce the dual geometric $K$-theory by

$$K^\text{top}_* (M, \mathcal{F}) = \lim_{\rightarrow} K_* (C^* (N/\mathcal{F}_N))$$

where the inverse limit is taken over triples $(N, \mathcal{F}_N, g)$ as above, subject to the relations

$$\xi_{(N, g \circ h)} = h! \xi_{(N, g)} = h! \xi_{(N, g)}$$

for $\xi = \{\xi_{(N, g)}\} \in K^\text{top}_* (M, \mathcal{F})$ and $h : N/\mathcal{F} \to N'/\mathcal{F}_{N'}$ a K-oriented map. Again, we suppress ‘top’ and the foliation $\mathcal{F}_N$ in order to simplify the notation.

The symbol map, seen as the composition

$$\sigma : K K_* (C^* (M/\mathcal{F}), \mathbb{C}) \to K_* (B\mathbb{Q}^*, S\mathbb{Q}^*) \to K_*(M, \mathcal{F})$$

is defined for any element $\eta \in K K_* (C^* (M/\mathcal{F}), \mathbb{C})$ by

$$\sigma(\eta) = \{g^!(\eta)\}_{(N, g)} = \{g^! \otimes \eta\}_{(N, g)} \in K_* (M, \mathcal{F}) .$$

Then there is a canonical pairing of geometric $K$-groups

$$\hat{\otimes} : K_* (M, \mathcal{F}) \otimes K_* (M, \mathcal{F}) \to \mathbb{Z}$$

or equivalently,

$$K_* (M, \mathcal{F}) = \lim_{\rightarrow} K_* (N) \to \lim_{\rightarrow} \text{Hom}(K_* (N), \mathbb{Z}) = \text{Hom}(K_* (M, \mathcal{F}), \mathbb{Z})$$

given by

$$y \otimes \xi = x \otimes \xi_{(N, g)}$$

for $y = [N, x, g] \in K_* (M, \mathcal{F})$ and $\xi = \{\xi_{(N, g)}\} \in K_* (M, \mathcal{F})$. This is well-defined, since for $(N, x, g \circ h) \sim (N', h!(x), g)$, we have

$$h!(x) \otimes \xi_{(N, g)} = (x \otimes h)! \otimes \xi_{(N, g)}$$

$$= x \otimes (h! \otimes \xi_{(N, g)})$$

$$= x \otimes (h!(\xi_{(N, g)}))$$

$$= x \otimes \xi_{(N, g \circ h)} .$$

**Proposition 5.5.** The symbol map $\sigma$ is dual to $\mu$ with respect to the above pairing. Explicitly,

$$\mu(y) \otimes \eta = y \otimes \sigma(\eta)$$

for $y \in K_* (M, \mathcal{F})$ and $\eta \in K K_* (C^* (M/\mathcal{F}), \mathbb{C})$.

**Proof.** For $y = [N, x, g]$, we have by direct calculation:

$$\mu(y) \otimes \eta = g!(x) \otimes \eta$$

$$= (x \otimes g!) \otimes \eta$$

$$= x \otimes (g! \otimes \eta)$$

$$= x \otimes \sigma(\eta)_{(N, g)}$$

$$= y \otimes \sigma(\eta) . \square$$
Recall that the index theorem for a twisted Dirac operator $D_E$ on a $\text{spin}^c$-manifold $N$ can be written as
\[
\text{Ind}(D_E) = [E] \hat{\otimes} [DN] = \langle \text{ch}^*(E), \text{ch}_*(DN) \rangle
\]
where $\text{ch}_*(DN) = \text{Td}(N) \cap [N]$. We should expect a similar formula for the pairing between $K^*(M,F)$ and $K_*(M,F)$ and this indeed turns out to be the case. On $K_*(M,F)$ we have a Chern character
\[
\text{ch}^* : K_*(M,F) \rightarrow H^*(B\hat{Q},S\hat{Q},\mathbb{Q}) \rightarrow H^*(BG,\mathbb{Q})
\]
which is related to the homology Chern character $\text{ch}_*$ on $K_*(N)$ by a formula ‘dual’ to (5.4), namely
\[
\text{ch}_*(\xi_{(N,g)}) = g^*(\text{Td}(\hat{Q}) \cup \text{ch}^*(\xi)) \cup \text{Td}(N) \cap [N]. \quad (5.15)
\]

We are now ready to establish the following theorem.

**Theorem 5.6.** For $y \in K^*(M,F)$ and $\eta \in K^*(C^*(M/F),\mathbb{C})$, we have the index formula
\[
\mu(y) \hat{\otimes} \eta = y \hat{\otimes} \sigma(\eta) = \langle \text{ch}_*(y), \text{ch}^*(\sigma(\eta)) \rangle \quad (5.16)
\]
and the latter expression is given by the index formula
\[
\langle \text{ch}_*(y), \text{ch}^*(\sigma(\eta)) \rangle = \langle \text{ch}^*(x), \text{ch}_*(\sigma(\eta))_{(N,g)} \rangle \quad (5.17)
\]
on the ‘test manifold’ $N$, whenever $y$ is represented by the triple $(N,x,g)$.

**Proof.** Taking into account (5.4) and Theorem 5.1, we calculate formally:
\[
\mu(y) \hat{\otimes} \eta = y \hat{\otimes} \sigma(\eta)
= x \hat{\otimes} \sigma(\eta)_{(N,g)}
= \langle \text{ch}^*(x), \text{ch}_*(\sigma(\eta)_{(N,g)}) \rangle
= \langle \text{ch}^*(x), g^*(\text{Td}(\hat{Q}) \cup \text{ch}^*(\sigma(\eta))) \cup \text{Td}(N) \cap [N] \rangle
= \langle g_* \text{ch}^*(x) \cup \text{Td}(N) \cap [N], \text{Td}(\hat{Q}) \cup \text{ch}^*(\sigma(\eta)) \rangle
= \langle \text{Td}(\hat{Q}) \cup g_* \text{ch}^*(x) \cup \text{Td}(N) \cap [N], \text{ch}^*(\sigma(\eta)) \rangle
= \langle \text{ch}_*(y), \text{ch}^*(\sigma(\eta)) \rangle. \quad \square
\]

Observe that for $\eta = p_V = [D_{T,V}]$ we have obtained the relationship
\[
\text{ch}^*(\sigma(p_V)) = \text{ch}(\hat{V}) \cup \text{Td}^{11}(\hat{Q}) \quad (5.18)
\]
on $BG$ and the formulas (5.16) and (5.17) in Theorem 5.6 can be seen to agree with the expressions in Theorem 5.1 ii). In fact, we note that:
\[
\text{ch}_*(\sigma(p_V)_{(N,g)}) = g_*(\text{ch}^*(p_V))
= g^* \text{ch}^*(\hat{V}) \cup \text{Td}(N) \cap [N]
= \text{ch}^*(g^*\hat{V}) \cup \text{Td}(N) \cap [N].
\]
Example 5.7. We now return to the context of Theorem 3.4 in respect of its hypotheses. On setting $M = N$, $F_N = 0$ and $g_0 : M \to M/F$ the K-oriented submersion, we take $y \in K^0(M, F)$ as given by $y = [M, x, g_0]$. From (5.2), we have $\mu(y) = x \hat{\otimes} (g_0)!$ which now reduces to

$$\mu(y) = x \hat{\otimes} \alpha$$

(5.19)

recalling that $\alpha = [D_L] \in KK(C(M), C^*(M/F))$. We assign $\mu(y) = [\text{Ind } D_L]$ where $D_L$ is the ‘test’ longitudinal operator and continue to find a possible choice for $x$ (which would be unique if we knew the Baum-Connes conjecture to be true). In proceeding, we shall use the shift class in §3 without stating it explicitly. Thus on pairing (5.19) with the dual Dirac class $\beta$ on both sides, we obtain

$$\mu(y) \hat{\otimes} \beta = [\text{Ind } D_L] \hat{\otimes} \beta = x \hat{\otimes} \alpha \hat{\otimes} \beta.$$

Since we have $\alpha \hat{\otimes} \beta = 1_M$ by Proposition 3.1, we see that a possible choice of $x$ is, in fact,

$$x = [E] = [\text{Ind } D_L] \hat{\otimes} \beta$$

as stated in (3.14). Now setting $\eta = [D_T]$ in Theorem 5.6, we derive the following version of formula (3.15) in Theorem 3.4:

Corollary 5.8. Under the hypotheses of Theorem 3.4, we have

$$\text{Ind } D_E = [\text{Ind } D_L] \hat{\otimes} [D_T]$$

$$= (\text{ch}_*(y), \text{Td}^{-1}(\tilde{Q}))$$

$$= (\text{ch}^*(E), g^*(\text{Td}(\tilde{Q}) \cup \text{ch}^*(\sigma_T)) \cup \text{Td}(M) \cap [M]).$$

Returning to the general case, for a K-oriented map $f : M/F \to M'/F'$, we have the map

$$f^! : K_*(M', F') \to K_*(M, F)$$

given by $f^!(\xi') = \xi$, where $\xi_{(N, g)} = \xi'^{(N, f g)}$. From this we obtain the following relationships:

Proposition 5.9. For $\zeta \in KK(C, C^*(M/F))$ and $\eta' \in KK(C^*(M'/F'), \mathbb{C})$, we have

$$f_!(\zeta) \hat{\otimes} \eta' = \zeta \hat{\otimes} f^!(\eta').$$

(5.20)

Further, for $y \in K^*(M, F)$ and $\xi' \in K_*(M', F')$ we have

$$f_!(y) \hat{\otimes} \xi' = y \hat{\otimes} f^!(\xi').$$

(5.21)

Proof. These follow from the relationships:

$$f_!(\zeta) \hat{\otimes} \eta' = (\zeta \hat{\otimes} f_!)(\hat{\otimes} \eta' \hat{\otimes} \eta)$$

$$= \zeta \hat{\otimes} (f'! \hat{\otimes} \eta')$$

$$= \zeta \hat{\otimes} f^!(\eta').$$
and for \( y = [N, x, g] \), we have

\[
\begin{align*}
\tilde{f}^i(y) \otimes \xi^i & = [N, x, f \circ g] \otimes \xi^i \\
& = x \otimes \xi^i_{(N,f \circ g)} \\
& = x \otimes f^i_1(\xi^i_{(N,f \circ g)}) \\
& = y \otimes f^i_1(\xi^i_1). 
\end{align*}
\]

\( \square \)

The symbol map \( \sigma \) is compatible with \( f^i \) and \( f^i_1 \) in the sense that the following diagram

\[
\begin{array}{ccc}
KK(C^*(M/F), \mathbb{C}) & \xrightarrow{\sigma} & K_\ast(M, \mathcal{F}) \\
\uparrow f^i & & \uparrow f^i_1 \\
KK(C^*(M'/F'), \mathbb{C}) & \xrightarrow{\sigma'} & K_\ast(M', \mathcal{F}')
\end{array}
\]

is commutative. Effectively, for \( \eta^i \in KK(C^*(M'/F'), \mathbb{C}) \), we have

\[
\sigma(f^i(\eta^i)) = g^i (f^i(\eta^i))
\]

\[
= (f \circ g)^i(\eta^i)
\]

\[
= \sigma'((\eta^i))(N,f \circ g)
\]

\[
= f^i_1(\sigma'(\eta^i)).
\]

\( \square \)

From Theorem 5.6, we now can now establish a Grothendieck-Riemann-Roch theorem for

\[
f^i = f^i ! \otimes : KK(C^*(M'/F'), \mathbb{C}) \rightarrow KK(C^*(M/F), \mathbb{C})
\]

given any \( K \)-oriented map \( f : M/F \rightarrow M'/F' \).

**Theorem 5.10.** For \( \eta^i \in KK(C^*(M'/F'), \mathbb{C}) \), we have

\[
\text{Td}(\tilde{Q}_F) \cup ch^*(\sigma(f^i(\eta^i))) = f^*(\text{Td}(\tilde{Q}_F') \cup ch^*(\sigma'(\eta^i))).
\]

**Proof.** For any \( y = [N, x, g] \in K^\ast(M, \mathcal{F}) \), we have by Theorem 5.6 :

\[
\langle \text{ch}_\ast(y), ch^*(\sigma(f^i_1(\eta^i))) \rangle
\]

\[
= y \otimes f^i_1 \sigma'(\eta^i)
\]

\[
= f^i_1(y) \otimes \sigma'(\eta^i)
\]

\[
= \langle \text{ch}_\ast(f^i_1(y)), ch^*(\sigma'(\eta^i)) \rangle
\]

\[
= \langle \text{Td}(\tilde{Q}_F) \cap ch_\ast(y), ch^*(\sigma'(\eta^i)) \rangle
\]

\[
= \langle \text{Td}^{11}(\tilde{Q}_F) \cap ch_\ast(y), f^*(\text{Td}(\tilde{Q}_F') \cup ch^*(\sigma'(\eta^i))) \rangle
\]

\[
= \langle \text{ch}_\ast(y), \text{Td}^{11}(\tilde{Q}_F') \cup f^*(\text{Td}(\tilde{Q}_F') \cup ch^*(\sigma'(\eta^i))) \rangle
\]

and the result follows. \( \square \)
Corollary 5.11. For \( \eta' = p_V \in KK(C^*(M'/\mathcal{F}'), \mathbb{C}) \) and \( V \) as in Theorem 5.1, we have

\[
\text{Td}(\hat{Q}_\mathcal{F}) \cup \text{ch}^*(\sigma(p_{V'})) = f^*(\text{Td}(\hat{Q}_\mathcal{F}) \cup \text{ch}^*(\sigma'(p_{V}))).
\]

Proof. Effectively, for \( \eta' = p'_{V'} \), we have:

\[
f'^*(p'_{V'}) = f! \hat{\otimes} ([V'] \hat{\otimes} p'!).
\]

\[
= (f! \hat{\otimes} [V']) \hat{\otimes} p'!.
\]

\[
= (f^*[V'] \hat{\otimes} f!) \hat{\otimes} p'!.
\]

\[
= f^*[V'] \hat{\otimes} (p' \circ f!).
\]

\[
= [f^*[V']] \hat{\otimes} p!.
\]

\[
= p_{f' \cdot V'}.
\]

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