Asymptotic Completeness
of N-particle systems

Chapter 3

Quantum time dependent 2-body Hamiltonians

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OF $N$–PARTICLE SYSTEMS

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Chapter 3

Quantum time-dependent 2-body Hamiltonians

3.0 Introduction

Our presentation of the classical 2-body scattering was divided into two chapters. In the first chapter we studied scattering in the presence of forces that decay in time. In the second chapter we investigated potentials that are time-independent but decay in space. In the quantum case we will also consider separately two analogous classes of 2-body systems.

In this chapter we will treat time-dependent Hamiltonians of the form

\[ H(t) := \frac{1}{2} D^2 + V(t, x). \]

(3.0.1)

We will make assumptions on the temporal decay of \( \partial_x^2 V(t, x) \) which are uniform in \( x \). We will study various objects that describe the asymptotics of the evolution defined by (3.0.1) for \( t \to \infty \).

In the literature scattering theory for time-dependent Hamiltonians of the form (3.0.1) is rarely studied as an end in itself. They appear usually as auxiliary objects useful in the study of time-independent Hamiltonians

\[ H := \frac{1}{2} D^2 + V(x). \]

(3.0.2)

that decay in space. As a matter of fact, results obtained in this chapter will be useful in the next chapter devoted to scattering theory for Hamiltonians of the form (3.0.2). Nevertheless, we think that time-dependent Hamiltonians deserve our attention. As we will see throughout this chapter, under suitable conditions on \( V(t, x) \) scattering theory for such Hamiltonians has very good mathematical properties and can serve as an excellent “training ground” to learn some of the concepts of scattering theory for time-independent Hamiltonians.

Let us briefly describe the contents of this chapter.

In Section 3.1 we describe what we mean by saying that an evolution \( U(t, s) \) is generated by a Hamiltonian \( H(t) \). We will not study conditions on \( H(t) \) that guarantee the existence of \( U(t, s) \) (apart from a certain rather simple example).

In Section 3.2 we introduce the asymptotic momentum, that is the self-adjoint operator defined by

\[ D^+ := \lim_{t \to \infty} \frac{U(0, t) DU(t, 0)}{t}, \]

(3.0.3)

where the limit (3.0.3) has to be understood in a special sense described in Appendix C.3. The existence of (3.0.3) is true under quite general assumptions on the potential, e.g. if \( V(t, x) = V_0(t, x) + V_1(t, x), \)
where
\[ |V_0(t, x)| \leq C(t)^{-\mu_0}, \quad \mu_0 > 1, \]
\[ |\nabla_x V_1(t, x)| \leq C(t)^{-1-\mu_1}, \quad \mu_1 > 0. \]

Another, equivalent definition of \( D^+ \) is possible:
\[
D^+ = \lim_{t \to \infty} U(0, t) \frac{\mathbf{x}}{t} U(t, 0).
\]

A large part of Section 3.1 is devoted to a proof of (3.0.4). This part of Section 3.1 will not be used in this chapter.

Section 3.3 is devoted to the short-range scattering theory. In the short-range case one can compare the dynamics with the evolution generated by the free Hamiltonian
\[
H_0 := \frac{1}{2} D^2.
\]

We prove that if we assume for instance that
\[ |V(t, x)| \leq C(t)^{-\mu}, \quad \mu > 1, \]
then the short-range wave operator
\[
\lim_{t \to \infty} U(0, t) e^{-itH_0} := \Omega^+_\text{sr}
\]
exists and is unitary. The unitarity of the wave operator goes under the name of asymptotic completeness.

The wave operator implements the unitary equivalence of \( D^+ \) and \( D \):
\[
D^+ = \Omega^+_\text{sr} \Omega^+_\text{sr}^+.
\]

Similarly as in the classical case, if we assume the hypothesis
\[
|\partial_\xi^\alpha V(t, x)| \leq C(t)^{-\mu-|\alpha|}, \quad \mu > 1, \quad |\alpha| \geq 1,
\]
then short-range wave operators have especially good properties. It turns out that (3.0.7) implies that \( \Omega^+_\text{sr} \) is a bounded pseudodifferential operator in the following sense: there exists a function \( a^+(x, \xi) \) such that
\[
\Omega^+_\text{sr} \phi(x) = (2\pi)^{-n} \int a^+(x, \xi) e^{i(x-y)\xi} \phi(y) dy d\xi,
\]

Moreover, if we set
\[
x^+ := \Omega^+_\text{sr} x \Omega^+_\text{sr}^+,
\]
then \( D^+ - D \) and \( x^+ - x \) are bounded pseudodifferential operators in the above sense. These properties of the short-range scattering theory are proven in Section 3.4.

Section 3.5 is devoted to the short-range scattering theory for a different class of time-dependent potentials. Roughly speaking the short-range potentials considered in Section 3.3 belong to \( L^1(\mathbb{R}^d, L^\infty(X)) \). Here we prove that the short-range wave operators exist and are unitary if \( V(t, x) \) belongs to \( L^\alpha(\mathbb{R}^d, L^p(X)) \) for some exponents \( \alpha > 1 \) and \( p < \infty \). The proof of this fact relies on \( L^p - L^q \) estimates for the free propagator which are of a purely quantum nature. Consequently this section is rather disconnected from the rest of the chapter and can be skipped on the first reading.

In the long-range case the limit (3.0.5) in general does not exist. We need to replace in (3.0.5) the free dynamics \( e^{-itH_0} \) with a modified one \( e^{-iS(t, D)} \). It turns out that if we chose appropriately the function \( S(t, \xi) \), then the modified wave operator
\[
\lim_{t \to \infty} U(0, t) e^{-iS(t, D)} := \Omega^+_\text{lr}
\]
exists and is unitary. It satisfies
\[ D^* = \Omega^+_l D \Omega^+_r. \]

The function \( S(t, \xi) \) is not uniquely defined. The choice that we usually make is a solution of the Hamilton-Jacobi equation with a certain potential which is close to the potential \( V(t, x) \).

The construction of the modified wave operator and the proof that it is unitary are described in Sections 3.6 and 3.7. In Section 3.6 we impose very weak assumption on the potential, roughly speaking we demand that
\[ |\partial^a_x V(t, x)| \leq C(t)^{-\mu - |\alpha|}, \quad \mu > 0, \quad |\alpha| = 1, 2. \]

In Section 3.7 we develop long-range scattering theory under more restrictive hypotheses:
\[ |\partial^a_x V(t, x)| \leq C(t)^{-\mu - |\alpha|}, \quad \mu > 0, \quad |\alpha| \geq 1. \]

The proof of Section 3.6 is quite technical. Therefore we decided for the convenience of the reader to give an independent treatment of this subject in Section 3.7. We recommend the reader to skip Section 3.6 on the first reading (the results of this section are not used in the remaining part of this chapter).

In practice it is useful to know how to construct a modified free dynamics which can be put in (3.0.9) without solving the Hamilton-Jacobi equation. Examples of such dynamics, that is the Dollard dynamics and the family of Bushlev-Matveev dynamics, are described in Section 3.0.7. We show in particular, that the Dollard dynamics can be used in the case when the system has some “internal degrees of freedom” and \( \mu > \frac{1}{2} \) in (3.0.11).

Just as in the classical case, scattering theory under condition (3.0.12) has good regularity properties. For example, this condition implies that \( D^* - D \) is a bounded pseudodifferential operator. If we introduce the long-range asymptotic position
\[ x^+_l := \Omega^+_l e^{\Omega^+_r}, \]
then in general we cannot claim that \( x^+_l - x \) is a bounded pseudodifferential operator. Instead we have
\[ x^+_l - x = Ax + B, \]
where \( A \) and \( B \) are bounded pseudodifferential operators. These facts are proven in Section 3.11. The proof of these facts relies on a version of the Egorov’s theorem, proved in Section 3.10 which describes properties of a solution of the Heisenberg equation in terms of pseudodifferential operators.

In Section 3.12 we present a construction of the modified wave operator which is an adaptation to the time-dependent case of a construction of Isozaki-Kitada [IK1]. This construction uses a Fourier integral operator
\[ J^+(s) \phi(x) := (2\pi)^{-n} \int \int e^{i\Phi^+(s, x, \xi) - i(y, \xi)} \phi(y) dy d\xi, \]
where \( \Phi^+(s, x, \xi) \) is a solution of the eikonal equation. We show that
\[ \Omega^+_l = \lim_{s \to \infty} U(0, s) J^+(s). \]

Note that (3.0.9) was a strong limit, whereas (3.0.15), under the hypotheses on potentials that we use, is a norm limit! If the choices of \( S(t, \xi) \) and \( \Phi^+(s, x, \xi) \) are related to one another in the way described in Chapter 1, then the modified wave operators defined by (3.0.9) and (3.0.15) are equal.

It turns out that the condition (3.0.12) does not imply that \( \Omega^+_l \) is a pseudodifferential operator in the sense of (3.0.8). Instead, we show in Section 3.0.16 that the time-translated modified wave operator is a Fourier integral operator in the following sense: there exists an amplitude \( a^+(s, x, \xi) \) such that
\[ U(s, 0) \Omega^+_l \phi(x) = (2\pi)^{-2n} \int a^+(s, x, \xi) e^{i\Phi^+(s, x, \xi) - i(y, \xi)} \phi(y) dy d\xi, \]
\[ |\partial^a_x \partial^a_\xi a^+(s, x, \xi)| \leq o(s^{-|\alpha|}). \]
This ends the main part of this chapter. Further on we give a number of appendices of a more general character. The most important ones are probably Appendix C.6 on pseudodifferential operators and Appendix C.7 on Fourier integral operators.

Let us note that some of the sections of this chapter contain a relatively specialized material and can be skipped on the first reading. In particular, a reader who needs just a short introduction to the basic construction of modified wave operators can restrict himself just to the first part of Section 3.2 and Section 3.6. On the first reading it is also a good idea to learn the alternative constructions of modified wave operators given in Sections 3.8 and 3.12.

The literature on quantum scattering theory, as we mentioned earlier, is devoted mainly to time-independent Hamiltonians of the form (3.0.2). It is very rich and contains numerous techniques. The reader will find a bibliographical review of this subject up to the early eighties in [RS3].

Main approaches to the problem of the asymptotic completeness can be divided into the abstract Kato-Birman approach, the stationary approach and the time-dependent approach.

The Kato-Birman approach uses the invariance of wave operators under the replacement of \( H_0 \) and \( H \) with \( f(H_0) \) and \( f(H) \). It applies only to a rather narrow class of fast decaying short-range potentials. The reader can find its description in [RS3].

The stationary method is usually more efficient. It is based on the study of the resolvent \((z - H)^{-1}\). A modern exposition can be found in [Hö2] vol II (the short-range case) and vol IV (the long-range case), see also [A1], [A2] and [RS3] and [RS4].

Note, however, that in our book we will use neither of the above two approaches. In particular, they seem to be useless in the case of time-dependent Hamiltonians that we consider in this chapter (see however [How] and [DKY] for the short-range case).

In the time-dependent approach the main object under study is the dynamics generated by the Hamiltonian \( H(t) \). It was used in the first proof of the existence of wave operators in [Cook]. On the contrary, in the problem of the asymptotic completeness time-dependent techniques entered the literature quite late with papers of V. Enss ([E1], [E2]). Ideas of Enss aroused a considerable interest and inspired a number of papers by other authors, among them: [Dav], [Ki.Ya1], [Ki.Ya2], [Pe] and [Sim].

The time-dependent approach to the scattering theory proved to be the most successful. It led to a series of remarkable results about \( N \)-body systems beginning with [E5] and [S.S1], which we discuss later on. There exist various techniques related to the time-dependent approach. In particular, one of the techniques which proved to be important is the so-called method of positive commutators (or, more exactly, the method of positive Heisenberg derivatives)—see Appendix C.1 for references.

In Section 3.1 we give references on the problem of defining a dynamics by a time-dependent Hamiltonian. This is actually a difficult problem, although it is essentially irrelevant for the main subject of this chapter.

The existence of the asymptotic momentum and short-range scattering theory are very easy in the case of Hamiltonians considered in this chapter. The most difficult subject of this chapter is the long-range scattering theory. In fact, a desire to give a clear exposition of the main technical difficulties of the long-range problem led us to write a separate chapter on time-dependent Hamiltonians.

Let us sketch the history of the long-range scattering theory. The definition of a modified wave operator in the case of the Coulomb potential was first given by Dollard in [Do1]. It was extended to a much larger class of potentials in [Bu-Ma]. Other early papers on the subject include [AK] and [AMM]. In [Hö1] Hörmander introduced modified free dynamics defined by exact solutions of the Hamilton-Jacobi equation.

The asymptotic completeness for long-range two-body systems was first proven by Saito ([Sa1], [Sa2]) and Kitada ([K3], [K4], [K5]). Their proofs used the stationary approach.

A fully time-dependent proof was first given in [E5] and later in [Pe] in the case \( \mu > \frac{1}{2} \) (see (3.0.12)). A time-dependent proof for potentials with a slower decay was given in [Ki.Ya1] and [Ki.Ya2]. Let us note that this proof allowed for time-dependent potentials.
A different definition of modified wave operators that uses a Fourier integral modifier whose phase is a solution of the eikonal equation was given in [Kako] and [IK1].

A time-independent proof of the asymptotic completeness using very weak assumptions on the decay of the potential was given in [H62] vol IV. This proof used (mostly unpublished) ideas of Agmon.

A very appealing time-dependent proof of the asymptotic completeness was given in [S].

The proof given in this chapter in Section 3.7 follows to a great extent that of [S] in its slightly simplified form contained in [De]. In Section 3.6 we present an improved version of this proof which works under much less restrictive hypotheses on the potential. In this proof some of the ideas of [H62, Vol. IV] are incorporated into the method of Sigal.

Regularity properties of wave operators were considered in [Iso3], [Iso4], [A2] and, recently, in [JenNa].

Among numerous other papers on the long-range scattering theory and related subjects let us mention [Ar], [BC], [Com1], [Com2], [Geo], [GGNT], [E6], [Ike1], [Ike2], [Ike-Iso2], [Iso1], [Iso2], [IK2], [IK3] and [IK4].

Pseudodifferential operators play an important role in our treatment of the long-range scattering. The class of pseudodifferential operators that we use in this paper is not the one most widely used in the literature and goes under the name of operators with the symbols of class $S^0_{0,0}$ in the notation of [H62, Vol. III]. This class is very natural from the mathematical point of view. Its basic properties are the boundedness on $L^2(R^n)$—the Calderon-Vaillancourt theorem—(see [CV] and [Ta]) and the so-called Beals criterion (see [Bea]). For more information on the literature on pseudodifferential operators we refer the reader to [H62, Vol. III], [Ta] and [Bo-Ch].

Another tool that we use in this chapter is Fourier integral operators. Once again, Fourier integral operators used in the literature are usually of a different kind. For instance, they usually have a phase homogeneous of degree one with respect to one of the variables (see [H62] vol IV). In the context of this chapter a different condition on the phase seems to be more natural—roughly speaking, the boundedness of the derivatives of order $\geq 2$. Fourier integral operators with similar conditions on the phase were probably first considered in [AF], [Full] and [Fu2]. Applications of Fourier integral operators to the Schrödinger equation are contained in [Ki1], [Ki2], [K-K], [Ki.Ya1], [Ki.Ya2] and [Ya2].

3.1 Existence of the dynamics

In this chapter we will consider dynamics generated by time-dependent Hamiltonians. In most respects it will make possible to simplify the exposition of various points of scattering theory. Unfortunately, it leads also to a technical question, which in general is not easy: when does a time-dependent Hamiltonian generate a dynamics and what does it mean. In the time-independent case the answer is simple: the Hamiltonian has to be self-adjoint. However, not every time-dependent self-adjoint Hamiltonian generates a dynamics.

In this section we would like to state some general properties of time-dependent Hamiltonians and dynamics that we will use in this chapter. We will not describe (apart from a rather simple example) sufficient conditions that one might impose on the Hamiltonian to guarantee that these properties are true.

Let $H$ be a Hilbert space. The scalar product of $\phi, \psi \in H$ will be denoted by $(\phi | \psi)$.

We need first to recall some basic facts about the measurability and integration for operator-valued functions. We will say that a function $[T_1, T_2] \ni t \mapsto B(t) \in B(H)$ is Bochner integrable iff there exists a sequence $B_n(t)$ of measurable step functions (i.e. functions whose range is a finite set and every preimage is measurable) such that

$$\lim_{n \to \infty} \int_{T_1}^{T_2} ||B(t) - B_n(t)||dt = 0.$$  

We will denote the set of Bochner integrable functions by $L^1_{loc}(R, B(H))$. Note that for such $B(t)$ the
integral
\[ \int_{t_1}^{t_2} B(s)ds = \lim_{n \to \infty} \int_{t_1}^{t_2} B_n(s)ds \]
is well defined and does not depend on the choice of \( B_n(t) \).

We will say that \( A(t) \in W^{1,1}(\mathbb{R}, B(\mathcal{H})) \) if there exists \( B(t) \in L^1_{\text{loc}}(\mathbb{R}, B(\mathcal{H})) \) such that for any \( t_1, t_2 \in \mathbb{R} \)

\[ A(t_2) - A(t_1) = \int_{t_1}^{t_2} B(s)ds. \]

We will write
\[ \frac{d}{dt} A(t) := B(t). \]

Let us note that if \( A_1(t), A_2(t) \in W^{1,1}(\mathbb{R}) \), then \( A_1(t)A_2(t) \in W^{1,1}(\mathbb{R}, B(\mathcal{H})) \) and

\[ \frac{d}{dt} A_1(t)A_2(t) = \left( \frac{d}{dt} A_1(t) \right) A_2(t) + A_1(t) \frac{d}{dt} A_2(t). \]

Let \( B \) be a fixed positive operator with a dense domain. We will say that the time-dependent Hamiltonian

\[ t \to H(t) \]
generates a dynamics

\[ (t, s) \mapsto U(t, s) \]
if the following conditions are satisfied:

1) \( U(s, s) = 1; \)
2) \( U(t, u)U(u, s) = U(t, s); \)
3) \( U(t, s)(1 + B)^{-1} \) and \( (1 + B)^{-1}U(t, s) \) belong to \( W^{1,1}(\mathbb{R}, B(\mathcal{H})); \)
4) \( H(t) \) is a self-adjoint operator on \( \mathcal{H} \) for almost all \( t, \mathcal{D}(H(t)) \subset \mathcal{D}(B) \) and \( H(t)(1 + B)^{-1} \) belongs to \( L^1_{\text{loc}}(\mathbb{R}, B(\mathcal{H})); \)
5) \[
\partial_t(1 + B)^{-1}U(t, s) = -(1 + B)^{-1}iH(t)U(t, s), \quad \partial_s U(t, s)(1 + B)^{-1} = U(t, s)iH(s)(1 + B)^{-1};
\]
6) \[
\lim_{\epsilon \to 0} \int_{T_1}^{T_2} ||H(t), (1 + \epsilon B)^{-1}||dt = 0.
\]

**Proposition 3.1.1** Under the above assumptions \( U(t, s) \) is unitary.

**Proof.** Clearly, by 1) and 2) \( U(t, s)^{-1} = U(s, t) \).

Let \( \phi \in \mathcal{H} \). It is enough to show that

\[ \lim_{\epsilon \to 0} \phi[U(s, t)(1 + \epsilon B)^{-1}U(t, s)\phi] = ||\phi||^2. \]

By the self-adjointness of \( H(t) \)

\[ \phi[U(s, t)(1 + \epsilon B)^{-1}U(t, s)\phi] = ||\phi||^2 \]

\[ = \int_s^t \phi[U(s, u)[H(u), (1 + \epsilon B)^{-1}U(u, s)\phi]du. \]

But by 6) the right hand side of (3.1.2) goes to zero as \( \epsilon \to 0 \).

Let us note another consequence of the above conditions that we will often use in this chapter.
Proposition 3.1.2 Let $U_i(t, s), \ i = 1, 2$ be two dynamics generated by two time-dependent Hamiltonians $H_i(t)$ that satisfy the conditions 1)–6) described above. Suppose that $\Phi(t)$ belongs to $W^{1,1}(\mathbb{R}, B(H))$ and

$$H_2(t)\Phi(t) - \Phi(t)H_1(t)$$

originally defined as a quadratic form on $\mathcal{B}(B)$ extends to an element of $L^1_{\text{loc}}(\mathbb{R}, B(H))$. Then

$$U_2(s, t_2)\Phi(t_2)U_1(t_2, s) = U_2(s, t_1)\Phi(t_1)U_1(t_1, s) + \int_{t_1}^{t_2} U_2(s, u) \left( iH_2(u)\Phi(u) - i\Phi(u)H_1(u) + \frac{d}{du}\Phi(u) \right) U_1(u) du.$$ 

Proof. We can write

$$U_2(s, t_2)\Phi(t_2)U_1(t_2, s) = U_2(s, t_1)\Phi(t_1)U_1(t_1, s) + \int_{t_1}^{t_2} U_2(s, u) \left( iH_2(u)\Phi(u) - i\Phi(u)H_1(u) + \frac{d}{du}\Phi(u) \right) (1 + \epsilon B)^{-1} U_1(u, s) du + \int_{t_1}^{t_2} U_2(s, u) \Phi(u)(1 + \epsilon B)^{-1} U_1(u, s) du$$

By 6) the last two terms on the right are zero. □

Let us note one simple example of a situation when we can construct a dynamics that satisfies the conditions 1)–6). Suppose that $H_0$ is a fixed self-adjoint operator and

$$t \rightarrow V(t)$$

is a function with values in bounded self-adjoint operators that belongs to $L^1_{\text{loc}}(\mathbb{R}, B(H))$. Then all the conditions 1)–6) are satisfied if we define $U(s, t)$ by the following convergent expansion:

$$U(t, s) = \sum_{n=0}^{\infty} \int \cdots \int e^{i(t-s)H_0} V(u_1) \cdots V(u_n) e^{i(u_{n+1})H_0} du_1 \cdots du_n.$$ 

Throughout this chapter we will always assume that

$$B = D^2 + x^2$$

and

$$H(t) = \frac{1}{2}D^2 + V(t, x),$$

where $V(t, x)$ is a real function. We will always assume that $H(t)$ generates a dynamics $U(t, s)$ in the sense defined in 1)–6).

Let us warn the reader that most probably the conditions 4)–6) do not guarantee the existence of a dynamics satisfying 1)–6). Actually, the question of when $H(t)$ generates a dynamics is irrelevant for the problems that we are interested in this chapter.

The problem of finding sufficient conditions on $V(t, x)$ to guarantee the well definedness of $U(t, s)$ has been attacked by many authors using a variety of methods. Among all these works we would like to mention just a few. The papers [K1] and [K2] used the abstract theory of evolution equations in Banach spaces. A rather different approach by reduction to a time-independent problem has been given by Howland [H1]. Another method based on Fourier integral operators has been introduced by Fujiiwara and developed in [F1], [F2], [K-K] and [K2]. A third approach based on the study of the associated integral equation satisfied by $U(t, s)$ by perturbation from the free evolution has been introduced by Yajima [Y1]. We will describe this last method in Section 3.5.
3.2 Asymptotic momentum

As in the classical case, we will start our exposition of the scattering theory for time-dependent potentials with the construction of the asymptotic momentum, which is a basic asymptotic quantity common to the short- and long-range cases. We will denote by $\nabla_x V(t, x)$ the distributional derivative of $V(t, x)$, it is equal to the (possibly unbounded) operator $[D, iV(t, x)]$. Our first result will be the quantum analog of Theorem 1.2.1.

**Theorem 3.2.1** Suppose that

\[
(3.2.1) \quad \int_0^\infty \|(1 + D^2)^{-1}, V(t, x)\|dt < \infty.
\]

Then there exists the limit

\[
(3.2.2) \quad s-C \lim_{t \to \infty} U(0, t)DU(t, 0) =: D^+.
\]

$D^+$ is a vector of commuting self-adjoint operators that satisfies

\[
[D^+, H] = 0.
\]

If we assume in addition that

\[
V(t, x) = V_0(t, x) + V_1(t, x)
\]

such that

\[
(3.2.3) \quad \int_0^\infty \|V_0(t, x)\|dt < \infty,
\]

\[
(3.2.4) \quad \lim_{\epsilon \to 0} \int_0^\infty \|(1 + \epsilon D^2)^{-1}, V_1(t, x)\|dt = 0,
\]

then $D^+$ is densely defined.

**Remark 3.2.2** Let us note that the conditions (3.2.3) and (3.2.4) imply (3.2.1). Moreover, (3.2.4) follows from the following condition: for some $\sigma < \frac{1}{2}$

\[
(3.2.5) \quad \int_0^\infty \|(D)^{-\sigma} \partial^\alpha_x V_1(t, x)(D)^{-\sigma}\|dt < \infty, \quad |\alpha| = 1.
\]

**Remark 3.2.3** If we assume that

\[
\int_0^+ \|\nabla_x V(t, x)\|dt < \infty,
\]

then $D^+ - D$ is a bounded operator.

This fact can be seen as an analog of Theorem 1.2.1. To prove this statement it is enough to note that one has

\[
U(0, t)DU(t, 0)\phi - D\phi = \int_0^t U(0, s)\nabla_x V(t, x)U(s, 0)\phi ds,
\]

which proves the result by letting $t$ go to $+\infty$.

**Proof of Theorem 3.2.1.** Let us first prove the existence of the limit in (3.2.2). By a density argument, it is enough to show the existence of

\[
(3.2.6) \quad s-C \lim_{t \to \infty} U(0, t)g(D)U(t, 0)
\]
for $g \in C_0^\infty(X)$.

Now
\[
\frac{d}{dt} U(0, t) g(D) U(t, 0) = U(0, t) [iV(t, x) , g(D)] U(t, 0).
\]

By Lemma C.5.2,
\[
[[iV(t, x) , g(D)]] \leq C \frac{1}{(1 + D^2)^{-1}}.
\]

This is integrable. So the existence of (3.2.6) follows by integration.

Next let us prove that $D^+$ has a dense domain. Let $U_1(t, s)$ denote the dynamics generated by the Hamiltonian
\[
H_1(t) := \frac{1}{2} D^2 + V_1(t, x).
\]

Then it is very easy to see that there exists
\[
\lim_{t \to \infty} U(0, t) U_1(t, 0).
\]

Now
\[
\begin{aligned}
&\lim_{t \to \infty} U_1(0, t) (1 + \epsilon D^2)^{-1} U_1(t, 0) - (1 + \epsilon D^2)^{-1} \\
&= \epsilon \int_0^\infty U_1(0, t) [(1 + \epsilon D^2)^{-1} , V_1(t, x)] U_1(t, 0) dt.
\end{aligned}
\]

This converges to zero as $\epsilon \to 0$. Therefore
\[
\begin{aligned}
&\lim_{t \to 0} (1 + \epsilon D^2)^{-1} \\
&= \lim_{t \to 0} \left( \lim_{t \to \infty} U_1(0, t) (1 + \epsilon D^2)^{-1} U_1(t, 0) \right) = 1.
\end{aligned}
\]

Applying Remark C.3.4 from Appendix C.3, this implies that the self-adjoint operator $D^+$ has a dense domain. \(\square\)

The observable $D^+$ classifies the states in $L^2(X)$ according to their asymptotic behavior in momentum space. It turns out that there exists an alternative method of constructing $D^+$ using the operator $\hat{F}$ instead of $D$. Consequently $D^+$ describes also the asymptotic behavior of states in position space.

**Theorem 3.2.4** Assume (3.2.3) and (3.2.4). Then there exists the strong $C_\infty$-limit
\[
(3.2.8) \quad s-C_\infty \lim_{t \to \infty} U(0, t) \frac{2}{t} U(t, 0)
\]

which is equal to $D^+$.

To prove this theorem we will need some additional techniques, which will be further developed in a somewhat different situation in the next chapter. Note that the remaining part of this section will not be used in this chapter.

**Proposition 3.2.5** Assume (3.2.1). Suppose that $j, g \in C_0^\infty(X)$ and $\text{supp} j \cap \text{supp} g = \emptyset$. Then
\[
(3.2.9) \quad \int_1^\infty \left\| \frac{2}{t} \frac{d}{dt} \right\| \phi \left\| \frac{dt}{t} \right\| \leq C ||\phi||^2, \phi \in L^2(X).
\]

If moreover $J \in C_0^\infty(X)$ such that $J = 1$ on a neighborhood of $\text{supp} g$ then
\[
(3.2.10) \quad s- \lim_{t \to \infty} U(0, t) J \left( \frac{2}{t} \right) g(D) U(t, 0) = g(D^+).
\]
Proof. We will prove the proposition by constructing a suitable propagation observable and applying Lemma C.1.1 of the Appendix C.1. By a covering argument, we may assume that the support of $g$ and $j$ are very close respectively to $\xi_0 \in \mathcal{X}$ and $x_0 \in \mathcal{X}$, with $\xi_0 \neq x_0$. We can then find $v \in \mathcal{X}$ and $\theta_1 < \theta_2$ such that
\[
\text{supp} g \subset \{ x | \langle v, x \rangle > \theta_1 \},
\]
\[
\text{supp} j \subset \{ x | \langle v, x \rangle < \theta_2 \}.
\]
Choose a function $\tilde{j} \in C^\infty(\mathbb{R})$ such that $\tilde{j} \in C_0^\infty(\mathbb{R})$ and
\[
\nabla_x \tilde{j}(\langle v, x \rangle) \geq j^2(x).
\]
Set $J(x) := \tilde{j}(\langle v, x \rangle)$.

We consider the following propagation observable
\[
g(D)J \left( \frac{x}{t} \right) g(D),
\]
which is uniformly bounded in $t$, and compute its Heisenberg derivative. We obtain
\[
\begin{align}
\mathbf{D}_t g(D) & = [V(t, x), ig(D)] J(\xi) g(D) + \text{hc} \\
& \quad + \frac{1}{2t} g(D)(D - \xi) \nabla_x J(\xi) g(D) + \text{hc}.
\end{align}
\]
We claim that for some $C_0 > 0$ we have
\[
\frac{1}{2t} g(D) \left( D - \frac{x}{t} \right) \nabla_x J \left( \frac{x}{t} \right) g(D) + \text{hc} \geq C_0 \frac{1}{t} g(D) j^2 \left( \frac{x}{t} \right) g(D) + O(t^{-3}).
\]

One way to prove (3.2.12) is to use the pseudodifferential calculus. In fact, we can rewrite the left hand side of (3.2.12) as
\[
\frac{1}{t} \rho^w(t, x, D) + O(t^{-3}),
\]
where
\[
r(t, x, \xi) = g^2(\xi) \left( \xi - \frac{x}{t} \right) \nabla_x J \left( \frac{x}{t} \right).
\]
The right hand side of (3.2.12) can be rewritten as
\[
\frac{1}{t} p^w(t, x, D) + O(t^{-3}),
\]
where
\[
p(t, x, \xi) = g^2(\xi) j^2 \left( \frac{x}{t} \right).
\]
Both $r(t, x, \xi)$ and $p(t, x, \xi)$ are symbols of the class $\mathcal{S}((t)^{-2}dx^2 + \tilde{d}\xi^2)$ (see Appendix C.6). We clearly have
\[
r(t, x, \xi) \geq (\theta_2 - \theta_1)p(t, x, \xi).
\]
Using (3.2.13) and the sharp Gårding inequality (see Proposition C.6.6) we get
\[
\frac{1}{t} \rho^w(t, x, D) \geq \frac{1}{t} (\theta_2 - \theta_1)p^w(t, x, D) + O(t^{-3}),
\]
which proves (3.2.12).
An alternative way to show (3.2.12) is to write the left hand side of (3.2.12) as

\[
\frac{1}{t}g(D) \left( \langle D, v \rangle - \left( \frac{\langle x, v \rangle}{t} \right) \right) \left( \frac{\langle x, v \rangle}{t} \right) g(D) + \text{hc}
\]

(3.2.14)

\[
= \frac{1}{t}g(D) \left( \langle D, v \rangle - \theta_2 \right) \left( \frac{\langle x, v \rangle}{t} \right) g(D) + \text{hc}
\]

\[
+ \frac{1}{t}g(D) \left( \theta_2 - \left( \frac{\langle x, v \rangle}{t} \right) \right) \left( \frac{\langle x, v \rangle}{t} \right) g(D) + \text{hc}.
\]

The second term on the right of (3.2.14) is greater than

\[
\frac{1}{t}(\theta_2 - \theta_1) g(D) \left( \frac{\langle x, v \rangle}{t} \right) g(D).
\]

In the first term we can commute functions of \(D\) and of \(x\) so that it becomes equal to

\[
\frac{1}{t}g(D) \left( \langle D, v \rangle - \theta_2 \right)^{1/2} \left( \frac{\langle x, v \rangle}{t} \right) \left( \langle D, v \rangle - \theta_2 \right)^{1/2}g(D) + O(t^{-3}) \geq O(t^{-3}).
\]

This ends the second proof of (3.2.12).

Applying Lemma C.1.1 we see that (3.2.12) implies (3.2.9). Let us now consider \(J \in C^\infty(X)\) such that \(\nabla_x J \in C_0^\infty(X)\) and \(\supp \nabla_x J \cap \supp g = \emptyset\). We will prove now that there exists

\[
(3.2.15)
\]

\[
\text{s-} \lim_{t \to \infty} U(0, t)J \left( \frac{\langle x, v \rangle}{t} \right) g(D)U(t, 0).
\]

In fact, take \(\tilde{j}, \tilde{g} \in C_0^\infty(X)\) such that \(\tilde{j} \nabla_x J = \nabla_x J, \tilde{g} g = g\) and \(\supp \tilde{j} \cap \supp g = \emptyset\). Then we can estimate

\[
(3.2.16)
\]

\[
\left\| \phi \left\{ D \left( \frac{\langle x, v \rangle}{t} \right) g(D) \phi \right\} \right\| \leq C t^{-1} \left\| \tilde{j} \left( \frac{\langle x, v \rangle}{t} \right) \tilde{g}(D) \phi \right\|^2 + O \left( \frac{1}{t^2} \right).
\]

This is integrable along the evolution by (3.2.9). Hence (3.2.15) exists.

If we assume in addition that \(J \in C_0^\infty(X)\) and \(\supp J \cap \supp g = \emptyset\), then we know by (3.2.9) that

\[
(3.2.17)
\]

\[
\int_1^\infty \left\| J \left( \frac{\langle x, v \rangle}{t} \right) g(D)U(t, 0) \phi \right\|^2 \frac{dt}{t} < \infty.
\]

Clearly, if a function \(f(t)\) satisfies

\[
\begin{cases}
\lim_{t \to \infty} f(t) \text{ exists,} \\
\int_1^\infty f(t) \frac{dt}{t} < \infty,
\end{cases}
\]

then \(\lim_{t \to \infty} f(t) = 0\). Hence (3.2.17) and the existence of (3.2.15) imply that (3.2.15) is zero if \(J \in C_0^\infty(X)\), \(\supp J \cap \supp g = \emptyset\).

Unfortunately, this is not end of the proof of (3.2.10), since we need to show that there is no propagation for large \(\frac{\|f\|}{t}\). To this end, take functions \(F \in C^\infty(\mathbb{R}), f \in C_0^\infty(\mathbb{R})\) such that \(F = 0\) on a neighborhood of \(0\), \(F = 1\) on a neighborhood of \(\infty\), and \(F' = f^2\). Now

\[
(3.2.18)
\]

\[
-Dg(D)F \left( \frac{\langle x, v \rangle}{t} \right) g(D)
\]

\[
= [V(t, x), g(D)]F \left( \frac{\langle x, v \rangle}{t} \right) g(D) + \text{hc}
\]

\[
+ \frac{1}{t}g(D)f^2 \left( \frac{\langle x, v \rangle}{t} \right) \frac{dt}{t} g(D)
\]

\[
+ \frac{1}{t^2}g(D)D \left( \frac{\langle x, v \rangle}{t} \right) f^2 \left( \frac{\langle x, v \rangle}{t} \right) g(D) + \text{hc}.
\]
Let \( \tilde{g} \in C_0^\infty(X) \) such that \( \tilde{g}g = g \). The third term on the right hand side of (3.2.18) equals

\[
\frac{1}{t^2 R^2} (D) f \left( \frac{|x|}{R} \right) g(D) + \frac{1}{t R} f \left( \frac{|x|}{R} \right) g(D) + O(t^{-2} R^{-2})
\]

Hence for \( R \geq C_0 \)

(3.2.19) \[
-Dg(D)F \left( \frac{|x|}{R} \right) g(D) \geq -C ||V(t, x), g(D)|| + O(t^{-2} R^{-2})
\]

Therefore, for \( R > C_0 \) and any \( t_0 \geq 0 \)

\[
\lim_{R \to \infty} U(0, t) g(D) F \left( \frac{|x|}{R} \right) g(D) U(t, 0)
\]

(3.2.20) \[
\leq U(0, t_0) g(D) F \left( \frac{|x|}{R} \right) g(D) U(t_0, 0)
\]

\[
+ C \int_{t_0}^{\infty} \left| ||V(t, x), g(D)|| \right| d\tau + O(t_0^{-1} R^{-2}).
\]

By choosing \( t_0 \) big enough we can make the integral on the right hand side of (3.2.20) as small as we wish. For a fixed \( t_0 \) the first and third terms on the right hand side of (3.2.20) go to zero as \( R \to \infty \). Hence

(3.2.21) \[
\lim_{R \to \infty} \left( \lim_{t \to \infty} U(0, t) g(D) F \left( \frac{|x|}{R} \right) g(D) U(t, 0) \right) = 0
\]

But we already know that for big enough \( R_1, R_2 \)

\[
\lim_{t \to \infty} U(0, t) g(D) \left( F \left( \frac{|x|}{R_1} \right) - F \left( \frac{|x|}{R_2} \right) \right) g(D) U(t, 0) = 0,
\]

since the function \( F \left( \frac{|x|}{R_1} \right) - F \left( \frac{|x|}{R_2} \right) \) has a compact support. Therefore, for big \( R \)

\[
\lim_{t \to \infty} U(0, t) g(D) F \left( \frac{|x|}{R} \right) g(D) U(t, 0) = 0
\]

This ends the proof of (3.2.10).

**Proposition 3.2.6** Assume (3.2.1). Suppose that \( g, J \in C_0^\infty(X) \) and \( \text{supp} g \cap \text{supp} \nabla J = 0 \). Then

(3.2.22) \[
\int_1^{\infty} \left\| \frac{x}{t} - D \right\| J \left( \frac{x}{t} \right) g(D) U(t, 0) \| \phi \|_2 dt \leq C \| \phi \|_2, \phi \in L^2(X).
\]

and

(3.2.23) \[
\lim_{t \to \infty} \frac{x}{t} - D \left[ J \left( \frac{x}{t} \right) g(D) U(t, 0) = 0.
\]

**Proof.** We consider the following propagation observable

\[
\Phi(t) := -g(D) \left( \frac{x}{t} \right) \left( \frac{x}{t} - D \right) J \left( \frac{x}{t} \right) g(D)
\]

\[
= \left( \frac{x}{t} - D \right) J \left( \frac{x}{t} \right) g(D) U(t, 0) = 0.
\]

\[
\Rightarrow \frac{x}{t} - D \left[ J \left( \frac{x}{t} \right) g(D) U(t, 0) = 0.
\]

This ends the proof of (3.2.22)
which is bounded uniformly in \( t \), and compute its Heisenberg derivative. We obtain

\[
D \Phi(t) = -\frac{i}{2} g(D)D(J(\frac{x}{t}))(\frac{x}{t} - D)^2 J(\frac{x}{t})g(D) + hc
\]

\[
-\left[ V(t, x), ig(D) \right] J(\frac{x}{t})(\frac{x}{t} - D)^2 J(\frac{x}{t})g(D) + hc
\]

\[
+\frac{1}{2} g(D)J(\frac{x}{t})(\frac{x}{t} - D)^2 J(\frac{x}{t})g(D)
\]

\[
+g(D)J(\frac{x}{t})(\frac{x}{t} - D)[V(t, x), iD]J(\frac{x}{t})g(D) + hc.
\]

By Lemma C.5.2, the terms in the second and fourth line of the right hand side of (3.2.24) are integrable in norm. Let \( j \in C_0^\infty(X) \) is such that \( j = 1 \) on \( \text{supp} \nabla_x J \) and \( \text{supp} g \cap \text{supp} j = \emptyset \). Then the first line of the right hand side of (3.2.24) can be written as

\[
\frac{1}{2t} g(D) \nabla J(\frac{x}{t}) \left( \frac{x}{t} - D \right)^3 j \left( \frac{x}{t} \right) g(D) + hc + O(t^{-3})
\]

This is integrable along the evolution by Proposition 3.2.5. Now by Lemma C.1.1 we obtain (3.2.22).

To prove (3.2.23), we observe that there exists

\[
s = \lim_{t \to \infty} U(0, t) \Phi(t) U(t, 0),
\]

because the Heisenberg derivative of \( \Phi(t) \) is integrable by Proposition 3.2.5. Moreover by (3.2.22), we have

\[
\int_1^\infty (\phi, U(0, t) \Phi(t) U(t, 0) \phi) \frac{dt}{t} < \infty.
\]

Therefore,

\[
\lim_{t \to \infty} U(0, t) \Phi(t) U(t, 0) = 0,
\]

which proves (3.2.23).

\[\square\]

\textbf{Proof of Theorem 3.2.4} Let \( f, g \in C_0^\infty(X) \) and \( \phi \in L^2(X) \). Since the domain of \( D^+ \) is dense, it suffices to show that

\[
\lim_{t \to \infty} U(0, t) f \left( \frac{x}{t} \right) U(t, 0) g(D^+) \phi - f(D^+) g(D^+) \phi = 0.
\]

Choose \( J \in C_0^\infty(X) \) such that \( J = 1 \) on a neighborhood of \( \text{supp} g \). Then by (3.2.10) the left hand side of (3.2.27) is equal to

\[
\lim_{t \to \infty} U(0, t) \left( f \left( \frac{x}{t} \right) - f(D) \right) J \left( \frac{x}{t} \right) g(D) U(t, 0) \phi
\]

\[
= \lim_{t \to \infty} U(t, 0) \int_0^1 \nabla f \left( \tau \frac{x}{t} \right) (1 - \tau) D \left( \frac{x}{t} \right) J \left( \frac{x}{t} \right) g(D) U(t, 0) \phi
\]

\[
+ \lim_{t \to \infty} \frac{1}{t} U(0, t) \int_0^1 \Delta f \left( \tau \frac{x}{t} \right) (1 - \tau) D \left( \frac{x}{t} \right) J \left( \frac{x}{t} \right) g(D) U(t, 0) \phi.
\]

This equals zero by (3.2.23).

\[\square\]

\textbf{3.3 The short-range case}

The asymptotic momentum constructed in Theorem 3.2.1 gives a classification of the states in \( L^2(X) \) according to their behavior under the dynamics \( U(t, 0) \). Clearly to understand the evolution \( U(t, 0) \) for large \( t \) we need first of all to study the spectral properties of \( D^+ \). In this respect a very natural question
to ask is whether the asymptotic momentum is unitarily equivalent to the momentum. The answer is positive only if we assume some additional conditions on the potential. In the short-range case one can construct a unitary operator that intertwines the momentum and the asymptotic momentum in a particularly simple way.

Let us start this section with some general remarks. Assume for the moment that \( U(t, 0) \) is a rather arbitrary family of unitary operators on a Hilbert space \( \mathcal{H} \), which we want to compare with the flow generated by a self-adjoint operator \( H_0 \) on \( \mathcal{H} \). It is possible to do it in at least two ways: either we compare the automorphisms generated by these dynamics, or we compare the dynamics themselves.

To be more precise, let us introduce the following two definitions. For a given Hilbert space \( \mathcal{H} \), we will denote by \( \mathcal{J}_\infty(\mathcal{H}) \) the ideal of compact operators in \( B(\mathcal{H}) \).

**Definition 3.3.1** Assume that for a certain subspace \( \mathcal{H}_1 \subset \mathcal{H} \) and any \( A \in \mathcal{J}_\infty(\mathcal{H}_1) \) there exists the norm limit
\[
\lim_{t \to \infty} e^{itH_0} U(t, 0) AU(0, t) e^{-itH_0}.
\]

Suppose also that the mapping
\[
\mathcal{J}_\infty(\mathcal{H}_1) \ni A \mapsto \lim_{t \to \infty} e^{itH_0} U(t, 0) AU(0, t) e^{-itH_0} \in \mathcal{J}_\infty(\mathcal{H})
\]
is onto. Clearly, the mapping (3.3.2) is an injective homomorphism of \( C^* \)-algebras. We call its inverse the wave homomorphism and denote it by \( \omega^+_{br} \).

**Definition 3.3.2** We say that the wave operator exists if there exists
\[
s_\infty^0 U(0, t) e^{-itH_0} = \Omega^+_{br}.
\]

Note that \( \Omega^+_{br} \) is automatically an isometry.

The relationship between these two definitions is explained in the following proposition.

**Proposition 3.3.3** a) Suppose that the wave operator \( \Omega^+_{br} \) exists and its range equals \( \mathcal{H}_1 \). Then the wave homomorphism from \( \mathcal{J}_\infty(\mathcal{H}) \) to \( \mathcal{J}_\infty(\mathcal{H}_1) \) exists and equals
\[
\omega^+_{br}(A) = \Omega^+_{br} A \Omega^+_{br}.
\]
b) Suppose that the wave homomorphism from \( \mathcal{J}_\infty(\mathcal{H}) \) to \( \mathcal{J}_\infty(\mathcal{H}_1) \) exists. Then there exists a real function \( R^+ \ni t \mapsto \theta(t) \) such that the following limit exists
\[
s_\infty^0 \lim_{t \to \infty} U(0, t) e^{-itH_0 - i\theta(t)} = \Omega^+_{br, \theta}.
\]

Moreover, the range of \( \Omega^+_{br, \theta} \) equals \( \mathcal{H}_1 \) and
\[
\omega^+_{br}(A) = \Omega^+_{br, \theta} A \Omega^+_{br, \theta}.
\]

**Proof.** To see a) it is enough to note that
\[
\lim_{t \to \infty} e^{itH_0} U(t, 0)
\]
extists on \( \mathcal{H}_1 \) and the strong convergence of compact operators implies the norm convergence.
To prove b) choose an orthonormal basis \{\phi_i \mid i \in I\} of \mathcal{H}_1. For any \( i \in I \) there exists
\[
\lim_{t \to \infty} e^{i t H_0} U(t, 0) |\phi_i \rangle \langle \phi_i| U(0, t) e^{-i t H_0}. \]
Therefore, we will find \( \theta_i(t) \) such that
\[
\lim_{t \to \infty} U(0, t) e^{-i t H_0 - i \theta_i(t)} |\phi_i \rangle
\]
exists. Considering the compact operator \( |\phi_i \rangle \langle \phi_j| \) we see that \( e^{i\theta_i(t) - \theta_j(t)} \) has a limit. Therefore, it is enough to fix a \( i \in I \) and set \( \theta(t) := \theta_i(t) \). □

One can argue that wave homomorphisms are better motivated physically. Nevertheless, as the above proposition shows, if a wave homomorphism exists then by renormalizing the Hamiltonian with an appropriate constant we can make sure that the wave operators exist too.

We return to our usual framework where \( H(t) = \frac{1}{4} D^2 + V(t, x) \) as in Sections 3.1 and 3.2. We will give conditions on the potential that guarantee the existence of wave operators. We will denote by \( H_0 \) the free Hamiltonian
\[
H_0 = \frac{1}{2} D^2
\]
which generates the free dynamics \( e^{-i t H_0} \). We will often denote \( e^{-i t H_0} \) by \( U_0(t) \).

**Theorem 3.3.4** Suppose that the potential can be written as
\[
V(t, x) = V_0(t, x) + V_1(t, x),
\]
such that
\[
\begin{align*}
(3.3.5) & \quad \int_0^\infty \|V_0(t, x)\| dt < \infty, \\
(3.3.6) & \quad \int_0^\infty \| (\nabla_x V_1(t, x)) |(t) dt < \infty,
\end{align*}
\]
and \( V_1(t, 0) \in L^1_{\text{loc}}(dt) \).

Set
\[
\theta(t) := \int_0^t V_1(s, 0) ds.
\]
Then there exist
\[
(3.3.7) \quad s- \lim_{t \to \infty} U(0, t) e^{-i t H_0 - i \theta(t)}
\]
and
\[
(3.3.8) \quad s- \lim_{t \to \infty} e^{i t H_0 + i \theta(t)} U(t, 0).
\]
If we denote (3.3.7) by \( \Omega_{\mu, \delta}^+ \) then (3.3.8) equals \( \Omega_{\mu, \delta}^{++} \). Moreover,
\[
D^+ = \Omega_{\mu, \delta}^+ D \Omega_{\mu, \delta}^{++}.
\]
Under the above conditions, the limit
\[
(3.3.10) \quad s-C_\infty = \lim_{t \to \pm \infty} U(0, t)(x - t D) U(t, 0) =: x_{\mu, \delta}^+
\]
exists in strong resolvent sense. The observable \( x_{\mu, \delta}^+ \) is a vector of commuting densely defined self-adjoint operators and will be called the asymptotic position. Moreover one has
\[
(3.3.11) \quad x_{\mu, \delta}^+ = \Omega_{\mu, \delta}^+ x \Omega_{\mu, \delta}^{++}.
\]
Remark 3.3.5 Note that the hypothesis imposed on the potential in the above theorem imply the hypotheses of Theorem 3.2.1. Therefore the asymptotic momentum exists.

Proof. It is enough to assume that \( V_1(t, 0) = 0 \) and \( \theta(t) = 0 \).

Clearly,

\[
x e^{-i t H_0} = t D e^{-i t H_0} + e^{-i t H_0} x.
\]

Hence

\[
\|xe^{-i t H_0}(D)^{-1}(x)^{-1}\| \leq C(t).
\]

Now let \( \phi = \langle D \rangle^{-1}(x)^{-1}\psi \). Then

\[
\frac{d}{dt} U(0, t) e^{-i t H_0} \phi = U(0, t) V(t, x) e^{-i t H_0} \phi
\]

\[
= U(0, t) V_0(t, x) e^{-i t H_0} \phi
\]

\[
+ U(0, t) \int_0^1 \nabla V_1(t, \tau x) d\tau x e^{-i t H_0} \langle D \rangle^{-1}(x)^{-1}, \psi,
\]

which is integrable by (3.3.5), (3.3.6) and (3.3.12). This shows the existence of (3.3.7).

To prove the existence of (3.3.8) we need to introduce an auxiliary dynamics \( U_1(s, t) \) generated by

\[
H_1(t) := \frac{1}{2} D^2 + V_1(t, x).
\]

First we see that there exists

\[
\lim_{t \to \infty} U_1(0, t) U(t, 0).
\]

Next we note that the following identity is true:

\[
U_1(0, t) x U(t, 0) = x + t D + \int_0^t U_1(0, s) \nabla x V_1(s, x) U_1(s, 0)(t - s) ds.
\]

Hence

\[
\|xe^{-i t H_0}(D)^{-1}(x)^{-1}\| \leq C(t)
\]

Let us now prove the existence of

\[
s - \lim_{t \to \infty} e^{-i t H_0} U_1(t, 0).
\]

Consider a vector \( \phi = \langle D \rangle^{-1}(x)^{-1}\psi \). Then

\[
\frac{d}{dt} e^{-i t H_0} U_1(t, 0) \phi
\]

\[
= e^{-i t H_0} \int_0^1 \nabla V_1(t, \tau x) d\tau U_1(t, 0) \langle D \rangle^{-1}(x)^{-1} \psi,
\]

which is integrable by (3.3.6) and (3.3.16). Hence (3.3.17) exists.

Now the existence of (3.3.14) and (3.3.17) imply the existence of (3.3.8). \( \square \)

3.4 Smoothness of short-range wave operators

In this section we will study the scattering theory for potentials that satisfy the so-called smooth short-range condition. More precisely, we will assume, as in Section 1.3, that the potential satisfies

\[
\int_0^\infty (t)^{\nu} ||\partial_0^\alpha V(t, .)||_\infty dt < \infty, \quad |\alpha| \geq 1.
\]
To simplify, we will also suppose that

\[(3.4.2) \quad \int_0^\infty |V(t, 0)| dt < \infty,\]

which implies that we do not need to renormalize the free dynamics in order to define wave operators.

In the classical case under these assumptions we showed that the wave transformation is smooth and all its derivatives are bounded. In the quantum case there is an analog of this property, which can be expressed using an appropriate class of pseudodifferential operators. In Appendix we will give a short introduction to the pseudodifferential calculus. For the convenience of the reader some of the definitions from this appendix are repeated below.

Let \( S(X \times X') \) denote the set of functions on \( X \times X' \) such that

\[ |\partial_x^\alpha \partial_{\xi}^\beta a(x, \xi)| \leq C_{\alpha, \beta}, \ \alpha, \beta \in \mathbb{N}^n. \]

Let \( \Psi(L^2(X)) \) denote the set of operators \( A \) such that

\[ \text{ad}_D^\alpha \text{ad}_x^\beta A \in B(L^2(X)) \ \alpha, \beta \in \mathbb{N}^n. \]

or, equivalently,

\[ A = a^w(x, D) \text{ for some } a \in S(X \times X'). \]

One can regard the algebra \( \Psi(L^2(X)) \) as a non-commutative analog of \( S(X \times X') \). In this section we would like to show that various operators constructed so far for short-range potentials belong to this algebra.

The main result of this section can be formulated in the following theorem.

**Theorem 3.4.1** Assume (3.4.1) and (3.4.2). Then

\[ \Omega^x_+ \in \Psi(L^2(X)), \]

\[ D^+ - D \in \Psi(L^2(X)), \]

\[ x^+_w - x \in \Psi(L^2(X)). \]

The class of operators \( \Psi(L^2(X)) \) is associated in a natural way with the problem we are looking at in this chapter. Unfortunately, in this class we do not have a “semiclassical parameter”, and hence no symbolic calculus is available. A natural semiclassical parameter \( s^{-1} \) appears if we allow our quantities to depend on the initial time \( s \).

Let us introduce some classes of operators and symbols that are natural in the context of this chapter. Let \( g_0(t) \) denote the metric

\[(3.4.3) \quad g_0(t) := \langle t \rangle^{-2} dx^2 + d\xi^2.\]

We will write

\[ a(t, x, \xi) \in S(\rho(\langle t \rangle^m), g_0(t)) \]

if

\[ \|\partial_x^\alpha \partial_{\xi}^\beta a(t, \cdot, \cdot)\|_{\infty} \leq o(\langle t \rangle^{m-|\alpha|}), \ \alpha, \beta \in \mathbb{N}^n. \]

We will write

\[ a(t) \in \Psi(\rho(\langle t \rangle^m), g_0(t)) \]

if one of the following equivalent conditions holds:

\[(3.4.4) \quad \|\text{ad}_D^\alpha \text{ad}_x^\beta a(t)\| \leq o(\langle t \rangle^{m-|\alpha|}), \ \alpha, \beta \in \mathbb{N}^n, \]

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(3.4.5) \[ a(t) = a^m(t, x, D) \] for some \( a \in S(o(t^m), g_0(t)) \).

Let us define

\[
D(t, s) := U(s, t)DU(t, s),
\]

\[
x(t, s) := U(s, t)xU(t, s),
\]

\[
x^s(t, s) := x(t, s) - tD(t, s),
\]

\[
\Omega^s(t, s) := U(s, t)U(t, s).
\]

We extend the definition of \( D(t, s) \), \( x^s(t, s) \) and \( \Omega^s(t, s) \) to \( t = \infty \) in the obvious way.

The following theorem is an extension of Theorem 3.4.1.

**Theorem 3.4.2** Assume (3.4.1) and (3.4.2). Then uniformly for \( s \leq t \leq \infty \) we have

(3.4.6) \[ \Omega^s(t, s) - 1 \in \Psi(o(s^0), g_0(s)), \]

(3.4.7) \[ D(t, s) - D \in \Psi(o(s^{-1}), g_0(s)), \]

(3.4.8) \[ x^s(t, s) - x + sD \in \Psi(o(s^0), g_0(s)). \]

We first prove the following lemma. We refer the reader to Section C.6 for the notation.

**Lemma 3.4.3** Let

\[
\int_0^\infty dt \sup_{\{s \mid s \leq t\}} ||\text{ad}_\alpha^D \text{ad}_\beta^D P(t, s)|| < \infty,
\]

or using the notation of Section C.6, \( P(t, s) \in L^1(dt, \Psi(g_0(t))) \). Let \( W(t, s) \) be the unique solution of

\[
\partial_t W(t, s) = W(t, s) P(t, s),
\]

\[
W(s, s) = 1, \ s \leq t
\]

and \( \bar{W}(s, t) \) be the unique solution of

\[
\partial_s \bar{W}(s, t) = \bar{W}(s, t) P(t, s),
\]

\[
\bar{W}(t, t) = 1, \ s \leq t
\]

Then \( W(t, s) - 1 \) and \( \bar{W}(s, t) - 1 \) belong to \( \Psi(o(s^0), g_0(s)) \).

**Proof.** Let us prove the statement concerning \( W(t, s) \), the proof of the statement concerning \( \bar{W}(s, t) \) being similar. Clearly

\[
W(t, s) - 1 = o(s^0).
\]

Let us now prove by induction on \( |\alpha| + |\beta| \) that

(3.4.9) \[ \text{ad}_\alpha^D \text{ad}_\beta^D (W(t, s) - 1) = o(s^{-|\alpha|}). \]

Assume that (3.4.9) holds for \( |\alpha| + |\beta| \leq n - 1 \). Using Leibniz rule and (3.4.11), we obtain

(3.4.10) \[
= \sum_{(\gamma_1, \delta_1) + (\gamma_2, \delta_2) = (\alpha, \beta), |\gamma_1| + |\delta_1| \geq 1} C_{\gamma_1, \delta_1, \delta_2} \text{ad}_{\gamma_2}^D \text{ad}_{\delta_2}^D W(t, s) \text{ad}_{\gamma_2}^D \text{ad}_{\delta_2}^D P(t, s)
\]

\[ \in L^1(dt)O(s^{-|\alpha|}). \]
This implies (3.4.9) for $|\alpha| + \beta| = n$ and proves the desired result. \(\Box\)

**Proof of Theorem 3.4.2.** One has

$$ \partial_t \Omega_{\mathcal{S}}(t, s) = -iU(s, t)V(t, x)U_0(t - s) $$

(3.4.11)

$$ = -\Omega_{\mathcal{S}}(t, s)V^w(t, x + (t - s)D), $$

$$ \Omega_{\mathcal{S}}(s, s) = 1. $$

It follows from (3.4.1) that

$$ V^w(t, x + (t - s)D) \in L^1(dt, \psi(g_0(t))), $$

Applying Lemma 3.4.3 to $\Omega_{\mathcal{S}}(t, s)$ gives (3.4.6).

Next note that

$$ D(t, s) = D $$

$$ = [\Omega_{\mathcal{S}}(t, s), D] \Omega_{\mathcal{S}}^*(t, s) $$

This belongs to $\Psi(o(s^{-1}), g_0(s))$, because

$$ [\Omega_{\mathcal{S}}(t, s), D] \in \Psi(o(s^{-1}), g_0(s)), $$

$$ \Omega_{\mathcal{S}}^*(t, s) \in \Psi(1, g_0(s)). $$

Similarly, we have

$$ x_{\mathcal{S}}(t, s) - x + sD $$

$$ = [\Omega_{\mathcal{S}}(t, s), x] \Omega_{\mathcal{S}}^*(t, s) $$

This belongs to $\Psi(o(1), g_0(s))$, because

$$ [\Omega_{\mathcal{S}}(t, s), x] \in \Psi(o(1), g_0(s)), $$

$$ \Omega_{\mathcal{S}}^*(t, s) \in \Psi(1, g_0(s)). $$

\(\Box\)

**Remark 3.4.4** Since $\Omega_{\mathcal{S}}^+$ is a pseudodifferential operator in $\Psi(L^2(X))$, we can write

$$ \Omega_{\mathcal{S}}^+ \phi(x) = (\pi)^{-2n} \int a^+(x, \xi)e^{i(x-y)\xi}\phi(y)d\xi, $$

where

$$ |\partial_x^\alpha \partial_{\xi}^\beta a^+(x, \xi)| \leq C_{\alpha, \beta}. $$

Set

$$ \phi_{0, \xi}(x) := e^{ix\xi}, $$

$$ \phi_{\xi}^+(x) := a^+(x, \xi)e^{ix\xi}. $$

Note that $\phi_{0, \xi}$ is a generalized eigenvector of $D$ with the eigenvalue $\xi$. Similarly, $\phi_{\xi}$ is a generalized eigenvector of $D^+$ with the eigenvalue $\xi$. Note that by Theorem 3.4.1 the wave operator $\Omega_{\mathcal{S}}^+$ is bounded on all weighted spaces

$$ \langle x \rangle^m L^2(X) := \{ \phi \mid \langle x \rangle^{-m}\phi \in L^2(X) \}. $$

Therefore, the following identity makes sense if we treat $\phi_{0, \xi}^+, \phi_{\xi}$ as elements of $\langle x \rangle^m L^2(X)$ for $m \geq \frac{2n}{2}$:

$$ \Omega_{\mathcal{S}}^+ \phi_{0, \xi} = \phi_{\xi}^+. $$

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3.5 A class of short-range singular potentials

In this section we will prove existence and completeness of wave operators for a class of time-dependent short-range potentials which is rather different from the one considered in section 3.3. The results of Sections 3.3 and 3.4 are inspired to a large extent by the corresponding classical problem, even though the conditions on the potentials are usually weaker. In Section 3.3, a typical assumption on the potential \( V(t, x) \) was that \( V \) belongs to \( L^4(\mathbb{R}_t, L^\infty(X)) \). Under this assumption the proof of existence and completeness of the wave operators is elementary. It turns out that this result still holds if \( V \) belongs to \( L^\alpha(\mathbb{R}_t, L^p(X)) \) for some exponents \( \alpha > 1 \) and \( p < \infty \). Intuitively a decay of \( V \) at infinity in the \( t \) variable which is slower than \( L^1 \) can be compensated by a decay at infinity in the \( x \) variables which is faster than \( L^\infty \).

The proof of existence and unitarity of the wave operators is however longer and relies on \( L^p - L^q \) estimates for the free propagator \( e^{-itH_0} \). These estimates can be considered as a manifestation of diffusion for solutions of the Schrödinger equation, which has no analog in classical mechanics.

Many of the arguments used below are inspired by the proof by Yajima of the existence of a unitary propagator for a class of time-dependent potentials belonging to some \( L^p - L^q \) spaces. Rather similar results on completeness of wave operators were obtained before by Howland [How] and Deich-Korotyaev-Yafaev [DKY].

We start this section by recalling Yajima’s theorem. We will denote below by \( U_0(t) \) the free propagator,

\[
U_0(t) := e^{-it\Delta^2}.
\]

and by \( V(t) \) the operator of multiplication by \( V(t, x) \).

**Theorem 3.5.1 [Yaj]** Assume that the potential \( V \) satisfies

\[
\begin{align*}
V(t, x) &= V_1(t, x) + V_2(t, x), \text{ a.e in } (t, x), \\
V_1(t, x) &\in L^1_{\text{loc}}(\mathbb{R}_t, L^\infty(X)), \\
V_2(t, x) &\in L^\alpha_{\text{loc}}(\mathbb{R}_t, L^p(X)),
\end{align*}
\]

for \( p \geq 1, p > n/2, \alpha = \frac{2p}{n+p} \). Then

i) there exist a family of unitary operators \( U(t,s) \) such that

\[
U(t,s)U(s,r) = U(t,r), \quad U(t, t) = 1, \quad \forall t, s, r \in \mathbb{R}.
\]

ii) \( U(t,s) \) is strongly continuous in \( t, s \) on \( L^2(\mathbb{R}_t) \).

iii) \( \forall u \in L^2(\mathbb{R}_t) \), \( U(t,s)u \in C^0(\mathbb{R}_t, L^2(\mathbb{X})) \cap L^q_{\text{loc}}(\mathbb{R}_t, L^3(\mathbb{X})) \) for \( q = \frac{2p}{p-1} \quad \theta = \frac{4p}{n} \).

iv) \( U(t,s)u \) satisfies the following integral equation:

\[
U(t,s)u = U_0(t-s)u - i \int_t^s U_0(\tau-s)V(\tau)U(\tau,s)u d\tau.
\]

The main result of this section is the following theorem:

**Theorem 3.5.2** Assume that \( V(t,x) \) can be written as

\[
V(t, x) = V_1(t, x) + V_2(t, x) \text{ a.e in } (t, x),
\]

\[
U(t,s)u = U_0(t-s)u - i \int_t^s U_0(\tau-s)V(\tau)U(\tau,s)u d\tau.
\]
where
\[ V_1 \in L^1(\mathbb{R}_+, L^\infty(X)), \quad V_2 \in L^\alpha(\mathbb{R}_+, L^p(X)), \]
for \( p \geq 1, \ p > n/2 \) and \( \alpha = 2p(2p - n)^{-1} \). Then the following results hold:

i) the limit
\[ (3.5.3) \quad \Omega^+ := \lim_{t \to +\infty} U(0, t) e^{-izH_0} \]
e exists.

ii) the limit
\[ (3.5.4) \quad s- \lim_{t \to +\infty} e^{-izH_0} U(t, 0) \]
exists and is equal to \( \Omega^{++} \).

iii) if \( v_0 = \Omega_+ u_0 \), then
\[ ||U(\cdot)u_0 - U(\cdot, 0)v_0||_{L^\theta([T, +\infty], L^4(X))} \to 0, \]
when \( T \) tends to \( +\infty \), for \( l = 2p(p - 1)^{-1}, \ \theta = 4q(n - 1)(q - 2)^{-1} \).

Before starting the proof of Theorem 3.5.2, we will introduce some notations. We will prove Theorem 3.5.2 using integral equations and the fixed point theorem on the Banach space \( Z_T \) defined by
\[ (3.5.5) \quad Z_T := C_0^\infty([T, +\infty[, L^2(X)) \cap L^\theta([T, +\infty[, L^4(X)), \]
with the norm
\[ ||u||_{Z_T} = \sup_{[T, +\infty[} ||u(t)||_{L^2(X)} + \left( \int_T^{+\infty} dt \left( \int_X |u(t, x)|^4 dx \right)^{\theta/(4\theta)} \right)^{1/\theta}, \]
for the exponents \( \theta, l \) defined in Theorem 3.5.2. For simplicity of notations, we will denote by
\[ ||u||_{L^\theta_T^m} = \left( \int_T^{+\infty} dt \left( \int_X |u(t, x)|^\frac{m}{\theta} dx \right)^{m/\theta} \right)^{1/m}, \]
the norm of \( u \) in the space \( L^m([T, +\infty[, L^\theta(X)) \), with the obvious modification if \( p = \infty \). We will denote by
\[ ||u||_{L^\theta_T^m + L^\theta_T^r} \]
the norm
\[ ||u||_{L^\theta_T^m} + ||u||_{L^\theta_T^r} \]
in the space \( L^m([T, +\infty[, L^\theta(X)) \cap L^r([T, +\infty[, L^\theta(X)) \). Finally we will denote by
\[ ||u||_{L^\theta_T^{m + L^\theta_T^r}} \]
the norm
\[ \inf \{ ||u_1||_{L^\theta_T^m} + ||u_2||_{L^\theta_T^r} \mid u = u_1 + u_2 \}, \]
on the space \( L^\theta_T^m + L^\theta_T^r \).

For an exponent \( l \geq 1 \), we will denote by \( l' \) the dual exponent equal to \( l(l - 1)^{-1} \).

We will need the following lemma due to Yajima \([\text{Ya} 1]\).

Lemma 3.5.3 Let \( n(1/2 - 1/l) < 1, \ \theta = 4l/n(l - 2) \). Then
i) \( ||U_0(\cdot)u_0||_{Z_T} \leq C ||u_0||_{L^2(X)} \).

Let \( S \) be the operator
\[ Su(t) := \int_{-\infty}^{+\infty} K(t, \tau) U_0(t - \tau) u(\tau) d\tau, \]
where $K(t, \tau)$ is a piecewise continuous function in $L^\infty(\mathbb{R}^2)$. Then

$$ii) \|Su\|_{L^2_t L^{2,\infty}_x} \leq C\|u\|_{L^{2,\infty}_x + L^{2,\infty}_x}.$$  

For the proof of Lemma 3.5.3, we will need the following well-known estimate:

**Lemma 3.5.4** Let $2 \leq l \leq \infty$, and $l'$ be its conjugate exponent. Then

$$\|U_0(t)f\|_m \leq C|t|^{-2/l'}\|f\|_{m'}, \quad 0 = 4l/n(l - 2).$$

**Proof.** the lemma is immediate for $l' = 2$ and $l' = 1$. The general case follows by interpolation. □

**Proof of Lemma 3.5.3.** We start by proving $ii)$, which will easily imply $i)$. Clearly to prove $ii)$ we have to establish the following four estimates:

$$\begin{align*}
\text{(3.5.6)} \quad & ii) \|Su\|_l \leq C\|u\|_{l', \theta }, \\
& ii) \|Su\|_{2,\infty} \leq C\|u\|_{l', \theta }, \\
& ii) \|Su\|_{2,\infty} \leq C\|u\|_{2,1}, \\
& ii) \|Su\|_{2,\theta} \leq C\|u\|_{2,1}.
\end{align*}$$

Note that the estimate $ii)$ is immediate. We have:

$$\|Su\|_l \leq C\|K\|_l \left( \int \left( t - s \right)^{-2/l} \|u(s)\|_{l'} \|u\|_{l'} \right)^{1/l},$$

by Lemma 3.5.4. Using Sobolev's inequality (see [RS2, Vol II], this implies $ii$). Let us now prove $ii)$. We have for $K(t, s) = K(r, t)K(r, s)$:

$$\begin{align*}
\|Fu\|_{2,\infty} & \leq \sup_t \int (K(t, s)U_0(t - s)u(s))ds \| u(t) \| dt \\
& \leq C\|u\|_{l', \theta},
\end{align*}$$

using Hölder inequality and $ii)$. Finally $ii)$ follows from $ii)$ by duality. To prove $i)$ we write for $K(t, s) \equiv 1$:

$$\begin{align*}
\|Su\|_{L^\infty(\mathbb{R}^2)} & = \| (Su) \|_1 \\
& \leq C\|u\|_{L^\infty(\mathbb{R}^2)} \|u\|_{L^\infty(\mathbb{R}^2)} \\
& \leq C\|u\|_{L^\infty(\mathbb{R}^2)} \|F\|_{2,\infty} \leq C\|u\|_{L^\infty(\mathbb{R}^2)} \|u\|_{l', \theta},
\end{align*}$$

which proves $i)$. This completes the proof of the lemma. □

We will need some additional technical lemmas. For $T \leq t \leq +\infty$, we will denote by $K_\tau$ the operator defined on $Z_T$ by

$$\begin{align*}
K_\tau z(s) & := i \int_{[0, t]} 1_{[0, \tau]}(\tau)U_0(s - \tau)V(\tau)z(\tau)d\tau. \\
\text{(3.5.8)}
\end{align*}$$

The next lemma will be needed to prove the existence of wave operators.
Lemma 3.5.5 Under the assumptions of Theorem 3.5.2, we have
\[ \| K_z \|_{L_T^2} \leq C(\| V_1 \|_{L_T^{p,1}} + \| V_2 \|_{L_T^{p,s}}) \| z \|_{L_T^s}, \]
uniformly for \( T \leq t \leq +\infty. \)

Proof. we consider for \( z \in L_T^s \)

\[ \begin{align*}
V(t)z(t) &= V_1(t)z(t) + V_2(t)z(t).
\end{align*} \]

Let us first estimate the first term in the right hand side of (3.5.9). Clearly, one has

\[ \| V_1(\cdot)z(\cdot) \|_{L_T^{p,1}} \leq C\| V_2(\cdot) \|_{L_T^{p,s}} \| z \|_{L_T^s}. \]

Let us next consider the second term. Using first Hölder’s inequality in the \( x \) variables, we have

\[ \begin{align*}
\| V_2(\cdot)z(\cdot) \|_{L_T^{p',s'}} &= \left( \int_{t}^{+\infty} dt \int_X |V_2(t, x)|^{p'} |z(t, x)|^{s'/s} \| dx \|^{s'/s} \right)^{1/s'} \\
&\leq \left( \int_{t}^{+\infty} dt \int_X |z(t, x)|^{p'/s'} \| dx \|^{p'/s} \times \left( \int_X |V_2(t, x)|^{p} \| dx \|^{p'} \right)^{1/s'} \right)^{1/s'},
\end{align*} \]

for \( p = l(l - 2)^{-2}, p^{-1} = l^{-1} - l^{-1} \). Using then Hölder’s inequality in the \( t \) variables, we obtain

\[ \begin{align*}
\| V_2(\cdot)z(\cdot) \|_{L_T^{p',s'}} &\leq \| z \|_{L_T^{p,s}} \times \| V_2(\cdot) \|_{L_T^{p,s}},
\end{align*} \]

for \( \alpha^{-1} = \theta^{(l-1)} - \theta^{-1} \). Since \( \theta = 4l(n^{-1}(l - 2)^{-1}, \) we see that \( \alpha = 2p(2p - n)^{-1} \). So from (3.5.11), we obtain

\[ \begin{align*}
\| V_2(\cdot)z(\cdot) \|_{L_T^{p',s'}} \leq \| V_2(\cdot) \|_{L_T^{p,s}} \| z \|_{L_T^{l,s}}.
\end{align*} \]

Using again lemma 3.5.3, we deduce from (3.5.10) and (3.5.12) that

\[ \begin{align*}
\| K_z \|_{L_T^s} \leq C(\| V_1 \|_{L_T^{p,1}} + \| V_2 \|_{L_T^{p,s}}) \| z \|_{L_T^s}.
\end{align*} \]

This completes the proof of the lemma. \( \square \)

The next lemma will be an analog to Lemma 3.5.5 with the free evolution \( U_0(t - s) \) replaced by the perturbed one \( U(t, s). \)

Lemma 3.5.6 Under the assumptions of Theorem 3.5.2, we have

\[ i)\ \| U(\cdot, s)u_0 \|_{L_\infty} \leq C\| u_0 \|_{L_2(X)}. \]

Let \( \tilde{K}_s \) be the operator

\[ \tilde{K}_s u(t) := i \int_s^t U_0(t - \tau)V(\tau)u(\tau) d\tau. \]

Then one has

\[ ii)\ \| \tilde{K}_s \|_{L_T^s} \leq C_0(T_0^0) \| u \|_{L_T^s}. \]

Let \( M_t \) be the operator

\[ M_t u(s) := i \int_s^t U(s, \tau)V(\tau)u(\tau) d\tau. \]
Then one has

\[ iii) \|M_t u\|_{Z_T} \leq C_0(T^0)\|u\|_{L^2(X)}; \]

uniformly for \( t \geq T \).

iv) \( U_0(t - s) \) satisfies the following integral equation:

\[ (3.5.14) \quad U_0(t - s)u_0 = U(t, s)u_0 + \int_s^t U(t, \tau)V(\tau)U_0(\tau - s)u_0 d\tau. \]

**Proof.** Let us first prove i). By the integral equation satisfied by \( u(t) = U(t, s)u_0 \), one has

\[ u(t) = U_0(t - s)u_0 - i \int_s^t U_0(t - \tau)V(\tau)u(\tau) d\tau = iU_0(t - s)u_0 - \tilde{K}_s u(t). \]

By Lemma 3.5.5, we know that for \( T \) large enough, the operator norm of \( \tilde{K}_s \) on \( Z_T \) is strictly less than 1, which using Lemma 3.5.3 implies that

\[ \|U(\cdot, s)u_0\|_{Z_T} \leq C\|u_0\|_{L^2(X)}. \]

To complete the proof of ii), it suffices then to use the group property of \( U(t, s) \) and Theorem 3.5.1. ii) is already proven in Lemma 3.5.5 ii).

Let us now prove iii). We choose \( T \) large enough to be able to use Dyson’s expansion for \( U(t, s) \). Indeed for \( T \gg 1 \), we have

\[ (3.5.15) \quad U(t, s)u_0 = \sum_{n=0}^{\infty} (-i\tilde{K}_s)^n U(\cdot, s)u_0 = \sum_{n=0}^{\infty} f_n(t, s)u_0, \]

By ii), we know that the series (3.5.15) converges in \( Z_T \), for \( T \) large enough. We have

\[ f_n(s, \tau)u_0 \]

\[ = (-i)^n \int_s^\tau dt_1 U_0(s - t_1)V(t_1) \int_{t_1}^{t_2} \cdots \int_{t_{n-1}}^{t_n} U_0(t_{n-1} - t_n)V(t_n)U_0(t_n - \tau)u(t_n) dt_n. \]

So we obtain

\[ M_t u(s) = \int_s^t U(s, \tau)V(\tau)u(\tau) d\tau = \sum_{n=0}^{\infty} (-i)^n g_n(t, s), \]

where \( g_n \) is equal to

\[ \int_s^\tau d\tau \int_s^\tau dt_1 U_0(s - t_1)V(t_1) \cdots \int_{t_{n-1}}^{t_n} U_0(t_{n-1} - t_n)V(t_n)U_0(t_n - \tau)V(\tau)u(\tau) dt_n. \]

The above integral is over the domain

\[ s \leq t_1 \leq \cdots \leq t_n \leq \tau \leq t. \]

By Fubini’s theorem, we may rewrite it as

\[ \int_s^t U_0(s - t_1)V(t_1)dt_1 \int_{t_1}^t U_0(t_1 - t_2)V(t_2)dt_2 \cdots \int_{t_{n-1}}^{t_n} U_0(t_{n-1} - t_n)V(t_n)dt_n \int_{t_n}^t U_0(t_n - \tau)V(\tau)u(\tau) d\tau. \]

Applying then Lemma 3.5.3, we obtain that

\[ (3.5.16) \quad \|g_n\|_{Z_T} \leq C_0(C(T))^{n+1}\|u\|_{Z_T}. \]

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where

\[ C(T) = ||V_1||_{L^\infty_T} + ||V_2||_{L^p_T} = o(T^0). \]

This proves \( iii \).

Let us now prove \( iv \). Using the group property of \( U_0(t-s) \) and \( U(t,s) \), it clearly suffices to prove \( iv \) when \( s, t \in [0,a] \) for \( 0 < a \ll 1 \). We first observe that this equation is immediate if \( V(t,x) \in C^4(\mathbb{R}_t, L^\infty(X)) \), since we may assume by density that \( u_0 \in H^2(X) \) and differentiate both sides of the equation. Next we approximate \( V(t,x) \) in the space \( L^1(\mathbb{R}_t, L^\infty(X)) + L^a(\mathbb{R}_t, L^1(X)) \) by a sequence \( V_\varepsilon \in C^4(\mathbb{R}_t, L^\infty(X)) \), and we denote by \( U_\varepsilon(t,s) \) the unitary propagator for the potential \( V_\varepsilon \). We claim that

\[ \begin{align*}
  & i) \ U_\varepsilon(t,s)u \to U(t,s)u, \\
  & ii) \ \int_s^t U_\varepsilon(t,\tau)V_\varepsilon(\tau)U_\varepsilon(\tau-s)u_0d\tau \to \int_s^t U(t,\tau)V(\tau)U(\tau-s)u_0d\tau,
\end{align*} \]

when \( \varepsilon \) tends to 0, in the space \( C^0([0,a], L^2(X)) \cap L^a([0,a], L^1(X)) \). Part \( i \) of (3.5.17) is proven in [Ya], Lemma 3.1. To prove (3.5.17) \( ii \), we use again the Dyson expansion (3.5.15) which converges if \( a \) is small enough. As in the proof of \( iii \), we write

\[
\int_s^t U_\varepsilon(t,\tau)V_\varepsilon(\tau)U_\varepsilon(\tau-s)u_0d\tau = \sum_{n=0}^{+\infty} \int_s^t f_n(t,\tau)V_\varepsilon(\tau)U_\varepsilon(\tau-s)u_0d\tau.
\]

We use then Fubini’s theorem as in the proof of \( ii \) to obtain (3.5.17) \( ii \). This completes the proof of the Lemma. \( \square \)

**Proof of Theorem 3.5.2.**

We start by proving \( i \). To prove \( i \), we will first find for \( u_0 \in L^2(X) \) and \( t \gg 1 \) a function \( u_t(.) \in Z_T \) such that

\[ u_t(s) = U_0(s)u_0 + t \int_s^t U_0(s-\tau)V(\tau)u_t(\tau)d\tau. \]

Note that by the integral equation (3.5.2) satisfied by \( U(t,s) \), the function \( u_t(s) \) is equal to

\[ u_t(s) = U(s,t)e^{-i\hbar a}u_0, \]

for \( s \leq t \).

The equation (3.5.18) can be rewritten as

\[ u_t = K_t u_t = U_0(.)u_0, \]

where \( K_t \) is defined in (3.5.8). By Lemma 3.5.3, we know that \( U_0(.)u_0 \) belongs to \( Z_T \). Using then Lemma 3.5.6, we obtain that there exists for \( T \) large enough a unique solution \( u_t = (1 - K_t)^{-1}U_0(.)u_0 \) in \( Z_T \) of (3.5.20). Moreover, since for \( z \in Z_T \), \( K_tz \) is continuous in \( t \) up to \( t = +\infty \), there exists

\[ u_\infty = (1 - K_{\infty})^{-1}(U_0(.)u_0), \]

and

\[ u_\infty = \lim_{t \to +\infty} u_t. \]

We have

\[ u_\infty(s) = U_0(s)u_0 + \int_s^{+\infty} U_0(s-\tau)V(\tau)u_\infty(\tau)d\tau, \]
for $s \geq T$, which easily implies that for $t > T$, $u_\infty(t)$ solves the integral equation (3.5.2). Using the group property of $U(t, s)$, we can then extend $u_\infty(s)$ for all $s \in [-\infty, T]$, and we have $u_\infty(t) = U(t, 0)v_0$, for some $v_0 \in L^2(X)$. Finally using (3.5.22), we obtain that

$$
\lim_{T \to +\infty} \|u_\infty - U(\cdot)v_0\|_{L^2} = 0,
$$

which complete the proof of i). We also deduce from (3.5.23) that ii) holds.

Let us now prove ii). By exchanging the roles of $U_0(t-s)$ and $U(t, s)$, we will first find for $v_0 \in L^2(X)$ and $t \gg 1$ a function $u_t(\cdot) \in Z_T$ such that

$$
u_t(s) = U(s, 0)v_0 - i \int_s^t U(s, \tau)V(\tau)u_t(\tau)d\tau.
$$

We claim that one has

$$
u_t(s) = U_0(s-t)U(t, 0)v_0.
$$

Indeed, by Lemma 3.5.6 iv), we have

$$
U_0(t-s)v_0 = U(t, s)v_0 - i \int_s^t U(t, \tau)V(\tau)U_0(\tau-s)v_0d\tau
$$

The identity (3.5.25) can then be obtained from (3.5.26) as (3.5.19) was obtained from (3.5.2). The equation (3.5.24) can be rewritten as

$$
u_t(s) = M_t u_t(s) = U(s, 0)v_0,
$$

where $M_t$ is defined in Lemma 3.5.3. By Lemma 3.5.6 i), we know that $U(\cdot, 0)v_0 \in Z_T$, and by Lemma 3.5.6 ii), we obtain for large enough $T$ a unique solution $u_t = (1 + M_t)^{-1}U(\cdot, 0)v_0$ of (3.5.27). As above we consider

$$
u_\infty = \lim_{t \to +\infty} u_t,
$$

which solves

$$
u_\infty(s) = U(s, 0)v_0 - i \int_s^{+\infty} U(s, \tau)V(\tau)\nu_\infty(\tau)d\tau,
$$

for $s \geq T$. This easily implies that for $s > T$, $\nu_\infty(s)$ solves the integral equation (3.5.14), so that by the group property of $U_0(t-s)$, we can extend $\nu_\infty(s)$ to $[-\infty, T]$, and we have $\nu_\infty(t) = U_0(t)v_0$, for some $v_0 \in L^2(X)$. As above we have

$$
\|\nu_\infty(s) - U(s, 0)v_0\|_{L^2} \to 0, \text{ when } s \to +\infty,
$$

which completes the proof of the Theorem. \(\Box\)

### 3.6 Long-range scattering for Hörmander potentials.

In this section we begin our study of the scattering theory in the long-range case. In this case the asymptotic momentum is well defined, although the usual wave operators, characteristic of the short-range case, in general do not exist.

As pointed out in the beginning of Section 3.3, the main problem of the long-range scattering theory is whether the asymptotic momentum is unitarily equivalent to the momentum operator. One can view this property as a definition of what is traditionally called the asymptotic completeness of modified wave operators which is independent of a specific choice of a modified free evolution. In this section we present
a construction of unitary operators that intertwine the momentum and the asymptotic momentum. It resembles the construction of usual short-range wave operators. Therefore, these unitary operators are usually called modified wave operators.

We prove that modified wave operators exist and are unitary under quite general assumptions on the potential. This proof is unfortunately rather involved. The reader who prefers an easier exposition of the long-range case under more restrictive hypotheses should go directly to Section 3.7.

For long-range potentials satisfying only the hypotheses of Section 3.2, the asymptotic momentum $D^+$ can have a spectral measure which is very different from the one of $D$. In fact we will give in Section 3.9 an example of a time-dependent long-range potential for which $E_{[0]}(D^+)$ is an infinite dimensional projection.

The main result of this section is the following theorem.

**Theorem 3.6.1** Assume that

\[
V(t, x) = V_\delta(t, x) + V_\xi(t, x)
\]

such that

\[
\int_0^\infty \|V_\delta(t, \cdot)\|_\infty dt < \infty,
\]

\[
\int_0^\infty \langle t \rangle^{a-\frac{1}{2}} \|\partial_x^a V_\xi(t, \cdot)\|_\infty dt < \infty, \quad |a| = 1, 2.
\]

Then there exists a function $S(t, \xi)$ such that

\[
\lim_{t \to \infty} U(0, t) e^{-iS(t, D)}
\]

and

\[
\lim_{t \to \infty} e^{iS(t, D)} U(t, 0).
\]

exist. If we denote (3.6.2) by $\Omega_\xi^+$, then (3.6.3) equals $\Omega_{\xi^+}^+$. Moreover,

\[
D^+ = \Omega_{\xi^+}^+ D \Omega_{\xi^+}^+.
\]

There exists also the limit

\[
\lim_{t \to \infty} U(0, t)(x - \nabla_\xi S(t, D)) U(t, 0) =: x_{\xi+}^+.
\]

The observable $x_{\xi+}^+$ is a vector of commuting densely defined self-adjoint operators and is called the asymptotic position. One has

\[
x_{\xi^+}^+ = \Omega_{\xi^+}^+ x_{\xi^+}^+.
\]

Let us mention that using a regularization as in Lemma 3.6.5 a potential $V(t, x)$ can satisfy the conditions of Theorem 3.6.1 without being twice differentiable. For example, if $V_\xi(t, x)$ satisfies

\[
\int_0^\infty \langle t \rangle^{1/2} \|\partial_x^a V_\xi(t, \cdot)\|_\infty dt < \infty, \quad |a| = 1,
\]

then the hypotheses of Theorem 3.6.1 are satisfied. Note that under the above conditions one can also use the Dollard construction of modified wave operators (see Section 3.8).

One might want to know what is the relationship of the function $S(t, \xi)$ and the potentials that appear in the statement of the theorem. It is natural to ask whether as this function we can take a solution of the Hamilton-Jacobi equation with the potential $V_\xi(t, x)$. It turns out that this is possible if we strengthen the assumptions of the theorem, as we describe in the following proposition.
**Proposition 3.6.2** If instead of (3.6.1) the potential $V_1(t, x)$ satisfies one of the following two hypotheses:

(3.6.7) \[ \int_0^\infty (t)^{|\alpha| - 1} ||\partial_x^\alpha V_1(t, \cdot)||_\infty dt < \infty, \quad |\alpha| = 1, 2, 3, \]

or

(3.6.8) \[ \int_0^\infty (t)^{|\alpha| - 1} ||\partial_x^\alpha V_1(t, \cdot)||_\infty dt < \infty, \quad |\alpha| = 1, \]

\[ \int_0^\infty (t)^{|\alpha| - 1} ||\partial_x^\alpha V_1(t, \cdot)||_\infty dt < \infty, \quad |\alpha| = 2, \]

then as the function $S(t, \xi)$ in Theorem 3.6.1 we can take the solution of the Hamilton-Jacobi equation

(3.6.9) \[
\begin{align*}
\partial_t S(t, \xi) &= \frac{1}{2} \xi^2 + V_1(t, \nabla \xi S(t, \xi)), \\
S(T, \xi) &= 0,
\end{align*}
\]

which exists for large enough $T$.

**Proof.** The proposition follows from Theorems 3.6.1 and C.2.2. \(\square\)

The proof of Theorem 3.6.1 will be divided into a series of lemmas. First we need some additional analysis of classical scattering that was not contained in Chapter 1.

Recall that under the assumption

(3.6.10) \[
\int_0^\infty ||\partial_x^\alpha F(t, \cdot)||_\infty (t)^{|\alpha| - 1} dt < \infty, \quad |\alpha| = 0, 1,
\]

in Theorem 1.4.1 we constructed solutions $\hat{y}(s, t_1, t_2, x, \xi)$ of the classical boundary problem where we fixed the initial position and the final momentum. In Theorem 1.8.1 assuming the so-called smooth long-range condition we showed some estimates on the derivatives of these solutions. Unfortunately, in this section we will deal with a much wider class of potentials and we need to generalize a part of Theorem 1.8.1.

Note that in the following proposition we do not assume the force to be conservative.

**Proposition 3.6.3** Suppose that for $n = 0, 1, \ldots$ we fix positive numbers $\kappa(n)$ that satisfy

(3.6.11) \[ \kappa(n) + \kappa(m) \leq \kappa(n + m). \]

(Note that this implies $\kappa(0) = 0$). Assume that

(3.6.12) \[
\int_0^\infty ||\partial_x^\alpha F(t, \cdot)||_\infty (t)^{|\alpha| - 1} dt < \infty, \quad |\alpha| \geq 1,
\]

Then uniformly for $T \leq t_1 \leq s \leq t_2 < \infty$ we have the estimate

(3.6.13) \[
|\partial_x^\alpha (\hat{y}(s, t_1, t_2, x, \xi) - x - (s - t_1)\xi)| \leq o(t_1) |s - t_1| (t_2) \kappa(|\alpha| - 1).
\]

**Proof.** Recall from the proof of Theorem 1.4.1 that

\[
\hat{z}(s) := \hat{y}(s) - x - (s - t_1)\xi
\]

satisfies

(3.6.14) \[
\hat{z}(s) = - \int_{t_1}^t \zeta_{t_1} u F(u, \hat{y}(u)) du.
\]
We will prove our proposition by induction wrt $|\beta|$. The induction hypothesis $H(n)$ will be

$$ |\partial_\xi^\beta \check{z}(s)| \leq o(t_1^{|\beta|} |s - t_1| |t_2|^{\epsilon(1 - |\beta| - 1)}, \quad 1 \leq |\beta| \leq n. \tag{3.6.15} $$

Let us assume that $H(n - 1)$ is true. Consider $\beta$ such that $|\beta| = n$. We use the Faà di Bruno formula to compute $\partial_\xi^\beta \check{z}(s)$ and we obtain

$$ \partial_\xi^\beta \check{z}(s) + \int_{t_1}^{t_2} \partial_\xi^\beta \xi_i,\sigma(u) \nabla \check{g} F(u, \check{y}(u)) \partial_\xi^\beta \check{z}(u) \, du $$

$$ = \int_{t_1}^{t_2} \partial_\xi^\beta \xi_i,\sigma(u) \nabla \check{g} F(u, \check{y}(u)) \partial_\xi^\beta (x + (u - t_1) \xi) \, du $$

$$ - \sum_{\gamma \neq \beta} C_{\gamma} \int_{t_1}^{t_2} \partial_\xi^\gamma \xi_i,\sigma(u) \nabla \check{g} F(u, \check{y}(u)) \partial_\xi^\beta \check{y}(u) \partial_\xi^\gamma \check{y}(u) \, du. \tag{3.6.16} $$

(3.6.16) can be rewritten as

$$ \partial_\xi^\beta \check{z} = - \nabla_\xi \check{P}(\xi) \partial_\xi^\beta \check{z} = \sum g_\delta, \tag{3.6.17} $$

where the map $\check{P}$ was introduced in the proof of Theorem 1.1.1. The induction hypothesis $H(n - 1)$ implies

$$ |\partial_\xi^\beta \check{y}(u)| \leq O(t_1^{|\beta|} |u - t_1| |t_2|^{\epsilon(1 - |\beta| - 1), \quad 1 \leq |\delta| \leq n - 1. $$

Therefore,

$$ |g_\delta(s)| $$

$$ \leq C(t_2)^{\epsilon(1 + |\delta| - 1)} \int_{t_1}^{t_2} \partial_\xi^\beta \xi_i,\sigma(u) \|\nabla \check{g} F(u, \cdot)\|\infty \|u\|^\delta \, du $$

$$ \leq C(t_2)^{\epsilon(1 + |\delta| - 1)} \int_{t_1}^{t_2} \|\nabla \check{g} F(u, \cdot)\|\infty \|u\|^\delta \, du $$

Moreover, we know that $(1 - \nabla_\xi \check{P}(\xi))$ is uniformly invertible on $Z_{t_1}^1$ for $T \leq t_1 \leq t_2 \leq \infty$. Therefore, we can use the identity

$$ \partial_\xi^\beta \check{z} = (1 - \nabla_\xi \check{P}(\xi))^{-1} \sum g_\delta $$

to show that (3.6.15) is true. \( \square \)

From now on we assume that the force is conservative and $F(t, x) = -\nabla_x V(t, x)$. Recall that the functions $\check{y}(s, t_1, t_2, x, \xi)$ are used to define the function $S(t, \xi)$, which is the unique solution of the problem

$$ \begin{cases} 
\partial_t S(t, \xi) = \frac{\xi^2}{2} + V(t, \nabla_\xi S(t, \xi)), \\
S(T, \xi) = 0.
\end{cases} \tag{3.6.18} $$

Below we give estimates on $S(t, \xi)$ that follow from Proposition 3.6.3.

**Corollary 3.6.4** Suppose that

$$ \int_0^\infty \|\partial_\xi^\alpha V(t, \cdot)\|\infty \|t\|^{\alpha - 1} \, dt < \infty, \quad |\alpha| = 1, 2, \tag{3.6.19} $$

$$ \int_0^\infty \|\partial_\xi^\alpha V(t, \cdot)\|\infty \|t\|^{\alpha - 1 - \epsilon(1 - \alpha)} \, dt < \infty, \quad |\alpha| \geq 2. \tag{3.6.20} $$

Then

$$ \left| \partial_\xi^\beta \left( S(t, \xi) - \frac{1}{2} \xi^2 \right) \right| \leq o(t), \quad |\beta| = 1, 2, \tag{3.6.21} $$

$$ \left| \partial_\xi^\beta \left( S(t, \xi) - \frac{1}{2} \xi^2 \right) \right| \leq o(t^{1 + \epsilon(1 - \alpha - 2)}), \quad |\beta| \geq 2. \tag{3.6.22} $$
Below we will show how we can change the splitting of the potential into a long-range and a short-range part such that the results of Proposition 3.6.3 will be applicable to $V_1(t, x)$.

**Lemma 3.6.5** i) Suppose that $V_1(t, x)$ satisfies

$$
\int_0^\infty \langle t \rangle^{1/2} \| \partial_x^\alpha V_1(t, \cdot) \|_{\infty} dt < \infty, \quad |\alpha| = 1.
$$

Then there exists a splitting

$$
V_1(t, x) = \tilde{V}_1(t, x) + \check{V}_1(t, x)
$$

such that

$$
\int_0^\infty \| \tilde{V}_1(t, \cdot) \|_{\infty} dt < \infty,
$$

and

$$
\int_0^\infty \langle t \rangle^{1/2} \| \partial_x^\alpha V_1(t, \cdot) \|_{\infty} dt < \infty, \quad |\alpha| = 1.
$$

ii) Suppose that $V_1(t, x)$ satisfies

$$
\int_0^\infty \langle t \rangle^{1/2} \| \partial_x^\alpha V_1(t, \cdot) \|_{\infty} dt < \infty, \quad |\alpha| = 1, 2.
$$

Then there exists a splitting

$$
V_1(t, x) = \tilde{V}_1(t, x) + \check{V}_1(t, x)
$$

such that

(3.6.23) \hspace{1cm} \int_0^\infty \| \tilde{V}_1(t, \cdot) \|_{\infty} dt < \infty,

(3.6.24) \hspace{1cm} \int_0^\infty \| \partial_x^\alpha \tilde{V}_1(t, \cdot) \|_{\infty} \langle t \rangle^{1/2} dt < \infty, \quad |\alpha| = 1, 2

(3.6.25) \hspace{1cm} \int_0^\infty \| \partial_x^\alpha \tilde{V}_1(t, \cdot) \|_{\infty} \langle t \rangle^{1/2} dt < \infty, \quad |\alpha| = 1.

iii) Suppose that $V_1(t, x)$ satisfies

$$
\int_0^\infty \langle t \rangle^{1/2} \| \partial_x^\alpha \tilde{V}_1(t, \cdot) \|_{\infty} dt < \infty, \quad |\alpha| = 1, 2, 3.
$$

Then in addition to (3.6.23) the potential $V_4(t, x)$ satisfies

(3.6.26) \hspace{1cm} \int_0^\infty \langle t \rangle \| \partial_x^\alpha \tilde{V}_4(t, \cdot) \|_{\infty} dt < \infty, \quad |\alpha| = 1.

**Proof.** Consider first i). Choose $j \in C_0^\infty(X)$ such that $\int j(x) dx = 1$. Set

$$
\tilde{V}_1(t, x) := \int V_1(t, x + t^2y) j(y) dy
$$

and

$$
\check{V}_1(t, x) := V_1(t, x) - \tilde{V}_1(t, x).
$$
Now
\[ \mathcal{V}_x(t, x) = \int (V_1(t, x) - V_1(t, x + t \dot{x} y)) dy \]
\[ = \int \int_0^1 \nabla V_1(t, x + t \dot{x} y) t \dot{x} y j(y) d\tau dy. \]

Moreover,
\[ \nabla^2_x \mathcal{V}_1(t, x) = (-1)^{k-1} t^{-\frac{k-1}{2}} \int \nabla V_1(t, x + t \dot{x} y) \nabla \nabla^{-1} j(y) dy. \]

Let us now prove \( \text{ii} \). This time we assume additionally that \( \int j(x) x dx = 0 \). We define \( \mathcal{V}_2(t, x) \) and \( \mathcal{V}_1(t, x) \) as above. Now
\[ \mathcal{V}_2(t, x) = \int (V_1(t, x) - V_1(t, x + t \dot{x} y)) dy \]
(3.6.27)
\[ = \int \nabla V_1(t, x) t \dot{x} y j(y) dy + O(t^{-1}) ||\nabla^2 V_1(t, \cdot)||_\infty, \]

The first term on the rhs of (3.6.27) is zero. Hence (3.6.23) is true.
Moreover,
\[ \nabla^k \mathcal{V}_1(t, x) = (-1)^{k-2} t^{-\frac{k-1}{2}} \int \nabla^{k} V_1(t, x + t \dot{x} y) \nabla \nabla^{-2} j(y) dy. \]

This implies (3.6.25).

The proof of \( \text{iii} \) is similar. \( \Box \)

**Corollary 3.6.6** Suppose that we are given a potential \( V(t, x) \) satisfying the assumptions of Theorem 3.6.1. Then we can change the splitting
\[ V(t, x) = \mathcal{V}_x(t, x) + \mathcal{V}_1(t, x) \]
such that \( \mathcal{V}_x(t, x) + \mathcal{V}_1(t, x) \) satisfies the assumptions of Theorem 3.3.4 and \( \mathcal{V}_1(t, x) \) satisfies

(3.6.28)
\[ \int_0^\infty ||\nabla^\alpha \mathcal{V}_1(t, \cdot)||_\infty(t)^{\alpha-1} dt < \infty, \quad |\alpha| = 1, 2, \]

(3.6.29)
\[ \int_0^\infty ||\nabla^\alpha \mathcal{V}_1(t, \cdot)||_\infty(t)^{\frac{1}{2}+\alpha} dt < \infty, \quad |\alpha| \geq 2. \]

For \( T \) big enough let \( S(t, \xi) \) be the solution of the Hamilton-Jacobi equation (3.6.18) with this new \( \mathcal{V}_1(t, x) \). Then
\[ \left| \frac{\partial^\beta}{\partial \xi} \left( S(t, \xi) - \frac{1}{2} \xi^2 \right) \right| \leq o(t) \quad |\beta| = 1, 2, \]
(3.6.30)
\[ \left| \frac{\partial^\beta}{\partial \xi} \left( S(t, \xi) - \frac{1}{2} \xi^2 \right) \right| \leq o(t^{\frac{1}{2}+|\beta|}) , \quad |\beta| \geq 2. \]
(3.6.31)

Finally, if we set
\[ g(t, x, \xi) := \int_0^1 F_1(t, \tau x + (1 - \tau) \nabla \xi S(t, \xi)) d\tau, \]
where \( F_1(t, x) = -\nabla_x \mathcal{V}_1(t, x) \), then
\[ \int_0^\infty ||g(t, \cdot, \cdot)||_\infty dt < \infty, \]
(3.6.32)
\[ \int_0^\infty ||\nabla^\alpha \nabla^\beta g(t, \cdot, \cdot)||_\infty(t)^{\frac{1}{2}+|\alpha|+|\beta|} dt < \infty, \quad |\alpha| + |\beta| \geq 1. \]
(3.6.33)
Proof. Lemma 3.6.5 implies immediately that we can change the splitting of $V(t, x)$. The estimates on $S(t, \xi)$ follow immediately from Corollary 3.6.4 with $\kappa(n) = \frac{1}{2n}$.

It remains to show the estimates on $g(t, x, \xi)$. By the Faa di Bruno formula

$$\partial_x^\alpha \partial_{\xi}^\beta F_i(t, \tau x + (1 - \tau)\nabla_{\xi} S(t, \xi))$$

$$= \sum C_\delta \partial_x^\alpha \nabla_{\xi}^\beta F_i(t, \tau x + (1 - \tau)\nabla_{\xi} S(t, \xi)) \partial_x^\delta \nabla_{\xi} S(t, \xi) \cdots \partial_x^\delta \nabla_{\xi} S(t, \xi) = \sum f_\delta(t, x, \xi).$$

Now

$$|f_\delta(t, x, \xi)| \leq C\|\nabla_{\xi}^{n+1+1} v_i(t, \cdot)|\infty(t)^{\frac{1}{2}(1+1+\ldots+1)}$$

$$= C\|\nabla_{\xi}^{n+1+1} v_i(t, \cdot)|\infty(t)^{\frac{1}{2}(n+q)}.$$ 

This implies (3.6.33). \(\Box\)

Now let

$$r(t, x, \xi) := \text{div}_\xi g(t, x, \xi).$$

Set

$$G(t) := g(t, x, D),$$

$$R(t) := r(t, x, D).$$

Lemma 3.6.7 One has

$$V_i(t, x) - V_i(t, \nabla_{\xi} S(t, D)) = G(t)(x - \nabla_{\xi} S(t, D)) + R(t).$$

Moreover

$$\int_0^\infty \|G(t)\|dt < \infty,$$

(3.6.35)

$$\int_0^\infty \|[x, G(t)]\|dt < \infty,$$

(3.6.36)

$$\int_0^\infty \||\nabla_{\xi} S(t, D), G(t)\|dt < \infty,$$

(3.6.37)

$$\int_0^\infty \|R(t)\|dt < \infty.$$

(3.6.38)

Proof. First note that

$$V_i(t, x) - V_i(t, \nabla_{\xi} S(t, D))$$

$$= xG(t) - G(t)\nabla_{\xi} S(t, D)$$

$$= G(t)(x - \nabla_{\xi} S(t, D)) + \sum_{i=1}^n [x_i, G_i(t)].$$

This implies (3.6.34).

Then we note that

$$g(t, x, \xi) \in L^1(dt, S((t)^{-1}dx^2 + \langle t \rangle d\xi^2)),$$

where we used the notations of Appendix, Section C.6. Hence

$$G(t) \in L^1(dt, \Psi((t)^{-1}dx^2 + \langle t \rangle d\xi^2)).$$
This implies (3.6.35).

Moreover,
\[ \nabla g(t, x, \xi) \in L^1(dt, S((t)^{-1}dx^2 + \langle t \rangle d\xi^2)). \]

Therefore
\[ R(t), [x, G(t)] \in L^1(dt, \Psi((t)^{-1}dx^2 + \langle t \rangle d\xi^2)), \]
which implies (3.6.36) and (3.6.38).

Finally note that
\[ \nabla g(t, x, \xi) \in L^1((t) dt, S((t)^{-1}dx^2 + \langle t \rangle d\xi^2)) \]
and
\[ \nabla^2 S(t, \xi) \in S((t)^{-1}dx^2 + \langle t \rangle d\xi^2). \]

Hence by (C.6.31)
\[ [\nabla gS(t, D), G(t)] \in L^1(dt, \Psi((t)^{-1}dx^2 + \langle t \rangle d\xi^2)). \]

This implies (3.6.38). \( \square \)

We define
\[ H(t) := \frac{1}{2} D^2 + V(t, x), \]
\[ \tilde{H}(t) := \frac{1}{2} D^2 + V(t, x) + R(t) \]
\[ \quad = \frac{1}{2} D^2 + V(t, \nabla gS(t, D)) + G(t)(x - \nabla gS(t, D)). \]

We define also \( U(t, s) \) to be the dynamics generated by \( H(t) \).

Note that the operator \( \tilde{H}(t) \) is in general not self-adjoint. We will write \( \tilde{D} \) for
\[ \frac{d}{dt} + i[\tilde{H}(t), \cdot]. \]

**Lemma 3.6.8** There exists a two-parameter family of uniformly bounded operators \( \tilde{U}(t, s) \) defined by

\[
\begin{aligned}
&i\partial_t \tilde{U}(t, s) = \tilde{H}(t)\tilde{U}(t, s) \\
&\tilde{U}(t, t) = 1.
\end{aligned}
\]

**Proof.** Let
\[ W(t, s) := U(t, s)\tilde{U}(t, s). \]

It satisfies
\[
\begin{aligned}
&i\partial_t W(t, s) = Z(t, s)W(t, s) \\
&W(t, t) = 1.
\end{aligned}
\]

where \( Z(t, s) := U(t, s)R(t)\tilde{U}(t, s). \) We observe that \( ||Z(\cdot, s)|| \in L^1(dt) \), which directly implies the lemma. \( \square \)

**Lemma 3.6.9**

(3.6.39)
\[ (x - \nabla gS(t, D))\tilde{U}(t, T)\langle x \rangle^{-1} \]
is uniformly bounded.
Proof. We compute:

\[ \tilde{D}_i(x - \nabla \xi S(t, D)) = D + \nabla \xi \tilde{V}_i(t, \nabla \xi S(t, D)) \nabla _\xi S(t, D) \]
\[ + i[\hat{G}(t), (x - \nabla \xi S(t, D))\nabla \xi S(t, D) - \partial_t \nabla \xi S(t, D)] \]
\[ = i[\hat{G}(t), (x - \nabla \xi S(t, D))\nabla \xi S(t, D)]. \]

Let

\[ f(t) := \|\tilde{U}_i(T, t)(x - \nabla \xi S(t, D))\tilde{U}_i(t, T)(x)^{-1}\| \]

Then

\[ \frac{d}{dT} f(t) \leq \|\frac{d}{dT} \tilde{U}_i(T, t)(x - \nabla \xi S(t, D))\tilde{U}_i(t, T)(x)^{-1}\| \]
\[ = \|\tilde{U}_i(T, t)\tilde{D}_i(x - \nabla \xi S(t, D))\tilde{U}_i(t, T)(x)^{-1}\| \]
\[ \leq g(t)f(t), \]

where

\[ g(t) := \|\tilde{U}_i(T, t)[(x - \nabla \xi S(t, D), \hat{G}(t)]\tilde{U}_i(t, T)\| \]

is integrable by (3.6.37). Therefore, by the Gronwall inequality

\[ f(t) \leq Cf(T). \]

This completes the proof of the lemma. □

Proof of Theorem 3.6.1. We observe that by the arguments of Theorem 3.3.4 in Section 3.3 the norm limit

\[ \lim_{t \to +\infty} U_i(t, t)U(t, T) \]

exists. Moreover,

\[ \|H_i(t) - \tilde{H}_i(t)\| = \|R(t)\| \in L^1(dt), \]

which implies that the limit

\[ \lim_{t \to +\infty} U_i(t, t)\tilde{U}_i(t, T) \]

exist. By the chain rule of wave operators, it suffices to prove that the limits

\[ \lim_{t \to +\infty} \tilde{U}_i(t, t)e^{-iS(t, D)} \]

and

\[ \lim_{t \to +\infty} e^{iS(t, D)}\tilde{U}_i(t, T) \]

exist.

We first observe that

\[ e^{iS(t, D)}(x - \nabla \xi S(t, D))e^{-iS(t, D)} = x, \]

which implies

\[ \| (x - \nabla \xi S(t, D))e^{-iS(t, D)}(x)^{-1}\| \leq C, \quad t \geq T. \]

We have for \( \phi \in \mathcal{D}(x) \):

\[ \frac{d}{dt} \tilde{U}_i(T, t)e^{-iS(t, D)}\phi = \tilde{U}_i(T, t)G(t, x, D)(x - \nabla \xi S(t, D))e^{-iS(t, D)}\phi \]

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Hence
\[ \left\| \frac{d}{dt} \tilde{U}_1(T, t) e^{-iS(t, D)} \phi \right\| \]
\leq \left\| \tilde{U}_1(T, t) G(t, x, D) \right\| \left\| (x - \nabla \xi S(t, D)) e^{-iS(t, D)} \phi \right\| \in L^1(dt),

by (3.6.42), which proves that the limit (3.6.2) exists. Similarly we compute
\[ \left\| \frac{d}{dt} e^{iS(t, D)} \tilde{U}_1(t, T) \phi \right\| \]
\leq \left\| G(t, x, D) \right\| \left\| (x - \nabla \xi S(t, D)) \tilde{U}_1(t, T) \phi \right\| \in L^1(dt),

by Lemma 3.6.9 which proves the existence of (3.6.3).

The proof of (3.6.4) is immediate. The statements (3.6.5) and (3.6.6) are easy consequences of (3.6.2) and (3.6.3) and of the identity
\[ e^{iS(t, D)} f(x) e^{-iS(t, D)} = f(x + \nabla \xi S(t, D)). \]

This completes the proof of the theorem. \( \square \)

### 3.7 Long-range scattering for smooth potentials.

In this section we give an independent treatment of the topics discussed in Section 3.6 if the long-range part of the potentials satisfies the smooth long-range condition. For potentials from this class one can avoid some of the technicalities of the previous section and give a simpler proof of the existence and completeness of wave operators.

The main result of this section is the following analog of Theorem 3.6.1.

**Theorem 3.7.1** Assume that
\[ V(t, x) = V_2(t, x) + V_1(t, x) \]
such that
\[ \int_0^\infty \| V_2(t, \cdot) \|_{\infty} dt < \infty \]
and \( V_1(t, x) \) satisfies
\[ \int_0^\infty \| \partial_\alpha V_1(t, \cdot) \|_{\infty} dt < \infty, \quad |\alpha| \geq 1. \tag{3.7.1} \]
For \( T \) big enough let \( S(t, \xi) \) be the solution of the problem
\[ \left\{ \begin{array}{l} \partial_t S(t, \xi) = \frac{1}{\xi^2} + V_1(t, \nabla \xi S(t, \xi)), \\ S(T, \xi) = 0. \end{array} \right. \tag{3.7.2} \]
Then the limits
\[ \lim_{t \to \infty} \text{s-} U(0, t) e^{-iS(t, D)} \]
and
\[ \lim_{t \to \infty} e^{iS(t, D)} U(t, 0). \tag{3.7.3} \]
exist. If we denote (3.6.2) by \( \Omega^+_r \), then (3.6.3) equals \( \Omega^+_r \). Moreover,
\[ D^+ = \Omega^+_r D \Omega^+_r. \tag{3.7.5} \]
The limit
\[ s - C_\infty = \lim_{t \to \infty} U(0, t)(x - \nabla_\xi S(t, D))U(t, 0) =: x^+_t \]
exists and defines a vector of commuting densely defined self-adjoint operators and will be called the asymptotic position. One has
\[ x^+_t = \Omega^+_t x^+_t. \]

Recall from Appendix C.6 that
\[ L^1((t)^m dt, S(g_0(t))) \]
denotes the set of the functions
\[ [0, \infty] \times X \times X^t \ni (t, x, \xi) \mapsto a(t, x, \xi) \in \mathbb{C} \]
such that
\[ \int \| \partial_\beta^a \partial_\alpha^b a(t, \cdot, \cdot) \| (t)^{m+4}|dt < \infty, \ \alpha, \beta \in \mathbb{N}^n. \]
The space
\[ L^1((t)^m dt, \Psi(g_0(t))) \]
is defined as the set of functions
\[ [0, \infty] \ni t \mapsto f(t) \in B(L^2(X)) \]
such that one of the following equivalent conditions holds:
\[ \int \| [ad_{\partial_\beta^a} \partial_\alpha^b f(t)](t)^{m+4}|dt < \infty, \ \alpha, \beta \in \mathbb{N}^n, \]
\[ f(t) = f^0(t, x, D) \text{ for some } f \in L^1((t)^m dt, S(g_0(t))). \]

Note that the force \( F_i(t, x) = -\nabla_\xi V_i(t, x) \) can be regarded both as an element of \( L^1(dt, S(g_0(t))) \) and as an element of \( L^1(dt, \Psi(g_0(t))) \).

We set
\[ g(t, x, \xi) := \int_0^1 F_i(t, \tau x + (1 - \tau)\nabla_\xi S(t, \xi))(d\tau, \]
\[ r(t, x, \xi) := \text{div} g(t, x, \xi). \]

We define
\[ G(t) := g(t, x, D), \]
\[ R(t) := r(t, x, D). \]

**Lemma 3.7.2** One has
\[ V_i(t, x) - V_i(t, \nabla_\xi S(t, D)) = G(t)(x - \nabla_\xi S(t, D)) + R(t). \]
Moreover \( G(t) \) and \( R(t) \) belong to \( L^1(dt, \Psi(g_0(t))) \).

**Proof.** First note that
\[ V_i(t, x) - V_i(t, \nabla_\xi S(t, D)) = xG(t) - G(t)\nabla_\xi S(t, D) = G(t)(x - \nabla_\xi S(t, D)) + \sum_{i=1}^n [x_i, G_i(t)]. \]
Let us show the following estimate on $g(t, x, \xi)$:

\begin{equation}
\int_0^\infty \|\partial_\xi^\alpha \partial_\eta^\beta g(t, \cdot, \cdot)\|_\infty(t)^{\alpha_1} dt < \infty, \quad \alpha, \beta \in \mathbb{N}^n.
\end{equation}

Let us recall that in Proposition 1.8.2 we proved that

\begin{equation}
|\partial_\xi^\beta S(t, \xi)| \leq C_\beta(t), \quad |\beta| \geq 1.
\end{equation}

By the Faa di Bruno formula

\[
\partial_\xi^\alpha \partial_\eta^\beta F_i(t, \tau x + (1 - \tau)\nabla_\xi S(t, \xi)) = \sum C_\beta \partial_\xi^\alpha \nabla_\xi^\beta F_i(t, \tau x + (1 - \tau)\nabla_\xi S(t, \xi)) \partial_\xi^\beta \nabla_\xi^\beta S(t, \xi) \ldots \partial_\xi^\beta \nabla_\xi^\beta S(t, \xi) = \sum f_\beta(t, x, \xi).
\]

Therefore,

\[
|f_\beta(t, x, \xi)| \leq C\|\nabla_\xi^{\alpha_1 + \beta_1} V_i(t, \cdot)\|_\infty(t)^{\beta}.
\]

Therefore, $g(t, x, \xi)$ and $r(t, x, \xi)$ belong to $L^1(dt, S(g_0(t)))$, which implies that $G(t)$ and $R(t)$ belong to $L^1(dt, \Psi(g_0(t)))$. \qed

We define $H_i(t) := \frac{1}{2}D^2 + V_i(t, x)$.

We will write $D_i$ for

\[
\partial_t + i[H_i(t), \cdot].
\]

Let $U_i(t, s)$ be the unitary propagator associated with $H_i(t)$.

**Lemma 3.7.3**

\begin{equation}
(x - \nabla_\xi S(t, D))U_i(t, T)(x)^{-1}
\end{equation}

is uniformly bounded.

**Proof.** We compute:

\[
D_i(x - \nabla_\xi S(t, D)) = D + [V_i(t, x), \nabla_\xi S(t, D)] - \partial_t \nabla_\xi S(t, D)
\]

\[
= i[V_i(t, x) - V_i(t, \nabla_\xi S(t, D), (x - \nabla_\xi S(t, D))]
\]

\[
= i[G(t), (x - \nabla_\xi S(t, D))](x - \nabla_\xi S(t, D)) + i[R(t), (x - \nabla_\xi S(t, D)).
\]

Let

\[
f(t) := \|U_i(T, t)(x - \nabla_\xi S(t, D))U_i(t, T)(x)^{-1}\|
\]

Then

\[
\frac{d}{dt} f(t)
\]

\[
\leq \|\frac{d}{dt} U_i(T, t)(x - \nabla_\xi S(t, D))U_i(t, T)(x)^{-1}\|
\]

\[
= \|U_i(T, t)D_i(x - \nabla_\xi S(t, D))U_i(t, T)(x)^{-1}\|
\]

\[
\leq g(t)f(t) + h(t),
\]

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where the functions
\[ g(t) := \|U_1(T, t)[(x - \nabla \xi S(t, D), G(t)]U_1(t, T)] \|
\]
\[ h(t) := \|U_1(T, t)[(x - \nabla \xi S(t, D), R(t)]U_1(t, T)(x)^{-1}] \|
\]
are integrable by Lemma 3.7.2. Therefore, by the Gronwall inequality
\[ f(t) \leq C + Cf(T) \]
is bounded. □

Proof of Theorem 3.7.1. We start with the proof of the existence of (3.7.3) and (3.7.4). We first observe that by the arguments of Theorem 3.3.4 in Section 3.3 the norm limit
\[ \lim_{t \to +\infty} U_1(T, t)U(t, T) \]
exists. So by the chain rule of wave operators, it suffices to prove that the limits
\[ \text{(3.7.14)} \quad s- \lim_{t \to +\infty} U_1(T, t)e^{-iS(t, D)} \]
and
\[ \text{(3.7.15)} \quad s- \lim_{t \to +\infty} e^{iS(t, D)}U_1(t, T) \]
extist.
We first observe that
\[ e^{iS(t, D)}(x - \nabla \xi S(t, D))e^{-iS(t, D)} = x, \]
which implies
\[ \text{(3.7.16)} \quad \| (x - \nabla \xi S(t, D))e^{-iS(t, D)}(x)^{-1} \| \leq C, \quad t \geq T. \]
We have
\[ \frac{d}{dt}U_1(T, t)e^{-iS(t, D)} = U_1(T, t)(V_1(t, x) - V_1(t, \nabla \xi S(t, D)))e^{-iS(t, D)} \phi. \]
Hence
\[ \| \frac{d}{dt}U_1(T, t)e^{-iS(t, D)} \phi \|
\]
\[ \leq \|U_1(T, t)G(t)(x - \nabla \xi S(t, D))e^{-iS(t, D)} \phi \| + \|U_1(T, t)R(t)e^{-iS(t, D)} \phi \| \in L^1(dt), \]
by (3.7.16), which proves that the limit (3.7.14) exists. Similarly we compute
\[ \| \frac{d}{dt}e^{iS(t, D)}U_1(t, T) \phi \|
\]
\[ \leq \|e^{iS(t, D)}G(t)(x - \nabla \xi S(t, D))U_1(t, T) \phi \| + \|e^{iS(t, D)}R(t)U_1(t, T) \phi \| \in L^1(dt), \]
which proves the existence of (3.7.15).

The proof of (3.7.5) is immediate. The statements (3.7.6) and (3.7.7) are easy consequences of (3.7.3) and (3.7.4) and of the identity
\[ e^{iS(t, D)}f(x)e^{-iS(t, D)} = f(x + \nabla \xi S(t, D)). \]

This completes the proof of the theorem. □
3.8 Other modified free dynamics

While the modified free dynamics considered in Section 3.6 and 3.7 work well for a very large class of potentials, it is often convenient for practical purposes to consider other modified free dynamics. In this section, which is parallel to Section 1.9 in Chapter 1, we give some examples of such dynamics.

Another problem which we would like to consider in this section is scattering theory for systems with some “internal degrees of freedom” (e.g., spin). More precisely, let us assume that the dynamics of our system is generated by the following time-dependent Hamiltonian

\[ H(t) := \frac{1}{2} \dot{\mathbf{x}}^2 + V(t, \mathbf{x}), \]

where

\[ V^*(t, x) = V(t, x) \in \mathcal{B}(\mathcal{H}_1) \]

and \( \mathcal{H}_1 \) is a certain auxiliary Hilbert space and the Hamiltonian \( H(t) \) generates a flow \( U(t, s) \) in the sense described in Section 3.1. One can try to study scattering theory for such systems.

Scattering theory for such \( H(t) \) in the short-range case is completely analogous to what we described in Section 3.3. In the long-range case however, if the potential couples the internal degrees of freedom in a nontrivial way the constructions of Sections 3.6 and 3.7 do not go through. In fact, we cannot even write the Hamilton-Jacobi equation. Nevertheless, it turns out that the so-called Dollard modified wave operators, which we are going to present in the following theorem, work in the case of internal degrees of freedom.

We will assume that the time-dependent potential \( V(t, x) \) is equal to

\[ V(t, x) = V_0(t, x) + V_1(t, x), \]

where for almost all \((t, x)\) the operators \( V_0(t, x) \) and \( V_1(t, x) \) are self-adjoint on \( \mathcal{B}(\mathcal{H}_1) \),

\[
\int_0^\infty \| V_0(t, x) \| dt < \infty, \quad \int_0^\infty \| V_1(t, x) \| \|D^\alpha V_1(t, x)\| \| \mathcal{B}(\mathcal{H}_1) \| dt < \infty, \quad |\alpha| = 1.
\]

By Lemma 3.6.5 we can change the splitting of \( V(t, x) \) so that

\[
\int_0^\infty \| V_0(t, x) \| dt < \infty, \quad \int_0^\infty \| V_1(t, x) \| \|D^\alpha V_1(t, x)\| \| \mathcal{B}(\mathcal{H}_1) \| dt < \infty, \quad |\alpha| = 1, \quad |\alpha| = 2.
\]

Now we can introduce the Dollard modified dynamics. For \( \xi \in \mathcal{X} \), we denote by

\[
T \left( e^{-i \int_0^t V(t, s, \xi) ds} \right)
\]

the unitary propagator on \( \mathcal{H}_1 \) for the time-dependent Hamiltonian \( V(t, t\xi) \). The symbol \( T \) stands for ‘time-ordered product’.

**Definition 3.8.1** We define the Dollard modified dynamics \( U_D(t) \) by

\[
U_D(t) := e^{-i \int_0^t \dot{\mathbf{x}}^2 dt} T \left( e^{-i \int_0^t V(t, s, \xi) ds} \right).
\]
The modified free dynamics $U_D(t)$ was essentially first introduced by Dollard [Do1].
Then we have the following result:

**Theorem 3.8.2** Under the above conditions, the limits

\[(3.8.5) \quad \lim_{t \to \pm \infty} U(0, t)U_D(t) =: \Omega_D^\pm \]

and

\[(3.8.6) \quad \lim_{t \to \pm \infty} U_D(T)^* U(t, 0) \]

exist and the limit in (3.8.6) is equal to $\Omega_D^{\pm_*}$. Moreover,

\[D^* = \Omega_D^{\pm_*} \Omega_D^D.\]

**Proof.** Let us denote by $U_1(t, 0)$ the unitary propagator generated by the time-dependent Hamiltonian

\[H(t) = \frac{1}{2}D^2 \otimes 1_{\mathbb{R}_1} + V_1(t, x).\]

We first claim that

\[(3.8.7) \quad ||(x - tD)U_1(t, 0)(x)^{-1}|| = O(t^\frac{1}{2}), \]

and

\[(3.8.8) \quad ||(x - tD)U_D(t)(x)^{-1}|| = O(t^\frac{1}{2}).\]

Indeed we compute

\[
\frac{d}{dt}U_1(0, t)(x - tD)U_1(t, 0)(x)^{-1}
= iU_1(0, t)t\nabla_x V_1(t, x)U_1(t, 0)(x)^{-1} = O(t^\frac{1}{2})v(t),
\]

for $v(t) \in L^1(dt)$, which proves (3.8.7) by integration from 0 to $t$. Similarly

\[
\frac{d}{dt}U_D(t)^*(x - tD)U_D(t)(x)^{-1}
= iU_D(t)^*t\nabla_x V(t, tD)U_D(t)(x)^{-1} = O(t^\frac{1}{2})v(t),
\]

which proves (3.8.8).

Let us now prove the existence of the limit (3.8.6). By the chain rule of wave operators and the arguments used in the proof of Theorem 3.3.4, we may replace in $H(t)$ the potential $V_1$ by 0, and assume that $U(t, 0) = U_1(t, 0)$. For $\phi \in \mathcal{D}(x)$, we compute

\[(3.8.9) \quad \frac{d}{dt}U_D(t)^*U(t, 0)\phi
= U_D(t)^*(V_1(t, tD) - V_1(t, x))U(t, 0)\phi.\]

Using the Baker-Campbell-Hausdorff formula, we get:

\[(3.8.10) \quad V_1(t, tD) - V_1(t, x)
= \int_0^1 \nabla V_1(t, \tau tD + (1 - \tau)x) (tD - x) d\tau
+ \frac{it}{2} \int_0^1 \Delta V_1(t, \tau tD + (1 - \tau)x) d\tau.\]

Now let $\phi \in \mathcal{D}(x)$. Then using the estimates (3.8.11) satisfied by $V_1(t, x)$, we obtain

\[
||U_D(t)^*(V_1(t, tD) - V_1(t, x))U(t, 0)\phi||
\leq C ||\nabla V_1(t, \cdot)||_\infty ||(x - tD)U(t, 0)(x)^{-1} || ||(x)\phi|| + Ct||\Delta V_1(t, \cdot)||_\infty ||\phi||
\in L^1(dt),
\]
which proves the existence of the limit (3.8.6). The proof of the existence of the limit (3.8.5) is analogous except that we use (3.8.8) instead of (3.8.7). □

For the potential that decays more slowly than (3.8.3) the Dollard wave operators in general do not exist. In this case, if one does not want to construct precise solutions of the Hamilton-Jacobi equation one can define wave operators using the Buslæv-Matveev construction. This construction uses functions $Z_N(t, \xi)$, which were introduced in Section 1.9. For the convenience of the reader let us recall their definition.

**Definition 3.8.3** Let $V_i(t, x)$ be a potential such that for some $N \in \mathbb{N}, N \geq 2$

$$
\int_0^\infty dt |t| \| \partial^\alpha_x \nabla_x V_i(t, \cdot) \|_{\infty} dt < \infty, \forall |\alpha| = N - 2,
$$

$$
\int_0^\infty t^{N-1} |\partial^\alpha_x \nabla_x V_i(t, \cdot) \|_{\infty} dt < \infty, \forall |\alpha| = N - 1.
$$

(3.8.11)

Let

$$
P(Z)(t, \xi) := \frac{1}{2} t \xi^2 + \int_0^t V_i(u, \nabla_\xi Z(u, \xi)) du.
$$

(3.8.12)

Then we define inductively

$$
Z_1(t, \xi) := \frac{1}{2} t \xi^2,
$$

$$
Z_n(t, \xi) := P(Z_{n-1})(t, \xi), \text{ for } 2 \leq n \leq N.
$$

(3.8.13)

**Theorem 3.8.4** Assume that $V(t, x) = V_0(t, x) + V_i(t, x)$ such that

$$
\int_0^\infty \| V_0(t, \cdot) \|_{\infty} dt < \infty
$$

and $V_i(t, x)$ satisfies (3.8.11) for some $N \in \mathbb{N}, N \geq 2$. Then the limits

$$
\lim_{t \to \pm \infty} U(0, t) e^{-i Z_N(t, D)} =: \Omega_+^N
$$

and

$$
\lim_{t \to \pm \infty} e^{i Z_N(t, D)} U(t, 0)
$$

(3.8.14)

exist and the limit in (3.8.15) is equal to $\Omega^+_N$. Moreover

$$
D^+ = \Omega^+_N D \Omega^+_N.
$$

(3.8.15)

The dynamics $e^{-i Z_N(t, D)}$ was first defined by Buslæv-Matveev [Bu-Ma].

**Proof.** The conditions of the above theorem imply the hypotheses of Proposition 3.6.2. Hence in Theorem 3.6.1 in order to construct a modified wave operator we can use a solution $S(t, \xi)$ of the Hamilton-Jacobi equation with the potential $V_i(t, x)$. But in Proposition 1.9.2 we showed that the uniform limit

$$
\lim_{t \to \infty} (S(t, \xi) - Z_N(t, \xi))
$$

exists. Hence there exists also

$$
\lim_{t \to \infty} e^{i S(t, D)} e^{-i Z_N(t, D)} \text{ and } \lim_{t \to \infty} e^{i Z_N(t, D)} e^{-i S(t, D)}.
$$

(3.8.16)

Now Theorem 3.6.1 and (3.8.16) imply the existence of (3.8.14) and (3.8.15). □
3.9 A counterexample to asymptotic completeness

There exists a large class of time-dependent potentials for which the asymptotic momentum $D^+$ and the wave operator $\Omega^+_{\text{sc}}$ are well defined but the asymptotic completeness breaks down, that is $\text{Ran} \Omega^+_{\text{sc}} \neq L^2(X)$. In this section we construct a potential with such a property. Similar examples were first studied by Yafaev [Ya].

Let us fix a certain cutoff function $\chi \in C^\infty_c(\mathbb{R})$ equal to 1 near the origin. Let us consider the time-dependent potential

(3.9.1) \[ V(t, x) := C|x|^2 \chi \left( \frac{|x|}{ax} \right), \]

where $C > \frac{1}{8}$ and $a > 0$. We denote by $U(t, s)$ the unitary evolution generated by the Hamiltonian

\[ H(t) = \frac{1}{2} D^2 + V(t, x). \]

Note that one has

\[ \|\nabla_x V(t, \cdot)\|_\infty \sim t^{-3/2} (\ln t)^{3/2}, \]

\[ \|\nabla_x^2 V(t, \cdot)\|_\infty \sim t^{-2}, \]

so $V(t, x)$ almost (but not quite) satisfies the conditions of the existence and completeness of modified wave operators for general long-range potentials in Theorem 3.6.1 or of the existence and completeness of Dollard wave operators in Theorem 3.8.2. As we will show below, the usual wave operators exist for this Hamiltonian, but they are not complete.

We have

**Theorem 3.9.1** For the potential $V(t, x)$ defined in (3.9.1) the eigenprojection $E_{\{0\}}(D^+)$ has infinite multiplicity. Moreover, there exists the (usual) wave operator

(3.9.2) \[ s- \lim_{t \to \infty} U(1, t)e^{-itH_0} =: \Omega^+_{\text{sc}}, \]

and it satisfies

\[ D^+ \Omega^+_{\text{sc}} = \Omega^+_{\text{sc}} D. \]

**Proof.** We start with the observation that the unitary evolution $U_Y(t)$ generated by the time-dependent Hamiltonian

\[ H_Y(t) := \frac{1}{2} D^2 + \frac{C|x|^2}{t^2} \]

can be constructed explicitly. Indeed if we put

\[ T_t \phi(x) := e^{-t^{1/4} \phi(x)} \theta \left( \frac{x}{t^{1/4}} \right), \]

\[ U_0 \phi(x) := e^{-ix^2/4} \phi(x), \]

\[ H_Y = \frac{1}{2} D^2 + (C - \frac{1}{8}) x^2, \]

then it is an easy computation to check that

\[ U_Y(t, 1) \phi = T_t U_0 e^{-it H_Y} \phi. \]

We will show the existence of

(3.9.3) \[ s- \lim_{t \to \infty} U(1, t)U_Y(t, 1) =: \Omega_Y. \]

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In fact, suppose that \( \psi \) is an eigenfunction of \( H_Y \) for the eigenvalue \( \lambda \). Then

\[
U_Y(t, 1)\psi(x) = t^{-\alpha/2} e^{-i \lambda \log t} t^{\alpha/2} \chi \left( \frac{1}{\alpha t^4} \right) \psi \left( \frac{x}{t^2} \right),
\]

Moreover,

\[
\frac{d}{dt} U_Y(t, 1)\psi = U(1, t) \frac{g^2}{1 - \chi \left( \frac{1}{\alpha t^4} \right)} U(t, 1)\psi
\]

If we use (3.9.4), (3.9.5) and the exponential decay of \( \psi \) (which is an eigenfunction of a harmonic oscillator), then we see that

\[
\| \frac{d}{dt} U_Y(t, 1)\psi \| \leq C_1 t \exp \left( -C_2 \alpha \log t \right),
\]

which is integrable.

It is easy to see that \( \text{Ran} \Omega_Y^+ \subset \text{Ran} E_{[0]}(D^+) \). Since \( \Omega_Y^+ \) is an isometry from \( L^2(X) \), its range is infinite dimensional. This proves the first statement of the theorem.

Let us now show the existence of (3.9.2)

Let \( g, J \in C_0^\infty (X) \) such that \( 0 \not\subset \text{supp} J \) and \( J = 1 \) on a neighborhood of \( \text{supp} g \). Then we have

\[
g(D) = \lim_{t \to \infty} e^{-itH_0} J \left( \frac{x}{t} \right) g(D) e^{-itH_0}.
\]

Therefore,

\[
\lim_{t \to \infty} U_Y(1, t)e^{-itH_0} g^2(D)
\]

\[
\lim_{t \to \infty} U_Y(1, t)g(D)J \left( \frac{x}{t} \right) g(D) e^{-itH_0}
\]

Now let

\[
D_Y A(t) := \frac{d}{dt} A(t) + i H(t) A(t) - i A(t) H_Y(t).
\]

For large \( t \) we have

\[
D_Y g(D) J \left( \frac{x}{t} \right) g(D)
\]

\[
= \left[ g(D), V(t, \cdot) \right] J \left( \frac{x}{t} \right) g(D)
\]

\[
+ g(D) \left( D_Y J \left( \frac{x}{t} \right) \right) g(D).
\]

The first term on the right of (3.9.7) is integrable in norm and the second is integrable along the evolution. Therefore the limit (3.9.6) exists. \( \square \)

### 3.10 Egorov theorem for time-dependent Schrödinger operators

In this section we investigate the properties of solutions of the Heisenberg equation. We consider a solution of the Heisenberg equation, which at time \( t \) is given by a pseudodifferential operator \( M(t) \) and study its properties for times \( s \leq t \). It is easy to see that for any fixed time this observable is a pseudodifferential operator in the class \( \Psi(L^2(X)) \). It is more difficult to obtain more precise estimates which are uniform in time.

We will assume that the potential satisfies the so-called smooth long-range condition, that is

\[
\int_0^\infty |\alpha|^{-1} ||\partial_\alpha^\alpha V(t, \cdot)||_{\infty} dt < \infty, \quad |\alpha| \geq 1.
\]

We start with a rather crude lemma.
Lemma 3.10.1 Assume (3.10.1). Suppose that
\[
P(t) \in L^1((t)^m dt, \Psi(dx^2 + d\xi^2)).
\]
Set
\[
P(t, s) := U(s, t)P(t)U(t, s).
\]
Then uniformly for \(0 \leq s \leq t\)
(3.10.2) \[
P(t, s) \in L^1((t)^m dt, \Psi(dx^2 + (t^2d\xi^2)),
\]
or in other words
\[
\int_0^\infty \sup_{0 \leq s \leq t} \|\text{ad}_D^\alpha \text{ad}_x^\beta P(t, s)\|(t)^m-1|^{\beta} dt < \infty, \quad \alpha, \beta \in \mathbb{N}^n.
\]
Proof. Assume (3.10.1) and let
\[
\Omega_{\text{sr}}(t, s) := U(s, t)U_0(t - s).
\]
We will prove by induction on \(|\alpha| + |\beta|\) that
\[
\text{ad}_D^\alpha \text{ad}_x^\beta \Omega_{\text{sr}}(t, s) \in O(|t|).
\]
This follows easily from (3.4.10) if we use the fact that
\[
\text{ad}_D^\alpha \text{ad}_x^\beta \Psi(t, x + (t - s)D) \in t^{|\alpha| - 1 |\beta|} L^1(dt), \quad \text{for } |\alpha| + |\beta| \geq 1.
\]
So we know that
\[
U(s, t)U_0(t - s) \in \Psi(1, dx^2 + t^2d\xi^2).
\]
It is also easy to see that
\[
U_0(s - t)P(t)U_0(t - s) \in L^1(t^m dt, \Psi(dx^2 + t^2d\xi^2)).
\]
Therefore (3.10.2) follows from:
\[
P(t, s) = \Omega_{\text{sr}}(t, s)U_0(s - t)P(t)U_0(t - s)\Omega_{\text{sr}}^*(t, s).
\]
\[\square\]

The main result of this section is the following theorem.

Theorem 3.10.2 Assume (3.10.1). Suppose that
\[
M(t) \in \Psi(1, g_0(t)).
\]
Set
\[
M(t, s) := U(s, t)M(t)U(t, s).
\]
Then uniformly for \(0 \leq s \leq t\)
(3.10.3) \[
M(t, s) \in \Psi(1, g_0(s)),
\]
or, in other words,
\[
\|\text{ad}_D^\alpha \text{ad}_x^\beta M(t, s)\| \leq C_{\alpha, \beta}(s)^{-|\alpha|}, \quad \alpha, \beta \in \mathbb{N}^n.
\]
The proof of the above theorem will consist in constructing an approximation of \( M(t, s) \) in terms of explicit pseudodifferential operators. Let us describe this approximation.

Let
\[
M(t) = m^w(t, x, D), \\
M(t, s) = m^w(t, s, x, D).
\]

Define inductively the operators
\[
M_k(t, s) = \sum_{j=0}^k m^w_j(t, s, x, D)
\]
where
\[
\begin{aligned}
m_0(t, s, y, \eta) &:= m(t, x(t, s, y, \eta), \xi(t, s, y, \eta)), \\
m_k(t, s, y, \eta) &:= \int_s^t p_{k-1}(t, u, x(u, s, y, \eta), \xi(u, s, y, \eta)) du, \\
p_k(t, s, x, \xi) &:= \sum_{|\alpha|=3} \frac{1}{2} \int_{-1/2}^{1/2} d\tau (|\tau| - \frac{1}{2})^2 \partial_\xi^\alpha V(s, x + \tau D_\xi) D_\xi^\alpha m_k(t, s, x, \xi).
\end{aligned}
\]

The following proposition is a by-product of the proof of Theorem 3.10.2.

**Proposition 3.10.3** Uniformly for \( 0 \leq s \leq t \) we have:

\[
M(t, s) - M_k(t, s) \in \Psi((s)^{-2k-2}, g_0(s)).
\]

---

**Proof of Theorem 3.10.2 and Proposition 3.10.3.**

Define
\[
v_N(t) := \max_{1 \leq \alpha \leq N} \langle t | [H(s), ] M(t, s) \rangle.
\]

Note that for any \( N \) the function \( v_N(\cdot) \) belongs to \( L^1(dt) \).

The equation
\[
\begin{aligned}
(\partial_s + i[H(s), ]) M(t, s) &= 0, \\
M(t, t) &= M(t),
\end{aligned}
\]

is equivalent to
\[
\begin{aligned}
(\partial_s + \{ h(s, x, \xi), \}) \\
- \sum_{|\alpha|=3} \frac{1}{2} \int_{-1/2}^{1/2} d\tau (|\tau| - \frac{1}{2})^2 \partial_\xi^\alpha V(s, x + \tau D_\xi) D_\xi^\alpha m(t, s, x, \xi) = 0, \\
m(t, t, x, \xi) &= m(t, x, \xi).
\end{aligned}
\]

Let us introduce the operator
\[
\Phi(t, s) : S(X \times X') \rightarrow S(X \times X')
\]
given by
\[
(\Phi(t, s) a)(y, \eta) := a(x(t, s, y, \eta), \xi(t, s, y, \eta))
\]
and
\[
\Gamma(s) : S(X \times X') \rightarrow S(X \times X')
\]
given by
\[
\Gamma(s) := \sum_{|\alpha|=3} \frac{1}{2} \int_{-1/2}^{1/2} d\tau (|\tau| - \frac{1}{2})^2 \partial_\xi^\alpha V(s, x + \tau D_\xi) D_\xi^\alpha.
\]
Now (3.10.7) can be written as

\begin{align}
\Phi(t, s) \partial_s \Phi(t, s) m(t, s) &= \Gamma(s) m(t, s), \\
m(t, t) &= m(t),
\end{align}

or in the integral form

\begin{align}
m(t, s) &= \Phi(t, s) m(t) + \int_s^t \Phi(u, s) \Gamma(u) m(t, u) du.
\end{align}

Let us show the following properties of the maps \( \Phi(t, s) \) and \( \Gamma(s) \).

**Lemma 3.10.4** Let \( a \in S(X \times X') \). Then

\begin{align}
\| \Phi(t, s)a \|_{(1, \rho_0(s))}, N \leq C_N \| a \|_{(1, \rho_0(t))}, N.
\end{align}

and for any \( N \) there exists \( N_1 \) such that

\begin{align}
\| \Gamma(s)a \|_{(1, \rho_0(s)), N} \leq C_N \langle s \rangle^{-2} v_N(s) \| a \|_{S(1, \rho_0(s)), N_1},
\end{align}

where the seminorms \( \| \cdot \|_{S(1, \rho_0(s)), N} \) are defined in (C.6.26).

**Proof.** We have

\begin{align}
\partial^\zeta_y \partial^\xi_x \Phi(t, s) a(y, \eta) \\
= \sum C_{\gamma \rho \xi} \nabla^\gamma_x \nabla^\xi_y a(x(t, s, y, \eta), \xi(t, s, y, \eta)) \\
\times \prod_{\gamma=1}^{d_{\gamma}} \partial^\gamma_x \partial^\gamma_y x(t, s, y, \eta) \prod_{\xi=1}^{d_{\zeta}} \partial^\zeta_x \partial^\zeta_y \xi(t, s, y, \eta)
\end{align}

Using Theorem 1.3.1 in Chapter 1, we obtain

\[ |g_{\gamma \rho \xi}(t, s, y, \eta)| \leq O(s^{-|\gamma|}1\ldots|\rho|\ldots|\zeta|) |t^\gamma \nabla^\gamma_x \nabla^\xi_y a(t, \cdot, \cdot)|_\infty. \]

Hence,

\[ \left| \partial^\zeta_y \partial^\xi_x \Phi(t, s) a(x, \xi) \right| \leq O(s^{-|\gamma|}) \|a\|_{S(1, \rho_0(t))} \|a\|_{S|s|}. \]

The estimate (3.10.11) follows from Proposition C.6.3 2) \( \Box \).

If we write \( m(t, s), m(t) \) for \( m(t, s, \cdot, \cdot) \) and \( m(t, \cdot, \cdot) \) respectively, then the equations (3.10.4) can be rewritten in a more compact notation:

\begin{align}
\begin{cases}
m_0(t, s) := \Phi(t, s)m(t), \\
m_k(t, s) := \int_s^t \Phi(s, u)m_{k-1}(t, u) du, \\
p_k(t, s) := \Gamma(s)m_k(t, s).
\end{cases}
\end{align}

By (3.10.10) we have

\begin{align}
|\partial^\zeta_y \partial^\xi_x m_0(t, s, x, \xi)| \leq O(s^{-|\gamma|}).
\end{align}

By (3.10.14) and (3.10.11) there exists \( N \) such that

\begin{align}
|\partial^\zeta_y \partial^\xi_x p_0(t, s, x, \xi)| \leq O(s^{-|\gamma|}v_N(s)),
\end{align}

We easily see by induction using (3.10.13) and the fact that \( v_N(s) \in L^1(ds) \) that for any \( k \) there exists \( N \) such that

\begin{align}
|\partial^\zeta_y \partial^\xi_x m_k(t, s, x, \xi)| \leq O(s^{-|\gamma|}v_N). \]

\[ \Box \]
and
\[ \left| \partial^2_x \partial^2_{\xi} P_k(t, s, x, \xi) \right| \leq O(s^{-|\alpha| - 2k - 2}) v_N(s). \]

Now define
\[ P_k(t, s) := p^n_k(t, s, x, D) \in L^1(s^{-2k - 2} ds, \Psi(g_0(s))). \]

Note that
\[ M(t, s) - M_k(t, s) = \int_s^t U(s, u)P_k(u, t)U(u, s)\,du. \]

By Lemma 3.10.1 for any \( \alpha \in \mathbb{N}^n \) and \( |\beta| \leq 2k + 2 \)
\[ \left\| \text{ad}^\alpha_{D_x} \text{ad}^\beta_{D_{\xi}} \int_s^t U(s, u)P_k(u, t)U(u, s)\,du \right\| \leq O(s^{\frac{1}{2} - 2k - 2}), \]

Clearly, for any \( \alpha, \beta \in \mathbb{N}^n, k \)
\[ \left\| \text{ad}^\alpha_{D_x} \text{ad}^\beta_{D_{\xi}} M_k(t, s) \right\| \leq O(s^{-|\alpha|}). \]

Therefore,
\[ \left\| \text{ad}^\alpha_{D_x} \text{ad}^\beta_{D_{\xi}} M(t, s) \right\| \leq O(s^{-\min\{|\alpha|, |k + 2\}}), \quad |\beta| \leq 2k + 2. \]

Since \( k \) was arbitrary, it implies
\[ \left\| \text{ad}^\alpha_{D_x} \text{ad}^\beta_{D_{\xi}} M(t, s) \right\| \leq O(s^{-|\alpha|}), \]

and therefore it ends the proof of (3.10.3).

It remains to prove Proposition 3.10.3. For any \( k_0 \geq k \) we can write
\[ M(t, s) - M_k(t, s) = M_{k_0}(t, s) - M_k(t, s) + \int_s^t U(s, u)P_{k_0}(u, t)U(u, s)\,du. \]

Now clearly
\[ \left\| \text{ad}^\alpha_{D_x} \text{ad}^\beta_{D_{\xi}} (M_{k_0}(t, s) - M_k(t, s)) \right\| \leq O(s^{-|\alpha| - 2k - 2}) \]
and
\[ \left\| \text{ad}^\alpha_{D_x} \text{ad}^\beta_{D_{\xi}} \int_s^t U(s, u)P_{k_0}(u, t)U(u, s)\,du \right\| \leq O(s^{\frac{1}{2} - 2k_0}), \quad |\beta| \leq 2k_0 + 2. \]

This implies (3.10.5) since \( k_0 \) is arbitrary. \( \square \)

### 3.11 Smoothness of long-range wave operators

In this section we study the smoothness properties of various objects that we constructed in the long-range case.

In the long-range case the problem of the regularity of wave operators is much more difficult than in the short-range case. In particular, the wave operator is no longer a pseudodifferential operator. A naive approach, which was possible in the short-range case, here does not work; one needs to apply finer techniques of the pseudodifferential calculus, which were described in the previous section.

The main result of this section can be described in the following theorem.
Theorem 3.11.1 Assume (3.10.1). Then  

\[(3.11.1) \quad D^+ + D \in \Psi(L^2(X)),\]

there exist \( A, B \in \Psi(L^2(X)) \) such that  

\[(3.11.2) \quad x^+_k - x = Ax + B,\]

\[(3.11.3) \quad \|\langle x^+_k \rangle^{-m} \langle x \rangle^m \| \leq C, \quad m \in \mathbb{R},\]

\[(3.11.4) \quad \|\langle x \rangle^{-m} \text{ad}_{(\Omega^+_k)}^\alpha (\langle x \rangle^m) \| \leq C, \quad m \in \mathbb{R}, \quad \alpha \in \mathbb{N}^n.\]

Similarly as in Section 3.3 we will actually prove a more general theorem with an explicit dependence on the initial time \( s \). We will use the notation  

\[x_{lt}(t, s) := x(t, s) - \nabla_\xi S(t, D(t, s)),\]

\[\Omega_{lt}(t, s) := U(s, t) e^{-iS(t, D)} e^{iS(s, D)}.\]

We extend the definitions of \( x_{lt}(t, s) \) and \( \Omega_{lt}(t, s) \) to \( t = \infty \) in the obvious fashion.

Theorem 3.11.2 Assume (3.10.1). Then  

\[(3.11.5) \quad D(t, s) - D \in \Psi(o(s^0), g_0(s))\]

\[(3.11.6) \quad x(t, s) - x - (t - s) D \in \Psi([t - s] o(s^0), g_0(s)),\]

there exist \( A(t, s), B(t, s) \in \Psi(o(s^0), g_0(s)) \) such that  

\[(3.11.7) \quad x_k(t, s) - x_{lt}(t, s, s) = A(t, s) x_{lt}(t, s, s) + B(t, s),\]

\[(3.11.8) \quad \|\langle x_k(t, s) \rangle^{-m} \langle x_{lt}(t, s, s) \rangle^m \| \leq C, \quad m \in \mathbb{R},\]

\[(3.11.9) \quad \|\langle x_{lt}(t, s, s) \rangle^{-m} \text{ad}_{x_{lt}(t, s, s)}^\alpha (\langle x_{lt}(t, s, s) \rangle^m) \| \leq o(s^0), \quad \alpha \in \mathbb{N}^n,\]

Proof. Clearly,  

\[(3.11.10) \quad D(t, s) - D = \int_s^t U(s, u) F(u, x) U(u, s) du,\]

\[(3.11.11) \quad x(t, s) - x - (t - s) D = \int_s^t (t - u) U(s, u) F(u, x) U(u, s) du,\]

where \( F(u, x) \) is the operator of the multiplication by the force \( -\nabla \xi V(u, x) \). Now  

\[F(u, x) \in L^1(du, \Psi(g_0(u))).\]

Hence by Theorem 3.10.2  

\[(3.11.12) \quad U(s, u) F(u, x) U(u, s) \in L^1(du, \Psi(g_0(s))).\]

Therefore (3.11.10) belongs to \( \Psi(o(s^0), g_0(s)) \) and (3.11.11) belongs to \( \Psi([t - s] o(s^0), g_0(s)) \).

Next we turn our attention to the operator of the asymptotic position. Recall from the proof of Lemma 3.7.3 that  

\[(3.11.13) = U(s, t) i[(V(t, x) = V(t, \nabla \xi S(t, D)), (x - \nabla \xi S(t, D))] U(t, s)\]

\[= U(s, t) i[G(t), (x - \nabla \xi S(t, D))] U(t, s) x_{lt}(t, s) + U(s, t) i[R(t), (x - \nabla \xi S(t, D))] U(t, s).\]
Therefore
\begin{equation}
    x_{t,s} = x_{t,s}^{(s)} + \int_s^t G(u,s)x_{t,u} \, du + \int_s^t R(u,s) \, du,
\end{equation}
where
\[ G(u,s) = U(s,u) \mathcal{L}[G(u), (x - \nabla_x S(t,D))U(u,s)], \]
\[ R(u,s) = U(s,u) \mathcal{L}[R(u), (x - \nabla_x S(t,D))U(u,s)]. \]

We know that
\[ [G(u), (x - \nabla_x S(t,D))], [R(u), (x - \nabla_x S(t,D))] \in L^1(du, \Psi(g_0(u))). \]

Hence by Theorem 3.10.2
\begin{equation}
    G(u,s), R(u,s) \in O(s^0) L^1(du, \Psi(g_0(s))).
\end{equation}

It is possible to solve (3.11.4). In fact, we define $A(t,s)$ by the convergent expansion
\begin{equation}
    A(t,s) := \sum_{m=0}^\infty \int_s^t \cdots \int_s^t G(u_{n+1},s) \cdots G(u_1,s) \, du_1 \cdots du_n
\end{equation}
and we set
\[ B(t,s) := \int_s^t A(t,u) R(u,s) \, du. \]

Then $A(t,s)x_{t,s}(s,s) + B(t,s)$ solves the equation (3.11.4). Besides, it is easy to see by applying the Beals criterion that $A(t,s), B(t,s) \in \Psi(g_0(s^0), g_0(s))$. This ends the proof of (3.11.7).

To prove (3.11.8) we note that it follows immediately from (3.11.7) for $m$ negative. It is more difficult to prove (3.11.8) for $m$ positive.

First we will prove by induction that there exists $C_m > 0$ such that
\begin{equation}
    \langle x_{t,s}(t,s) \rangle^{2m} \geq C_m \langle x_{t,s}(s,s) \rangle^{2m}
\end{equation}
In fact, it follows from (3.11.7) that
\begin{equation}
    x_{t,s}^2 = (x_{t,s}(s,s)(1 + A^*(t,s)) + B^*(t,s))(1 + A(t,s))x_{t,s}(s,s) + B(t,s)
\end{equation}
We use twice the inequality
\[ A^*_1 A_2 + A^*_2 A_1 \geq -c^2 A^*_1 A_1 - c^{-2} A^*_2 A_2 \]
and we estimate (3.11.18) from below by
\begin{equation}
    (1 - c^2)x_{t,s}(s,s)(1 - \delta^2 + (1 - \delta^{-2})A(t,s)^* A(t,s))x_{t,s}(s,s) + (1 - c^{-2})B^*(t,s)B(t,s).
\end{equation}

Hence, if we choose $T_1$ such that for $T_1 \leq s \leq t$
\[ \|A(t,s)\| \leq \frac{1}{2} \]
\[ c^2 < 1 \]
then for some $C_0 > 0$
\begin{equation}
    x_{t,s}^2 \geq C_0 x_{t,s}^2(s,s) - C_0,
\end{equation}
which proves (3.11.17) for $m = 1, T_1 \leq s \leq t$. 

\[ 1 - \delta^2 + (1 - \delta^{-2}) \frac{1}{4} > 0, \]
Next assume that (3.11.17) is true for $m$ replaced with $1, \ldots, m - 1$. The following identity is a consequence of (3.11.7):

\begin{equation}
    x^m(t, s) = (1 + A(t, s))x^m(s, s) + \sum_{j=0}^{m-1} A_j(t, s)x^j(s, s)
\end{equation}

where $A_j(t, s) \in \Psi(o(s^0), g_0(s))$. Hence

\begin{equation}
    x^{2m}(t, s) = x^m(s, s)(1 + A^*(t, s))(1 + A(t, s))x^m(s, s)
\end{equation}

\begin{equation}
    + \sum_{0 \leq j, k \leq m, \ j + k \leq 2m - 1} x^j(s, s)C^k_j(t, s)x^k(s, s)
\end{equation}

where $C^k_j(t, s) \in \Psi(o(s^0), g_0(s))$. Using the induction assumption and (3.11.22) we get (3.11.17).

Next we will prove (3.11.17) for $0 \leq s \leq t \leq T_1$. If we use the first identity of (3.11.13), then we can write

\begin{equation}
    x_k(t, s) = x_k(s, s)
\end{equation}

\begin{equation}
    + \int_s^t U(s, u) \left[ \left[ V(u, x), -\nabla \zeta S(u, D) \right] + i\left[ V(u, \nabla \zeta S(u, D)), -x^k \right] \right] U(u, s) du
\end{equation}

Hence there exists $\tilde{A}(t, s) \in \Psi([t-s][o(s^0), g_0(s))]$ such that

\begin{equation}
    x^k(t, s) = x^k(s, s) + \tilde{A}(t, s).
\end{equation}

Therefore,

\begin{equation}
    x^{2m}(t, s) = x^{2m}(s, s) + \sum_{k=0}^{m-1} \tilde{A}(t, s)x^k(s, s),
\end{equation}

where $\tilde{A}(t, s) \in \Psi([t-s][o(s^0), g_0(s))]$. This implies by induction that for a fixed $T_1$ and $0 \leq s \leq t \leq T_1$ (3.11.17) is true.

It remains to prove the case $0 \leq s \leq T_1 \leq t$. We already know that for some $C_m > 0$

\begin{equation}
    \langle x^{2m}(t, T_1) \rangle \geq C_m \langle x^{2m}(T_1, T_1) \rangle.
\end{equation}

We multiply (3.11.26) from the left by $U(s, T_1)$ and from the right by $U(T_1, s)$ and we get

\begin{equation}
    \langle x^k(t, s) \rangle \geq C_m \langle x^k(T_1, s) \rangle.
\end{equation}

But we already proved that for some $C_m > 0$

\begin{equation}
    \langle x^{2m}(T_1, s) \rangle \geq C_m \langle x^{2m}(s, s) \rangle.
\end{equation}

Now (3.11.27) and (3.11.28) imply (3.11.17) in the case $0 \leq s \leq T_1 \leq t$.

Now let us show (3.11.9).

First note that

\begin{equation}
    [D, \Omega_k(t, s)] = (D(t, s) - D)\Omega_k(t, s).
\end{equation}

By induction we show that

\begin{equation}
    \text{ad}_D\Omega_k(t, s) = \Omega_k(t, s)B_\alpha(t, s),
\end{equation}

where $B_\alpha(t, s) \in \Psi(o(s^0), g_0(s))$. Moreover

\begin{equation}
    x_k(t, s)\Omega_k(t, s) = \Omega_k(t, s)x_k(s, s)
\end{equation}
so
\[(3.11.31)\]
\[
\text{ad}_{e^{i s \phi}} \Omega_{\mathbf{k}}(t, s) = (x_{\mathbf{i}}(s, s) - x_{\mathbf{i}}(t, s)) \Omega(k, t).
\]

Now (3.11.30), (3.11.31) (3.11.7) and (3.11.8) imply that
\[
(x_{\mathbf{i}}(s, s))^{-1} \text{ad}_{e^{i s \phi}} \Omega_{\mathbf{k}}(t, s) = o(s^0).
\]
Since
\[
\text{ad}_{e^{i s \phi}} B_{\alpha} \in \Psi(o(s^0), g_0(s))
\]
this gives (3.11.9). \(\square\)

### 3.12 Isozaki-Kitada construction of long-range wave operators

In this section we introduce another construction of long-range wave operators — a time-dependent version of the one introduced in the time-independent case by Isozaki-Kitada [IK1]. We will prove that for smooth long-range time-dependent potentials the two wave operators coincide. This result is the quantum analog of Proposition 1.6.3 in Chapter 1.

First let us recall some facts from Chapter 1. In Proposition 1.6.3 we constructed a function \(\Phi^+(s, x, \xi)\) which for \(s \geq T\) solves the eikonal equation
\[
-\partial_s \Phi^+(s, x, \xi) = \frac{1}{2}(\nabla_x \Phi^+(s, x, \xi))^2 + V(t, x).
\]
By Proposition 1.8.2, the function \(\Phi^+(s, x, \xi)\) satisfies the estimates
\[
|\partial_x^\alpha \partial_\xi^\beta (\Phi^+(s, x, \xi) - \langle x, \xi \rangle + \frac{i}{s} \xi^2)| \leq o(s^{1-|\alpha|}), \quad |\alpha| \geq 1,
\]
\[
\int_T^\infty \sup_{x, \xi} |\partial_x^\alpha \partial_\xi^\beta (\Phi^+(s, x, \xi) - \langle x, \xi \rangle + \frac{i}{s} \xi^2)| |s|^{|\alpha|+2} ds < \infty, \quad |\alpha| \geq 2.
\]

We define next the following operator
\[
J_+(s) \phi(x) := (2\pi)^{-n} \int e^{i \Phi^+(s, x, \xi) - i \langle y, \xi \rangle} \phi(y) dyd\xi.
\]

In Chapter 1 we introduced the function \(S(s, t, x, \xi)\), which is the solution of the Hamilton-Jacobi equations
\[
\begin{align*}
\partial_s S(s, t, x, \xi) &= \frac{1}{2}(\nabla_x S(s, t, x, \xi))^2 + V(s, x), \\
S(t, t, x, \xi) &= \langle x, \xi \rangle.
\end{align*}
\]
The function \(S(s, t, x, \xi)\) satisfies the estimates:
\[
|\partial_x^\alpha \partial_\xi^\beta (S(s, t, x, \xi) - \langle x, \xi \rangle)| \leq o(s^{1-|\alpha|}), \quad |\alpha| \geq 1,
\]
\[
\int_T^\infty \sup_{t, \xi} |\partial_x^\alpha \partial_\xi^\beta (S(s, t, x, \xi) - \langle x, \xi \rangle)| |s|^{|\alpha|+2} ds < \infty, \quad |\alpha| \geq 2,
\]
by Proposition 1.8.2 in Chapter 1.

For \(T \leq s \leq t < +\infty\), we denote by \(I(s, t)\) the operator
\[
I(s, t) \phi(x) := (2\pi)^{-n} \int e^{i S(s, t, x, \xi) - i \langle y, \xi \rangle} \phi(y) dyd\xi
\]
Recall from Proposition 1.6.3 that the functions \(S(s, \xi)\) and \(\Phi^+(s, x, \xi)\) are related by the identity
\[
\Phi^+(s, x, \xi) = \lim_{t \to +\infty} (S(s, t, x, \xi) - S(t, \xi)).
\]
The main result of this section is the following theorem:
Theorem 3.12.1 Assume that the potential $V(t, x)$ satisfies the estimates (3.10.1). Then the following results hold:

i) the operator $J^+(s)$ is uniformly bounded on $L^2(X)$ for $s \geq T$,

ii) the norm limit

$$
\lim_{s \to +\infty} U(0, t)J^+(s)
$$

exists.

iii) (3.12.6) is equal to $\Omega^+_{\nu}$ defined in Theorem 3.7.1.

To prove part iii) of Theorem 3.12.1, for large $s, t$ we will construct an approximation of $U(s, t)$ as a Fourier integral operator. Such construction first appeared in [Kil], where it was used to prove existence and completeness of wave operators for smooth time-dependent potentials. A more refined construction will be given in the next section.

Proposition 3.12.2 Assume that the potential satisfies (3.10.1). Then

$$
\sup_{s \leq t \leq \infty} ||U(s, t) - I(s, t)|| = o(s^0).
$$

Proof. We compute

$$
\frac{d}{dt} U(t, s) I(s, t)
$$

(3.12.8)

$$
= U(t, s)(iH(s) + \partial_s) I(s, t)
$$

$$
= U(t, s)P(s, t),
$$

where $P(t, s)$ is the operator defined as

$$
P(t, s)\phi(x) := (2\pi)^{-n} \int \int e^{is(x, \xi) - iy(\eta, \xi)} \Delta_{\xi} S(s, t, x, \xi) \phi(\eta) d\eta d\xi.
$$

Note that

$$
I(t, t) = U(t, t) = 1.
$$

Therefore

$$
I(s, t) - U(s, t) = -\int_s^t U(s, u) P(u, t) du,
$$

and

$$
\sup_{t \leq s \leq t} ||I(s, t) - U(s, t)|| \leq \int_s^\infty \sup_{u \leq t} ||P(u, t)|| du.
$$

(3.12.10)

Now (3.12.3) implies that for $T_0 \leq s \leq t$

$$
\int_{T}^{s} \sup_{t \leq s \leq t} ||\partial_{\xi}^a \partial_{\epsilon}^b \Delta_{\xi} S(s, t, \cdot, \cdot)||_{\infty} ds < \infty,
$$

(3.12.11)

$$
|\partial_{\xi}^a \partial_{\xi}^b S(s, t, x, \xi)| \leq C_{\alpha, \beta}, |\alpha| \geq 1,
$$

$$
|\nabla_{\xi} \nabla_{\xi} S(s, t, x, \xi) - 1| \leq \frac{1}{2}.
$$

From the estimates (3.12.11) and Theorem C.7.2 of the Appendix, we infer that the right hand side of (3.12.10) is less than $o(s^0)$. □

Proof of Theorem 3.12.1. Using Theorem C.7.2, we see that i) follows directly from the estimates (3.12.1). Let us now prove ii). We compute the derivative

$$
\frac{d}{ds} U(0, s)J^+(s)
$$

(3.12.12)

$$
= U(0, s)Q^+(s),
$$
where
\[ Q^+(s) \phi(y) := (2\pi)^{-n} \int \int \Delta_x \Phi^+(s, x, \xi)e^{i\Phi^+(s, x, \xi) - i(y, \xi)} \phi(y) d\xi dy. \]

We have the following estimates for \( s \geq T_0 \)
\[ \int_T^\infty \| \partial_x^\alpha \partial_\xi^\beta \Delta_x \Phi^+(s, \cdot, \cdot) \|_\infty ds < \infty, \]
\[ |\partial_x^\alpha \partial_\xi^\beta \Phi^+(s, x, \xi)| \leq C_{\alpha, \beta}, |\alpha| \geq 1, \]
\[ |\nabla_x \nabla_\xi \Phi^+(s, x, \xi) - 1| \leq \frac{1}{s}. \]

Therefore, by Theorem C.7.2
\[ \int_T^\infty \|Q^+(s)\| ds < \infty, \]
which proves that the norm limit
\[ \lim_{s \to +\infty} U(0, s)J^+(s) \]
e xists.

Let us now prove iii). We first claim that
\[ J^+(s)^* = \lim_{t \to \infty} e^{iS(t, D)} I(s, t)^*. \]

To see this it is enough to prove that if \( \chi \in C_0^\infty(X) \), then
\[ \lim_{t \to \infty} (e^{iS(t, D)} I(s, t)^* - J^+(s)^*) \chi(x) = 0 \]

But
\[ (e^{iS(t, D)} I(s, t)^* - J^+(s)^*) \chi(x) \phi(x) \]
\[ = (2\pi)^{-n} \int e^{-i\Phi^+(s, x, \xi) + i(y, \xi)} b(s, t, x, \xi) \phi(y) d\xi dy \]
where
\[ b(s, t, x, \xi) = \left( e^{-iS(s, t, x, \xi) + i\Phi^+(s, x, \xi)} - 1 \right) \chi(x). \]

By (3.12.5) the amplitude \( b(s, t, x, \xi) \) goes uniformly to zero as \( t \to \infty \) together with all its derivatives. Therefore, (3.12.16) is true.

Let us now fix a vector \( \phi \in L^2(X) \). Using first (3.12.15) and then Proposition 3.12.2 we see that for \( s \leq t \) one has
\[ J^+(s)^* U(s, 0) \phi \]
\[ = e^{iS(t, D)} I(s, t)^* U(s, 0) \phi + o(t^0) \]
\[ = e^{iS(t, D)} U(t, s) U(s, 0) \phi + o(s^0) + o(t^0) \]

Therefore, if we let \( t \to \infty \) in (3.12.18) we obtain
\[ J^+(s)^* U(s, 0) \phi = \Omega_{\phi}^+ \phi + o(s^0). \]

Letting then \( s \) go to \( +\infty \), we obtain
\[ \lim_{s \to +\infty} J^+(s)^* U(s, 0) \phi = \Omega_{\phi}^+ \phi, \]
which is the desired result. \( \square \)
3.13 Wave operator as a Fourier integral operator

In this section we will show that the time translated wave operator

$$\Omega^+_t(s) := U(s, 0)\Omega^+_t,$$

and the evolution $U(s, t)$ are Fourier integral operators associated with the canonical transformation $\phi(s, 0) \circ F^+_t$ and $\phi(s, t)$ resp. in the sense described by the following theorem.

**Theorem 3.13.1** For $T_0 \leq s \leq t$ there exist functions $a(s, t, x, \xi)$ and $a^+(s, x, \xi)$ such that

$$a^+(s, x, \xi) := \lim_{t \to +\infty} a(s, t, x, \xi),$$

(3.13.1)

$$|\partial_s^2 \partial_\xi^2(a(s, t, x, \xi) - 1)| \leq o(s^{-|\alpha|}), \quad T_0 \leq s \leq t \leq +\infty,$$

$$|\partial_s^2 \partial_\xi^2(a^+(s, x, \xi) - 1)| \leq o(s^{-|\alpha|}), \quad T_0 \leq s \leq t \leq +\infty,$$

such that

(3.13.2) $$U(s, t)\phi(x) := (2\pi)^{-n} \int \int e^{iS(s,t,x,\xi) - i(y, \xi)} a(s, t, x, \xi)\phi(y)dyd\xi,$$

(3.13.3) $$\Omega^+_t(s)\phi(x) = (2\pi)^{-n} \int \int e^{i\Phi^+(s,x,\xi) - i(y, \xi)} a^+(s, x, \xi)\phi(y)dyd\xi.$$

**Proof.** Recall that the following identity is true:

(3.13.4) \[
\begin{cases}
\frac{\partial}{\partial s} U(t, s)I(s, t) = U(t, s)P(s, t), \\
U(t, t) = I(t, t) = 1,
\end{cases}
\]

where $P(s, t)$ was defined in (3.12.9).

Note that it follows from Proposition C.7.5 that

$$I(s, t)I^+(s, t) - 1 \in \Psi(o(s^0), g_0(s)).$$

Hence for $s \geq T_0$

$$\|I(s, t)I^+(s, t) - 1\| \leq C_0 < 1,$$

$I(s, t)I^+(s, t)$ is invertible. Using a Neumann series and Beals criterion (see Theorem C.6.1) we obtain

(3.13.5) 

$$(I(s, t)I^+(s, t))^{-1} - 1 \in \Psi(o(s^0), g_0(s)).$$

Set

$$W(s, t) := U(t, s)I(s, t).$$

Then it follows from (3.13.4) that for $T_0 \leq s \leq t$

(3.13.6) \[
\begin{cases}
\frac{\partial}{\partial s} W(s, t) = W(s, t)\tilde{P}(s, t), \\
W(t, t) = 1,
\end{cases}
\]

where

$$\tilde{P}(s, t) := I^{-1}(s, t)P(s, t) = I^+(s, t)(I(s, t)I^+(s, t))^{-1} P(s, t).$$
By (3.13.5) and Proposition C.7.6 and C.7.5 we have
\[ \tilde{P}(s, t) \in L^1(ds, \Psi(g_0(s))). \]
Therefore by Lemma 3.4.3
\[ W(s, 1t) = (I(s, t))^{-1} I(s, t) W^*(s, t). \]
We know that
\[ (s, t) = (I(s, t))^{-1} I(s, t) W^*(s, t). \]
Now (3.13.5), (3.13.7), (3.13.8) and Proposition C.7.6 imply the properties of \( U(s, t) \) stated in our theorem.

If we know that \( U(s, t) \) can be written as a Fourier integral operator (3.13.2), then the result that \( \Omega^k_k(s) \) can be written as a Fourier integral operator (3.13.3) is easy to prove. In fact, we note that
\[ e^{iS(t, D)}U(s, t)^* \phi(y) \]
\[ = (2\pi)^{-n} \int a(s, t, x, \xi) e^{i(y, \xi)-iS(t, x, \xi)+iS(t, \xi)} \phi(x) d\xi dx. \]
Using similar arguments as in the proof of (3.12.15) we see that (3.13.9) goes to
\[ \Omega^k_k(s) \phi(y) = (2\pi)^{-n} \int a^*(s, x, \xi) e^{i(y, \xi)-i\Phi^+(s, x, \xi)} \phi(x) d\xi dx. \]
\[ \square \]

If we treat the inverse of \( s \) as a kind of a Planck’s constant, then we can obtain more precise information on the amplitudes of \( \Omega^k_k(s) \) and \( U(s, t) \).

to this end it is natural to factor these amplitudes as follows:
\[ a(s, t, x, \xi) = (\det \nabla_x \nabla_\xi S(s, t, x, \xi))^{1/2} b(s, t, x, \xi), \]
\[ a^*(s, x, \xi) = (\det \nabla_x \nabla_\xi \Phi^+(s, x, \xi))^{1/2} b^+(s, x, \xi). \]

Let us remark that
\[ \lim_{t \to \infty} (\det \nabla_x \nabla_\xi S(s, t, x, \xi))^{1/2} = (\det \nabla_x \nabla_\xi \Phi^+(s, x, \xi))^{1/2}. \]
Moreover,
\[ \left| \frac{\partial^2_\xi \partial^2_\xi}{(\det \nabla_x \nabla_\xi S(s, t, x, \xi))^{1/2} - 1} \right| \leq o(s^{-1}) \]

Let us now introduce approximations to the amplitudes of \( \Omega^k_k(s) \) and \( U(s, t) \).

We define inductively \( b_n(s, t, x, \xi), a_n(s, t, x, \xi) \) and \( p_n(s, t, x, \xi) \) such that
\[ b_0(s, t, x, \xi) = 1, \]
\[ b_m(s, t, x, \xi) = \int_{\xi_{m-1}}^\xi p_{m-1}(u, t, \hat{y}(u, t, x, \xi), \xi) du, \]
\[ a_m(s, t, x, \xi) = (\det \nabla_x \nabla_\xi S(s, t, x, \xi))^{1/2} b_m(s, t, x, \xi), \]
\[ p_m(s, t, x, \xi) = \frac{1}{2} (\det \nabla_x \nabla_\xi S(s, t, x, \xi))^{-1/2} \Delta_x a_m(s, t, x, \xi). \]

(Recall that the trajectories \( \hat{y}(u, t, x, \xi) \) were defined in Theorem 1.4.1). Note that the quantities \( b_n(s, t, x, \xi), a_n(s, t, x, \xi) \) and \( p_n(s, t, x, \xi) \) have a limit as \( t \to \infty \), which we denote \( b^+_n(s, x, \xi), a^+_n(s, x, \xi) \) and \( p^+_n(s, x, \xi) \). Let us introduce approximations of finite order
\[ I_m(s, t) \phi(x) := \sum_{j=0}^m (2\pi)^{-n} \int c^{iS(s, t, x, \xi)-i(y, \xi)} a_j(s, t, x, \xi) \phi(y) dy d\xi, \]
\[ 55 \]
\[ J^+_m(s) \phi(x) = \sum_{j=0}^{m} (2\pi)^{-n} \int \int e^{i\Phi(s, x, \xi) - i|y, \xi|} a^+_j(s, x, \xi) \phi(y) dy d\xi. \]

Then we have the following result.

**Proposition 3.13.2** The amplitudes \( a_m(s, t, x, \xi) \) satisfy uniformly for \( T \leq s \leq t \leq \infty \)

\[ \left| \partial_x^\alpha \partial_\xi^\beta \left( (a(s, t, x, \xi) - \sum_{j=0}^{m} a_j(s, t, x, \xi) \right) \right| \leq o(s^{-m-|\alpha| - 1}), \quad T \leq s \leq t \leq +\infty, \]

\[ \left| \partial_x^\alpha \partial_\xi^\beta \left( a^+(s, x, \xi) - \sum_{j=0}^{m} a_j^+(s, x, \xi) \right) \right| \leq o(s^{-m-|\alpha| - 1}), \quad T \leq s \leq t \leq +\infty. \]

**Proof.** The equation

\[ \begin{cases} \partial_t + H(s)U(s, t) = 0, \\ U(t, t) = 1, \end{cases} \]

can be rewritten as

\[ \begin{cases} \partial_t + \nabla_x S(s, t, x, \xi) \nabla_x + \frac{1}{2} \Delta_x S(s, t, x, \xi) - \frac{i}{2} \Delta_x a(s, t, x, \xi) = 0, \\ a(t, t, x, \xi) = 1. \end{cases} \]

Note that the following identity is true:

\[ (\partial_t + \nabla_x S(s, t, x, \xi) \nabla_x + \frac{1}{2} \Delta_x S(s, t, x, \xi))(\det \nabla_x \nabla_\xi S(s, t, x, \xi))^{1/2} = 0. \]

Hence (3.13.13) can be rewritten as

\[ \begin{cases} \partial_t + \nabla_x S(s, t, x, \xi) \nabla_x b(s, t, x, \xi) \\ - \frac{i}{2} (\det \nabla_x \nabla_\xi S(s, t, x, \xi))^{-1/2} \Delta_x \left( (\det \nabla_x \nabla_\xi S(s, t, x, \xi))^{1/2} b(s, t, x, \xi) \right) = 0, \\ b(t, t, x, \xi) = 1. \end{cases} \]

Introduce the operator

\[ \Psi(s, t) : S(X \times X') \rightarrow S(X \times X') \]

given by

\[ (\Psi(s, t)a)(y, \eta) := a(x(s, t, y, \eta), \eta), \]

and the operator

\[ \Gamma(s, t) : S(X \times X') \rightarrow S(X \times X') \]

given by

\[ (\Gamma(s, t)b)(x, \xi) := \frac{i}{2} (\det \nabla_x \nabla_\xi S(s, t, x, \xi))^{-1/2} \Delta_x (\det \nabla_x \nabla_\xi S(s, t, x, \xi))^{1/2} b(x, \xi). \]

Then the equation (3.13.14) can be rewritten as

\[ \begin{cases} \Psi^{-1}(s, t) \partial_t (\Psi(s, t)b(s, t)) = \Gamma(s, t)b(s, t) \\ b(t, t) = 1. \end{cases} \]

or in the integral form

\[ b(s, t) = 1 + \int_s^t \Psi^{-1}(s, u) \Psi(u, t) \Gamma(u, t)b(u, t) du. \]

Let us note the following properties of the above mappings.
Lemma 3.13.3

\[
\Psi^{-1}(s, t)\Psi(u, t)b(x, \xi) = b(\hat{y}(u, s, t, x, \xi), \xi).
\]

Moreover,

\[
\|\Psi^{-1}(s, t)\Psi(u, t)b\|_{S(g_0(s), N)} \leq C_N\|b\|_{S(g_0(u), N)}
\]

(3.13.17)

\[
\|\Gamma(s, t)b\|_{S(g_0(s), N)} \leq C_N\|b\|_{S(g_0(s), N+2)}
\]

(3.13.18)

Proof. We have

\[
\partial_x^\alpha \partial_\xi^\beta (\Psi^{-1}(s, t)\Psi(u, t)b)(x, \xi)
\]

\[
= \sum \partial_x^\alpha \partial_\xi^\beta \hat{y}(u, s, t, x, \xi) \partial_x^\alpha \partial_\xi^\beta \hat{y}(u, s, t, x, \xi) \ldots \partial_x^\alpha \partial_\xi^\beta \hat{y}(u, s, t, x, \xi)
\]

Therefore

\[
|g_{\partial_x \partial_\xi}(u, s, t, x, \xi)| \leq o(s^{-|\alpha|} \ldots |\beta|) |u - t|^{\|\nabla_x^\alpha \partial_\xi^\beta \hat{y}\|_{\infty}}.
\]

Hence

\[
\|\partial_x^\alpha \partial_\xi^\beta \Psi^{-1}(s, t)\Psi(u, t)b\|_\infty \leq o(s^{-|\alpha|}) \|b\|_{S(g_0(u), |\alpha|+|\beta|)}.
\]

square

The equations (3.13.10) can be rewritten in a more compact form

\[
b_0(s, t) = 1
\]

\[
b_m(s, t) = \int_s^t \Psi^{-1}(s, t)\Psi(u, t)b_{m-1}(u, t)du,
\]

\[
p_m(s, t) = \Gamma(s, t)b_m(s, t).
\]

Using Lemma 3.13.3 we easily prove by induction that for \(m = 1, 2, \ldots\) we have

(3.13.19)

\[
b_m(s, t) \in S(o(s^{-m}), g_0(s)), \quad p_m(s, t) \in S(o(s^{-m-2}), g_0(s)).
\]

Set

\[
P_m(s, t)\phi(x) := (2\pi)^{-n} \int \int e^{iS(s, t, x, \xi)} e^{-i(y, \xi)} p_m(s, t, x, \xi) \phi(y)dyd\xi.
\]

Then we have

\[
\begin{cases}
(\partial_x + iH(s))I_m(s, t) - U(s, t) = P_m(s, t), \\
I_m(t, t) - U(t, t) = 0.
\end{cases}
\]

Consequently,

(3.13.20)

\[
I_m(s, t) - U(s, t) = U(s, t) \int_s^t U(t, u)P_m(u, t)du.
\]

Recall that \(U(t, u)\) is a Fourier integral operator with the phase \(-S(u, t, x, \xi)\). Hence by Proposition C.7.5

\[
U(t, u)P_m(u, t) \in \Psi(o(s^{-m-2}, g_0(s)).
\]

Therefore

\[
\int_s^t U(t, u)P_m(u, t)du \in \Psi(o(s^{-m-1}, g_0(s)).
\]
Using again the fact that \( U(s, t) \) is a Fourier integral operator with the phase \( S(s, t, x, \xi) \) and Proposition C.7.6 we obtain that
\[
(\text{I}_m(s, t) - U(s, t))\phi(x) = (2\pi)^{-n} \int \int e^{i S(s, t, x, \xi)} e^{-i(y, \xi)} \tilde{a}_m(s, t, x, \xi) \phi(y) dy d\xi.
\]
with
\[
\tilde{a}_m \in S(\sigma(s^{-m-1}), g_0(s)),
\]
which ends the proof of our theorem. \( \Box \).
Appendix C

C.1 Propagation estimates

In this section, we describe certain abstract arguments that are used in scattering theory to prove propagation estimates, existence of asymptotic observables and of wave operators. These arguments do not depend on a concrete form of the time-dependent Hamiltonian.

In the first lemma we describe how two prove two types of so-called propagation estimates. This name is usually given to various estimates on the evolution. The first type of a propagation estimate is a direct consequence of the fundamental theorem of calculus. The second one is a version of the Putnam-Kato theorem developed by Sigal and Soffer (see [S.R4], [S.S1]).

Let $H(t)$ denote a time-dependent self-adjoint operator on a Hilbert space $\mathcal{H}$ and let $U(t)$ be the evolution generated by $H(t)$, that is:

$$
\begin{align*}
\frac{d}{dt}U(t) &= -iH(t)U(t), \\
U(0) &= 1.
\end{align*}
$$

We will assume that $H(t)$ and $U(t)$ satisfy the conditions 1)–6) of Section 3.1. We will denote by $D\Phi(t)$ the Heisenberg derivative associated with $U(t)$:

$$
D\Phi(t) := \frac{d}{dt}\Phi(t) + i[H(t), \Phi(t)].
$$

Lemma C.1.1 i) Suppose that $\Phi(t) \in W^{1,1}(\mathbb{R}^+, B(\mathcal{H}))$ is a family of self-adjoint operators such that $D\Phi(t) \in L^1(\mathbb{R}^+, B(\mathcal{H}))$. Then

(C.1.1) \[ \||\Phi(t)U(t)\phi\| \leq \||\Phi(0)\phi\| + \int_0^t \|D\Phi(s)\|ds. \]

ii) Suppose that $\Phi(t) \in W^{1,1}(\mathbb{R}^+, B(\mathcal{H}))$ is a uniformly bounded function with values in self-adjoint operators. Assume that there exist $C_0 > 0$ and operator valued functions $B(t)$ and $B_i(t)$ with $i = 1, \ldots, n$ such that

(C.1.2) \[ D\Phi(t) \geq C_0 B^*(t)B(t) - \sum_{i=1}^n B_i^*(t)B_i(t). \]

Suppose that for $i = 1, \ldots, n$

\[ \int_1^\infty \|B_i(t)U(t)\phi\|^2 dt \leq C\|\phi\|^2. \]
Then there exists $C_1$ such that
\[
\int_1^\infty \|B(t)U(t)\phi\|^2 dt \leq C_1\|\phi\|^2.
\]

The observable $\Phi(t)$ used to derive (C.1.3) is called a propagation observable. As we saw, the main idea of the proof of (C.1.3) is to find a propagation observable whose Heisenberg derivative is “essentially positive”.

Proof. i) follows directly from the fact that
\[
\frac{d}{dt} U(t)^* \Phi(t) U(t) = U(t)^* D \Phi(t) U(t).
\]

To prove ii), we consider
\[
C_0 \int_{t_1}^{t_2} \|B(t)U(t)\phi\|^2 dt
\leq \int_{t_1}^{t_2} (\Phi(t_2) U(t_2) \phi) (U(t)\phi) dt + \sum_i \int_{t_1}^{t_2} \|B_i(t)U(t)\phi\|^2 dt
\leq (\Phi(t_2) U(t_2) \phi) (U(t_1)\phi) + (\Phi(t_1) U(t_1) \phi) (U(t_2)\phi)
+ \sum_i \int_{t_1}^{t_2} \|B_i(t)U(t)\phi\|^2 dt
\leq C\|\phi\|^2, \forall 0 \leq t_1 \leq t_2,
\]
which proves the desired result. □

The observable $\Phi(t)$ used to derive (C.1.3) is called a propagation observable. As we saw above, the main idea of the proof of (C.1.3) is to find a propagation observable whose Heisenberg derivative is “essentially positive”.

Next we describe two methods of proving existence of wave operators and asymptotic observables. The first one is known as Cook's method and the second one is its variation due to Kato (see [RS4] and references therein).

Let $H_1(t)$ and $H_2(t)$ be two time-dependent self-adjoint operators with a fixed domain $D(H)$. Let $U_1(t)$ be the unitary evolutions generated by $H_1(t)$ in the sense of 1)-6) of Section 3.1. Denote
\[
D_1; 2\Phi(t) := \frac{d}{dt} \Phi(t) + iH_2(t)\Phi(t) - i\Phi(t)H_1(t).
\]

Lemma C.1.2 Suppose that $\Phi(t) \in W^{1,1}([R^+, B(\mathcal{H})])$ is a uniformly bounded function. Let $D_1 \subset \mathcal{H}$ be a dense subspace.

i) Assume that for $\phi \in D_1$
\[
\int_1^\infty \|D_1; 2\Phi(t)U_1(t)\phi\| dt < \infty.
\]

Then there exists
\[
(C.1.4) \quad \text{s-}\lim_{t \to \infty} U_2^*(t)\Phi(t)U_1(t).
\]

ii) Assume that
\[
|\langle \psi_2 | D_1; 2\Phi(t)\psi_1 \rangle| \leq \sum_{i=1}^n \|B_2(t)\psi_2\| |B_1(t)\psi_1|;
\]
\[
\int_1^\infty \|B_2(t)U_2(t)\phi\|^2 dt \leq c\|\phi\|^2;
\]
and if \( \phi \in D_1 \) then,  
\[
\int_1^\infty \|B_1(t)U_1(t)\phi\|^2 \, dt \leq c\|\phi\|^2.
\]

Then the limit (C.1.4) exists.

The proof of i) is easy and left to the reader. Let us show ii).

Let \( \phi \in D_1, \psi \in \mathcal{H} \). Then

\[
|\langle \psi | U_2^* (t_2) \Phi (t_2) U_1 (t_2) \phi \rangle - \langle \psi | U_2^* (t_1) \Phi (t_1) U_1 (t_1) \phi \rangle|
\]

\[
\leq \int_{t_1}^{t_2} |\langle \psi | U_2^* (t) \Phi (t) U_1 (t) \phi \rangle| \, dt
\]

\[
\leq \sum_{j=1}^n \left( \int_{t_1}^{t_2} \|B_j(t)U_2(t)\psi\|^2 \, dt \right)^{1/2} \left( \int_{t_1}^{t_2} \|B_j(t)U_2(t)\phi\|^2 \, dt \right)^{1/2}.
\]

Therefore

\[
\|U_2^* (t_2) \Phi (t_2) U_1 (t_2) \phi - U_2^* (t_1) \Phi (t_1) U_1 (t_1) \phi\|
\]

\[
= \sup_{\|\psi\|=1} |\langle \psi | U_2^* (t_2) \Phi (t_2) U_1 (t_2) \phi \rangle - \langle \psi | U_2^* (t_1) \Phi (t_1) U_1 (t_1) \phi \rangle|
\]

\[
\leq \sum_{j=1}^n C \left( \int_{t_1}^{t_2} \|B_j(t)U_2(t)\phi\|^2 \, dt \right)^{1/2}.
\]

If we choose \( T \) big enough and \( T \leq t_1 \leq t_2 \), then we can make (C.1.6) arbitrarily small. This proves the existence of

\[
s- \lim_{t \to \infty} U_2^*(t) \Phi (t) U_1 (t) \phi, \quad \phi \in D_1,
\]

and hence it implies the existence of (C.1.4). \( \square \)

### C.2 Comparison of classical dynamics

In this appendix we compare two different classical dynamics. In the first theorem we give conditions when trajectories of two long-range systems are asymptotic to each other.

**Theorem C.2.1** Suppose that the forces \( F_1(t, x), F_2(t, x) \) satisfy

\[
\int_0^\infty \|\partial_x^\alpha F_i(t, \cdot)\|_\infty(t) \, dt < \infty, \quad |\alpha| = 0, 1, \quad i = 1, 2,
\]

\[
\int_0^\infty \|F_1(t, \cdot) - F_2(t, \cdot)\|_\infty(t) \, dt < \infty.
\]

Let the time \( T \) be chosen such that the conditions of Theorem 1.4.1 be satisfied for both \( F_1(t, x) \) and \( F_2(t, x) \). Let \( Y_i(t, \xi) \) and \( \tilde{Y}_i(t, \xi) \) be defined as in Sections 1.4 and 1.5 using the force \( F_i(t, x) \). Then the following limits exist and are equal:

\[
\lim_{t \to \infty} (Y_1(t, \xi) - Y_2(t, \xi)),
\]

\[
\lim_{t \to \infty} (\tilde{Y}_1(t, \xi) - \tilde{Y}_2(t, \xi)).
\]
Proof. Let \( \tilde{y}_1(s, t_1, t_2, x, \xi) \) be the solution of the boundary value problem constructed in Theorem 1.4.1 for the force \( F_i(t, x) \). Let us show first that for \( T \leq t_1 \leq s \leq t_2 \leq \infty \) there exists a uniform bound

\[
|\tilde{y}_1(s, t_1, t_2, x, \xi) - \tilde{y}_2(s, t_1, t_2, x, \xi)| \leq C. \tag{C.2.3}
\]

In fact, we have the following identity:

\[
\tilde{y}_1(s) - \tilde{y}_2(s) = \int_t^s \zeta_t, \tau(u)(F_1(u, y_1(u)) - F_2(u, y_2(u)))du
\]

\[
= \int_t^s \zeta_t, \tau(u)(F_1(u, \tilde{y}_1(u)) - F_2(u, \tilde{y}_1(u)))du
\]

\[
+ \int_t^s \zeta_t, \tau(u)(F_2(u, \tilde{y}_1(u)) - F_2(u, \tilde{y}_2(u)))du,
\]

Hence

\[
|\tilde{y}_1(s) - \tilde{y}_2(s)| \leq \int_t^\infty |u - t| ||F_1(u, \cdot) - F_2(u, \cdot)||_{\infty}du
\]

\[
+ \int_t^\infty |u - t| ||\nabla F_2(u, \cdot)||_{\infty}||\tilde{y}_1(u) - \tilde{y}_2(u)||du.
\]

Therefore, the bound (C.2.3) follows by the Gronwall inequality.

Next recall that

\[
Y_1(t, \xi) = \tilde{y}(t, T, \infty, 0, \xi),
\]

\[
\tilde{Y}_1(t, \xi) = \tilde{y}(t, T, 0, \xi),
\]

Hence,

\[
Y_1(t) - Y_2(t) = \int_T^\infty \zeta_T, \tau(u)(F_1(u, \tilde{y}_1(u, T, \infty, 0, \xi)) - F_2(u, \tilde{y}_2(u, T, \infty, 0, \xi)))du
\]

\[
\tilde{Y}_1(t) - \tilde{Y}_2(t)
\]

\[
= \int_T^\infty ((u - T)(F_1(u, \tilde{y}_1(u, T, t, 0, \xi)) - F_2(u, \tilde{y}_2(u, T, t, 0, \xi)))du.
\]

Using the bound (C.2.3) and the assumptions on the potentials (C.2.1) and (C.2.2) we can estimate the integrands in (C.2.6) and (C.2.7) by an integrable function. Thus by Lebesgue’s theorem both (C.2.6) and (C.2.7) converge to

\[
\int_T^\infty ((u - T)(F_1(u, \tilde{y}_1(u, T, \infty, 0, \xi)) - F_2(u, \tilde{y}_2(u, T, \infty, 0, \xi)))du.
\]

\( \Box \)

For the quantum mechanical scattering it is useful to know when the difference of two solutions of the Hamilton-Jacobi equation converges.

**Theorem C.2.2** Suppose that the forces \( F_i(t, x) \) are conservative and \( F_i(t, x) = -\nabla_x V_i(t, x) \). Let \( S_i(t, \xi) \) be the solution of the Hamilton-Jacobi equation

\[
\begin{cases}
\partial_t S_i(t, \xi) = \frac{1}{2} \xi^2 + V_i(t, \nabla_\xi S_i(t, \xi)), \\
S_i(T, \xi) = 0.
\end{cases}
\]

Suppose that

\[
\int_0^\infty ||V_1(t, \cdot) - V_2(t, \cdot)||_{\infty}dt < \infty.
\]

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Assume in addition either one of the following hypotheses:
a)\[ \int_0^\infty \|\partial_\xi^\alpha F_i(t, \cdot)\|_{L^1(t)} dt < \infty, \quad |\alpha| = 0, 1, \quad i = 1, 2, \]
or
b)\[ \int_0^\infty \|F_i(t, \cdot) - F_j(t, \cdot)\|_{L^1(t)} dt < \infty, \quad i = 1, 2, \]
\[ \int_0^\infty \|\partial_\xi^\alpha F_i(t, \cdot)\|_{L^1(t)} dt < \infty, \quad |\alpha| = 1, \quad i = 1, 2. \]

Then there exists the uniform limit
\[ \lim_{t \to \infty} (S_1(t, \xi) - S_2(t, \xi)). \]

Proof. We have
\[ \partial_t (S_1(t, \xi) - S_2(t, \xi)) = V_1(t, \nabla_\xi S_1(t, \xi)) - V_2(t, \nabla_\xi S_2(t, \xi)). \]
Hence
\[ |\partial_t (S_1(t, \xi) - S_2(t, \xi))| \leq \|V_1(t, \cdot) - V_2(t, \cdot)\|_{L^1(t)} + \|\nabla_\xi V_2(t, \cdot)\|_{L^1(t)}|\tilde{Y}_1(t, \xi) - \tilde{Y}_2(t, \xi)|. \]
In the case a) (C.2.13) is integrable, because by Theorem C.2.1 \(|\tilde{Y}_1(t, \xi) - \tilde{Y}_2(t, \xi)|\) is bounded.

In the case 2) we first see that
\[ |\tilde{Y}_i(t, \xi) - t\xi| \leq C(t)^{1/2}, \quad i = 1, 2. \]
Hence
\[ |\tilde{Y}_1(t, \xi) - \tilde{Y}_2(t, \xi)| \leq C(t)^{1/2}. \]
Therefore, (C.2.13) is integrable also in this case. \(\square\)

C.3 Convergence of self-adjoint operators

Throughout this section we will assume \(B_n\) to be a sequence of vectors of commuting self-adjoint operators on a Hilbert space \(L\). More precisely,
\[ B_n = (B_1^n, \ldots, B_m^n), \]
and
\[ [B_i^n, B_j^n] = 0, \quad 0 \leq i, j \leq m, \quad n = 1, 2, \ldots. \]

We will not assume the boundedness of \(B_n\). We will study various concepts of the convergence of \(B_n\).

Proposition C.3.1 Suppose that for every \(g \in C_\infty(R^m)\) there exists
\[ \lim_{n \to \infty} g(B_n). \]

Then there exists a unique (possibly, non-densely defined) vector of self-adjoint operators
\[ B = (B^1, \ldots, B^m) \]
such that (C.3.1) equals \(g(B)\).
Definition C.3.2 Under the assumptions of the above proposition we will write

\[ B = s - C_\infty \lim_{n \to \infty} B_n. \]

If the limit in (C.3.1) is the norm limit, then we will write

\[ B = C_\infty \lim_{n \to \infty} B_n. \]

Remark C.3.3 If \( B_n \) are bounded, then

\[ s - \lim_{n \to \infty} B_n = s - C_\infty \lim_{n \to \infty} B_n. \]

and

\[ \lim_{n \to \infty} B_n = C_\infty \lim_{n \to \infty} B_n. \]

If \( n = 1 \), then the strong \( C_\infty \) convergence is equivalent to the strong resolvent convergence, that is

\[ s - \lim_{n \to \infty} (B_n \pm i)^{-1} = (B \pm i)^{-1}. \]

Likewise, the norm-\( C_\infty \) convergence is equivalent to the norm resolvent convergence, that is

\[ \lim_{n \to \infty} (B_n \pm i)^{-1} = (B \pm i)^{-1}. \]

Proof of Proposition C.3.1 Denote (C.3.1) by \( \gamma(g) \). Clearly, the strong limit preserves the multiplication and

\[ \lim_{n \to \infty} (\phi | g(B_n) \psi) = \lim_{n \to \infty} (\phi | \overline{g(B_n)} \psi). \]

Hence

\[ C_\infty(X) \ni g \mapsto \gamma(g) \in B(\mathcal{H}) \]

is a homomorphism of \( C^* \)-algebras.

For any open \( \Theta \subset X \) we define

\[ E_\Theta := \sup\{ \gamma(g) | g \in C_\infty(X), \ 0 \leq g \leq 1, \ \text{supp} g \subset \Theta \}. \]

Clearly, \( E_\Theta \) are orthogonal projections that satisfy

\[ E_\Theta E_\Theta' = E_{\Theta \cap \Theta'}. \]

In a standard way we can extend the definition of \( E_\Theta \) to arbitrary Borel subsets \( \Theta \). We obtain a map

\[ \Theta \mapsto E_\Theta \]

defined for every Borel subset \( \Theta \subset \mathbb{R}^m \) that satisfies the conditions:

1) \( E_\Theta \) is an orthogonal projection,

2) \( E_\emptyset = 0 \),

3) if \( \Theta = \bigcup_{n=1}^\infty \Theta_n \) and for \( j \neq n \) we have \( \Theta_j \cap \Theta_n = \emptyset \), then

\[ E_\Theta = s - \lim_{N \to \infty} \sum_{n=1}^N E_{\Theta_n}. \]

4) \( E_\Theta E_\Theta' = E_{\Theta \cap \Theta'}. \)
In a standard way for any Borel function $f$ on $\mathbb{R}^m$ we can define the integral

$$\int f(x) dE(x).$$

We can now set

$$B := \int x dE(x).$$

It is easy to check that $B$ satisfies the requirements of our proposition. □

**Remark C.3.4** Let us note that the operator $B$ is densely defined if and only if

$$E_{\mathbb{R}^n} = 1.$$

Let us now describe the relationship of the spectrum of $B$ with the spectra of $B_n$.

**Proposition C.3.5** a) Suppose that for any $g \in C_0(\mathbb{X})$

$$\text{s-} \lim_{n \to \infty} g(B_n) = g(B).$$

Let $\lambda \in \sigma(B)$. Then there exist $\lambda_n \in \sigma(B_n)$ such that $\lim_{n \to \infty} \lambda_n = \lambda$. 

b) Suppose that for any $g \in C_0(\mathbb{X})$

$$\lim_{n \to \infty} g(B_n) = g(B).$$

Then $\lambda \in \sigma(B)$ if and only if there exist $\lambda_n \in \sigma(B_n)$ such that $\lim_{n \to \infty} \lambda_n = \lambda$.

**Proof.** Let $\lambda \in \sigma(B)$. Let $\mathcal{U}$ be a neighborhood of $\lambda$ and $g \in C_0(\mathbb{X})$ such that $g(\lambda) = 1$ and $\text{supp} g \subset \mathcal{U}$. Clearly,

$$\lim \inf \|g(B_n)\| \geq \| \text{s-} \lim_{n \to \infty} g(B_n) \| = \|g(B)\| \geq 1.$$ 

Therefore, we will find $\lambda_n$ such that for $n \geq N$ we have $\lambda_n \in \sigma(B_n) \cup \mathcal{U}$. This proves a) and the $\Rightarrow$ part of b).

To show the $\Leftarrow$ part of b) consider $\lambda_n \in \sigma(B_n)$ such that $\lim_{n \to \infty} \lambda_n = \lambda$. Let $\mathcal{U}$ be a neighborhood of $\lambda$, $g \in C_0(\mathbb{X})$ such that $\text{supp} g \subset \mathcal{U}$ and for large enough $n$ we have $g(\lambda_n) = 1$. Because of the norm convergence we have

$$1 \leq \lim_{n \to \infty} \|g(B_n)\| = \| \lim_{n \to \infty} g(B_n) \| = \|g(B)\|.
$$

Therefore $\mathcal{U} \cap \sigma(B) \neq \emptyset$. □

### C.4 Almost analytic extensions

In this section we describe the concept of an almost analytic extension of a $C^\infty$ function. Such extensions can be used as a tool in some estimates on functions of operators.

We start with describing how to construct an almost analytic extension of a $C^\infty_0$ function in one-dimensional case. We imbed $\mathbb{R}$ in $\mathbb{C}$. We denote the variable of $\mathbb{R}$ by $x$ and the variable of $\mathbb{C}$ by $z = x + iy$. 

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Proposition C.4.1 Let $f \in C_0^\infty (\mathbb{R})$. Then there exist a function $\tilde{f} \in C_0^\infty (\mathbb{C})$ called an almost analytic extension of $f$ such that

\[
\tilde{f}|_{\mathbb{R}} = f;
\]

\[
|\partial_{\bar{z}, \bar{z}}^{\alpha} \tilde{f}(z)| \leq C_{\alpha, \beta} |\text{Im} z|^\beta, \quad \forall N \in \mathbb{N}, \quad \alpha \in \mathbb{N}^2.
\]

Moreover,

\[
f(z) = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z)(z - x)^{-1} \, dz \wedge d\bar{z}.
\]

Proof. Let $\chi \in C_0^\infty (\mathbb{R})$ be a cutoff function such that $\chi(x) = 1$ for $|x| \leq 1$. Then it is easy to check that for an appropriate sequence $C_n$ the series

\[
\sum_{n=0}^{+\infty} i^n \partial_n^\alpha f(x) \frac{y^n}{n!} \chi \left( \frac{y}{C_n} \right)
\]

converges uniformly with all derivatives. Taking as $\tilde{f}(x + iy)$ the series in (C.4.3) we obtain a function that satisfies (C.4.1).

Next we note that

\[
\frac{i}{2\pi} \int_{C_{\varepsilon}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z)(z - x)^{-1} \, dz \wedge d\bar{z}
\]

\[
= \lim_{\varepsilon \to 0} \frac{i}{2\pi} \int_{C_{\varepsilon}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z)(z - x)^{-1} \, dz \wedge d\bar{z},
\]

where $C_{\varepsilon}$ is the domain

\[
C_{\varepsilon} := \{ z \in \mathbb{C} \mid |\text{Im} z| > \varepsilon, \quad |z| < \varepsilon \},
\]

for some $c$ large enough. Using Green’s formula, we obtain

\[
\frac{i}{2\pi} \int_{C_{\varepsilon}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z)(z - x)^{-1} \, dz \wedge d\bar{z}
\]

\[
= \lim_{\varepsilon \to 0} \frac{i}{2\pi} \int_{C_{\varepsilon}} \tilde{f}(z)(z - x)^{-1} \, dz
\]

\[
= \lim_{\varepsilon \to 0} \frac{i}{2\pi} \int_{\mathbb{R}} \tilde{f}(\lambda + ie)(\lambda + ie - x)^{-1} - \tilde{f}(\lambda - ie)(\lambda - ie - x)^{-1} \, d\lambda,
\]

Using the fact that $|\lambda + ie - x|^{-1} \leq \varepsilon^{-1}$, and Lebesgue’s dominated convergence theorem, we have

\[
\lim_{\varepsilon \to 0} \frac{i}{2\pi} \int_{\mathbb{R}} \tilde{f}(\lambda + ie)(\lambda + ie - x)^{-1} - \tilde{f}(\lambda - ie)(\lambda - ie - x)^{-1} \, d\lambda
\]

\[
= \lim_{\varepsilon \to 0} \frac{i}{2\pi} \int_{\mathbb{R}} \tilde{f}(\lambda + ie - x)^{-1} - \tilde{f}(\lambda - ie - x)^{-1} \, d\lambda
\]

\[
= f(x),
\]

since

\[
\frac{i}{2\pi} ((\lambda + i0)^{-1} - (\lambda - i0)^{-1}) = \delta.
\]

This completes the proof of the (C.4.2). $\Box$

One can also construct almost-analytic extensions of functions that do not have a compact support.
Proposition C.4.2 Let $m, h$ be two non-negative functions such that there exists $C, \epsilon > 0$ with
\begin{equation}
(\epsilon) \quad h(x)^{-1}|x-y| \leq \epsilon \Rightarrow C^{-1}h(x) \leq h(y) \leq C(h(x), C^{-1}m(x) \leq m(y) \leq Cm(x).
\end{equation}
Let $f \in C^\infty(R)$ such that
\begin{equation}
|\partial_x^n f(x)| \leq C_n m(x) h(x)^{-n}, \quad n \in \mathbb{N}.
\end{equation}
Then there exists a function $\tilde{f} \in C^\infty(C)$ called an almost analytic extension of $f$ such that
\begin{equation}
\tilde{f}|_R = f,
\end{equation}
\begin{equation}
|\frac{\partial^N}{\partial x^N}(\tilde{f}(z))| \leq C_N m(x) h(x)^{-1-N} |\text{Im} z|^N, \quad \forall N \in \mathbb{N},
\end{equation}
\[
\text{supp} \tilde{f} \subset \{x + iy | |y| \leq Ch(x)\}.
\]
Moreover, if
\[
\int m(x) h^{-1}(x) dx < \infty,
\]
then the formula (C.4.2) is true.

Proof. using (C.4.4), one can by [Hö2, Lemma 18.4.4] find a sequence $(x_k)_{k \in \mathbb{N}}$ in $R$ and partition of unity
\[
1 = \sum_k \chi_k(x),
\]
such that $\text{supp} \chi_k \subset B(x_k, \epsilon h(x_k))$ and
\begin{equation}
|\partial^n_x (\chi_k f)(x)| \leq C_n m(x_k) h(x_k)^{-n}.
\end{equation}
If we put
\[
g_k(x) := m(x_k)^{-1} \chi_k f(x_k + h(x_k) x),
\]
we have
\begin{equation}
|\partial^n_x g_k(x)| \leq C_n, \quad n \in \mathbb{N}, \quad \text{supp} g_k \subset B(0, 1).
\end{equation}
By Proposition C.4.1, we can construct almost-analytic extensions $\tilde{g}_k$ to $g_k$ such that
\begin{equation}
|\partial^N_x \tilde{g}_k(z)| \leq C_N |\text{Im} z|^N, \quad N \in \mathbb{N},
\end{equation}
\begin{equation}
|\partial^\alpha_x \tilde{g}_k(z)| \leq C_\alpha, \quad \alpha \in \mathbb{N}^2.
\end{equation}
If we put
\[
\tilde{f}(z) := \sum_k m(x_k) \tilde{g}_k(h(x_k)^{-1}(z - x_k)),
\]
we see from (C.4.8) that $\tilde{f}$ satisfies (C.4.5). If
\[
\int m(x) h^{-1}(x) dx < \infty,
\]
then (C.4.2) can be proven as in Proposition C.4.1. □

Finally we have the following extension of Proposition C.4.1 to the multi-dimensional case.
Proposition C.4.3 Let $f \in C_0^\infty(\mathbb{R}^n)$. Then there exist a function $\tilde{f} \in C_0^\infty(\mathbb{C}^n)$ called an almost analytic extension of $f$ such that

$$(C.4.9) \quad f|_{\mathbb{R}^n} = \tilde{f},$$

$$(\partial_{z_1,\cdots,z_n}^\alpha \tilde{f})(z) \leq C_{N,\alpha}(\text{dist}(z, \mathbb{R}^n))^N, \quad \forall N \in \mathbb{N}, \alpha \in \mathbb{N}^n.$$

Moreover,

$$(C.4.10) \quad f(x_1, \cdots, x_n) = (\frac{i}{2\pi})^n \int_{\mathbb{C}^n} \frac{\tilde{f}(z)}{z_1-x_1, \cdots, z_n-x_n} (z_1-x_1)^{-1} \cdots (z_n-x_n)^{-1} dz_1 \wedge dz_2 \cdots dz_n.$$

Proof. The proof of (C.4.9) is similar to the proof of (C.4.1) and is left to the reader.

The identity (C.4.10) can be proven as (C.4.2) using Green’s formula successively in the variables $z_1, \cdots, z_n$. $\Box$

C.5 Estimates on functions of operators

If $A$ is a self-adjoint operator and $B$ is bounded, then by the commutator $[A, B]$ we will mean the quadratic form on $\mathcal{D}(A)$ such that if $\phi, \psi \in \mathcal{D}(A)$, then

$$(\phi|[A, B]|\psi) = (A \phi|B \psi) - (B^* \phi|A \psi).$$

If $B$ is self-adjoint and possibly unbounded, then we define $[A, B]$ as the quadratic form on $\mathcal{D}(A)$ such that if $\phi, \psi \in \mathcal{D}(A)$, then

$$(C.5.1) \quad (\phi|[A, B]|\psi) = \lim_{\epsilon \to 0} ((A \phi|B(1 + i\epsilon)B)^{-1}\psi) - (B(1 - i\epsilon)B)^{-1}\phi|A \psi),$$

whenever the right hand side is well defined.

We often need estimates on commutators of the form

$$(C.5.2) \quad ||[f(A), B]|,$$

where $A = (A_1, \ldots, A_n)$ is a vector of commuting self-adjoint operators and $f$ is a function on $\mathbb{R}^n$. There exist a number of different estimates of such commutators, which can be obtained by different methods. Probably the simplest one is the following estimate.

Lemma C.5.1 Suppose that $F' = f$ and

$$(C.5.3) \quad \hat{f} \in L^1(\mathbb{R}^n)$$

(the Fourier transform of the derivative of $F$ is integrable) and $[A, B]$ is bounded. Then

$$(C.5.4) \quad ||[F(A), B]| \leq ||f||_1||[A, B]|.$$
(we use the integral to denote the pairing between test functions and distributions). Clearly
\[
\hat{F} = \text{Pr} \left( \frac{\hat{f}_\xi}{\xi} \right) + C\delta_0.
\]
Hence, both \( \hat{f} \) and \( \hat{F} \) are distributions that are well defined on test functions that are bounded together with their first derivative. Moreover, if \( \phi \in \text{Ran} E_\Theta(A) \) for some compact \( \Theta \), then \( e^{i\xi A}\phi \) is bounded together with all its derivatives. Therefore, the following identity is true:

(C.5.5)
\[
F(A)\phi = (2\pi)^{-n} \int \hat{F}(\xi) e^{i\xi A} d\xi \phi.
\]

Therefore, in the sense of quadratic forms on \( \text{Ran} E_\Theta(A) \) for some compact \( \Theta \), we can write

(C.5.6)
\[
[F(A), B] = (2\pi)^{-n} \int \hat{F}(\xi)\xi d\xi \int_0^1 e^{i\tau \xi A} i[A, B] e^{i(1-\tau)\xi A} d\tau,
\]

from which the estimate (C.5.4) follows immediately. \( \Box \)

Sometimes one needs to regularize the commutator \( [A, B] \) on the right hand side of (C.5.4) by using the inverse of \( A \). Below we give an example of how this can be done.

**Lemma C.5.2** Suppose that \( f \in C^\infty_0(\mathbb{R}^n) \). Then there exists a \( C \) that depends on \( f \) such that

(C.5.7)
\[
|||[f(A), B]|| \leq C||(1 + A^2)^{-1}, B||.\]

**Proof.** We set
\[
 f(A) = (1 + A^2)^{-1} f_1(A)(1 + A^2)^{-1},
\]
where \( f_1 \in C^\infty_0(\mathbb{R}^n) \). We have

(C.5.8)
\[
[f(A), B] = [(1 + A^2)^{-1}, B] f_1(A)(1 + A^2)^{-1} + (1 + A^2)^{-1}[f_1(A), B](1 + A^2)^{-1} + (1 + A^2)^{-1} f_1(A)[(1 + A^2)^{-1}, B].
\]
The middle term on the right of (C.5.8) we treat as in the proof of Proposition C.5.1. \( \Box \)

It is possible to prove similar estimates on \( [f(A), B] \) by using an almost analytic extension of \( f \). In fact, for instance, if \( f \in C^\infty_0(\mathbb{R}^n) \), and \( \hat{f} \) is an almost analytic extension of \( f \) constructed in Proposition C.4.1, then we can use the representation

(C.5.9)
\[
f(A) = \left( \frac{\hat{f}(z)}{z} \right)^n \int_{\mathbb{C}^n} \frac{\partial^n \hat{f}(z)}{\partial z_1 \cdots \partial z_n} (z_1 - A_1)^{-1} \cdots (z_n - A_n)^{-1} dz_1 \wedge dz_2 \cdots dz_n \wedge d\bar{z}_n,
\]
which follows from (C.4.10) and the spectral theorem. (Note that since \( \partial \hat{f}(z) = O(|Im z|) \) and \( ||(z - A)^{-1}|| \leq |Im z|^{-1} \), the integral in (C.5.9) converges in norm.) Then the commutator (C.5.2) becomes

(C.5.10)
\[
[f(A), B] = \sum_{j=1}^n \left( \frac{\hat{f}(z)}{z} \right)^n \int_{\mathbb{C}^n} \frac{\partial^n \hat{f}(z)}{\partial z_1 \cdots \partial z_n} \times (z_1 - A_1)^{-1} \cdots (z_j - A_j)^{-1} \cdots (z_n - A_n)^{-1} dz_1 \wedge dz_2 \cdots dz_n \wedge d\bar{z}_n.
\]

Note, however, that this method of estimating commutators is most effective in the case of a single self-adjoint operator \( A \). In particular, if we have \( A = (A_1, \ldots, A_n) \) with \( n \geq 2 \), then the expression (C.5.9) is even not invariant with respect to linear transformations in \( \mathbb{R}^n \), whereas (C.5.5) is.

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C.6 Pseudodifferential operators

The name “pseudodifferential operators” is usually used in two different (although related) meanings. First, it is used to denote operators on $L^2(\mathbb{R}^n)$ defined by certain integral formulas. The main ingredient of these formulas is a function that is called the symbol of a pseudodifferential operator, which encodes the phase space properties of the operator.

In its second meaning the word “pseudodifferential operators” is used to describe certain classes of operators that can usually be defined by describing properties of their symbols.

There exist a number of different notions of a symbol of a pseudodifferential operator. Here we consider the two most commonly used: the standard (or $x$-$D$) symbol and the Weyl symbol.

Let $\hat{S}'(X) \otimes S'(X)$ denote the space of sesquilinear forms on the space of Schwartz test functions $S(X)$. We will view $\hat{S}'(X) \otimes S'(X)$ as a kind of an extension of the set of linear operators on $L^2(X)$. We will treat all the elements of this space as “pseudodifferential operators” and we will define their symbols. Note that by the Schwartz’s kernel theorem (see eg. Appendix to chapter V3 of vol I of [RS]) elements of this set can be defined by a kernel $K \in \hat{S}'(X \times X)$ with help of the following equation:

$$(\phi|A\psi) = \int \int K(x, y)\tilde{\phi}(x)\psi(y)dx dy.$$  

(C.6.1)

Let $A \in \hat{S}'(X) \otimes S'(X)$. Then we say that $a_1 \in S'(T^*X)$ is the $x$-$D$-symbol of $A$ if for any $\phi, \psi \in S(X)$

$$(\phi|A\psi) = \int a_1(x, \xi)\tilde{\phi}(x)\psi(y)e^{i(x-y, \xi)} \frac{dx d\xi dy}{(2\pi)^n}.$$  

We will write

(C.6.2)  

$$A = a_1(x, D).$$  

We say that $a_2 \in S'(T^*X)$ is the Weyl symbol of $A$ if for any $\phi, \psi \in S(X)$

$$(\phi|A\psi) = \int a_2 \left( \frac{x+y}{2}, \xi \right)\tilde{\phi}(x)\psi(y)e^{i(x-y, \xi)} \frac{dx d\xi dy}{(2\pi)^n}.$$  

We will write

(C.6.3)  

$$A = a_2^w(x, D).$$  

Using basic properties of the Fourier transform on $S'(T^*X)$ and the Schwartz’s kernel theorem we easily see that every element of $\hat{S}'(X) \otimes S'(X)$ possesses a unique $x$-$D$-symbol and a unique Weyl symbol. Conversely, with any symbol in $S'(T^*X)$ we can associate a unique $x$-$D$-pseudodifferential operator and a unique Weyl pseudodifferential operator.

The following well known identity [Hö1] allows one to go from the $x$-$D$-symbol to the Weyl symbol:

$$(C.6.5) \quad e^{\frac{i}{2}(\xi \cdot D_x)}a_1 = a_2.$$  

Note that (C.6.5) makes sense for symbols in $S'(T^*X)$.

Let $S(X \times X')$ denote the set

$$\{ a \in C^\infty(X \times X') \mid |\partial_\alpha^\beta a(x, \xi)| \leq C_{\alpha, \beta}, \quad \alpha, \beta \in \mathbb{N}^n \}.$$  

It is a Fréchet space with the family of seminorms

$$||a||_{S(X \times X')N} = \sum_{|\alpha| + |\beta| \leq N} \||\partial_\alpha^\beta a||_\infty.$$  

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More generally, if \( Y \) is any Euclidean space, then we will sometimes use the notation
\[
S(Y) = \{ f \in C^\infty(Y) \mid |\partial_y^\alpha f(y)| \leq C_\alpha, \ \alpha \in \mathbb{N}^n \}.
\]
with the family of seminorms
\[
||f||_{S(Y),N} = \sum_{|\alpha| \leq N} ||\partial_y^\alpha f||_{\infty}.
\]
(The space \( S(Y) \) itself does not depend on the scalar product in \( Y \), but the family of seminorms \( || \cdot ||_{S(Y),N} \) does). Pseudodifferential operators associated with the class of symbols \( S(X \times X') \), which we introduce in the following theorem, play an important role in this chapter.

**Theorem C.6.1** The following statements are equivalent:

1) \( A \) is an operator on \( L^2(X) \) such that
\[
ad_D^\alpha \ad_x^\beta(A) \in B(L^2(X)), \ \alpha, \beta \in \mathbb{N}^n;
\]

2) the operator \( A \) is of the form
\[
A = a^\alpha(x, D), \ a \in S(X \times X');
\]

3) the operator \( A \) is of the form
\[
A = a(x, D), \ a \in S(X \times X');
\]

4) the operator \( A \) can be written as
\[
(C.6.6) \quad (\psi|A\phi) = \int \int \int \tilde{\psi}(x) b(x, \xi, y) \phi(y) e^{i(y-x, \xi)} dx d\xi dy,
\]
for some function \( b \in S(X \times X' \times X) \).

The set of operators satisfying any of the above conditions will be called \( \Psi(L^2(X)) \). It is a Fréchet space with the family of seminorms
\[
||A||_{\Psi(L^2(X))} = \sum_{|\alpha| + |\beta| \leq N} ||\ad_D^\alpha \ad_x^\beta A||.
\]

Note that in the literature usually the properties 2) or 3) are used to define this class. The implications 2)\( \rightarrow 1) \) and 3)\( \rightarrow 1) \) are essentially equivalent to the Calderón-Vaillancourt theorem (see [CV], [Ta] and [H62] vol III). The implications 1)\( \rightarrow 2) \) and 1)\( \rightarrow 3) \) go under the name of the Beals criterion (see [Bea]). The maps
\[
S(X \times X') \ni a \mapsto a(x, D) \in \Psi(L^2(X)),
\]
\[
S(X \times X') \ni a \mapsto a(x, D) \in \Psi(L^2(X))
\]
are linear homeomorphisms of Fréchet spaces. More precisely, there exists \( M \) that depends just on the dimension of \( X \) such that for any \( N \)
\[
\begin{align*}
||a(x, D)||_{\Psi(L^2(X))} & \leq C_N ||a||_{S(X \times X'), N + M}; \\
||a^\alpha(x, D)||_{\Psi(L^2(X))} & \leq C_N ||a||_{S(X \times X'), N + M}; \\
||a||_{S(X \times X'), N} & \leq C_N ||a(x, D)||_{\Psi(L^2(X))}, N + M; \\
||a||_{S(X \times X'), N} & \leq C_N ||a^\alpha(x, D)||_{\Psi(L^2(X))}, N + M; \\
\end{align*}
\]
(C.6.7)

The condition 1) (the “Beals criterion”) and Lemma C.5.1 easily imply the following proposition.
Proposition C.6.2 $\Psi(L^2(X))$ is a $*$-algebra. If $A \in \Psi(L^2(X))$ is invertible in $B(L^2(X))$, then
$$A^{-1} \in \Psi(L^2(X)).$$

If $A \in \Psi(L^2(X))$ is self-adjoint and $f$ is a $C^\infty$ function on the spectrum of $A$, then
$$f(A) \in \Psi(L^2(X)).$$

The following fact is useful when we want to make sense of various formulas in the calculus of pseudodifferential operators from $\Psi(X \times X')$.

Proposition C.6.3 1) Let $R(y)$ be a non-degenerate quadratic form on $Y$. Let $a \in S(Y)$. Then for some $N$
\begin{equation}
|\int e^{iR(y)}a(y)dy| \leq ||a||_{S(Y),N}.
\end{equation}

2) Let $Q(\eta)$ be a quadratic form on $Y'$. Then the map
$$e^{iQ(D_\eta)} : S(dy^2) \to S(dy^2)$$
is continuous.

3) Let $A \in \Psi(L^2(X))$. Then the map
$$A : S(X) \to S(X)$$
is continuous.

Proof. Let us prove 1). It is enough to assume that $Y = \mathbb{R}$ and $R(y) = y^2$. Consider an operator $L$ defined as
$$(Lb)(y) := (1 + 2y^2)^{-1} \left( -iy \frac{\partial}{\partial y} + 1 \right) b(y).$$
Then
$$Le^{iy} = e^{iy},$$
Hence
$$\int e^{iy}a(y)dy$$
(C.6.9)
$$= \int L^2(e^{iy})a(y)dy$$
$$= \int e^{iy}(L^*)^2a(y)dy,$$
where
$$L^*b(y) := i\frac{\partial}{\partial y}(1 + 2y^2)^{-1}b(y).$$
But
$$(L^*)^2a(y) \in L^1(\mathbb{R}).$$
Hence the integral (C.6.9) is finite. This ends the proof of 1).

2) follows easily from 1).

Let us prove 3). Let $A = a(x, D)$. Then
\begin{equation}
(A\phi)(x) = \int e^{i(x-y, \xi)}a(x, \xi)\phi(y)d\xi dy
= \int e^{i(y, \xi)}a(x, \xi)\phi(y + x)d\xi dy.
\end{equation}
Therefore by 1), treating $x$ as a parameter we conclude that (C.6.10) and all its derivatives are bounded.
\[\square\]
Note that if $A = a^\omega (x, D)$ and $A$ is given by (C.6.6), then we can compute its Weyl symbol from the formula

\[(C.6.11) \quad a(x, \xi) = e^{\frac{i}{\hbar}(P_\alpha \xi + (P_\alpha D_\xi))} b(x, \xi, \eta) \bigg|_{\eta = \xi = \eta}.
\]

Thanks to Proposition C.6.3.2) the expression in (C.6.11) is well defined.

The equivalence of 2), 3) and 4) of Theorem C.6.1 follows from (C.6.5), (C.6.11) and from 2) of Proposition C.6.3.

There exists a number of apparently different formulas that can be used to compute the product of two pseudodifferential operators. Namely, if

\[(C.6.12) \quad a^\omega(x, D)b^\omega(x, D) = c^\omega(x, D),
\]

then

\[(C.6.13) \quad c(x, \xi) = e^{\frac{i}{\hbar}(P_\alpha \xi - (P_\alpha D_\xi))} a(x, \xi, \eta) b(y, \eta) \bigg|_{\eta = \xi = \eta}.
\]

\[(C.6.14) \quad = a \left( x - \frac{1}{2} D_\xi, \xi + \frac{1}{2} D_x \right) b(x, \xi)
\]

\[(C.6.15) \quad = a^\omega \left( x + \frac{1}{2} D_\xi, \xi - \frac{1}{2} D_x \right) b(x, \xi)
\]

\[(C.6.16) \quad = b \left( x + \frac{1}{2} D_\xi, \xi - \frac{1}{2} D_x \right) a(x, \xi)
\]

\[(C.6.17) \quad = b^\omega \left( x + \frac{1}{2} D_\xi, \xi - \frac{1}{2} D_x \right) a(x, \xi).
\]

(C.6.13) can be found e.g. in [Hö]; other expressions follow immediately from (C.6.13).

All these formulas make sense for $a, b \in S(X \times X')$. In fact, to see that (C.6.13) is well defined we use 2) of Proposition C.6.3, for the remaining expressions we use 3) of Proposition C.6.3.

From the above identities we can obtain a formula for the symbol of a commutator. Namely, if $a, b \in S(X \times X')$ and

\[\left[ a^\omega(x, D), b^\omega(x, D) \right] = c^\omega(x, D),
\]

then

\[(C.6.18) \quad c(x, \xi)
\]

\[= \frac{1}{2} \int_{|r| < |x|} \langle r, \xi \rangle^{1/2} |r| \Gamma(b(x, \xi))
\]

where

\[(C.6.19) \quad \Gamma := \frac{1}{2} \int_{|r| < |x|} \frac{1}{|r|^2} d^2 r
\]

\[= \sum_{|\alpha| = 3} \partial_\l a(x + \tau D_\xi, \xi - \tau D_\xi) D_\xi a(x + \tau D_\xi, \xi - \tau D_\xi) D_\xi
\]

\[+ 3 \sum_{|\alpha| = 1, |\beta| = 2} \partial_\l \partial^{\beta}_r a(x + \tau D_\xi, \xi - \tau D_\xi) D_\xi a(x + \tau D_\xi, \xi - \tau D_\xi) D_\xi
\]

It is often useful to know that in a certain sense the symbol of the product of two pseudodifferential operators is approximately equal to the product of their symbols. More precisely, it follows easily from (C.6.14) and (C.6.16) that there exist $C$ and $N$ depending just on the dimension of $X$ such that

\[||a^\omega(x, D)b^\omega(x, D) + b^\omega(x, D)a^\omega(x, D) - 2(ab)^\omega(x, D)||
\]

\[\leq C||\nabla_{x, \xi} a||_{S(X \times X'), N} ||\nabla_{x, \xi} b||_{S(X \times X'), N}.
\]

Let us mention also the so-called sharp Gårding inequality. For operators of the class $\Psi(\mathcal{L}^2(X))$ it can be formulated as follows.
Theorem C.6.4 There exists $C$ and $N$ that depend just on the dimension of $X$ such that if $a \in S(X \times X')$ and
\[ a(x, \xi) \geq 0, \]
then
\[ a^\pi(x, D) \geq -C\|\nabla_x^2 a\|_{L^2(X \times X'), N}. \]  

Proof. Let $P$ denote the projection onto the vector
\[ \phi(x) := \pi^{-n/4} e^{-\frac{1}{4} x^2}. \]
The Weyl symbol of $P$ equals
\[ p(x, \xi) := 2^{-n} e^{-\frac{1}{2} (x^2 + \xi^2)}. \]
Let $P_{y, \eta}$ denote
\[ e^{i(y, x) + i(\eta, D_\xi)} p e^{-i(y, x) - i(\eta, D_\xi)}. \]
The Weyl symbol of $P_{y, \eta}$ equals $p(x - y, \xi - \eta)$. Therefore the Weyl symbol of
\[ \int \int a(y, \eta) P_{y, \eta} dy d\eta \]
equals the convolution $a * p$. Now,
\[ (a - \pi^{-n} a * p)(x, \xi) \]
\[ = \pi^{-n} \int \int p(y, \eta)(a(x, \xi) - a(x - y, \xi - \eta)) dy d\eta \]
\[ = \pi^{-n} \int \int p(y, \eta) \frac{1}{2} \int_0^1 \int_0^1 (y, \eta)^2 \nabla_x^2 a(x - \tau y, \xi - \tau \eta) d\tau dy d\eta, \]
Hence for some $N$
\[ \| \int \int a(y, \eta) P_{y, \eta} dy d\eta - a^\pi(x, D) \| \leq \| \nabla_x^2 a \|_{L^2(X \times X'), N}. \]
Now theorem follows from (C.6.23), and the positivity of (C.6.21). \qed

Suppose that $T_{r}$ is the operator of dilations defined by
\[ (T_{r} \phi)(x) := r^{-\frac{1}{2}} \phi(r^{-1} x). \]
Then it is easy to see that
\[ T_{r} a(x, D) T_{r} = a(r x, r^{-1} D), \]
\[ T_{r} a^\pi(x, D) T_{r} = a^\pi(r x, r^{-1} D). \]
(Note that in the case of Weyl symbols a similar covariance of symbols is true for a much larger group called the metaplectic group).

Now we would like to consider symbols and operators that depend on a parameter $t$. Suppose that $f(t)$, $c_x(t)$ and $c_\xi(t)$ are some non-negative functions. Then we define the spaces
\[ S(f(t), c_x^2(t) dx^2 + c_\xi^2(t) d\xi^2), \]
\[ S(c(t), c_x^2(t) dx^2 + c_\xi^2(t) d\xi^2) \]
and
\[ L^1(f(t) dt, S(c_x^2(t) dx^2 + c_\xi^2(t) d\xi^2) \]
to be the spaces of functions
\[ t \mapsto a(t, x, \xi) \in C^\infty(X \times X') \]
such that
\[
|\partial_\alpha^\beta \partial_\xi^\gamma a(t, x, \xi)| \leq C_{\alpha \beta} f(t) c_\xi^{[\alpha]}(t) c_\xi^{[\beta]}(t), \quad \alpha, \beta \in \mathbb{N}^n,
\]
\[
|\partial_\xi^\gamma a(t, x, \xi)| \leq o(f(t)) c_\xi^{[\alpha]}(t) c_\xi^{[\beta]}(t), \quad \alpha, \beta \in \mathbb{N}^n,
\]
\[
\int |\partial_\alpha^\beta \partial_\xi^\gamma a(t, x, \xi)| f(t) c_\xi^{-[\alpha]}(t) c_\xi^{-[\beta]}(t) dt < \infty, \quad \alpha, \beta \in \mathbb{N}^n
\]
respectively.

Similarly, we define the algebras
\[
\Psi(f(t), c_\xi^2(t)dx^2 + c_\xi^2(t)d\xi^2),
\]
\[
\Psi(o(f(t)), c_\xi^2(t)dx^2 + c_\xi^2(t)d\xi^2)
\]
and
\[
L^1(f(t) dt, \Psi(c_\xi^2(t)dx^2 + c_\xi^2(t)d\xi^2)
\]
to be the spaces of operator-valued functions
\[
t \to A(t) \in B(L^2(X))
\]
such that
\[
||\text{ad}_\alpha^\beta \text{ad}_\xi^\gamma A(t)|| \leq C_{\alpha \beta} f(t) c_\xi^{[\alpha]}(t) c_\xi^{[\beta]}(t), \quad \alpha, \beta \in \mathbb{N}^n,
\]
\[
||\text{ad}_\xi^\gamma A(t)|| \leq o(f(t)) c_\xi^{[\alpha]}(t) c_\xi^{[\beta]}(t), \quad \alpha, \beta \in \mathbb{N}^n,
\]
\[
\int ||\text{ad}_\alpha^\beta \text{ad}_\xi^\gamma A(t)|| f(t) c_\xi^{-[\alpha]}(t) c_\xi^{-[\beta]}(t) dt < \infty, \quad \alpha, \beta \in \mathbb{N}^n
\]
respectively.

To simplify the notation, in what follows we will put
\[
g(t) := c_\xi^2(t)dx^2 + c_\xi^2(t)d\xi^2.
\]
We will denote by
\[
(C.6.26) \quad || \cdot ||_{S(f(t), g(t)), N} \text{ etc}
\]
the seminorms naturally associated with the spaces of operators introduced above.

In applications to scattering theory described in this chapter a special role will be played by the metric
\[
g_0(t) := \langle t \rangle^2 dx^2 + d\xi^2.
\]

The number \(c_\xi(t)c_\xi(t)\) has the interpretation of the “effective Planck constant”. The following proposition can serve as an alternative definition of the algebras defined above if this Planck constant is bounded.

**Proposition C.6.5** Suppose that
\[
(C.6.27) \quad c_\xi(t)c_\xi(t) \leq C.
\]
Then
\[
\Psi(f(t), g(t)),
\]
\[
\Psi(o(f(t)), g(t))
\]
and
\[
L^1(f(t) dt, \Psi(g(t)),
\]

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are the sets of operator-valued functions
\[ t \to a^\nu(t, x, D) \]
such that
\[ a \in S(f(t), g(t)), \]
\[ a \in S(\phi(f(t)), g(t)) \]
and
\[ a \in L^1(f(t) dt, S(g(t)) \]
respectively.

Proof. If both \( c_x(t) \) and \( c_\xi(t) \) are bounded, then the proposition follows immediately from Theorem C.6.1.

If not, we conjugate our operators with a \( t \)-dependent generator of dilations \( T_{r(t)}(t) \), where we can take for example
\[ r(t) := c_x^{-1/2}(t)c_\xi^{1/2}(t) \]
The metric becomes \( c_x(t)c_\xi(t)dx^2 + c_x(t)c_\xi(t)d\xi^2 \). Thus the coefficients in the metric become bounded. Now we can apply (C.6.25). \( \square \)

The same trick of conjugating with \( t \)-dependent dilatations can be used to obtain the following version of the sharp Gårding inequality.

**Proposition C.6.6** Let \( a \in S(f(t), g(t)) \) and
\[ a(t, x, \xi) \geq 0. \]
Then
\[ a^\nu(x, D) \geq -C f(t)c_x(t)c_\xi(t). \]

If
\[ A_i(t) \in \Psi(f_i(t), g(t)), \ i = 1, 2, \]
then it is easy to see that
\[ A_1(t)A_2(t) \in \Psi(f_1(t)f_2(t), g(t)), \]
and
\[ [A_1(t), A_2(t)] \in \Psi(f_1(t)f_2(t)c_x(t)c_\xi(t), g(t)). \]
To show (C.6.29) it is enough to use the definition of \( \Psi \) (the "Beals criterion"). (C.6.30) is much deeper — one needs to use for example (C.6.18).

Sometimes it is possible to improve (C.6.30). Namely, suppose that we know that
\[ [x, A_i(t)] \in \Psi(f_i(t)c_\xi(t), g(t)), \ i = 1, 2, \]
\[ [D, A_i(t)] \in \Psi(f_i(t)c_x(t), g(t)), \ i = 1, 2, \]
then
\[ [A_1(t), A_2(t)] \in \Psi(f_1(t)f_2(t)c_x(t)c_\xi(t), g(t)). \]
C.7 Fourier integral operators.

In this section we recall some results on Fourier integral operators associated with symbols from $S(X \times X')$. We start with the following auxiliary fact, known as Schwartz’s global inversion theorem (see [Sch]).

**Proposition C.7.1** Suppose that the function

(C.7.1) \( \mathbb{R}^n \ni y \mapsto x(y) \in \mathbb{R}^n \)

satisfies

(C.7.2) \( |\det \nabla_y z| \geq C_0 > 0 \)

and

(C.7.3) \( |\partial_y^\alpha x| \leq C_\alpha \) for \( |\alpha| = 1, 2 \).

Then the function (C.7.1) is bijective.

**Proof.** First note that by the local inverse function theorem there exist \( \epsilon, \delta > 0 \) such that for any \( y_0 \in \mathbb{R}^n \)

(C.7.4) \( x(B(y_0, \epsilon)) \supset B(x(y_0), \delta) \).

Therefore \( x(\mathbb{R}^n) = \mathbb{R}^n \).

We will now prove that the function (C.7.1) is injective. Fix \( y_1, y_2 \in \mathbb{R}^n \) such that \( x(y_1) = x(y_2) = x_0 \).

Note that given a curve \( \Gamma \) that starts at \( x(y_1) \) we can find a unique curve \( \tilde{\Gamma} \) that starts at \( y_1 \) and \( x(\tilde{\Gamma}) = \Gamma \) (we start at \( y_1 \) and extend it by continuity).

Join \( y_1 \) and \( y_2 \) with a curve \( [0, 1] \ni \tau \mapsto x_\tau \). Set \( x(\tilde{\Gamma}) = \Gamma \). The curve \( \Gamma \) can be continuously deformed to form a family of curves \( \Gamma_\tau, \tau \in [0, 1] \), with the following properties:

\[
\begin{align*}
\Gamma_1 &= \Gamma, \\
\Gamma_\tau(0) &= \Gamma_\tau(1) = x_0, \quad 0 \leq \tau \leq 1, \\
\Gamma_0(s) &= x_0, \quad 0 \leq s \leq 1.
\end{align*}
\]

The corresponding curves \( \tilde{\Gamma}_\tau \) are also continuously deformed. But \( \tilde{\Gamma}_\tau(0) = y_0 \) and \( \tilde{\Gamma}_\tau(1) = y_1 \) for all \( \tau \in [0, 1] \), hence \( y_1 = y_2 \). \( \square \)

The main result of this appendix is the following theorem on the boundedness of a certain class of Fourier integral operators.

**Theorem C.7.2** i) Suppose that

(C.7.5) \[ |\partial_x^\alpha \partial_\xi^\beta \Phi(x, \xi)| \leq M_{\alpha, \beta} \text{ for } |\alpha| + |\beta| \geq 2, |\alpha| \geq 1 \]

and

(C.7.6) \[ |\det M| \geq C_0 > 0, \forall M \in \text{ch} \left\{ \nabla_x \nabla_\xi \Phi^+(x, \xi) \mid (x, \xi) \in X \times X' \right\} \]

where \( \text{ch}(\Theta) \) denotes the convex hull of the set \( \Theta \). Let

(C.7.7) \[ |\partial_x^\alpha \partial_\xi^\beta a_i(x, \xi)| \leq C_{\alpha, \beta}, \quad i = 1, 2. \]

Define \( A_i \) to be the operator given by

\[ A_i \phi(x) := (2\pi)^{-n} \int a_i(x, \xi) e^{i\Phi(x, \xi) - i(y, \xi)} \phi(y) dy, \quad i = 1, 2. \]
Then
(C.7.8) \[ A_1 A_2^* \in \Psi(L^2(X)). \]

ii) Suppose that
(C.7.9) \[ \left| \partial_x^\alpha \partial_{\xi}^\beta \Phi(x, \xi) \right| \leq M_{\alpha, \beta} \text{ for } |\alpha| + |\beta| \geq 2, |\beta| \geq 1. \]
Assume also (C.7.6) and (C.7.7). Let the operators \( A_1 \) be defined as in ii). Then
(C.7.10) \[ A_1^* A_2 \in \Psi(L^2(X)). \]

iii) Let \( \Phi(x, \xi) \) satisfy either the hypotheses of i) or of ii), Let
\[
\left| \partial_x^\alpha \partial_{\xi}^\beta a(x, \xi) \right| \leq C_{\alpha, \beta},
\]
and
\[
A \phi(x) := (2\pi)^{-n} \int a(x, \xi) e^{i \Phi(x, \xi)} e^{-i [y, \xi]} \phi(y) dy.
\]
Then \( A \) is bounded on \( L^2(\mathbb{R}^n) \) and there exists an integer \( N \) and a constant \( C \) depending on \( C_0 \) and \( M_{\alpha, \beta}, |\alpha| + |\beta| \leq N \), such that
(C.7.11) \[ \| A \| \leq C \sum_{|\alpha|+|\beta| \leq N} C_{\alpha, \beta}. \]

**Proof.** Let us prove i). The kernel of \( A_1 A_2^* \) equals
(C.7.12) \[
K(x_1, x_2) = \int a_1(x_1, \xi) \tilde{a}_2(x_2, \xi) e^{i \Phi(x_1, \xi) - i [y, \xi]} d\xi
\]
For fixed \( x_1, x_2 \) the map
\[
X' \ni \xi \mapsto \eta(x_1, x_2, \xi) := \int_0^1 \nabla_x \Phi(\tau x_1 + (1 - \tau)x_2, \xi) d\tau \in X'
\]
satisfies the assumptions of Proposition C.7.1, hence it is invertible. Let us denote its inverse by
\[
\eta \mapsto \xi(x_1, x_2, \eta).
\]
We note that
(C.7.13) \[ \left| \partial_{x_1}^\alpha \partial_{\xi}^\beta \xi(x_1, x_2, \eta) \right| \leq C_{\alpha_2, \beta} \text{ for } |\alpha_1| + |\alpha_2| + |\beta| \geq 1. \]
We rewrite (C.7.12) as
\[
K(x_1, x_2) = \int b(x_1, x_2, \eta) e^{i [\eta, (x_1 - x_2)]} d\eta,
\]
where
\[
b(x_1, x_2, \eta) := a_1(x_1, \xi(x_1, x_2, \eta)) \tilde{a}_2(x_2, \xi(x_1, x_2, \eta)) |\det \nabla_\eta \xi(x_1, x_2, \eta)|.
\]
We easily see that all the seminorms of \( b(x_1, x_2, \eta) \) in \( S(X' \times X' \times X') \) can be estimated by the right-hand side of (C.7.11). Therefore (C.7.11) follows from Theorem C.6.1.

ii) follows from i) by conjugation with the Fourier transformation.

iii) is an immediate consequence of i) and ii). \( \square \)

The function \( a(x, \xi) \) is usually called the *amplitude* of \( A \) and \( \Phi(x, \xi) \) is called the *phase* of \( A \).
Remark C.7.3  In practice instead of (C.7.6) we will use a simpler condition on $\nabla_x \nabla_\xi \Phi(x, \xi)$. We will just assume that

$$
\|\nabla_x \nabla_\xi \Phi(x, \xi) - 1\| \leq C_0 < 1.
$$

The class of Fourier integral operators described in Theorem C.7.2 is invariant with respect to multiplication on the left and right with pseudodifferential operators of the class $\Psi(L^2(X))$.

Proposition C.7.4  i) Suppose that $a(x, \xi), \Phi(x, \xi)$ and $A$ satisfy the assumptions of Theorem C.7.2 i). Let $B = b(x, D) \in \Psi(L^2(X))$. Then the operator $BA$ can be written as

$$
BA\phi(x) := (2\pi)^{-n} \int c(x, \xi) e^{i\Phi(x, \xi) - i(y, \xi)} \phi(y) dx dy,
$$

where

$$
\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial \xi^\beta} c(x, \xi) \leq C_{\alpha, \beta}.
$$

ii) Suppose that $a(x, \xi), \Phi(x, \xi)$ and $A$ satisfy the assumptions of Theorem C.7.2 ii). Let $B = b(x, D) \in \Psi(L^2(X))$. Then the operator $AB$ can be written as

$$
AB\phi(x) := (2\pi)^{-n} \int d(x, \xi) e^{i\Phi(x, \xi) - i(y, \xi)} \phi(y) dx dy,
$$

where

$$
\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial \xi^\beta} d(x, \xi) \leq C_{\alpha, \beta}.
$$

Proof. To prove i) we compute the kernel of $BA$:

$$
K(x_1, y) = \int \int b(x_1, \xi_1) a(x_2, \xi_2) e^{i(x_1 - r z_1, \xi_1) + i\Phi(x_2, \xi_2) - i(y, \xi_2)} d\xi_1 d x_2 d\xi_2.
$$

This can be rewritten as

$$
\int c(x_1, \xi_2) e^{i\Phi(x_1, \xi_2) - i(x, y)} d\xi_2,
$$

where

$$
c(x_1, \xi_2) = \int \int b(x_1, \xi_1) a(x_2, \xi_2) e^{i(x_1 - r z_1, \xi_1) + i\Phi(x_2, \xi_2) - i(y, \xi_2)} d\xi_1 d x_2 d\xi_2
$$

$$
= \int \int b(x_1, \xi_1) a(x_2, \xi_2) e^{i(x_1 - r z_1, \xi_1) - \int_0^1 \int \nabla_\xi \Phi(x_1 + (1 - \tau) z, \xi_2 + \eta) a(x_1 + z, \xi_2) e^{-i(x, \eta)} d\eta} d\xi_2 d x_2
$$

Now from (C.7.18) we see that $c(x, \xi) \in S(X \times X')$.

To prove ii), we first see that using the same arguments as above the kernel of $F BF^{-1} FA^* F^{-1}$, where $F$ denotes the Fourier transform, is equal to:

$$
K(\xi, \eta) = (2\pi)^{-n} \int e^{-i\Phi(x, \xi) + i\xi, \eta} d(x, \xi) dx,
$$

where $d \in S(X \times X')$, which implies directly ii).  \(\square\)

Let us consider also Fourier integral operators depending on a parameter. The following proposition is a parameter-dependent version of Theorem C.7.2 i).
Proposition C.7.5  Suppose that \( c_x(t)c_\xi(t) \leq C \),

\[
\left| \frac{\partial^\alpha \partial^\beta \Phi(t, x, \xi)}{\partial \xi^{\beta}} \right| \leq C_{\alpha, \beta} |\xi|^{|\beta|-1}(t)|\xi|^{|\beta|-1}(t), \quad \text{for } |\alpha| + |\beta| \geq 2, \ |\alpha| \geq 1, \\
|| \nabla_x \nabla_\xi \Phi(x, \xi) - 1|| \leq C_0 < 1.
\]

Let \( a_i(t, x, \xi) \in S(f_1(t), c_x^2(t)dx^2 + c_\xi^2(t)d\xi^2), \ i = 1, 2. \)

Define \( A_i(t) \phi(x) := (2\pi)^{-n} \int a_i(t, x, \xi)e^{i\Phi(t, x, \xi) - i(x_1, \xi)}\phi(y)d\xi dy. \)

Then \( A_1(t)A_2(t) \in \Psi(f_1(t)f_2(t), c_x^2(t)dx^2 + c_\xi^2(t)d\xi^2). \)

Finally, let us state a parameter dependent version of Proposition C.7.4.

Proposition C.7.6 i) Suppose that \( c_x(t)c_\xi(t) \leq C, \) and

\[
\left| \frac{\partial^\alpha \partial^\beta \Phi(t, x, \xi)}{\partial \xi^{\beta}} \right| \leq C_{\alpha, \beta} |\xi|^{|\beta|-1}(t)|\xi|^{|\beta|-1}(t), \quad \text{for } |\alpha| + |\beta| \geq 2, \ |\alpha| \geq 1, \\
|| \nabla_x \nabla_\xi \Phi(x, \xi) - 1|| \leq C_0 < 1.
\]

Let \( a(t, x, \xi) \in S(f_2(t), c_x^2(t)dx^2 + c_\xi^2(t)d\xi^2), \)

\( A(t) \phi(x) := (2\pi)^{-n} \int a(t, x, \xi)e^{i\Phi(t, x, \xi) - i(x_1, \xi)}\phi(y)d\xi dy. \)

and \( B(t) \in \Psi(f_1(t), c_x^2(t)dx^2 + c_\xi^2(t)d\xi^2). \)

Then \( B(t)A(t) \phi(x) := (2\pi)^{-n} \int c(t, x, \xi)e^{i\Phi(t, x, \xi) - i(x_1, \xi)}\phi(y)d\xi dy, \)

such that \( c(t, x, \xi) \in S(f_1(t)f_2(t), c_x^2(t)dx^2 + c_\xi^2(t)d\xi^2). \)

ii) If we assume instead that

\[
\left| \frac{\partial^\alpha \partial^\beta \Phi(t, x, \xi)}{\partial \xi^{\beta}} \right| \leq C_{\alpha, \beta} c_x^{|\beta|-1}(t)c_\xi^{|\beta|-1}(t), \quad \text{for } |\alpha| + |\beta| \geq 2, \ |\beta| \geq 1, \\
|| \nabla_x \nabla_\xi \Phi(x, \xi) - 1|| \leq C_0 < 1,
\]

then \( A(t)B(t) \phi(x) := (2\pi)^{-n} \int d(t, x, \xi)e^{i\Phi(t, x, \xi) - i(x_1, \xi)}\phi(y)d\xi dy, \)

such that \( d(t, x, \xi) \in S(f_1(t)f_2(t), c_x^2(t)dx^2 + c_\xi^2(t)d\xi^2). \)
Bibliography


