The spectrum of Schrödinger operators
in $L_p(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$

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Supported by Federal Ministry of Science and Research, Austria
The Spectrum of Schrödinger Operators
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Introduction

The aim of this paper is to present results on the independence of the spectrum of Schrödinger operators in different spaces. We treat Schrödinger operators of a very general kind, namely $-\frac{1}{2}\Delta$ perturbed by certain measures $\mu$.

In Section 1 we recall what measures can be used and we review results stating the $p$-independence of the spectrum of the realizations of $-\frac{1}{2}\Delta + \mu$ in $L_p(\mathbb{R}^d)$, $1 \leq p \leq \infty$.

In Section 2 we show that the realizations of $-\frac{1}{2}\Delta + \mu$ in spaces of continuous functions, e.g., the bounded uniformly continuous functions or the continuous functions vanishing at infinity, again have the same spectrum, for suitable $\mu$. In fact, this is derived in a much more general context, utilizing the semigroup dual of a Banach space with respect to a strongly continuous semigroup.

In Section 3 it is shown that Shnol’s method of constructing singular sequences can also be employed in a proof of the inclusions $\sigma(H_{2,V}) \subset \sigma(H_{p,V})$ and $\sigma(H_{2,V}) \subset \sigma(H_{C_0,V})$, for suitable potentials $V$. This establishes the connection between the spectrum in $L_p$ and $C_0$ and the existence of polynomially bounded generalized eigenfunctions.

*Presented at the meeting by J. Voigt

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1. Review of $L_p$-results.

In order to state the results we have to recall some notations. Let

$$M_0 := \{ \mu : \mathcal{B} \to [0, \infty] : \mu \text{ } \sigma\text{-additive, } \mu(B) = 0 \text{ for all sets } B \in \mathcal{B} \text{ with capacity zero} \},$$

where $\mathcal{B}$ denotes the $\sigma$-algebra of Borel subsets of $\mathbb{R}^d$.

For the definition of the extended Kato class $^\ast S_K \subseteq M_0$ of measures and of the constant $c(\mu)$ defined for $\mu \in ^\ast S_K$ we refer to [StV]. We recall that for $\mu_+ \in M_0$, $\mu_- \in ^\ast S_K$ with $c(\mu_-) < 1$ a closed form in $L_2(\mathbb{R}^d)$ is defined by

$$(h - \mu_+\mu_+)[u, v] := \frac{1}{2} \int \nabla u \cdot \nabla v dx - \int u^- v^- d\mu_- + \int u^- v^+ d\mu_+,$$

with domain

$$D(h - \mu_+\mu_+) = \{ u \in W^1_2(\mathbb{R}^d) : \int |u^+|^2 d\mu_+ < \infty \}$$

($u^-$ denoting a quasi-continuous version of $u$). The closure of $D(h - \mu_+\mu_+)$ in $L_2(\mathbb{R}^d)$ is of the form $L_2(Y)$, for a suitable set $Y \in \mathcal{B}$. The operator $H_\mu := H_{\mu_+\mu_-}$ is the self-adjoint operator in $L_2(Y)$ associated with $h - \mu_+\mu_+$. It is shown in [StV; Corollary 4.2] that the semi-group $e^{tH_\mu}$ on $L_2(Y)$ acts also as a strongly continuous semigroup $U_{p,\mu}^t$ on $L_p(Y)$, for all $p \in [1, \infty)$; the generators of these semigroups will be denoted by $-H_{p,\mu}$. Also, $H_{\infty,\mu} := H_{1,\mu}^+$. The corresponding unperturbed operators (for $\mu = 0$) will be denoted by $H_p$.

1.1. **Theorem.** With the notations introduced so far, we have

$$\sigma(H_{p,\mu}) = \sigma(H_{2,\mu})$$

for all $p \in [1, \infty]$.

We are going to give an outline of the proof of this result. In order to do so we first collect several facts which are needed in the proof.

1.2. **Remark.** (a) Let $\varepsilon > 0$. There exist constants $C, \omega$ such that

$$\| e^{t \varepsilon x} e^{-iH_{p,\mu} t} e^{-\varepsilon x} \|_{p, q} \leq C t^{-\gamma} e^{\omega t}$$

for all $t > 0, 1 \leq p \leq q \leq \infty$, $x \in \mathbb{R}^d$ with $|\xi| \leq \varepsilon$, where $\gamma = \frac{1}{2} (\frac{1}{p} - \frac{1}{q})$. (Here $\| \cdot \|_{p, q}$ denotes the norm in $L(L_p, L_q)$.)

The proof of this fact consists in two steps. In both of these steps it is essential that there exists $a > 1$ such that $a\mu$ is also in the class considered above (in particular, $c(a\mu) < 1$).

(i) One shows the inequality for $x = 0$, using Stein interpolation; cf. [StV; Theorem 5.1 (b)].
(ii) From the fact that the desired statement is true for the unperturbed heat semigroup \((\mu = 0)\) one concludes it for the perturbed semigroup, again using Stein interpolation; cf. [ScV; Remark 3.4 (b), (c)].

(b) Let \(\epsilon > 0, \omega \) be as in (a). Then there exists \(C\) such that

\[
\|e^{\xi x}(H_\mu - w)^{-1}e^{-\xi x}\|_{p,q} \leq C \left( \frac{1}{1 - \gamma} + \frac{1}{-w - \omega} \right)
\]

for all \(w \in \mathbb{R}\) with \(w < -\omega\), \(p \leq q\) with \(\gamma = \frac{d}{2}(\frac{1}{p} - \frac{1}{q}) < 1\), \(|\xi| \leq \epsilon\). Further, \((-\infty, -\omega) \subset \rho(H_{p,\mu})\) for all \(p \in [1, \infty]\), and

\[
(H_{p,\mu} - w)^{-1} = (H_\mu - w)^{-1}
\]

on \(L_p(Y) \cap L_2(Y)\), for \(w < -\omega\).

The proof consists in integrating the inequality in (a) after multiplying by \(e^{wt}\); cf. [HV; Proposition 3.7], [ScV; Remark 3.4 (d)].

1.3. Lemma. ([ScV; Corollary 3.3]) Let \(1 \leq p \leq q \leq \infty\), \(0 < \epsilon' < \epsilon''\). Then there exists \(C \geq 0\) such that for each linear operator

\[
A : L_{\infty,c}(\mathbb{R}^d) \to L_{\infty, loc}(\mathbb{R}^d)
\]

\((L_{\infty,c} \) denoting \(L_{\infty}\)-functions with compact support) satisfying

\[
\|e^{\xi x}Ae^{-\xi x}\|_{p,q} \leq 1 \quad \text{for all} \quad \xi \in \mathbb{R}^d \quad \text{with} \quad |\xi| \leq \epsilon''
\]

one has

\[
\|e^{\xi x}Ae^{-\xi x}\|_{r,s} \leq C
\]

for \(p \leq r \leq q\), \(|\xi| \leq \epsilon'\).

The inclusion \(\rho(H_{p,\mu}) \subset \rho(H_{2,\mu})\) in Theorem 1.1 is obtained as in [HV; section 2], using Remark 1.2 (a) for \(\xi = 0\).

Sketch of the proof of the inclusion \(\rho(H_{2,\mu}) \subset \rho(H_{p,\mu})\) (compare [ScV]). It is sufficient to prove the assertion for all \(p \in [1, 2]\). According to Remark 1.2 (b) we find \(w (-\omega)\), \(C\) such that

\[
\|e^{\xi x}(H_\mu - w)^{-1}e^{-\xi x}\|_{p,q} \leq C
\]

whenever \(1 \leq p \leq q \leq 2, \frac{d}{2}(\frac{1}{p} - \frac{1}{q}) \leq \frac{1}{2}, |\xi| \leq 1\).

Let \(K \subset \rho(H_{2,\mu})\) be compact, \(\bar{K}\) connected, \(K = \bar{K}\), \(w \in K\). Then there exist \(\epsilon \in (0, 1]\) and a constant \(C'\) such that \(K \subset \rho(e^{\xi x}H_{2,\mu}e^{-\xi x})\) for \(|\xi| \leq \epsilon\), and

\[
\|e^{\xi x}(H_{2,\mu} - z)^{-1}e^{-\xi x}\| = \|(e^{\xi x}H_{2,\mu}e^{-\xi x} - z)^{-1}\| \leq C' \quad (|\xi| \leq \epsilon, z \in K).
\]
This follows from perturbation theory and analytic continuation. (Note that the equality
\[ e^{\xi z}(H_{2,\mu} - z)^{-1}e^{-\xi z} = (e^{\xi z}H_{2,\mu}e^{-\xi z} - z)^{-1} \]
on \(L_2(Y) \cap L_{2,c}(\mathbb{R}^d)\), whose validity for \(z = w\) is obtained by Laplace transform, has to be extended to \(K\) by analytic continuation. The absence of this argument in [HV] was pointed out to the authors by W. Arendt.)

Using the resolvent equation
\[ (H_{2,\mu} - z)^{-1} = (I + (z - w)(H_{2,\mu} - z)^{-1})(H_{2,\mu} - w)^{-1} \]
together with Lemma 1.3 one concludes the existence of \(C^{''}\) such that
\[ k e^{\xi z}(H_{2,\mu} - z)^{-1}e^{-\xi z}k_{p,q} \leq C^{''} \]
for \(z \in K, 1 \leq p \leq 2\) with \(\frac{q}{2}(\frac{1}{p} - \frac{1}{q}) \leq \frac{1}{2}, |\xi| \leq \frac{\xi}{2} \).

Iterating this argument one obtains the last inequality for all \(p \in [1,2]\) and small \(|\xi|\). Using this estimate for \(\xi = 0\) and the fact that
\[ (H_{2,\mu} - w)^{-1} = (H_{p,\mu} - w)^{-1} \]
on \(L_p \cap L_2(Y)\) one obtains \(K \subset \rho(H_{p,\mu})\).

### 1.4. Remarks.
(a) A slightly different situation has been treated in [ScV]. In this paper the perturbation \(\mu\) is the sum of a form small distributional part \(\mu_0\) (cf. [HS]) and \(\mu_+ \in M_0\). This implies that the semigroup \((e^{-tH_{\mu}}; t \geq 0)\) acts as a strongly continuous semigroup on \(L_p(Y)\) for \(p_0 \leq p \leq p_0'\) where \(p_0 \in [1,2]\) depends on the form bound of \(\mu_0\) (cf. [BS]). It is then shown that \(\sigma(H_{p,\mu}) = \sigma(H_{2,\mu})\) for all \(p \in (p_0, p_0')\).

(b) The \(p\)-independence of the \(L_p\)-spectrum of elliptic operators on certain Riemannian manifolds was shown in [Stu]. In a similar context the \(p\)-independence for \(1 < p < \infty\) was shown in [Sh; Proposition 2.6].

(c) The \(p\)-independence of spectra has been shown in [Al] for perturbations of certain translation invariant operators.

(d) If \(U(\cdot)\) is a strongly continuous semigroup on \(L_2(\Omega)\) (where \(\Omega \subset \mathbb{R}^d\)) satisfying a Gaussian estimate, then it was shown in [Ar] that the spectra of the generators of the corresponding semigroups on \(L_p(\Omega)\) are \(p\)-independent.

### 2. The spectrum of \(-\frac{1}{2}\Delta + \mu\) in spaces of continuous functions
We want to show that under suitable hypotheses the spectrum of \(-\frac{1}{2}\Delta + \mu\) in
\[ C_0(\mathbb{R}^d) = \{ f \in C(\mathbb{R}^d); f(x) \to 0 (|x| \to \infty) \} \]
(or in other spaces of bounded continuous functions) is the same as the \(L_p\)-spectrum.
It turns out that the main point which is specific about this situation is the question whether \((e^{-tH}; \ t \geq 0)\) acts as a strongly continuous semigroup on \(C_0(\mathbb{R}^d)\). The fact that then coincidence of spectra can be concluded will follow from very general considerations presented next.

Let \(X\) be a Banach space, \((U(t); t \geq 0)\) a strongly continuous semigroup on \(X\), and \(T\) its generator. The semigroup dual of \(X\) is then defined by

\[
X^\odot := \{ x^* \in X^* ; \ t(t)x^* \to x^* (t \to 0) \};
\]

see, e.g., [HP; Chap. XIV], [BB; Sec. 1.4] (where \(X^\odot\) is denoted by \(X_0^\ast\)), [Ne]. (We use the adjoint space \(X^*\) of continuous conjugate linear functionals on \(X\) in order to stay consistent with duality in \(L_2\).)

2.1. Theorem. Let \(Y \subset X^\odot\) be a closed subspace which is invariant under \(U^*(t)\) \((t \geq 0)\). Denote by \(U_Y(\cdot)\) the part of the semigroup \(U^*(\cdot)\) in \(Y\), and by \(T_Y\) the generator of \(U_Y(\cdot)\).

(a) Then \(T_Y\) is the part of \(T^*\) in \(Y\),

\[
D(T_Y) = \{ x^* \in Y \cap D(T^*); \ T^*x^* \in Y \},
\]

\[
T_Y = T^*|D(T_Y).
\]

(b) \(\rho_\infty(T) \subset \rho_\infty(T_Y)\), and \((\lambda - T_Y)^{-1}\) is the part of \(((\lambda - T)^{-1})^*\) in \(Y\), for \(\lambda \in \rho_\infty(T)\). (Here \(\rho_\infty(T)\) denotes the component of \(\rho(T)\) containing a right half plane; and similarly for \(T_Y\).)

(c) If additionally \(Y\) is equi-norming for \(X\), i.e., the norm

\[
\|x\|_Y := \sup \{ | < x^*, x > ; \ x^* \in Y, \ \|x^*\| \leq 1 \} \quad (x \in X)
\]

is equivalent to the original norm in \(X\), then

\[
\rho_\infty(T) = \rho_\infty(T_Y).
\]

Proof. (a) This is known for \(Y = X^\odot\), and the proof carries over to our case (cf. [BB: p. 51], [Ne; Theorem 1.3.3]).

(b) For \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda\) larger than the type of \(U(\cdot)\), the resolvents of \(T\) and \(T_Y\) are given by the Laplace transform of \(U(\cdot)\) and \(U_Y(\cdot)\), respectively, and therefore

\[
< x^*, (\lambda - T)^{-1}x > = < (\lambda - T_Y)^{-1}x^*, x >
\]

for all \(x \in X\), \(x^* \in Y\). Therefore \((\lambda - T_Y)^{-1}\) is the part of \(((\lambda - T)^{-1})^*\) in \(Y\). This implies that \(((\lambda - T)^{-1})^*\) maps \(Y\) to \(Y\) for all \(\lambda \in \rho_\infty(T)\). By uniqueness we obtain the claimed assertions.

(c) The equivalence of \(\| \cdot \|\) and \(\| \cdot \|_Y\) implies that there exists a constant \(c\) such that
\[ \|(\lambda - T)^{-1}\| \leq c \|(\lambda - T_Y)^{-1}\| \quad \text{for all} \quad \lambda \in \rho_\infty(T). \]

This implies \( \partial(\rho_\infty(T)) \subset \sigma(T_Y) \), and therefore \( \rho_\infty(T) = \rho_\infty(T_Y) \).

### 2.2. Remark

The assumptions made in the previous theorem are satisfied, in particular, for \( Y = X^\odot \). For this case, however, one has \( \rho(T^\odot) = \rho(T) \); cf. [Ne; Theorem 1.4.2].

### 2.3. Corollary

Assume that \( \mu \) satisfies the hypotheses of Theorem 1.1. Let \( Y \) be a closed subspace of \( L_\infty \) which is equi-norming for \( L_1 \), invariant under \( (e^{-tH_1,\mu})^* \) (\( t \geq 0 \)) and such that

\[ \|(e^{-tH_1,\mu})^* f - f\|_\infty \to 0 \quad (t \to 0) \]

for all \( f \in Y \). Denote by \( -H_{Y,\mu} \) the generator of the strongly continuous semigroup on \( Y \) induced by \( ((e^{-tH_1,\mu})^*; t \geq 0) \). Then

\[ \sigma(H_{Y,\mu}) = \sigma(H_{2,\mu}). \]

### 2.4. Remarks

(a) The semigroup dual of \( L_1(\mathbb{R}^d) \) for the unperturbed Schrödinger semigroup \( (e^{-tH_1}; t \geq 0) \) is

\[ C_{b,u}(\mathbb{R}^d) = \{ f \in C(\mathbb{R}^d); \text{\textit{f} bounded and uniformly continuous}\}. \]

The generator is then the part of \( -H_\infty \) in \( C_{b,u} \),

\[ D(H_{C_{b,u}}) = \{ f \in C_{b,u}(\mathbb{R}^d); H_{C_{b,u}} f = -\frac{1}{2} \Delta f \in C_{b,u} \}. \]

For \( V \in C_{b,u}(\mathbb{R}^d) \), the multiplication operator by \( V \) is a bounded operator in \( C_{b,u}(\mathbb{R}^d) \), and therefore Theorem 2.3 is applicable to \( H + V \) with \( Y = C_{b,u}(\mathbb{R}^d) \).

(b) The space \( C_0(\mathbb{R}^d) \) is invariant under the unperturbed Schrödinger semigroup, and

\[ D(H_{C_0}) = \{ f \in C_0(\mathbb{R}^d); H_{C_0} f = -\frac{1}{2} \Delta f \in C_0 \}. \]

For bounded \( V \in C(\mathbb{R}^d) \) the multiplication by \( V \) is a bounded operator on \( C_0(\mathbb{R}^d) \). Therefore Theorem 2.3 is applicable to \( H + V \) with \( Y = C_0(\mathbb{R}^d) \).

(c) For \( V = V_+ - V_-; V_+ \geq 0, V_- \in K_d, V_+ \in K_{d,loc} \) it is shown in [S; Theorem B.3.1] that \( e^{-tH_V} \) maps \( L_\infty \)-functions to continuous functions, for \( t > 0 \). As a consequence,

\[ Y := L_1(\mathbb{R}^d)^\odot \]

consists of continuous functions, in this case.
3. An application of Shnol’s method.

In order to establish a connection with the PDE-world, we will now discuss an alternative proof of the inclusions

\[ \sigma(H_{p,V}) \supset \sigma(H_{2,V}), \quad \sigma(H_{C_0,V}) \supset \sigma(H_{2,V}). \]  

(3.1)

To this end, we will produce rather explicit “Weyl sequences” in \( L_p \) and also in \( C_0 \) which are obtained by applying suitably chosen cut-offs to generalized eigenfunctions associated with the expansion theorem for \( H_{2,V} \) ([B], [S], [PStW]); this requires some mild modifications of Shnol’s method (cf. [Shn], [S; Section C.4], and [HSt]). Therefore, we learn that properties of the Schrödinger operator in Hilbert space \( L_2 \) fully determine the spectra in \( L_p \) and even in \( C_0 \); while estimates for the resolvent kernel \( (H_{2,V} - z)^{-1}(x,y) \) give the inclusion \( \varrho(H_{p,V}) \supset \varrho(H_{2,V}) \), the converse inclusion will now be a consequence of the eigenfunction expansion theorem for \( H_{2,V} \). Related ideas are also discussed in [Sh].

It should be stressed, however, that the approach proposed here requires more restrictive assumptions on the potential \( V \), as compared with the “duality and interpolation”-proof described in Section 2. In the following, we will restrict the discussion to the case \( V \in L_\infty(\mathbb{R}^d) \) where it is easy to obtain \( L_p \)-bounds for the gradient of a generalized eigenfunction.

We first collect a few facts (where we always assume that \( V \) is bounded):

1. For \( 1 \leq p \leq \infty \), we have ([HV1])

\[ D(H_{p,V}) = D(H_p) = \{ u \in L_p; \Delta u \in L_p \}. \]  

(3.2)

If, more strongly, \( V \) is bounded and continuous, then (cf. Section 2)

\[ D(H_{C_0,V}) = D(H_{C_0}) = \{ u \in C_0; \Delta u \in C_0 \}. \]  

(3.3)

2. From the generalized eigenfunction expansion theorem for \( H_{2,V} \) ([B], [S], [PStW]), we can draw the following conclusion: for any \( \mu \in \sigma(H_{2,V}) \) and any \( \varepsilon > 0 \), there exists a \( \lambda \in (\mu - \varepsilon, \mu + \varepsilon) \) and a (non-trivial) distributional solution \( u \) of the PDE

\[ -\frac{1}{2}\Delta u + Vu = \lambda u, \]  

(3.4)

satisfying a polynomial growth bound

\[ |u(x)| \leq c_1(1 + |x|)^K, \]  

(3.5)

with some constants \( c_1 > 0 \) and \( K \in \mathbb{N} \). For \( V \) bounded, it is also known that \( u \) is (equivalent to) a continuous function (cf., e.g., [S]).

3. To control the cut-off errors, we need an \( L_p \)-bound on \( \nabla u \), for \( u \) satisfying (3.4), (3.5). Note that there is no \( L_p \)-analogue of the \( L_2 \)-gradient bound given in
Here we proceed as in [HV1], using an argument of L. Schwartz, to obtain the following lemma.

3.1. Lemma. Let \( p \in [1, \infty] \), and suppose that \( \Omega \subseteq \Omega' \) are open sets in \( \mathbb{R}^d \) with the property that \( \text{dist}(\Omega, \partial\Omega') \geq 1 \). Then there exists a constant \( C = C(p) \), which is independent of both \( \Omega \) and \( \Omega' \), such that

\[
\| \nabla u \|_{L_p(\Omega)} \leq C \left( \| u \|_{L_p(\Omega')} + \| \Delta u \|_{L_p(\Omega')} \right),
\]

for all \( u \in L_p(\Omega) \) with the property that \( \Delta u \in L_p(\Omega) \).

Proof. We proceed as in [HV1]: letting \( T \) denote the usual fundamental solution for \( -\Delta \), and picking some \( \phi \in C^\infty_c(\mathbb{R}^d) \) with support in the unit ball and \( \phi(x) = 1 \) for \( |x| \leq 1/2 \), we have

\[
\nabla u = (\nabla(\phi T)) \ast \Delta u - \nabla \phi \ast u,
\]

(3.7)

(\text{where } \zeta = (\Delta \phi)T + 2\nabla \phi \cdot \nabla T \in C^\infty_c(\mathbb{R}^d))\), and the required estimate follows from Young’s inequality ([RS]). Furthermore, it is clear from eq. (3.7) that \( \nabla u \) is continuous, provided \( u \) and \( \Delta u \) are continuous functions.

Now let \( u \) be a (continuous) generalized eigenfunction of \( H_2 \) and \( \varphi \in C^\infty_c(\mathbb{R}^d) \). Then it follows from Lemma 3.1 and \( \Delta(\varphi u) = \varphi \Delta u + 2\nabla \varphi \nabla u + (\Delta \varphi)u \) that \( \varphi u \) will belong to the domain of \( H_p \), for \( 1 \leq p \leq \infty \). Similarly, if \( V \) is bounded and continuous, then \( \varphi u \) will belong to the domain of \( H_{C_0;V} \).

(4) Central to Shnol’s method is the observation that the growth bound (3.5) implies that the \( L_2 \)-norm of \( u \), considered on a suitable sequence of balls, will not grow too rapidly (cf. [S]). While the exposition given in [S; Section C.4] can directly be carried over to the \( L_p \)-case for \( 1 \leq p < \infty \), it has to be modified for \( p = 1 \) and, similarly, also for the space \( C_0 \). We therefore change the scenario used in [S] and consider

\[
\mathcal{E}_n = \{ x \in \mathbb{R}^d; \ |x| < 2^n \}, \quad \mathcal{F}_n = \mathcal{E}_{n+1} \setminus \mathcal{E}_n \quad (n \in \mathbb{N}).
\]

(3.8)

We then have the following lemma.

3.2. Lemma. Let \( 1 \leq p \leq \infty \), and let \( u \) be as in (3.5). Let \( a > 2 \) and set \( c_2 = c_2(p) = a^{N+\frac{2}{p}} \). Then there exists a sequence \( (n_j)_{j \in \mathbb{N}} \subseteq \mathbb{N}, \ n_j \to \infty \), such that

\[
\left\| u_{|\mathcal{F}_{n_j}} \right\|_p \leq c_2 \left\| u_{|\mathcal{E}_{n_j}} \right\|_p \quad (j \in \mathbb{N}).
\]

(3.9)

Proof. If the statement of the lemma were not true, there would exist some \( n_0 \) such that

\[
\left\| u_{|\mathcal{F}_{n}} \right\|_p \geq c_2 \left\| u_{|\mathcal{E}_{n}} \right\|_p > 0 \quad (n \geq n_0),
\]

(3.10)
so that
\[ \|u|_{V_n}\|_p \geq \|u|_{F_n-1}\|_p \geq c_2 \|u|_{V_{n-1}}\|_p \quad (n > n_0). \] (3.11)

This leads to
\[ \|u|_{V_n}\|_p \geq c_2^{n-n_0} \|u|_{V_{n_0}}\|_p \quad (n > n_0), \] (3.12)
in contradiction with the polynomial growth bound of \(u\). \]

With these preparations, it is now easy to prove the inclusions stated in eq. (3.1).

**Proposition 3.3.** Let \(V \in L^\infty(R^d)\). Then \(\sigma(H_{p,V}) \supset \sigma(H_{2,V})\), for all \(p \in [1, \infty]\). If, moreover, \(V\) is (bounded and) continuous, then \(\sigma(H_{C_0,V}) \supset \sigma(H_{2,V})\).

**Proof.** We first choose a function \(\varphi \in C_c^\infty(-2, 2)\) with the property that \(\varphi(x) = 1\), for \(|x| \leq 4/3\), and \(\varphi(x) = 0\), for \(|x| \geq 5/3\), and we define
\[ \varphi_n(x) = \varphi(2^{-n}|x|), \quad x \in R^d. \]

Then \(G_n := \text{supp}(\nabla \varphi_n) \subset F_n\) and \(\text{dist}(G_n, \partial F_n) \geq 1\), for \(n \geq 2\). Furthermore, we have \(|\nabla \varphi_n|_\infty \leq c_1 2^{-n}\) and \(|\Delta \varphi_n|_\infty \leq c_1 2^{-2n}\).

Now let \(\mu \in \sigma(H_{2,V})\) be given, and let \(\varepsilon > 0\). By what was said in point (2), there exists some \(\lambda \in (\mu - \varepsilon, \mu + \varepsilon)\) and a (non-trivial) generalized eigenfunction \(u\) of \(H_{2,V}\) that satisfies (3.4), (3.5). For given \(p \in [1, \infty]\), we will prove that there exists a sequence \((n_j) \subset N\) so that
\[ \|\!(H_{p,V} - \lambda)(\varphi_{n_j} u)\!\|_p / \|\varphi_{n_j} u\|_p \to 0, \quad j \to \infty. \] (3.13)

Therefore, \(H_{p,V} - \lambda\) does not have a bounded inverse, whence \(\lambda \in \sigma(H_{p,V})\). Taking \(\varepsilon \to 0\) then gives \(\mu \in \sigma(H_{p,V})\).

Applying Lemma 3.2 to \(u\), we find a constant \(c_2\) and a sequence \((n_j)\) such that (3.9) holds. As \(\varphi_{n_j} u \in D(H_{p,V})\) and \((H_{p,V} - \lambda)(\varphi_{n_j} u) = -\langle \Delta \varphi_n, \Delta u \rangle\), we have
\[
\|\!(H_{p,V} - \lambda)(\varphi_{n_j} u)\!\|_p \leq \|\nabla \varphi_{n_j}\|_\infty \|\nabla u|_{G_n}\|_p + \|\Delta \varphi_{n_j}\|_\infty \|u|_{G_n}\|_p \\
\leq c_3 2^{-n_j} \left(\|u|_{F_n}\|_p + \|\Delta u|_{F_n}\|_p\right),
\]
by Lemma 3.1. From \(V \in L^\infty\) and \(\frac{1}{2}\Delta u = (V - \lambda)u\) we now conclude that
\[
\|\!(H_{p,V} - \lambda)(\varphi_{n_j} u)\!\|_p \leq c_6 2^{-n_j} \|u|_{F_n}\|_p \leq c_7 2^{-n_j} \|u|_{\mathcal{F}_n}\|_p \leq c_8 2^{-n_j} \|\varphi_{n_j} u\|_p,
\]
and the result follows.

The proof in the case of the space \(C_0\) is essentially identical with the \(p = \infty\) proof and omitted. \]

**Acknowledgements.** R. Hempel would like to thank T. Hoffmann-Ostenhof for the kind invitation to the Erwin Schrödinger Institute at Vienna.
References.


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