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Orthomodular Lattices and a Quantum Algebra

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Abstract. We show that one can formulate an algebra with lattice ordering so as to contain one quantum and five classical operations as opposed to the standard formulation of the Hilbert space subspace algebra. The standard orthomodular lattice is embeddable into the algebra. To obtain this result we devised algorithms and computer programs for obtaining expressions of all quantum and classical operations within an orthomodular lattice in terms of each other, many of which are presented in the paper. For quantum disjunction and conjunction we prove their associativity in an orthomodular lattice for any triple in which one of the elements commutes with the other two and their distributivity for any triple in which a particular one of the elements commutes with the other two. We also prove that the distributivity of symmetric identity holds in Hilbert space, although it remains an open problem whether it holds in all orthomodular lattices, as it does not fail in any of over 50 million Greechie diagrams we tested.

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1 Introduction

Closed subspaces of Hilbert space form an algebra called a Hilbert lattice. A Hilbert lattice is a kind of orthomodular lattice which we, in the next section, introduce starting with an ortholattice which is a still simpler structure. In any Hilbert lattice the operation *meet*, $a \cap b$, corresponds to set intersection, $\mathcal{H}_a \bigcap \mathcal{H}_b$, of subspaces $\mathcal{H}_a, \mathcal{H}_b$ of Hilbert space $\mathcal{H}$, the ordering relation $a \le b$ corresponds to $\mathcal{H}_a \subseteq \mathcal{H}_b$, the operation *join*, $a \cup b$, corresponds to the smallest closed subspace of $\mathcal{H}$ containing $\mathcal{H}_a \bigcup \mathcal{H}_b$, and a' corresponds to $\mathcal{H}'_a$, the set of

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vectors orthogonal to all vectors in $\mathcal{H}_a$. Within Hilbert space there is also an operation which has no a parallel in the Hilbert lattice: the sum of two subspaces $\mathcal{H}_a + \mathcal{H}_b$ which is defined as the set of sums of vectors from $\mathcal{H}_a$ and $\mathcal{H}_b$. We also have $\mathcal{H}_a + \mathcal{H}'_a = \mathcal{H}$. One can define all the lattice operations on Hilbert space itself following the above definitions ($\mathcal{H}_a \cap \mathcal{H}_b = \mathcal{H}_a \bigcap \mathcal{H}_b$, etc.). Thus we have $\mathcal{H}_a \cup \mathcal{H}_b = \overline{\mathcal{H}_a + \mathcal{H}_b} = (\mathcal{H}_a + \mathcal{H}_b)^{\perp\perp} = (\mathcal{H}'_a \bigcap \mathcal{H}'_b)'$,[8, p. 175] where $\overline{\mathcal{H}_c}$ is a closure of $\mathcal{H}_c$, and therefore $\mathcal{H}_a + \mathcal{H}_b \subseteq \mathcal{H}_a \cup \mathcal{H}_b$. When $\mathcal{H}$ is finite dimensional or when the closed subspaces $\mathcal{H}_a$ and $\mathcal{H}_b$ are orthogonal to each other then $\mathcal{H}_a + \mathcal{H}_b = \mathcal{H}_a \cup \mathcal{H}_b$. [6, pp. 21-29], [10, pp. 66,67], [13, pp. 8-16]

In the past, scientists, starting with Birkhoff and von Neumann, wanted to find parallels with a possible logic lying underneath the orthomodular lattice and operations defined on such a logic. A possible candidate for the logic was formulated [13, 9, 10, 5, 2]. However, it has recently been shown [21] that the logic can have at least two models: Hilbert space and another model which is not orthomodular—so there is no *proper* quantum logic.[3] One can still consider operations within the model itself: the orthomodular lattice. The problem of finding quantum operations which would reduce to classical ones for compatible observables has been attacked many times in the past. In particular, it has been shown that one can start with unique *classical* conjunction, disjunction, and implication and by means of them define five *quantum* conjunctions, disjunctions, and implications [which collapse into former classical ones for commuting (compatible, commensurable) observables]. In this paper we show that one can start with unique quantum operations and arrive at five classical ones. Thus it turns out that the usual way of defining orthomodular lattice by means of unique classical conjunction and disjunction is a consequence of a direct translation of meet and join from Hilbert space. We also express all possible quantum and classical operations by each other, even a chosen either classical or quantum one by means of all other quantum and classical ones in single equations. We do so with the help of a computer program which reduces two-variable expressions to each other.

In Sec. 5 we prove that in an orthomodular lattice the associativity of both quantum disjunctions and conjunctions holds for any triple of lattice elements as soon as one of them commutes with the other two.

In the the end, we partially solve an open problem from Ref. [12] by proving that the "distributive law," for a quantum identity holds in the Godowski lattices and therefore in Hilbert space. It remains an open problem whether the law holds in all orthomodular lattices.

2 Quantum and classical lattice operations

One usually defines an ortholattice in the following way.

Definition 2.1. *An ortholattice* (OL) *is an algebra* $\langle \mathcal{L}_{\rm O}, ', \cup \rangle$ *such that the following condi-*

[3]Consequently, the papers that are now appearing and claim as, e.g., Dalla Chiara and Giuntini [3], that quantum logic, defined as a genuine logical system, characterizes orthomodular lattices are simply incorrect. For, obviously, one cannot claim that a statement holds in *proper* quantum logic if and only if it is true in orthomodular lattices and at the same time that a statement holds in *proper* quantum logic if and only if it is true in non-orthomodular lattices. The same holds for *proper* classical logic and the Boolean algebra. All previous such papers and books on quantum as well as classical logic are outdated by the result.

tions are satisfied for any $a, b, c \in \mathcal{L}_{\rm O}$*:*

$$\text{L1} \qquad a = a'' \tag{2.1}$$

$$\text{L2} \qquad a \le a \cup b \quad \& \quad b \le a \cup b \quad \& \quad b \le a \cup a' \tag{2.2}$$

$$\text{L3} \qquad a \le b \quad \& \quad b \le a \quad \Rightarrow \quad a = b \qquad ; \qquad a = b \quad \Rightarrow \quad a \le b \tag{2.3}$$

$$\text{L4} \qquad a \le b \quad \Rightarrow \quad b' \le a' \tag{2.4}$$

$$\text{L5} \qquad a \le b \quad \& \quad b \le c \quad \Rightarrow \quad a \le c \tag{2.5}$$

$$\text{L6} \qquad a \le c \quad \& \quad b \le c \quad \Rightarrow \quad a \cup b \le c \tag{2.6}$$

where

$$a \le b \stackrel{\rm def}{=} a \cup b = b, \qquad 1 \stackrel{\rm def}{=} a \cup a', \qquad 0 \stackrel{\rm def}{=} a \cap a'. \tag{2.7}$$

Then we can define six operations of implication:

Definition 2.2. $a \to_0 b \ =^{\rm def} \ a' \cup b$, $a \to_1 b \ =^{\rm def} \ a' \cup (a \cap b)$, $a \to_2 b \ =^{\rm def} \ b' \to_1 a'$, $a \to_3 b \ =^{\rm def} \ ((a' \cap b) \cup (a' \cap b')) \cup (a \cap (a' \cup b))$, $a \to_4 b \ =^{\rm def} \ a' \to_3 b'$, $a \to_5 b \ =^{\rm def} \ ((a \cap b) \cup (a' \cap b)) \cup (a' \cap b')$, *where* $\to_0$ *is called* classical implication *and* $\to_i$, $i = 1, \ldots, 5$ quantum implication*.*

Quantum implications reduce to the classical one whenever a and b commute.

Definition 2.3. *We say that* a *and* b commute *and write* aCb *when any and therefore all of the following equations hold:* [13, 22, 7] $(a \cap b) \cup (a \cap b') \cup (a' \cap b) \cup (a' \cap b') = 1$, $a \cap (a' \cup b) \le b$, $a = (a \cap b) \cup (a \cap b')$.

We can also define:

Definition 2.4.

$$a \cup_i b \stackrel{\rm def}{=} a' \to_i b, \qquad a \cap_i b \stackrel{\rm def}{=} (a \to_i b')'; \qquad i = 0, \ldots, 5 \tag{2.8}$$

$$a \equiv_i b \stackrel{\rm def}{=} (a \to_i b) \cap (b \to_0 a), \qquad i = 0, \ldots, 5, \tag{2.9}$$

where $a \cup_0 b = a \cup b$, $a \cap_0 b = a \cap b$ and $a \equiv_0 b$ are classical disjunction, conjunction, and identity, respectively, while $a \cup_i b$, $a \cap_i b$, and $a \equiv_i b$, $i = 1, \ldots, 5$ are quantum ones, respectively. The latter obviously reduce to the former when a and b commute.

For the above operations the following theorems hold. In them, we can also pick any one of the conditions in Theorem 2.5 as our definition of an orthomodular lattice and in Theorem 2.6 as our definition of a distributive lattice (Boolean algebra).

Theorem 2.5. *An ortholattice in which any one of the following conditions holds is an orthomodular lattice and vice versa.* [15, 16, 17, 18, 20]

$$a \to_i b = 1 \quad \Leftrightarrow \quad a \le b, \qquad i = 1, \ldots, 5, \tag{2.10}$$

$$a \cup_i b = 1 \quad \Leftrightarrow \quad a' \perp b', \qquad i = 1, \ldots, 5, \tag{2.11}$$

$$a \cap_i b = 0 \quad \Leftrightarrow \quad a \perp b, \qquad i = 1, \ldots, 5, \tag{2.12}$$

$$a \equiv_i b = 1 \quad \Leftrightarrow \quad a = b, \qquad i = 1, \ldots, 5, \tag{2.13}$$

$$a \perp b \quad \& \quad a \cup b = 1 \quad \Rightarrow \quad a' \perp b', \tag{2.14}$$

where $a \perp b \stackrel{\rm def}{=} a \le b'$

Theorem 2.6. *An ortholattice in which any one of the following conditions holds is a distributive lattice and vice versa.* [15, 16, 17, 18, 20]

$$a \to_0 b = 1 \qquad \Leftrightarrow \qquad a \le b\,, \tag{2.15}$$

$$a \cup b = a \cup_0 b = 1 \qquad \Leftrightarrow \qquad a' \perp b'\,, \tag{2.16}$$

$$a \cap b = a \cap_0 b = 0 \qquad \Leftrightarrow \qquad a \perp b\,, \tag{2.17}$$

$$a \equiv_0 b = 1 \qquad \Leftrightarrow \qquad a = b. \tag{2.18}$$

Actually, in any orthomodular lattice all expressions with 2 variables are reducible to one of 96 Beran canonical forms.[1, Table 1, p. 82] The reader can easily reduce any 2 variable expression with the help of our program `beran` which we describe in Sec. 8. All 96 forms can be also viewed inside the source code of `beran.c`. In general we can divide them to 16 *classical* and 80 *quantum* ones. Classical expressions are: classical implication and its negation (disjunction and conjunction)—Beran expressions 2–5 and 92–95, classical identity and its negation—expressions 9 and 88, variables a, b and their negations—expressions 22, 39, 58, and 75, and "0" and "1"—expressions 1 and 96, respectively. Quantum expressions are all the other expressions, that reduce to classical ones whenever the variables commute: quantum implications and their negations (quantum disjunctions and conjunctions)—12–15, 18–21, 28–31, 34–37, 50–53, 60–63, 66–69, 76–79, and 82–85, quantum identities ($a \equiv_1 b = a' \equiv_3 b'$, $a \equiv_2 b = a' \equiv_4 b'$, $a \equiv_5 b$) and their negations—24, 25, 40, 41, 56, 57, 72, 73, 8, and 89, "quantum variables" (which reduce to "classical a, b") and their negations—a: 6, 38, 54, 70, 86, b: 7, 23, 55, 71, 87, a': 11, 27, 43, 59, 91, and b': 10, 26, 42, 74, 90, and "quantum 0,1"—"0": 17, 33, 49, 65, 81 and "1": 16, 32, 48, 64, 80. For some of these quantum expressions we give the following definitions and theorems.

Definition 2.7. Quantum unities *and* zeros *in an* OML *are:*

$$1_{1(a,b)} \stackrel{\rm def}{=} a' \cup (a \cap b') \cup (a \cap b) \tag{2.19}$$

$$1_{2(a,b)} \stackrel{\rm def}{=} b \cup (a \cap b') \cup (a' \cap b') \tag{2.20}$$

$$1_{3(a,b)} \stackrel{\rm def}{=} a \cup (a' \cap b) \cup (a' \cap b') \tag{2.21}$$

$$1_{4(a,b)} \stackrel{\rm def}{=} b' \cup (a' \cap b) \cup (a \cap b) \tag{2.22}$$

$$1_{5(a,b)} \stackrel{\rm def}{=} (a \cap b) \cup (a \cap b') \cup (a' \cap b) \cup (a' \cap b') \tag{2.23}$$

$$0_{1(a,b)} \stackrel{\rm def}{=} a \cap (a' \cup b) \cap (a' \cup b') \tag{2.24}$$

$$0_{2(a,b)} \stackrel{\rm def}{=} b' \cap (a' \cup b) \cap (a \cup b) \tag{2.25}$$

$$0_{3(a,b)} \stackrel{\rm def}{=} a' \cap (a \cup b') \cap (a \cup b) \tag{2.26}$$

$$0_{4(a,b)} \stackrel{\rm def}{=} b \cap (a \cup b') \cap (a' \cup b') \tag{2.27}$$

$$0_{5(a,b)} \stackrel{\rm def}{=} (a \cup b) \cap (a \cup b') \cap (a' \cup b) \cap (a' \cup b') \tag{2.28}$$

Some consequences of these definitions are straightforward:

Lemma 2.8. *Two variables commute iff any of the 80 two-variable quantum expressions is equal to its classical counterpart. Two variables also commute iff any two of the five different forms of each quantum expression are equal to each other.*

For example, $1_{i(a,b)} = 1$ or $0_{i(a,b)} = 0$, $i = 1, \ldots, 5$ is equivalent to aCb. In particular, $1_{5(a,b)} = 1$ is the first expression from Def. 2.3. Also, e.g., $a_{i(a,b)} = a$, $i = 1, \ldots, 5$, where $a_{i(a,b)}$ are given by Beran expressions: 6, 38, 54, 70, 86, respectively, are equivalent to aCb. In particular, $a_{1(a,b)} = (a \cap b) \cup (a \cap b') = a$ is the third expression from Def. 2.3. Examples for the second claim of the theorem are that $a \to_i b = a \to_j b$, $a \cup_i b = a \cup_j b$, and $a \cap_i b = a \cap_j b$, $i \neq j$, $i, j = 1, \ldots, 5$ are equivalent to aCb. [17, p. 1487] The same holds for $1_{i(a,b)} = 1_{j(a,b)}$, $a_{i(a,b)} = a_{j(a,b)}$, $a \equiv_i b = a \equiv_j b$, $i \neq j$, $i, j = 1, \ldots, 5$, etc.

Theorem 2.9. *An ortholattice in which any one of the following conditions holds is an orthomodular lattice and vice versa.*

$$a \to_i b = 1_{j(a,b)} \quad \Leftrightarrow \quad a \le b, \qquad i, j = 1, \ldots, 5;\ i \neq j; \tag{2.29}$$

$$a \equiv_i b = 1_{j(a,b)} \quad \Leftrightarrow \quad a = b, \qquad i, j = 1, \ldots, 5;\ i \neq j. \tag{2.30}$$

Proof. We will exemplify the proofs by proving the case $i = 1, j = 5$. Other cases the reader can prove analogously. We first use Eq. (2.13) to write the premise as $(a \to_1 b) \equiv_5 1_5 = 1$ ($\equiv_5$ should be used for all cases—in it the subscript 5 is not j) and then we find the canonical expression of

$$(a \to_1 b) \equiv_5 1_{5(a,b)} = (a \to_1 b) \equiv_5 (a \cap b) \cup (a \cap b') \cup (a' \cap b) \cup (a' \cap b')$$

by typing (see Sec. 8 for details on our program `beran`):

```
beran "((aIb)=(((a^b)v(a^-b))v((-a^b)v(-a^-b))))"
```

The program responds with:

```
30 ((-avb)^((av(-a^-b))v(-a^b)))
```

which is nothing but $a \to_3 b$. Using Eq. (2.10) we get the desired conclusion. □

3 Relations between operations

In this section we show how one can connect operations we defined in Sec. 2 with each other in an orthomodular lattice defined in a standard way given by Def. 2.1. In counting the cases for commuting operations below we disregard the order of a and b.

In Ref. [20] we have shown how one can express classical disjunction by quantum and classical implications within a single equation. (That equation was one of the four smallest ones. Below is another.)

Lemma 3.1. *(i) The equation*

$$a \cup b = ((((b \to_i a) \to_i (a \to_i b)) \to_i b) \to_i a) \to_i a \tag{3.1}$$

is true in all orthomodular lattices for $i = 1, \ldots, 5$ and in all distributive lattices for $i = 0, \ldots, 5$;

(ii) an ortholattice in which Eq. 3.1 holds is an orthomodular lattice for $i = 1, \ldots, 5$ and a distributive lattice for $i = 0$.

This equation does not contain negations and if we wanted to define an algebra by means of so merged implications and without using negation we should at least introduce 0. Alternatively one can use the negation and define 0. In this paper we adopt the latter approach. We do not give proofs of the lemmas in this section because all expressions can be trivially checked with the help of the computer program `beran` written by one of us (N. D. M.) which the reader can download from our web sites.

Lemma 3.2. *There is only one "smallest" (lowest number of occurrence of variables, 5, and negations, 2) expression of classical disjunction by means of quantum implications:*

$$a \cup b = ((a' \to_i b') \to_i b) \to_i a) \to_i a \ ; \qquad i = 1, \dots, 5, \tag{3.2}$$

and seven smallest (5 variables, 4 negations) expressions of classical conjunction by means of quantum implications, one of which is:

$$a \cap b = (a \to_i ((a \to_i b) \to_i (b' \to_i a')')' \ ; \qquad i = 1, \dots, 5. \tag{3.3}$$

There are two smallest (5 variables, 3 negations) expressions of classical disjunction by means of quantum disjunctions one of which is:

$$a \cup b = ((a \cup_i b') \cup_i (b' \cup_i a))' \cup_i a \ ; \qquad i = 1, \dots, 5, \tag{3.4}$$

and two (5,5) by means of quantum conjunctions one of which is:

$$a \cup b = (((a' \cap_i b) \cap_i (b \cap_i a'))' \cap_i a')' \ ; \qquad i = 1, \dots, 5. \tag{3.5}$$

An equal number of smallest expressions of classical conjunction by means of quantum disjunctions and conjunctions we get by using $a \cap b = (a' \cup b')'$ and $a \cap_i b = (a' \cup_i b')'$ (of course with reversed smallest number of negations).

Any of these equations when added to an ortholattice makes it orthomodular.

Lemma 3.3. *Here are samples of the smallest expressions (with their numbers being given in curly brackets) of classical conjunction and disjunction by means of both, classical ($i = 0$) and quantum ($i = 1, \dots, 5$) implications, disjunctions, and conjunctions in single equations in any orthomodular lattice:*

$$\begin{aligned}
a \cup b &= ((b \to_i a) \to_i (((a \to_i b') \to_i b') \to_i a)) && \{1\} && (3.6)\\
a \cup b &= (b \cup_i (a \cup_i ((a \cup_i b) \cup_i (b' \cup_i a))')) && \{16\} && (3.7)\\
a \cup b &= ((a' \cap_i b)' \cap_i (a' \cap_i (b \cap_i (b \cap_i a)')'))' && \{8\} && (3.8)\\
a \cap b &= (a \to_i ((a \to_i ((a \to_i b) \to_i b')) \to_i a')')' && \{23\} && (3.9)\\
a \cap b &= ((b' \cup_i (a \cup_i (a \cup_i b)')') \cup_i (b' \cup_i a)')' && \{8\} && (3.10)\\
a \cap b &= (b \cap_i (a \cap_i ((a \cap_i b) \cap_i (b' \cap_i a))')) && \{16\}\,, && (3.11)
\end{aligned}$$

where $i = 0, \dots, 5$. Any of these equations for $i = 1, \dots, 5$ and Eqs. (3.6), (3.8), (3.9), and (3.10) for $i = 0$ when added to an ortholattice makes it orthomodular (fails in O6*). For $i = 0$, there are no such smallest samples of the type given by Eqs. (3.7) and (3.11) and* *there are 18 samples that pass* O6 *of Eq. (3.9) type, 4 of (3.8) type and 4 of (3.10) type. Samples of the latter ones are:*

$$a \cap b = (a \to_i (b \to_i ((b \to_i a) \to_i (b' \to_i a')')))' \tag{3.12}$$
$$a \cup b = (b' \cap_i (a' \cap_i ((a' \cap_i b) \cap_i (b \cap_i a'))'))' \tag{3.13}$$
$$a \cap b = (b' \cup_i (a' \cup_i ((a' \cup_i b) \cup_i (b \cup_i a'))'))', \tag{3.14}$$

respectively.

Lemma 3.4. *The shortest expressions of some above defined operations by each other are:*

$$\begin{aligned} a \cup b &= a \cup_0 b = b \cup_1 (b \cup_1 a')' = a \cup_2 (b' \cup_2 a)' = b \cup_3 (b \cup_3 a) \\ &= a \cup_4 (b \cup_4 a) = b \cup_5 (b \cup_5 a')' \end{aligned} \tag{3.15}$$
$$a \cup_1 b = b \cup_2 a = (a \cup_3 b) \cup_3 b = b \cup_4 (b \cup_4 a) = a \cup_5 (b \cup_5 a) \tag{3.16}$$
$$a \cup_2 b = b \cup_1 a = (b \cup_3 a) \cup_3 a = a \cup_4 (a \cup_4 b) = b \cup_5 (b \cup_5 a) \tag{3.17}$$
$$\begin{aligned} a \cup_3 b &= b \cup_4 a = (a \cup_1 b')' \cup_1 (b \cup_1 a) = (a \cup_2 b) \cup_2 (b' \cup_2 a)' \\ &= (a \cup_5 b) \cup_5 (a \cup_5 (b \cup_5 a'))' \end{aligned} \tag{3.18}$$
$$\begin{aligned} a \cup_4 b &= b \cup_3 a = (b \cup_1 a')' \cup_1 (a \cup_1 b) = (b \cup_2 a) \cup_2 (a' \cup_2 b)' \\ &= (b \cup_5 a) \cup_5 (b \cup_5 (a \cup_5 b'))' \end{aligned} \tag{3.19}$$
$$\begin{aligned} a \cup_5 b &= b \cup_5 a = ((a \cup_1 b)' \cup_1 (a' \cup_1 (b \cup_1 a))')' = ((b \cup_2 a)' \cup_2 ((a \cup_2 b) \cup_2 a')')' \\ &= ((b \cup_3 a)' \cup_3 ((b \cup_3 a) \cup_3 a)')' = ((b \cup_4 a)' \cup_4 (b \cup_4 (b \cup_4 a))')' \end{aligned} \tag{3.20}$$
$$a \equiv_0 b = (a' \equiv_5 b)' = (b \cup_i a)' \cup_i (b' \cup_i a')'; \qquad i = 1, \ldots, 5 \tag{3.21}$$
$$\begin{aligned} a \equiv_1 b &= a' \equiv_3 b' = (a \cup_{1,3} b)' \cup_{1,3} (b' \cup_{1,3} a')' = (a' \cup_{2,4} b')' \cup_{2,4} (b \cup_{2,4} a)' \\ &= (a \cup_5 (a \cup_5 b))' \cup_5 (a' \cup_5 (b \cup_5 a')')' \end{aligned} \tag{3.22}$$
$$\begin{aligned} a \equiv_2 b &= a' \equiv_4 b' = (b' \cup_{1,3} a')' \cup_{1,3} (a \cup_{1,3} (b \cup_{1,3} a))' \\ &= ((a' \cup_2 b)' \cup_2 (a \cup_2 (b \cup_2 a)')')' = (a \cup_4 b)' \cup_4 (a' \cup_4 (b' \cup_4 a'))' \\ &= ((b \cup_5 (b \cup_5 a'))' \cup_5 (a \cup_5 (b \cup_5 a)')')' \end{aligned} \tag{3.23}$$

Dual expressions on both sides of equations we get by using $a \cap_i b = (a' \cup_i b')'$.

Lemma 3.5. *Samples of expressions of particular quantum disjunctions by means of all five of them together in single equations are*

$$a \cup_1 b = a \cup_i (b' \cup_i (b \cup_i a)')' \tag{3.24}$$
$$a \cup_2 b = b \cup_i (a' \cup_i (a \cup_i b)')' \tag{3.25}$$
$$a \cup_3 b = ((a' \cup_i (b \cup_i a))' \cup_i ((b \cup_i (a \cup_i b)) \cup_i (b' \cup_i a)')')' \tag{3.26}$$
$$a \cup_4 b = ((b' \cup_i (a \cup_i b))' \cup_i ((a \cup_i (b \cup_i a)) \cup_i (a' \cup_i b)')')' \tag{3.27}$$
$$a \cup_5 b = ((b \cup_i a)' \cup_i (b' \cup_i ((a \cup_i b) \cup_i (b \cup_i a')))')', \tag{3.28}$$

where $i = 1, \ldots, 5$. *Dual expressions* ($a \cup b$ *by means of* $a \cap_i b$ *and* $a \cap b$ *by means of* $a \cup_i b$ *and* $a \cap_i b$) *we get by using* $a \cap b = (a' \cup b')'$ *and* $a \cap_i b = (a' \cup_i b')'$.

4 Quantum algebra

In Lemma 3.2, Eq. (3.4) we have shown how one can express the classical disjunction by means of quantum ones in a single equation. So, we can substitute this expressions for the disjunctions in conditions which define an orthomodular lattice (Def. 2.1) and obtain five formally identical ways to writing those conditions by means of five quantum disjunctions. But we can do even more and define an algebra with a lattice ordering as follows.

Definition 4.1. *A quantum algebra* QA *is an algebra* $\langle \mathcal{A}_{\rm O}, ', \Cup \rangle$ *such that the following conditions are satisfied for any* $a, b, c \in \mathcal{A}_{\rm O}$*:*

$$\text{A1} \qquad a = a'' \qquad \& \qquad a \le 1 \tag{4.1}$$

$$\text{A2} \qquad a \le ((a \Cup b') \Cup (b' \Cup a))' \Cup a \qquad \& \qquad b \le ((a \Cup b') \Cup (b' \Cup a))' \Cup a \tag{4.2}$$

$$\text{A3} \qquad a \le b \quad \& \quad b \le a \quad \Rightarrow \quad a = b \qquad ; \qquad a = b \quad \Rightarrow \quad a \le b \tag{4.3}$$

$$\text{A4} \qquad a \le b \quad \Rightarrow \quad b' \le a' \tag{4.4}$$

$$\text{A5} \qquad a \le b \quad \& \quad b \le c \quad \Rightarrow \quad a \le c \tag{4.5}$$

$$\text{A6} \qquad a \le c \quad \& \quad b \le c \quad \Rightarrow \quad ((a \Cup b') \Cup (b' \Cup a))' \Cup a \le c \tag{4.6}$$

$$\text{A7} \qquad a \perp b \quad \& \quad ((a \Cup b') \Cup (b' \Cup a))' \Cup a = 1 \quad \Rightarrow \quad a' \perp b', \tag{4.7}$$

where

$$a \le b \stackrel{\rm def}{=} ((a \Cup b') \Cup (b' \Cup a))' \Cup a = b \tag{4.8}$$

$$1 \stackrel{\rm def}{=} ((a \Cup a) \Cup (a \Cup a))' \Cup a \qquad \& \qquad 0 \stackrel{\rm def}{=} (((a \Cup a) \Cup (a \Cup a))' \Cup a)'. \tag{4.9}$$

Substitution Rule. *Any valid condition or equation one can obtain in the standard formulation of* OML *containing only variables,* $\cup_i$ *(satisfied for all* $i = 1, \dots, 5$*), and negations written in* QA *with* $\Cup$ *substituted for* $\cup_i$ *is a valid condition or equation in* QA.

We can easily check that the above ordering is a proper ordering and that for $a, b, c \in \mathcal{A}_{\rm O}$ lower upper and greater lower bounds exist—they are given by $((a \Cup b') \Cup (b' \Cup a))' \Cup a$ and $(((a' \Cup b) \Cup (b \Cup a'))' \Cup a')'$, respectively. Obviously we can introduce the following definition $x \cup y \ =^{\rm def} \ ((a \Cup b') \Cup (b' \Cup a))' \Cup a$ and obtain the standard definition of OML as given in Sec. 2. This enables us to formulate the above *Substitution Rule*, which actually introduces an infinite number of conditions. Whether they can be replaced with a finite set of individual conditions is an open problem. Along this rule, A7 becomes Eq. (2.14). Eq. (2.14) is equivalent to Eq. (2.13) which for $j = 5$ reads [21]: $a \equiv_5 b = (a \cap b) \cup (a' \cap b') = 1 \quad \Leftrightarrow \quad a = b$. Since we have $a \equiv_5 b = ((b \cup_i a')' \cup_i (b' \cup a)')'$, $i = 1, \dots, 5$ [Eq. (3.23)], we get A8 below. Similarly, we get A9 etc. Of course, we can never arrive at $a \le a \Cup b$, $a \Cup (a \Cap b) = a$, $a \Cup (a' \Cap (a \Cup b)) = a \Cup b$, or many other equations we are used to in OML. For example, if we had had $a \Cup (a \Cap b) = a$, that would have reduced Eq. (4.12) below to Eq. (2.17) and therefore turn QA into a Boolean algebra.

Lemma 4.2.

$$\text{A8} \qquad (b \Cup a') \Cap (b' \Cup a) = 1 \qquad \Leftrightarrow \qquad a = b, \tag{4.10}$$

$$\text{A9} \qquad a \Cup a' = 1, \tag{4.11}$$

$$\text{A10} \qquad a \Cap (b \Cup (b \Cap a)) = 0 \qquad \Leftrightarrow \qquad a \perp b\,, \tag{4.12}$$
$$\text{A11} \qquad (((b \Cup a) \Cap a)' \Cup b) \Cap a = a \Cap (b \Cup (b \Cap a))\,, \tag{4.13}$$
$$\text{A12} \qquad a \Cup b = 1 \qquad \Leftrightarrow \qquad a' \perp b'\,, \tag{4.14}$$

where $a \Cap b \overset{\text{def}}{=} (a' \Cup b')'$.

On the other hand, Lemma 3.4 indicates that there might be different ways of expressing classical disjunctions by means of quantum ones. And indeed, $a \cup b = (a \cup_5 b) \cup_5 (b' \cup_5 a')'$ does not match any other $\cup_i$—meaning $a \cup b \neq (a \cup_i b) \cup_i (b' \cup_i a')'$, $i = 1, 2, 3, 4$. The same is true with Eq. (3.15) for $\cup_3$ and $\cup_4$, as well as with $a \cup b = ((a' \cup_1 b')' \cup_1 b')' \cup_1 ((a \cup_1 b')' \cup_1 a')'$ and $a \cup b = ((a' \cup_2 b')' \cup_2 a')' \cup_2 (b' \cup_2 (b \cup_2 a')')'$. Thus we arrive at

Theorem 4.3. *In* QA *one can express classical disjunction in the following five non-equivalent ways:*

$$a \cup_{cl1} b \;=\; ((a' \Cup b')' \Cup b')' \Cup ((a \Cup b')' \Cup a')' \tag{4.15}$$
$$a \cup_{cl2} b \;=\; ((a' \Cup b')' \Cup a')' \Cup (b' \Cup (b \Cup a')')' \tag{4.16}$$
$$a \cup_{cl3} b \;=\; b \Cup (b \Cup a) \tag{4.17}$$
$$a \cup_{cl4} b \;=\; a \Cup (b \Cup a) \tag{4.18}$$
$$a \cup_{cl5} b \;=\; (a \Cup b) \Cup (b' \Cup a')'\,. \tag{4.19}$$

Of course, there are many other such non-equivalent 5-tuples. Altogether, there are $(5^5)^5$ such 5-tuples.

In conclusion, by using the parallels with the standard orthomodular lattice theory, in QA we can derive all the equations that hold in the lattice theory in terms of $\cup_i$, $i = 1, \ldots, 5$ and negation, even those that cannot be obtained by the method presented in Section 3—for example, $a \cup_i (b \cap_i a) = a \cup_i (b' \cap_i a)$ or $a \cup_i (b \cup_i (a' \cap_i (a \cup_i b))) = a \cup_i b$ where neither sides of these equations are equal to particular Beran expressions for all $i = 1, \ldots, 5$ while the equations themselves do hold for all $i = 1, \ldots, 5$. On the other hand, by using $a \cup b = \, =^{\text{def}} ((a \Cup b') \Cup (b' \Cup a))' \Cup a$ and A1-A7 from Def. 4.1 we can embed the standard orthomodular lattice theory in QA.

5 Conditional associativity of quantum operations

Quantum disjunctions and conjunctions are not associative. However, a conditional associativity, similar to Foulis-Holland distributivity, does hold in any orthomodular lattice as proved in the theorem below. D'Hooghe and Pykacz [4, p. 648] the theorem for i =1, 2 and 5, and conjectured it for i =3 and 4. Below we confirm their conjecture by giving the proofs for i =3 and 4. By doing so we prove that the conditional associativity holds for the unified quantum disjunction and conjunction ($\Cup$ and $\Cap$) from the previous section. For this purpose, we may take aCb to be $a \Cup (a' \Cap b) = b \Cup a$, noting that in any OML $a \cup_i (a' \cap_i b) = b \cup_i a$ is equivalent to aCb for $i = 1, \ldots, 5$.

Theorem 5.1. *In any orthomodular lattice any triple* $\{a,b,c\}$ *in which one of the elements commutes with the other two is associative with respect to* $\cup_i$ *and* $\cap_i$, $i=1,\dots,5$*:*

$$aCb \quad \& \quad aCc \quad \Rightarrow \quad (a\cup_i b)\cup_i c = a\cup_i (b\cup_i c), \qquad i=1,\dots,5 \tag{5.1}$$

$$aCb \quad \& \quad bCc \quad \Rightarrow \quad (a\cup_i b)\cup_i c = a\cup_i (b\cup_i c), \qquad i=1,\dots,5 \tag{5.2}$$

$$aCc \quad \& \quad bCc \quad \Rightarrow \quad (a\cup_i b)\cup_i c = a\cup_i (b\cup_i c), \qquad i=1,\dots,5 \tag{5.3}$$

$$aCb \quad \& \quad aCc \quad \Rightarrow \quad (a\cap_i b)\cap_i c = a\cap_i (b\cap_i c), \qquad i=1,\dots,5 \tag{5.4}$$

$$aCb \quad \& \quad bCc \quad \Rightarrow \quad (a\cap_i b)\cap_i c = a\cap_i (b\cap_i c), \qquad i=1,\dots,5 \tag{5.5}$$

$$aCc \quad \& \quad bCc \quad \Rightarrow \quad (a\cap_i b)\cap_i c = a\cap_i (b\cap_i c), \qquad i=1,\dots,5 \tag{5.6}$$

Proof. Since D'Hooghe and Pykacz [4, p. 648] proved the cases $i=1,2,5$ we only give sketchy proofs for these cases for the sake of completeness.

For $i=1,2$, Eq. (5.1), given the premise (aCb) and the Foulis-Holland theorem (F-H) $[aCb\,\&\,aCc \;\Rightarrow\; (a\cup b)\cap c = (a\cap c)\cup(b\cap c)$, etc] we have (since $a'Ca$): $a\cup_1 b = a\cup(a'\cap b) = (a\cup a')\cap(a\cup b) = a\cup b$). Thus, the conclusion from Eq. (5.1) reads

$$(a\cup_1 b)\cup_1 c = a\cup b\cup c = a\cup_1 (b\cup_1 c).$$

Eq. (5.2) and Eq. (5.3) follow analogously. Since $a\cup_2 b = b\cup_1 a$ and $a\cap_{1,2} b = (a'\cup_{1,2} b')'$, we have proved the theorem for $i=1,2$.

For $i=5$, (again we have $aCb \;\Rightarrow\; a\cup_5 b = a\cup b$, etc.) both sides of the conclusion of Eqs. (5.1), Eqs. (5.2), and Eqs. (5.3) reduce to $a\cup(b\cup_5 c)$, $b\cup(a\cup_5 c)$, and $c\cup(a\cup_5 b)$, respectively.

Let us now consider the case $i=3$, Eq. (5.1). According to the first definition of aCb from Def. 2.3 we have, given the premises (aCb and aCc) and the orthomodularity property $[a\cup(a'\cap(a\cup b)) = a\cup b]$:

$$a\cup_3 b = (a\cap b)\cup(a\cap b')\cup(a'\cap(a\cup b)) = a\cup(a'\cap(a\cup b)) = a\cup b$$

and therefore, using Foulis-Holland theorem (F-H) and the second premise and Def. 2.3:

$$\begin{aligned}
(a\cup_3 b)\cup_3 c &= ((a\cup b)\cap c)\cup((a\cup b)\cap c')\cup((a'\cap b')\cap(a\cup b\cup c))\\
&= [\text{F-H}] = (a\cap c)\cup(b\cap c)\cup(a\cap c')\cup(b\cap c')\cup((a'\cap b')\cap(a\cup b\cup c))\\
&= [\text{Df.2.3}] = a\cup(b\cap c)\cup(b\cap c')\cup((a'\cap b')\cap(a\cup b\cup c))\\
&= (b\cap c)\cup(b\cap c')\cup((a\cup(a'\cap b'))\cap(a\cup b\cup c))\\
&= (b\cap c)\cup(b\cap c')\cup(((a\cup a')\cap(a\cup b'))\cap(a\cup b\cup c))\\
&= (b\cap c)\cup(b\cap c')\cup((a\cup b')\cap(a\cup b\cup c))
\end{aligned} \tag{5.7}$$

The right-hand side of the conclusion in Eq. (5.1) reads:

$$a\cup_3 (b\cup_3 c) = (a\cap(b\cup_3 c))\cup(a\cap(b\cup_3 c)')\cup a'(\cap(a\cup(b\cup_3 c)))\,. \tag{5.8}$$

Now $b\cup_3 c = (b\cap c)\cup(b\cap c')\cup(b'\cap(b\cup c))$ and since we also have aCb and aCc and therefore: $aC(b\cap c)$, $aC(b\cap c')$, and $aC(b'\cap(b\cup c))$, we have $aC(b\cup_3 c)$ as well. Hence, using Def. 2.3 we reduce Eq. (5.8) to:

$$\begin{aligned}
a\cup_3 (b\cup_3 c) &= a\cup a'\cap(a\cup(b\cup_3 c)) = a\cup(b\cap c)\cup(b\cap c')\cup(b'\cap(b\cup c)))\\
&= (b\cap c)\cup(b\cap c')\cup((a\cup b')\cap(a\cup b\cup c)))\,,
\end{aligned}$$

which is nothing but Eq. (5.7). Hence, Eq. (5.1) is proved.

Let us next consider Eq. (5.2). Here we have: $a \cup_3 b = a \cup b$ and $b \cup_3 c = b \cup c$ and therefore:

$$\begin{aligned}
(a \cup_3 b) \cup_3 c &= ((a \cup b) \cap c) \cup ((a \cup b) \cap c') \cup ((a' \cap b') \cap (a \cup b \cup c)) \\
&= [\text{F-H}] = (a \cap c) \cup (b \cap c) \cup (a \cap c') \cup (b \cap c') \cup ((a' \cap b') \cap (a \cup c)) \\
&= (a \cap c) \cup (a \cap c') \cup (b \cap c) \cup (b \cap c') \cup ((a' \cap b') \cap (a \cup c)) \\
&= [bCc] = (a \cap c) \cup (a \cap c') \cup b \cup ((a' \cap b') \cap (a \cup c)) \\
&= [bC(a' \cap b'),\ bC(b \cup c)] = (a \cap c) \cup (a \cap c') \cup ((a' \cup b) \cap (a \cup b \cup c)). \quad (5.9)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
a \cup_3 (b \cup_3 c) &= (a \cap (b \cup c)) \cup (a \cap b' \cap c') \cup (a' \cap (a \cup b \cup c)) \\
&= [\text{F-H}] = (a \cap b) \cup (a \cap c) \cup (a \cap b' \cap c') \cup (a' \cap (a \cup b \cup c)) \\
&= (a \cap c) \cup (a \cap b' \cap c') \cup (a \cap b) \cup (a' \cap b) \cup (a' \cap (a \cup c)) \\
&= [aCb] = (a \cap c) \cup (a \cap b' \cap c') \cup b \cup (a' \cap (a \cup c)) \\
&= [bC(a \cap c')] = (a \cap c) \cup ((a \cap c') \cup b) \cap (b' \cup b) \cup (a' \cap (a \cup c)) \\
&= (a \cap c) \cup (a \cap c') \cup b \cup (a' \cap (a \cup c)) \\
&= [bCa', bC(a \cup c)] = (a \cap c) \cup (a \cap c') \cup ((a' \cup b) \cap (a \cup b \cup c))
\end{aligned}$$

which is nothing but Eq. (5.9) and this proves Eq. (5.2).

As for Eq. (5.3), here we again have $cC(a \cup_3 b)$ and $b \cup_3 c = b \cup c$. Thus we get:

$$\begin{aligned}
(a \cup_3 b) \cup_3 c &= ((a \cup_3 b) \cap c) \cup ((a \cup_3 b) \cap c') \cup ((a \cup_3 b)' \cap ((a \cup_3 b) \cup c)) \\
&= (a \cup_3 b) \cup ((a \cup_3 b)' \cap ((a \cup_3 b) \cup c)) \\
&= [\text{OM property}] = (\mathrm{a} \cup_3 \mathrm{b}) \cup \mathrm{c} = (\mathrm{a} \cap \mathrm{b}) \cup (\mathrm{a} \cap \mathrm{b}') \cup (\mathrm{a}' \cap (\mathrm{a} \cup \mathrm{b})) \cup \mathrm{c} \\
&= (a \cap b) \cup (a \cap b') \cup ((a' \cup c) \cap (a \cup b \cup c)) \quad (5.10)
\end{aligned}$$

For the right-hand side we have:

$$\begin{aligned}
a \cup_3 (b \cup_3 c) &= (a \cap (b \cup c)) \cup (a \cap b' \cap c') \cup (a' \cap (a \cup b \cup c)) \\
&= (a \cap b) \cup (a \cap c) \cup (a \cap b' \cap c') \cup (a' \cap (a \cup b \cup c)) \\
&= (a \cap b) \cup (a \cap b' \cap c') \cup (a' \cup (a \cap c)) \cap ((a \cap c) \cup a \cup b \cup c)) \\
&= (a \cap b) \cup (a \cap b' \cap c') \cup (a' \cup c) \cap (a \cup b \cup c)) \\
&= (a \cap b) \cup (a \cap b' \cap c') \cup (a' \cap (a \cup b \cup c))) \cup c \\
&= (a \cap b) \cup ((a \cap b') \cup c) \cap (c \cup c') \cup (a' \cap (a \cup b \cup c))) \\
&= (a \cap b) \cup (a \cap b') \cup c \cup (a' \cap (a \cup b \cup c))) \\
&= (a \cap b) \cup (a \cap b') \cup ((a' \cup c) \cap (a \cup b \cup c)))
\end{aligned}$$

which is nothing but Eq. (5.10), what proves Eq. (5.3).

Since $a \cup_4 b = b \cup_3 a$ and $a \cap_{3,4} b = (a' \cup_{3,4} b')'$, we have proved the theorem for $i = 3, 4$. □

We conjecture that the theorem holds in any weakly orthomodular lattice, WOML [19] as well.

6 Conditional distributivity of quantum operations

The Foulis-Holland theorem for conditional distributivity does not in general hold for the quantum disjunctions and conjunctions. D'Hooghe and Pykacz show this for $\cup_5, \cap_5$ [4, p. 646] and state that (in our notation for i) "the same can be checked for $i = 1, 2, 3, 4$" (p. 647). While this is true for $i = 3, 4$, distributivity in the forms given by Theorems 6.1 and 6.2 does hold for $i = 1, 2$. Also, parts of the Foulis-Holland theorem, presented in Theorem 6.3 hold for any i and therefore for the unified quantum disjunction and conjunction ($\Cup$ and $\Cap$) from Sec. 4.

Theorem 6.1. *In any orthomodular lattice any triple $\{a, b, c\}$ in which one of the elements commutes with the other two is distributive with respect to $\cup_1$ and $\cap_1$ in the following sense:*

$$aCb \;\;\&\;\; aCc \;\;\Rightarrow\;\; a \cup_1 (b \cap_1 c) = (a \cup_1 b) \cap_1 (a \cup_1 c) \tag{6.1}$$
$$aCb \;\;\&\;\; bCc \;\;\Rightarrow\;\; a \cup_1 (b \cap_1 c) = (a \cup_1 b) \cap_1 (a \cup_1 c) \tag{6.2}$$
$$aCc \;\;\&\;\; bCc \;\;\Rightarrow\;\; a \cup_1 (b \cap_1 c) = (a \cup_1 b) \cap_1 (a \cup_1 c) \tag{6.3}$$

Proof. In this and all other proofs of this section, we will implicitly make use of the rules $aCb \Rightarrow a \cup_i b = a \cup_j b$, $aCb \Rightarrow a \cap_i b = a \cap_j b$, $aCb \;\&\; aCc \Rightarrow aCb{\cup_i},{\cap_i}c$, and $aCb \Rightarrow a, b, a \cup_i b, a \cap_i bCa \cup_j b, a \cap_j b$, $0 \le i, j \le 5$. Also, $aCa{\cup_{0,1,3,5}}, {\cap_{0,1,3,5}}b$, $bCa{\cup_{0,2,4,5}}, {\cap_{0,2,4,5}}b$, $a \cup_{0,1} bCa' \cup_{0,1} c$, $a \cap_{0,1} bCa' \cap_{0,1} c$, $a \cup_{0,1} bCc \cup_{0,2} a'$, and $a \cap_{0,1} bCc \cap_{0,2} a'$. We will use F-H implicitly. Recall that $\cup_0 = \cup$.

For (6.1), $a \cup (b \cap_1 c) = a \cup (b \cap (b' \cup c)) = (a \cup b) \cap ((a' \cap b') \cup a \cup c) = (a \cup b) \cap_1 (a \cup c)$.

For (6.2), $a \cup_1 (b \cap c) = a \cup (a' \cap b \cap c) = a \cup (b \cap a' \cap c) = (a \cup b) \cap (a \cup (a' \cap c)) = (a \cup b) \cap (a \cup_1 c)$.

For (6.3), $a \cup_1 (b \cap c) = a \cup (a' \cap b \cap c) = (a \cup (a' \cap b)) \cap (a \cap c) = (a \cup_1 b) \cap (a \cup c)$. □

Because $\cup_1, \cap_1$ are not commutative, the "reverse" distributivity $(a \cap_1 b) \cup_1 c = (a \cup_1 c) \cap_1 (b \cup_1 c)$ does not hold for all F-H hypotheses. However, it does hold for $\cup_2, \cap_2$:

Theorem 6.2. *In any orthomodular lattice any triple $\{a, b, c\}$ in which one of the elements commutes with the other two is distributive with respect to $\cup_2$ and $\cap_2$ in the following sense:*

$$aCb \;\;\&\;\; aCc \;\;\Rightarrow\;\; (a \cap_2 b) \cup_2 c = (a \cup_2 c) \cap_2 (b \cup_2 c) \tag{6.4}$$
$$aCb \;\;\&\;\; bCc \;\;\Rightarrow\;\; (a \cap_2 b) \cup_2 c = (a \cup_2 c) \cap_2 (b \cup_2 c) \tag{6.5}$$
$$aCc \;\;\&\;\; bCc \;\;\Rightarrow\;\; (a \cap_2 b) \cup_2 c = (a \cup_2 c) \cap_2 (b \cup_2 c) \tag{6.6}$$

Proof. Theorem 6.1 and the fact that $a \cup_2 b = b \cup_1 a$, $a \cap_2 b = b \cap_1 a$. □

For certain F-H hypotheses, distributive laws hold for all $i = 1, \ldots, 5$. In addition, a couple of other cases hold for $i = 1, 2$.

Theorem 6.3. *In any orthomodular lattice the following laws hold:*

$$aCb \;\;\&\;\; aCc \;\;\Rightarrow\;\; a \cup_i (b \cap_i c) = (a \cup_i b) \cap_i (a \cup_i c), \qquad i = 1, \ldots, 5 \tag{6.7}$$
$$aCc \;\;\&\;\; bCc \;\;\Rightarrow\;\; (a \cap_i b) \cup_i c = (a \cup_i c) \cap_i (b \cup_i c), \qquad i = 1, \ldots, 5 \tag{6.8}$$
$$aCb \;\;\&\;\; aCc \;\;\Rightarrow\;\; (a \cap_1 b) \cup_1 c = (a \cup_1 c) \cap_1 (b \cup_1 c) \tag{6.9}$$
$$aCc \;\;\&\;\; bCc \;\;\Rightarrow\;\; a \cup_2 (b \cap_2 c) = (a \cup_2 b) \cap_2 (a \cup_2 c) \tag{6.10}$$

Proof. For Eq. (6.7), using $aCb \ \Rightarrow \ a\cup_i b = a\cup b$ and $aCb \ \& \ aCc \ \Rightarrow \ aC(b\cup_i c)$ and F-H we can write the conclusion as

$$a\cup(b\cap_i c) = (a\cup b)\cap_i(a\cup c), \qquad i=1,\ldots,5 \tag{6.11}$$

To prove that the right-hand side boils down to the left-hand one is straightforward and can be done in a complete analogy to the case $i=1$ already done above—Eq. (6.1). For example, for $i=4$ we have:

$$\begin{aligned}(a\cup b)\cap_4(a\cup c) &= ((a'\cap c')\cup((a\cup b)\cap(a\cup c)))\cap(a\cup b\cup c)\cap((a'\cap b')\cup a\cup c)\\ &= ((a'\cap c')\cup a\cup(b\cap c))\cap(a\cup b\cup c)\cap(a\cup b'\cap c)\\ &= a\cup(b\cap_4 c)\,.\end{aligned}$$

For Eq. (6.8), the proof follows from Eq. (6.11) by symmetry.

The proof of (6.9) seems a little tricky, so we show it in some detail. First, we show that (under the hypotheses)

$$(a\cap b)\cup(a'\cap c) \ = \ (a\cup c)\cap(b\cup a')\,. \tag{6.12}$$

From $b\ge a\cap b = a\cap(b\cup a')$ and $c\ge c\cap(b\cup a')$ we have $b\cup c\ge(a\cap(b\cup a'))\cup(c\cap(b\cup a')) = (a\cup c)\cap(b\cup a')$. Therefore $(a\cup c)\cap(b\cup a') = (b\cup a')\cap(a\cup c)\cap(b\cup c) = ((a\cap b)\cup a')\cap((a\cap b)\cup c) = (a\cap b)\cup(a'\cap c)$, establishing (6.12). The left-hand side of (6.9) reduces to $(a\cap b)\cup_1 c = (a\cap b)\cup((a\cap b)'\cap c) = (a\cap b)\cup((a'\cup b')\cap c) = (a\cap b)\cup(a'\cap c)\cup(b'\cap c)$. The right-hand side reduces to $(a\cup c)\cap_1(b\cup_1 c) = (a\cup c)\cap((a\cup c)'\cup b\cup(b'\cap c)) = (a\cup c)\cap((a'\cap c')\cup b\cup(b'\cap c)) = (a\cup c)\cap(b\cup(a'\cap c')\cup(b'\cap c)) = (a\cup c)\cap(((b\cup a')\cap(b\cup c'))\cup(b'\cap c)) = (a\cup c)\cap((b'\cap c)\cup_1(b\cup a')) = (a\cup c)\cap((b\cup a')\cup(b'\cap c)) = ((a\cup c)\cap(b\cup a'))\cup((a\cup c)\cap b'\cap c) = ((a\cup c)\cap(b\cup a'))\cup(b'\cap c)$. Using (6.12), we see they are the same.

For (6.10) we use (6.9) and $a\cup_2 b = b\cup_1 a$, $a\cap_2 b = b\cap_1 a$.

□

Similar results can be stated for the dual operations ($\cup_i$ and $\cap_i$ interchanged). In all other cases not shown in the three theorems above, the distributive law does not hold: all of them fail in orthomodular lattice MO2 (Fig. 1).

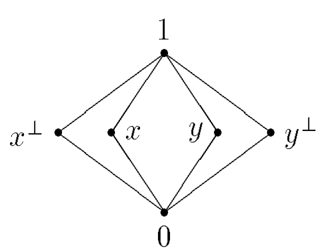

Figure 1: Lattice MO2.

If we allow a mixture of the different disjunctions and conjunctions, we can obtain a distributive law that holds unconditionally.

Theorem 6.4. *In any orthomodular lattice the following law holds:*

$$a \cup_1 (b \cap_0 c) = (a \cup_1 b) \cap_0 (a \cup_1 c) \tag{6.13}$$

Proof. Expanding definitions and using F-H, $a \cup_1 (b \cap_0 c) = a \cup (a' \cap b \cap c) = a \cup (a' \cap b \cap a' \cap c) = (a \cup (a' \cap b)) \cap (a \cup (a' \cap c)) = (a \cup_1 b) \cap_0 (a \cup_1 c)$. □

It is interesting that if we consider all equations of the form $a \cup_i (b \cap_j c) = (a \cup_k b) \cap_l (a \cup_m c)$ for all possible assignments $0 \le i, j, k, l, m \le 5$ ($6^5 = 7776$ possibilities), the equation holds in all OMLs for exactly the one case of (6.13): $i = 1, j = 0, k = 1, l = 0, m = 1$. All other 7775 cases fail in lattice MO2.

The "reverse" form of (6.13) holds with $\cup_2$ substituted for $\cup_1$. Dual results with $\cup_i$ and $\cap_i$ interchanged can also be stated.

7 An open problem

In Ref. [12] we opened an interesting problem on whether the "distributivity of symmetric identity," expressed by Equation (7.7) below, holds in all orthomodular lattices or not and whether a particular equation derivable from it in any orthomodular lattice characterizes the latter lattices. An indication that they might do so is that they pass all Greechie diagrams we let them run on—with up to 38 atoms and 38 blocks (more than 50 million lattices). We used our program `greechie` to obtain the diagrams and our program `latticeg` to check the equations on them. [11] On the other hand Equation (7.7) does not imply the orthomodularity property—it does not fail in the diagram O6 which characterizes any orthomodular lattice.

In Ref. [12] we proved several partial results for the above distributivity. In this section we prove that it holds in Hilbert space and in the Godowski lattices of the second lowest order (4GO). We recall from Ref. [12] that a 4GO is any OML (actually any OL) in which the following equation, which we call 4-Go, holds:

$$(a \to_1 b) \cap (b \to_1 c) \cap (c \to_1 d) \cap (d \to_1 a) \quad \le \quad a \to_1 d\,. \tag{7.1}$$

We define $a \equiv b \;\stackrel{\rm def}{=}\; (a \cap b) \cup (a' \cap b')$ and note that $a \equiv b = a \equiv_5 b$ holds in all OMLs.

Lemma 7.1. *In any* OML *we have:*

$$(a \equiv c) \cup (b \equiv c) \;=\; ((a \to_2 c) \cup (b \to_2 c)) \cap ((c \to_1 a) \cup (c \to_1 b)) \tag{7.2}$$

$$(a \equiv c) \cup (b \equiv c) \;\le\; ((a \cap b) \to_2 c) \cap (c \to_1 (a \cup b)) \tag{7.3}$$

$$((a \cup b) \equiv c) \cap (a \equiv b) \;=\; (a \equiv c) \cap (a \equiv b)\,. \tag{7.4}$$

In any 4GO *we have:*

$$(a \equiv b) \cap ((b \equiv c) \cup (a \equiv c)) \quad \le \quad a \equiv c\,. \tag{7.5}$$

Proof. For (7.2), we have

$$\begin{aligned}(a\equiv c)\cup(b\equiv c) &= (b\cap c)\cup(a'\cap c')\cup(b'\cap c')\cup(a\cap c)\\ &= (b\cap c)\cup(a'\cap c')\cup((b\to_2 c)\cap(c\to_1 a))\\ &= (b\cap c)\cup(((a'\cap c')\cup(b\to_2 c))\cap((a'\cap c')\cup(c\to_1 a)))\\ &= (b\cap c)\cup(((a'\cap c')\cup(b\to_2 c))\cap(c\to_1 a))\\ &= ((b\cap c)\cup(a'\cap c')\cup(b\to_2 c))\cap((b\cap c)\cup(c\to_1 a))\\ &= ((a'\cap c')\cup(b\to_2 c))\cap((b\cap c)\cup(c\to_1 a))\\ &= (((a'\cap c')\cup c)\cup(c\cup(b'\cap c')))\cap(((b\cap c)\cup c')\cup(c'\cup(c\cap a)))\\ &= ((a\to_2 c)\cup(b\to_2 c))\cap((c\to_1 b)\cup(c\to_1 a))\,.\end{aligned}$$

In the second step, we use Equation (3.20) from Ref. [12]. In the third and fifth steps we apply the Foulis-Holland theorem (F-H), and in the fourth and sixth steps we apply absorption laws.

For (7.3), $(a\to_2 c)\cup(b\to_2 c)\le(a\cap b)\to_2 c$ and $(c\to_1 a)\cup(c\to_1 b)\le c\to_1(a\cup b)$ in any OL, so $(a\equiv c)\cup(b\equiv c)=$ [from (7.2)]$((a\to_2 c)\cup(b\to_2 c))\cap((c\to_1 a)\cup(c\to_1 b))\le((a\cap b)\to_2 c)\cap(c\to_1(a\cup b))$.

For (7.4), we have

$$\begin{aligned}((a\cup b)\equiv c)\cap(a\equiv b) &= ((a\cup b)\cap c)\cup(a'\cap b'\cap c'))\cap((a\cap b)\cup(a'\cap b'))\\ &= (((a\cup b)\cap c)\cap((a\cap b)\cup(a'\cap b')))\cup((a'\cap b'\cap c')\cap((a\cap b)\cup(a'\cap b')))\\ &= (((a\cup b)\cap c)\cap(a\cap b))\cup(((a\cup b)\cap c)\cap(a'\cap b'))\cup\\ &\qquad ((a'\cap b'\cap c')\cap(a\cap b))\cup((a'\cap b'\cap c')\cap(a'\cap b'))\\ &= ((a\cap b\cap c)\cup 0\cup 0\cup(a'\cap b'\cap c'))=(a\equiv c)\cap(a\equiv b)\end{aligned}$$

where in the second and third steps we apply F-H and in the last step we apply Lemma 3.11 of Ref. [12].

Finally, (7.5) is proved as follows. Equation (3.30) of Ref. [12], which we repeat below as (7.6), was shown to hold in all 4GOs.

$$(a\equiv b)\cap((b'\cap c')\cup(a\cap c))\quad\le\quad a\equiv c\,. \tag{7.6}$$

Using Equation (3.20) of Ref. [12] and renaming variables, we see that this is the same as

$$(d\equiv e)\cap(e\to_2 c)\cap(c\to_1 d)\quad\le\quad d\equiv c\,.$$

Substituting $a\cup b$ for d and $a\cap b$ for e,

$$((a\cup b)\equiv(a\cap b))\cap((a\cap b)\to_2 c)\cap(c\to_1(a\cup b))\quad\le\quad(a\cup b)\equiv c\,.$$

Since $(a\cup b)\equiv(a\cap b)=a\equiv b$ holds in any OML, we have

$$\begin{aligned}(a\equiv b)\cap((a\cap b)\to_2 c)\cap(c\to_1(a\cup b)) &\le (a\cup b)\equiv c\\ (a\equiv b)\cap((a\cap b)\to_2 c)\cap(c\to_1(a\cup b)) &\le ((a\cup b)\equiv c)\cap(a\equiv b)\\ (a\equiv b)\cap((a\cap b)\to_2 c)\cap(c\to_1(a\cup b)) &\le (a\equiv c)\cap(a\equiv b)\\ (a\equiv b)\cap((a\cap b)\to_2 c)\cap(c\to_1(a\cup b)) &\le a\equiv c\\ (a\equiv b)\cap((a\equiv c)\cup(b\equiv c)) &\le a\equiv c\,.\end{aligned}$$

where in the third step we use (7.4) and in the last step we use (7.3). □

Theorem 7.2. *The following equation, which we call* distributivity of symmetric identity, *holds in all* 4GO*s (and therefore all* nGO, $n \geq 4$*) and thus in the lattice of all closed subspaces of finite or infinite dimensional Hilbert space:*

$$(a \equiv b) \cap ((b \equiv c) \cup (a \equiv c)) \quad = \quad ((a \equiv b) \cap (b \equiv c)) \cup ((a \equiv b) \cap (a \equiv c)) \,. \qquad (7.7)$$

Proof. The result follows immediately from (7.5) and Theorem 2.9 of Ref. [12]. □

Whether or not (7.7) holds in all OMLs or even in all WOMLs (since it does not fail in O6) is still an open question. However, the most important question from the point of view of quantum mechanics, which is whether it holds in Hilbert space, is answered by Theorem 7.2.

Since (7.6) also follows from (7.7), as shown in Ref. [12], the OML variety in which (7.6) holds is the same as the OML variety in which (7.7) holds. Thus if one of these holds in any OML (our open question), so does the other.

Another open question is whether the stronger-looking Equation (3.29) of Ref. [12], from which (7.6) follows and which we repeat below as (7.8),

$$(a \rightarrow_1 b) \cap (b \rightarrow_2 c) \cap (c \rightarrow_1 a) \quad \leq \quad (a \equiv c) \qquad (7.8)$$

can be derived (in an OML) from (7.6).

8 Algorithms for the programs

In an OML, any expression with 2 variables is equal to one of 96 canonical forms, corresponding to the 96 elements of the free OML F_2. We fix the 96 expressions of [1, Table 1, p. 82] as our canonical standard.

The program `beran` takes, as its input, an arbitrary two-variable expression and outputs the equivalent canonical form. The program can be used to prove or disprove any 2-variable conjecture expressed as an equation, simply by verifying that both sides of the equation reduce (or do not reduce) to the same canonical form.

Each element of OML F_2 can be separated into a "Boolean part" and an "MO2 part." [14] Each of them has relatively simple rules of calculation, and we use this method in the program `beran`.[4] This is implemented in the program by checking for either Boolean or MO2 lattice violation of the 96 equations formed by setting the input expression equal to each of the 96 canonical expressions, and the unique equation which violates neither lattice gives us the answer.

The program `beran` is contained in a single file, `beran.c`, and compiles on any platform with an ANSI C compiler such as `gcc`. The use of the program is simple. The operations $\cup$, $\cap$, and $'$ are represented by the characters `v`, `^`, and `-`. (Other operations are also defined and can be seen with the program's `--help` option). As an example, to see the canonical expression corresponding to $a \cup (a' \cap (a \cup b))$, we type

[4] The authors wish to thank Prof. Navara for suggesting this method. The reader can download this or any other afore-mentioned program from our ftp sites:
`ftp://users.shore.net/members/n/d/ndm/quantum-logic/` and
`ftp://m3k.grad.hr/pavicic/quantum-logic/programs`

```
beran "(av(-a^(avb)))"
```

and the program responds with `(avb)`.

A second program, `bercomb`, was used to find the minimal expressions shown in Section 3. This program is contained in the single file `bercomb.c`. Its input parameters include the number of variable occurrences and the number of negations (orthocomplementations), and it exhaustively constructs all possible expressions containing a single binary operation with these parameters fixed. For each expression it uses the algorithm from `beran.c` to determine the expression's canonical form which it prints out. When a set of operations is specified, such as $\to_1$ through $\to_5$, it prints out the canonical form only when all operations simultaneously result in that canonical form.

If $v \geq 2$ is the number of variable occurrences and $n \geq 0, \leq 2v-1$ the number of negations, then the number of possible expressions containing one or two different variables is as follows. The number of ways of parenthesizing a binary operation in an expression with v variables is the Catalan number C_{v-1}, where $C_i = \binom{2i}{i}/(i+1)$. There are 2^v possible ways to assign two variables to an expression. If we display an expression with no negations in Polish notation, it is easy to see that there are $2v-1$ symbols and therefore $\binom{2v-1}{n}$ ways to distribute n negations (disallowing double negations). Thus for fixed v and n, there are $2^v\binom{2v-1}{n}C_{v-1}$ possible expressions.

For example, if we type

```
bercomb 7 0 i n
```

then all $2^7\binom{2\cdot 7-1}{0}C_{7-1} = 16896$ expressions with 7 variable occurrences and 0 negations are scanned, and the output includes the four smallest implicational expressions resulting in $a \cup b$ that we mentioned before Lemma 3.1. We refer the reader to the program's `--help` option for the meaning and usage of the other `bercomb` parameters. In this example `i` means $\to_1$ through $\to_5$ and `n` means don't suppress duplicate canonical expressions.

9 Concluding remarks

In Ref. [21] we stressed that all the operations in an orthomodular lattice are fivefold defined and we illustrated this on the identity operations. The claim was based on Ref. [20] where we proved that "quantum" as well as "classical" operations can serve for a formulation of an orthomodular lattice underlying Hilbert space. In 1998 we also put on the web the computer program `beran` which reduces any two-variable expression in an orthomodular lattice to one of the 96 possible ones as given in Ref. [1].

In effect, in the standard orthomodular lattice formulation (where the "classical" operations are inherited from the Hilbert space formalism) there are 80 quantum expressions which for compatible variables reduce to 16 classical expressions. In general all quantum expressions (including "quantum 0" and "quantum 1") are fivefold defined. (Detailed presentation of them all we give in Sec. 2.) In our quantum algebraic approach the situation reverses and we have classical operations fivefold defined in a quantum algebra formulation.

Still, recently several papers on "some new operations on orthomodular lattices" appeared in press as, e.g., the one by D'Hooghe and Pykacz [4] in which they picked out Beran expressions 12, 18, 28, 34, 44, 50, 60, and 76 and looked at some of their properties. So, for example, in [4, p. 649, bottom] one reads (in our notation): "Theorem 7 allows one to express in many ways any of the studied operations by (any of) the other(s) orthocomplementation. However, the following example in which we express $\cup_1$ by $\cup_5$ and shows that the obtained formulas might be rather lengthy: $a \cup_1 b = (a \cup_5 ((a \cup_5 b)' \cup_5 a))' \cup_5 a$. It is an open question which of such formulas (if any) could be written in a more economical way." Our approach immediately closes this open question: all the formulas could be written in a more economical way and one gets all alternatives in seconds; e.g., in the afore-cited example, there are over 100 shorter expressions—one of 3 shortest ones is given by Eq. (3.16) above—and there are over 500 of them with the same (5) variable occurrences. On the other hand, Theorem 6 from Ref. [4, p. 648] is just a special case of our Theorem 2.5 from [17, p. 1487]. Also all the results from [4, Section 3.2, p. 646-8], can be trivially obtained using our computer program `latticeg`. [11] In addition, their two conjectures (p. 648) following from their Theorem 5 (p. 647) one can support by our program `latticeg` with millions of lattices. Hence, it appears necessary to present our results in detail, give explicit proofs of all our previous claims, present the most important and relevant outputs of our programs in some detail, and provide the reader with instructions on how to use our programs which give answers to virtually all questions one can have on algebraic properties of two variable orthomodular formulas in seconds.

Thus, in Sec. 3 we prove several lemmas in which we show how one can express operations in any standardly defined (in Sec. 2) orthomodular lattice by each other. Lemma 3.1 gives expressions of classical disjunction ($\cup$) by means of all five quantum implications $\to_i$, $i = 1, \ldots, 5$ and without negations in a shortest possible single equation—meaning that the equation preserves its form for all i's and that there are no simpler equations with such a property. Expressions of $\cup$ by means of quantum disjunctions ($\cup_i$, $i = 1, \ldots, 5$) and conjunctions ($\cap_i$, $i = 1, \ldots, 5$) follow from Def. 2.4. Lemma 3.2 gives the shortest expressions of $\cup = \cup_0$ and $\cap = \cap_0$ by means of $\to_i$, $\cup_i$, and $\cap_i$, $i = 1, \ldots, 5$ with negation. Lemma 3.4 gives the shortest expressions of $\cup_i$, $\cap_i$, and $\equiv_i$, $i = 0, \ldots, 5$ by means of $\cup_i$, and $\cap_i$, $i = 1, \ldots, 5$ with negation.

In Sec. 4 we start with the possibility—opened by Lemma 3.2—of expressing $\cup$ by means of $\cup_i$, $i = 1, \ldots, 5$ in five equations of the same form and define—in Def. 4.1—the orthomodular lattice by means of one unique quantum operation. We have chosen quantum disjunction $\Cup$, but the same, of course, can be done with quantum conjunction $\Cap$ or implication (the latter being just another way of writing disjunction)—quantum identity is the only quantum operation which cannot serve for the purpose as we proved in Ref. [18]. In such a formulation of orthomodular lattice everything reverses and now classical operations can be expressed in five different ways as shown by Theorem 4.3.

We stress that the quantum algebra QA (Def. 4.1) is actually completely defined by its Substitution Rule, and that "axioms" A1–A7 are merely some consequences of that rule. A1–A7 are important in that they show that standard OML can be embedded in QA and are included for this reason. However there are many other non-obvious consequences of QA such as those exemplified in Lemma 4.2. That lemma only touches the surface of the kind of conditions one can obtain from QA, and it is possible that QA provides a rich algebraic structure that has yet to be explored. It also remains an interesting open problem whether QA can be finitely axiomatized.

Lemma 3.3 shows the surprising result that classical disjunction can be expressed in a single equation that holds in any OML for *all 6* disjunctions $\cup_i$, $i = 0, \ldots, 5$. This opens the possibility of an even more general quantum algebra, with Eq. (3.7) used in place of Eq. (3.4) as the basis for A1–A7. In this case we would replace "$i = 0, \ldots, 5$" for "$i = 1, \ldots, 5$" in the Substitution Rule. The same kinds of open questions we brought up for QA would also apply to this more general algebra.

As for D'Hooghe and Pykacz's conjecture on a possible conditional associativity of $\cup_{3,4}$ and $\cap_{3,4}$ [4], we decided that its passage through millions of Greechie diagrams makes it worth proving it and we did so in Sec. 5. In that way we obtained the conditional associativity for the unified operations $\Cup$ and $\Cap$ from Sec. 5.

In Sec. 6 we prove several Foulis-Holland-type conditional distributivities some of which are valid for all standard quantum disjunctions and conjunctions and therefore for the unified quantum disjunction and conjunction ($\Cup$ and $\Cap$) from Sec. 4.

As for properties taken over from Hilbert space in Sec. 7 we present two, given by Eqs. (7.5) and (7.7) for which we proved to hold in a variety of orthomodular lattice 4GO (and therefore in nGO, $n \geq 4$). but which do not fail in any of over 50 million Greechie diagrams we tested the properties on. Thus it remains an open problem whether the properties hold in any orthomodular lattice and even more whether Eq. (7.7) holds in an even weaker ortholattice called weakly orthomodular lattice, WOML.

In Sec. 8 we give algorithms we used to obtain and check all our equations and proofs for properties involving two variables.

To conclude, the only genuine target that apparently remains for scientific investigation in algebraic properties of orthomodular lattices in the future are properties with 3 and more variables.

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