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Boltzmanngasse 9 A-1090 Wien, Austria

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María Laura Barberis

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HOMOGENEOUS HYPER-HERMITIAN METRICS WHICH ARE CONFORMALLY HYPER-KÄHLER

MARÍA LAURA BARBERIS

ABSTRACT. Let g be a hyper-Hermitian metric on a simply connected hypercomplex four-manifold $(M, \mathcal{H})$. We show that when the isometry group $I(M, g)$ contains a subgroup acting simply transitively on M by hypercomplex isometries then the metric g is conformal to a hyper-Kähler metric. We describe explicitely the corresponding hyper-Kähler metrics and it follows that, in four dimensions, these are the only hyper-Kähler metrics containing a homogeneous metric in its conformal class.

1. PRELIMINARIES

A hypercomplex structure on a $4n$-dimensional manifold M is a family $\mathcal{H} = \{J_\alpha\}_{\alpha=1,2,3}$ of fibrewise endomorphisms of the tangent bundle TM of M satisfying:

$$J_\alpha^2 = -I, \quad \alpha = 1,2,3 \qquad J_1 J_2 = -J_2 J_1 = J_3, \tag{1.1}$$

$$N_\alpha \equiv 0, \qquad \alpha = 1,2,3 \tag{1.2}$$

where I is the identity on the tangent space T_pM of M at p for all p in M and N_α is the Nijenhuis tensor corresponding to J_α:

$$N_\alpha(X, Y) = [J_\alpha X, J_\alpha Y] - [X, Y] - J_\alpha([X, J_\alpha Y] + [J_\alpha X, Y])$$

for all X, Y vector fields on M. A differentiable map $f : M \to M$ is said to be hypercomplex if it is holomorphic with respect to J_α, $\alpha = 1,2,3$. The group of hypercomplex diffeomorphisms on $(M, \mathcal{H})$ will be denoted by $\mathrm{Aut}(\mathcal{H})$.

A riemannian metric g on a hypercomplex manifold $(M, \mathcal{H})$ is called hyper-Hermitian when $g(J_\alpha X, J_\alpha Y) = g(X, Y)$ for all vectors fields X, Y on M, $\alpha = 1,2,3$.

Given a manifold M with a hypercomplex structure $\mathcal{H} = \{J_\alpha\}_{\alpha=1,2,3}$ and a hyper-Hermitian metric g consider the 2-forms $\omega_\alpha, \alpha = 1,2,3$, defined by

$$\omega_\alpha(X, Y) = g(X, J_\alpha Y). \tag{1.3}$$

The metric g is said to be hyper-Kähler when $d\omega_\alpha = 0$ for $\alpha = 1,2,3$.

It is well known that a hyper-Hermitian metric g is conformal to a hyper-Kähler metric $\tilde{g}$ if and only if there exists an exact 1-form $\theta \in \Lambda^1 M$ such that

$$d\omega_\alpha = \theta \wedge \omega_\alpha, \qquad \alpha = 1,2,3 \tag{1.4}$$

where, if $g = e^f\, \tilde{g}$ for some $f \in C^\infty(M)$, then $\theta = df$.

We prove the following result:

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Theorem 1.1. *Let $(M, \mathcal{H}, g)$ be a simply connected hyper-Hermitian 4-manifold. Assume that there exists a Lie group $G \subset I(M,g) \cap Aut(\mathcal{H})$ acting simply transitively on M. Then g is conformally hyper-Kähler.*

We conclude that one of the hyper-Kähler metrics constructed by the Gibbons-Hawking ansatz [2] contains a homogeneous hyper-Hermitian metric in its conformal class. This hyper-Hermitian metric is not symmetric and has negative sectional curvature [1].

As a consequence of Theorem 1.1 and the results in [1] we obtain that the following symmetric riemannian metrics are conformally hyper-Kähler:

- the riemannian product of the canonical metrics on $\mathbb{R} \times S^3$;
- the riemannian product of the canonical metrics on $\mathbb{R} \times \mathbb{R}H^3$, where $\mathbb{R}H^3$ denotes the real hyperbolic space;
- the canonical metric on the real hyperbolic space $\mathbb{R}H^4$.

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2. Proof of the main theorem

Proof of Theorem 1.1. Since G acts simply transitively on M then M is diffeomorphic to G and therefore the hypercomplex structure and hyper-Hermitian metric can be transferred to G and will also be denoted by $\{J_\alpha\}_{\alpha=1,2,3}$ and g, respectively. Since G acts by hypercomplex isometries it follows that both $\{J_\alpha\}_{\alpha=1,2,3}$ and g are left invariant on G. All such simply connected Lie groups were classified in [1], where it is shown that the Lie algebra $\mathfrak{g}$ of G is either abelian or isomorphic to one of the following Lie algebras (we fix an orthonormal basis $\{e_j\}_{j=1,\dots,4}$ of $\mathfrak{g}$):

1. $[e_2, e_3] = e_4, \quad [e_3, e_4] = e_2, \quad [e_4, e_2] = e_3, \quad e_1$ central;
2. $[e_1, e_3] = e_1, \; [e_2, e_3] = e_2, \; [e_1, e_4] = e_2, \; [e_2, e_4] = -e_1$;
3. $[e_1, e_j] = e_j, \; j = 2, 3, 4$;
4. $[e_3, e_4] = \frac{1}{2} e_2, \; [e_1, e_2] = e_2, \; [e_1, e_j] = \frac{1}{2} e_j, \; j = 3, 4$.

Observe that in case 1 above M is diffeomorphic to $\mathbb{R} \times S^3$ while in the remaining cases it is diffeomorphic to $\mathbb{R}^4$, therefore in all cases any closed form on M is exact. We now proceed by finding in each case a closed form $\theta \in \Lambda^1\mathfrak{g}^*$ satisfying (1.4). Note that we work on the Lie algebra level since g and ω_α are all left invariant on G. Let $\{e^j\}_{j=1,\dots,4} \subset \Lambda^1\mathfrak{g}^*$ be the dual basis of $\{e_j\}_{j=1,\dots,4}$. From now on we will write $e^{ij\cdots}$ to denote $e^i \wedge e^j \wedge \cdots$. In all the cases below the 2-forms ω_α are determined from (1.3) in terms of the hypercomplex structures constructed in [1].

Case 1. The 2-forms ω_α are given as follows:

$$\omega_1 = -e^{12} - e^{34}, \qquad \omega_2 = -e^{13} + e^{24}, \qquad \omega_3 = -e^{14} - e^{23}.$$

To calculate $d\omega_\alpha$ we obtain first de^j (recall that $d\sigma(x,y) = -\sigma[x,y]$ for $\sigma \in \Lambda^1\mathfrak{g}^*$):

$$de^1 = 0, \qquad de^2 = -e^{34}, \qquad de^3 = e^{24}, \qquad de^4 = -e^{23}. \tag{2.1}$$

These equations and the fact that $d(\sigma \wedge \tau) = d\sigma \wedge \tau + (-1)^r \sigma \wedge d\tau$ for all $\sigma \in \Lambda^r\mathfrak{g}^*$ give the following formulas:

$$d\omega_1 = -e^{134}, \qquad d\omega_2 = e^{124}, \qquad d\omega_3 = -e^{123}$$

from which we conlude that (1.4) holds for $\theta = e^1$, which is closed and therefore exact since G is diffeomorphic to $\mathbb{R} \times S^3$. We conclude that this hyper-Hermitian metric, which, as shown in [1], is homothetic to the riemannian product of the canonical metrics on $\mathbb{R} \times S^3$, is conformal to a hyper-Kähler metric.

Case 2. In this case we have the following equations for ω_α:

$$\omega_1 = e^{14} - e^{23}, \qquad \omega_2 = -e^{12} + e^{34}, \qquad \omega_3 = -e^{13} - e^{24}.$$

and we calculate

$$de^1 = -e^{13} + e^{24}, \quad de^2 = -e^{23} - e^{14}, \quad de^3 = 0, \quad de^4 = 0, \tag{2.2}$$

$$d\omega_1 = -2e^{134}, \qquad d\omega_2 = -2e^{123}, \qquad d\omega_3 = 2e^{234} \tag{2.3}$$

so that (1.4) is satisfied for $\theta = 2e^3$, which again is closed, so this hyper-Hermitian metric is also conformal to a hyper-Kähler metric. In this case the hyper-Hermitian metric is homothetic to the riemannian product of the canonical metrics on $\mathbb{R} \times \mathbb{R}H^3$, where $\mathbb{R}H^3$ denotes the real hyperbolic space.

Case 3. In this case the 2-forms ω_α are given as follows:

$$\omega_1 = -e^{12} - e^{34}, \qquad \omega_2 = -e^{13} + e^{24}, \qquad \omega_3 = -e^{14} - e^{23}$$

and a calculation of exterior derivatives gives:

$$de^1 = 0, \qquad de^j = -e^{1j}, \quad j = 2, 3, 4 \tag{2.4}$$

$$d\omega_1 = 2e^{134}, \qquad d\omega_2 = -2e^{124}, \qquad d\omega_3 = -2e^{123} \tag{2.5}$$

so that (1.4) is satisfied for $\theta = -2e^1$. This hyper-Hermitian metric is homothetic to the canonical metric on the real hyperbolic space $\mathbb{R}H^4$.

Case 4. In this case we have the following equations for ω_α:

$$\omega_1 = -e^{12} + e^{34}, \qquad \omega_2 = -e^{13} - e^{24}, \qquad \omega_3 = e^{14} - e^{23}$$

and we calculate

$$de^1 = 0, \quad de^2 = -e^{12} - \frac{1}{2}\, e^{34}, \quad de^j = -\frac{1}{2}\, e^{1j}, \quad j = 3, 4 \tag{2.6}$$

$$d\omega_1 = -\frac{3}{2}\, e^{134}, \qquad d\omega_2 = \frac{3}{2}\, e^{124}, \qquad d\omega_3 = \frac{3}{2}\, e^{123} \tag{2.7}$$

so that (1.4) is satisfied for $\theta = -\frac{3}{2}\, e^1$. This hyper-Hermitian metric is not symmetric and has negative sectional curvature (cf. [1]).

Remark 2.1. All the hyper-Hermitian manifolds $(M, \mathcal{H}, g)$ considered above admit a connection ∇ such that:

$$\nabla g = 0, \qquad \nabla J_\alpha = 0, \quad \alpha = 1, 2, 3$$

and the $(3, 0)$ tensor $c(X, Y, Z) = g(X, T(Y, Z))$ is totally skew-symmetric, where T is the torsion of ∇. Such a connection is called an HKT connection (cf. [3]). In case M is diffeomorphic to $\mathbb{R} \times S^3$ it can be shown that, moreover, the corresponding 3-form c is closed.

3. Coordinate description of the Hyper-Kähler metrics

In this section we will use global coordinates on each of the Lie groups considered in the previous section to describe the corresponding hyper-Kähler metrics. This will allow us to identify the hyper-Kähler metric in §2, Case 4, with one constructed by the Gibbons-Hawking ansatz [2].

Case 1. $G = \mathbb{H}^* = GL(1,\mathbb{H}) = \left\{ \begin{pmatrix} x & -y & -z & -t \\ y & x & -t & z \\ z & t & x & -y \\ t & -z & y & x \end{pmatrix} \ : \ (x,y,z,t) \in \mathbb{R}^4 - \{0\} \right\}.$

We obtain a basis of left invariant 1-forms on G as follows. Set $r^2 = x^2+y^2+z^2+t^2$, $r>0$, and $\Omega = g^{-1}dg$ for $g \in G$, that is,

$$\text{if} \quad g = \begin{pmatrix} x & -y & -z & -t \\ y & x & -t & z \\ z & t & x & -y \\ t & -z & y & x \end{pmatrix} \quad \text{then} \quad \Omega = \begin{pmatrix} \sigma_1 & -\sigma_2 & -\sigma_3 & -\sigma_4 \\ \sigma_2 & \sigma_1 & -\sigma_4 & \sigma_3 \\ \sigma_3 & \sigma_4 & \sigma_1 & -\sigma_2 \\ \sigma_4 & -\sigma_3 & \sigma_2 & \sigma_1 \end{pmatrix}$$

where

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{pmatrix} = \frac{1}{r^2} \begin{pmatrix} x & y & z & t \\ -y & x & t & -z \\ -z & -t & x & y \\ -t & z & -y & x \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \\ dt \end{pmatrix}.$$

Then σ_j, $1 \le j \le 4$, is a basis of left invariant 1-forms on G and it follows from $d\Omega + \Omega \wedge \Omega = 0$ that

$$d\sigma_1 = 0, \qquad d\sigma_2 = -2\sigma_3 \wedge \sigma_4, \qquad d\sigma_3 = 2\sigma_2 \wedge \sigma_4, \qquad d\sigma_4 = -2\sigma_2 \wedge \sigma_3.$$

Setting

$$e^1 = 2\sigma_1, \qquad e^2 = 2\sigma_2, \qquad e^3 = 2\sigma_3, \qquad e^4 = 2\sigma_4,$$

so that $\{e^j\}_{1 \le j \le 4}$ satisfy (2.1), the left-invariant hyper-Hermitian metric is

$$g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 = \frac{4}{r^2}(dx^2 + dy^2 + dz^2 + dt^2) \tag{3.1}$$

and since the Lee form is $\theta = e^1 = d(2\log r)$ the corresponding hyper-Kähler metric is $\tilde{g} = e^{-2\log r} g$, that is,

$$\tilde{g} = \frac{4}{r^2}\left(\frac{(dr)^2}{r^2} + (\sigma_2)^2 + (\sigma_3)^2 + (\sigma_4)^2 \right) = \frac{4}{r^4}(dx^2 + dy^2 + dz^2 + dt^2) \tag{3.2}$$

Case 2. Define a product on $\mathbb{R}^4$ as follows:

$$(x,y,z,t)(x',y',z',t') = \\ (x + e^z(x'\cos t - y'\sin t), y + e^z(x'\sin t + y'\cos t), z + z', t + t').$$

This defines a Lie group structure on $\mathbb{R}^4$ that makes it isomorphic to the Lie group considered in §2, Case 2. The following 1-forms are left-invariant with respect to the above product:

$$e^1 = e^{-z}\cos t\, dx + e^{-z}\sin t\, dy, \qquad e^3 = -dz, \tag{3.3}$$

$$e^2 = -e^{-z}\sin t\, dx + e^{-z}\cos t\, dy, \qquad e^4 = -dt \tag{3.4}$$

These forms satisfy relations (2.2). The hyper-Hermitian metric is therefore given as follows:

$$g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 = e^{-2z}(dx^2 + dy^2) + dz^2 + dt^2$$

and the Lee form is $\theta = 2e^3 = -2dz$, so that the hyper-Kähler metric becomes

$$\tilde{g} = e^{2z}g = (dx^2 + dy^2) + e^{2z}(dz^2 + dt^2).$$

Observe that the change of coordinates $s = e^z$ gives the following simple form for $\tilde{g}$ on $\mathbb{R}^+ \times \mathbb{R}^3$:

$$\tilde{g} = dx^2 + dy^2 + ds^2 + s^2 dt^2.$$

Case 3. We endow $\mathbb{R}^4$ with the following product:

$$(x, y, z, t)(x', y', z', t') = (x + e^t x', y + e^t y', z + e^t z', t + t')$$

thereby obtaining the Lie group structure considered in §2, Case 3, with corresponding left-invariant 1-forms:

$$e^1 = dt, \quad e^2 = e^{-t}dx, \quad e^3 = e^{-t}dy, \quad e^4 = e^{-t}dz.$$

The hyper-Hermitian metric is therefore

$$g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 = e^{-2t}(dx^2 + dy^2 + dz^2) + dt^2$$

with corresponding Lee form $\theta = -2e^1 = -2dt$, yielding the following hyper-Kähler metric:

$$\tilde{g} = e^{2t}g = dx^2 + dy^2 + dz^2 + e^{2t}dt^2.$$

Setting $s = e^t$, $\tilde{g}$ is the euclidean metric $ds^2 + dx^2 + dy^2 + dz^2$ on $\mathbb{R}^+ \times \mathbb{R}^3$.

Case 4. Consider the following product on $\mathbb{R}^4$:

$$(x, y, z, t)(x', y', z', t') = (x + e^{\frac{t}{2}}x', y + e^{\frac{t}{2}}y', z + e^t z' + \frac{e^{\frac{t}{2}}}{4}(xy' - yx'), t + t')$$

which yields the Lie group structure considered in §2, Case 4. It is easily checked that the following left-invariant 1-forms satisfy (2.6):

$$e^1 = dt, \quad e^2 = e^{-t}(dz - \frac{1}{4}xdy + \frac{1}{4}ydx), \quad e^3 = e^{-\frac{t}{2}}dx, \quad e^4 = e^{-\frac{t}{2}}dy.$$

The hyper-Hermitian metric is now obtained as in the above cases:

$$\begin{aligned} g = (e^1)^2 + (e^2)^2 + &(e^3)^2 + (e^4)^2 \\ &= dt^2 + e^{-t}(dx^2 + dy^2) + e^{-2t}(dz - \frac{1}{4}(xdy - ydx))^2 \end{aligned}$$

and the Lee form is $\theta = -\frac{3}{2}dt$, from which we obtain the hyper-Kähler metric as usual:

$$\tilde{g} = e^{-\frac{3}{2}t}dt^2 + e^{-\frac{t}{2}}(dx^2 + dy^2) + e^{-\frac{t}{2}}(dz - \frac{1}{4}(xdy - ydx))^2.$$

Setting $s = e^{\frac{t}{2}}$, $\tilde{g}$ becomes

$$\tilde{g} = s(ds^2 + dx^2 + dy^2) + \frac{1}{s}(dz - \frac{1}{4}(xdy - ydx))^2$$

on $\mathbb{R}^+ \times \mathbb{R}^3$, which allows us to identify $\tilde{g}$ with one of the hyper-Kähler metrics constructed by the Gibbons-Hawking ansatz [2]. The identification is easily obtained from [4], Proposition 1.

We can now rephrase Theorem 1.1 as follows, where $[h]$ denotes the conformal class of h:

Corollary 3.1. *Let h be a hyper-Kähler metric on a simply connected hypercomplex 4-manifold $(M,\mathcal{H})$ such that there exist $g \in [h]$ and a Lie group $G \subset I(M,g) \cap Aut(\mathcal{H})$ acting simply transitively on M. Then (M,h) is homothetic to either $\mathbb{R}^4$ with the euclidean metric or one of the following riemannian manifolds:*

1. $M = \mathbb{R}^4 - \{0\}, \;\; h = r^{-4}(dx^2 + dy^2 + dz^2 + dt^2)$,
2. $M = \mathbb{R}^+ \times \mathbb{R}^3, \;\; h = ds^2 + dx^2 + dy^2 + s^2 dt^2$,
3. $M = \mathbb{R}^+ \times \mathbb{R}^3, \;\; h = ds^2 + dx^2 + dy^2 + dz^2$,
4. $M = \mathbb{R}^+ \times \mathbb{R}^3, \;\; h = s(ds^2 + dx^2 + dy^2) + s^{-1}(dz - \frac{1}{4}(xdy - ydx))^2$.

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FaMAF, Universidad Nacional de Córdoba, Ciudad Universitaria, 5000 - Córdoba, Argentina

E-mail address: barberis@mate.uncor.edu