ESI The Erwin Schrödinger International Institute for Mathematical Physics

Boltzmanngasse 9
A-1090 Wien, Austria

A Gerbe Obstruction to Quantization of Fermions on Odd Dimensional Manifolds with Boundary

Alan Carey
Jouko Mickelsson

Vienna, Preprint ESI 797 (1999)

December 2, 1999

Supported by Federal Ministry of Science and Transport, Austria
Available via http://www.esi.ac.at

A GERBE OBSTRUCTION TO QUANTIZATION OF FERMIONS ON ODD DIMENSIONAL MANIFOLDS WITH BOUNDARY

Alan Carey and Jouko Mickelsson

Department of Mathematics, University of Adelaide, Adelaide
SA 5005, Australia, e-mail acarey@ maths.adelaide.edu.au.
Theoretical Physics, Royal Institute of Technology,
Stockholm SE-10044, Sweden, e-mail jouko@ theophys.kth.se

December 1, 1999

ABSTRACT We consider the canonical quantization of fermions on an odd dimensional manifold with boundary, with respect to a family of elliptic hermitean boundary conditions for the Dirac hamiltonian. We show that there is a topological obstruction to a smooth quantization as a function of the boundary conditions. The obstruction is given in terms of a gerbe and its Dixmier-Douady class is evaluated.

0. INTRODUCTION

In this paper we study the Hamiltonian quantization of massless fermions on a compact odd-dimensional manifold X with boundary $Y = \partial X$. Field theories on manifolds with boundary arise in several situations including gravitation on odd dimensional anti-de Sitter spacetimes [W]. Our aim is to investigate topological obstructions (or anomalies) arising from non-trivial topology in the boundary conditions. We assume that a Riemannian metric and spin structure is given on X. The Dirac field might also be coupled to a Yang-Mills potential but in this paper we shall concentrate only on the problem of the dependence of the canonical quantization on the boundary conditions.

As an additional technical assumption we require that the metric becomes a product metric near the boundary and that the normal component of the gauge potential A and the normal derivatives of A vanish at the boundary. Then near the boundary the Dirac equation can be written as

$$ic_t \partial_t \psi = -h_Y \psi,$$

where h_Y is the Dirac operator on the boundary, t is a local coordinate in the normal direction and c_t denotes Clifford multiplication by the unit normal in the t direction. The operator h_Y anticommutes with Clifford multiplication by c_t.

In order that the Dirac operator in the bulk X becomes an elliptic Fredholm operator we use Atiyah-Patodi-Singer type boundary conditions. These elliptic boundary conditions are labelled by projection operators P on the boundary Hilbert space $L^2(Y, S)$ (where S denotes the combined spin and gauge vector bundle) such that the difference $P - P_+$ is a trace-class operator. Here P_+ is the projection on to the spectral subspace on which h_Y is positive. The boundary condition is then written as

$$P\psi|_Y = 0.$$

The precise estimate on $P - P_+$ is actually not very critical in the following. Often one requires that $P - P_+$ is a smoothing operator. This has advantages when studying the analytical properties of Dirac determinants. However, for analyzing topological properties of the family of Dirac operators parametrized by such boundary conditions it turns out to be more convenient to work in the trace-class setting.

In order that D_P is hermitean we must have

$$Pc_t = c_t(1 - P).$$

A useful way to parametrize the boundary projections P above is to think of $P(L^2(Y, S))$ as the graph of an unitary operator

$$T : S^+ \to S^-,$$

where $S^\pm$ are the eigenspaces of the chirality operator c_t on the boundary. If T_0 is the unitary operator corresponding to the projection P_+ then $K = TT_0^{-1}$ should differ from the identity operator in $L^2(Y, S^-)$ by a trace-class operator.

Denote by G the group of all unitaries g in $L^2(Y, S^-)$ such that $g - 1$ is a trace-class operator. Thus G is the parameter space of the elliptic hermitean boundary conditions for the family of Dirac operators D_P and henceforth we write D_g, $g \in G$ to denote elements in this family.

For each $g \in G$ we would like to produce a fermionic Fock space $\mathcal{F}_g$ carrying an irreducible quasi-free representation of the canonical anticommutation relations (CAR) and a compatible action of the second quantized Dirac operator D_g. Here quasi-free means that there is a vacuum vector $|0> \in \mathcal{F}_g$ such that

$$a(v)|0>= 0 = a^*(u)|0>$$

for all $v \in H_+$ and $u \in H_-$ where $H_+ \oplus H_-$ is the polarization of the 1-particle Hilbert space $H = L^2(X, S)$ to positive and negative frequencies with respect to the Dirac operator D_g. The CAR algebra is generated by the relations

$$a^*(u)a(u') + a(u')a^*(u) =< u', u >$$

for $u, u' \in H$ with all other anticommutators zero and the Hilbert space inner product $< \cdot, \cdot >$ is antilinear in the first argument.

Note that in general one cannot expect that there would be a continuous choice $|0>= |0>_g$ of vacuum vectors parametrized by the boundary conditions $g \in G$; the vacuum line bundle (in case it is well-defined) could be nontrivial. In fact, we shall show that there is a more serious problem: There is a topological obstruction to the above quantization of fermions parametrized smoothly by elements of G.

The obstruction appears as follows. In order to define the quasi-free representation of the CAR algebra we need a polarization $H = H_+(g) \oplus H_-(g)$ of the 1-particle space H. This polarization should be a continuous function of the boundary condition $g \in G$. Furthermore, each Fock space $\mathcal{F}_g$ defined by the polarization should contain the vacuum vector for the Dirac operator $D_g = ic_t\partial_t + h_{Y,g}$, that is, the vacuum defined by the splitting to positive and negative spectral subspaces of D_g. This requirement is equivalent to the condition that the plane $H_+(g)$ lies in the infinite-dimensional Grassmannian Gr_g consisting of all closed subspaces $W \subset H$ such that the difference $P_g - Q_g$ is Hilbert-Schmidt; here P_g is the orthogonal projection onto $H_+(g)$ and Q_g is the orthogonal projection onto the positive spectral subspace of D_g.

We have now a bundle of infinite-dimensional Grassmannians $\{Gr_g | \; g \in G\}$ (modelled by the ideal of Hilbert-Schmidt operators) over the base G. We still need to show that this bundle is defined in terms of local trivializations and smooth transition functions. Once this is done we will show that the potential obstruction to quantization is the topological nontriviality of this bundle.

1. THE OBSTRUCTION TO CANONICAL QUANTIZATION

We first need to deal with a technicality.

Proposition 1. *The bundle Gr is smoothly locally trivial.*

Proof. Let T_0 be any fixed boundary condition, with $T_0 : S^+ \to S^-$ a unitary map. If T is another boundary condition then the graph of T is obtained from the graph of T_0 by the unitary transformation $(u_+, u_-) \mapsto (u_+, g \cdot u_-)$ with $g = TT_0^{-1} \in G$ and $u_\pm \in S^\pm$. In a small open neighborhood U of $1 \in G$ we can choose in a smooth way, for any $g \in U$, a smooth path $g(t)$ such that $g(0) = 1$ and $g(1) = g$; this is achieved for example by writing $g = \exp(Z)$ and setting $g(t) = \exp(tZ)$.

Near the boundary Y we may think of the L^2 functions on X as functions $f(t, y)$ on $[0, 1] \times Y$ (where t is a parameter in the normal direction at the boundary) and we can extend the action of g in the boundary Hilbert space $L^2(Y, S)$ to an action on $H = L^2(X, S)$ by setting

$$(R(g)f)(t, x) = (f_+(t, x), g(t) \cdot f_-(t, x))$$

in the tubular neighborhood $[0, 1] \times Y$ and $R(g)$ acts as an identity on f outside of this neighborhood. The map $g \mapsto R(g)$ is smooth with respect to the L^1 norm in G and the operator norm in the algebra of bounded operators in $L^2(X, S)$. This follows from the smoothness of the embedding of trace-class operators to the algebra of bounded operators and from the smoothness of the exponential mapping.

Clearly $R(g)$ is an unitary operator in $L^2(X, S)$ and it maps the domain $dom(D_g)$ onto the domain $dom(D_1)$ of the reference Dirac operator. Thus D_g is unitarily equivalent to the Dirac operator $R(g)D_gR(g)^{-1}$ in the fixed reference domain $dom(D_1)$.

Next we choose a smooth mapping from the space of polarizations ϵ in H to the unitary group $U(H)$ such that $\epsilon = F(\epsilon)\epsilon_0 F(\epsilon)^{-1}$ where ϵ_0 is the fixed reference polarization given by the sign of the Dirac operator D_1. This mapping exists because the space of polarizations $U(H)/(U(H_+) \times U(H_-))$ is a contractible Banach manifold (with respect to the operator norm) by Kuiper's theorem.

With these tools we can write an explicit local trivialization of the Grassmannian bundle Gr over G. Near the unit element in G the Hilbert-Schmidt Grassmannians are parametrized as

$$(g, \epsilon) \mapsto F(\epsilon_g)\epsilon F(\epsilon_g)^{-1},$$

where $\epsilon \in Gr_1$ and $\epsilon_g = R(g)\frac{D_g}{|D_g|}R(g)^{-1}$. We have assumed that zero does not belong to the spectrum of D_g; otherwise, we replace D_g by $D_g - \lambda$ for some real number λ in the neighborhood of $g = 1$.

Because of the potential non-triviality of the kernel of the operator D_g we cannot have a global trivialization of the bundle. However, for each real number λ the trivialization described above is well-defined in the open set G_λ consisting of those elements g for which $\lambda \notin Spec(D_g)$. The transition function on the overlap $G_\lambda \cap G_\mu$ is then given by

$$\epsilon \mapsto F(\epsilon_g(\lambda))^{-1}F(\epsilon_g(\mu))\epsilon F(\epsilon_g(\mu))^{-1}F(\epsilon_g(\lambda)),$$

where $\epsilon_g(\lambda)$ is defined as ϵ_g above but with the shifted operator $D_g - \lambda$. By the construction, the transition function $h_{\mu\lambda}(g) = F(\epsilon_g(\mu))^{-1}F(\epsilon_g(\lambda))$ satisfies

$$[\epsilon_0, h_{\mu\lambda}] = F(\epsilon_g(\mu))^{-1}\Delta_{\mu\lambda}F(\epsilon_g(\mu))$$

with $\Delta_{\mu\lambda} = \epsilon_g(\mu) - \epsilon_g(\lambda)$. Now on the overlap $G_\lambda \cap G_\mu$ the difference $\Delta_{\mu\lambda}$ has constant finite rank and therefore also $[\epsilon, h_{\mu\lambda}]$ has constant finite rank. A norm continuous mapping to operators of constant finite rank is continuous also with respect to the Hilbert-Schmidt norm (or with respect to any L_p norm) which proves the continuity of the transition functions. The same argument can then be used for the derivatives of the transition function.

A closer examination reveals that the discussion of the introduction is not quite accurate in the sense that often one would be satisfied with a determination of the CAR algebra representation without an explicit choice of the vacuum vector.

To explain this let us denote by U_{res} the group of unitaries T in a complex polarized Hilbert space $H = H_+ \oplus H_-$ such that the commutator $[P_+, T]$ is Hilbert-Schmidt; here P_+ is the orthogonal projection onto H_+. This group acts naturally and transitively on the Grassmannian $Gr(H_+)$ consisting of closed subspaces $W \subset$

H such that $P_W - P_+$ is Hilbert-Schmidt, where P_W is the orthogonal projection onto W.

Over $Gr(H_+)$ there is a canonical complex line bundle DET. When the projection from W to H_+ has Fredholm index equal to zero, the fiber at W is the set of equivalence classes $[q, \lambda]$, where $q : H_+ \to W$ is an isomorphism such that $P_+q - id_{H_+}$ is trace-class and $\lambda \in \mathbb{C}$. The equivalence is defined by $(q, \lambda) \sim (qt^{-1}, \lambda \det t)$, where $t : H_+ \to H_+$ is an isomorphism with $t - 1$ trace-class; for details, see [PrSe].

A central extension $\hat{U}_{res}$ of U_{res} acts in the total space of DET, [PrSe]. This extension acts unitarily in the fermionic Fock space corresponding to the given polarization $H = H_+ \oplus H_-$.

If a representation of the CAR is given with respect to a polarization $H = H_+ \oplus H_-$ then the set of (normalized) Fock vacua is a $\hat{U}_{res}$ orbit through some fixed vacuum $|H_+ >$ defined by the polarization. The orbit of H_+ under the U_{res} action is the infinite-dimensional Grassmannian $Gr(H_+)$ and the $\hat{U}_{res}$ orbit in the Fock space is the set of vectors of unit length in the canonical determinant bundle DET over $Gr(H_+)$.

In the case of a family of Grassmannians, the construction of the family of CAR represention can now be formulated as the problem of prolonging the Grassmannian bundle to a bundle with fiber equivalent to the determinant bundle DET. There is an illuminating alternative formulation of the prolongation problem which we shall now describe.

The Grassmannian $Gr(H_+)$ is a homogeneous space $U_{res}/(U(H_+) \times U(H_-))$. The topology of the block diagonal unitary group $U(H_+) \times U(H_-)$ is trivial by Kuiper's theorem. Thus U_{res} contracts to $Gr(H_+)$. It follows that the prolongation of a U_{res} bundle over some base manifold to a $\hat{U}_{res}$ bundle is equivalent to the problem of prolonging the associated Grassmann bundle to a bundle with model fiber DET. The relevance of the $\hat{U}_{res}$ bundle in Fock space quantization is immediate: selecting a model Fock space with a $\hat{U}_{res}$ action, one can construct a bundle of Fock spaces as an associated vector bundle to a given $\hat{U}_{res}$ bundle.

To close the circle, we note that starting from the given Grassmanian bundle over the parameter space G one constructs a natural U_{res} bundle $\mathcal{P}$ such that the Grassmannian bundle is recovered as an associated bundle. If $H = H_+ \oplus H_-$ is any fixed polarization then the fiber of $\mathcal{P}$ at $g \in G$ consists of all unitaries h in H such that $h \cdot H_+ \in Gr_g$.

Instead of looking at the specific construction of the U_{res} bundle over the family of boundary conditions G we can extend this to another universal construction over the space of all (bounded) self-adjoint Fredholm operators $\mathcal{F}_*$ such that the essential spectrum is neither negative nor positive. This means that the spectral subspaces both on the negative and positive side of the real axis are infinite-dimensional. The topology in $\mathcal{F}_*$ is defined by the operator norm. The fact that Dirac operators are unbounded need not bother us since we shall be really interested only on the spectral resolutions defined by the sign operators $(D-\lambda)/|D-\lambda|$, which are bounded.

The space $\mathcal{F}_*$ retracts onto the subspace $\hat{\mathcal{F}}_*$ consisting of operators with essential spectrum at ± 1, [AS]. Thus it is sufficient to study the U_{res} bundle $\mathcal{P}'$ over $\hat{\mathcal{F}}_*$. The fiber $\mathcal{P}'_A$ at $A \in \hat{\mathcal{F}}_*$ is defined as the set of unitary operators g in $H_+ \oplus H_-$ such that $g \cdot H_+$ belongs to the Grassmannian $Gr(W_A)$ where W_A is the positive spectral subspace of the operator A.

The proof of the local triviality of the bundle $\mathcal{P}'$ is a slight extension of the argument which was used in the proof of Proposition 1. First, we can choose an operator norm continuous (smooth) section $F : U(H)/(U(H_+) \times U(H_-)) \to U(H)$ (again by Kuiper's theorem). Defining U_λ, for $1 > \lambda > -1$, as the open set in $\hat{\mathcal{F}}_*$ consisting of operators A such that $\lambda \neq Spec(A)$ then $g_\lambda(A) = F((A-\lambda)/|A-\lambda|)$ is a local section of $\mathcal{P}'$ with local norm continuous transition functions $g_{\lambda\mu} = g_\lambda^{-1} g_\mu$. The Hilbert-Schmidt norm continuity (and smoothness) of the off-diagonal blocks follows as in Proposition 1 since, by our assumption about the essential spectrum of A, the spectral subspaces corresponding to the open intervals $]\mu, \lambda[$ are finite-dimensional for $-1 < \mu < \lambda < 1$.

2. A UNIVERSAL CONSTRUCTION

In this section G denotes the group of unitary operators g in H such that $g-1$ is trace-class.

Let $\mathcal{P}$ be a locally trivial principal U_{res} bundle over G. The Lie algebra of $\hat{U}_{res}$

is defined by the standard 2-cocycle

$$c(X,Y) = \frac{1}{4}\mathrm{tr}\,\epsilon[\epsilon,X][\epsilon,Y]. \tag{1}$$

Let $(\mathcal{U}_\alpha)$ be a family of open contractible sets covering the base G. Let $\mathcal{L} \to U_{res}$ be the complex line bundle associated to the circle bundle $1 \to S^1 \to \hat{U}_{res} \to U_{res} \to 1$. Let $\phi_\alpha : \mathcal{U}_\alpha \to \mathcal{P}$ be a family of local trivializations of the bundle $\mathcal{P}$. Let $g_{\alpha\beta}$ be the corresponding family of U_{res} valued transition functions. We define a family of local line bundles over the open sets $\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ by pull-back, $\mathcal{L}_{\alpha\beta} = {\phi_{\alpha\beta}}^* \mathcal{L}$.

Since $\hat{U}_{res}$ is a group we have a natural isomorphism

$$\mathcal{L}_g \otimes \mathcal{L}_f \equiv \mathcal{L}_{gf} \tag{2}$$

for all $g, f \in U_{res}$. This gives a family of isomorphisms

$$\mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} = \mathcal{L}_{\alpha\gamma} \tag{3}$$

over the common intersections $\mathcal{U}_{\alpha\beta\gamma} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$. Thus we have a bundle gerbe $\mathcal{Q}$ over the base G. The product structure gives also a natural isomorphism $\mathcal{L}_g \equiv {\mathcal{L}_{g^{-1}}}^{-1}$ and therefore an isomorphism $\mathcal{L}_{\alpha\beta} \equiv {\mathcal{L}_{\beta\alpha}}^{-1}$. Combining this with (3) we obtain a natural trivialization $f_{\alpha\beta\gamma}$ of the product bundle $\mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \otimes \mathcal{L}_{\gamma\alpha}$ over $\mathcal{U}_{\alpha\beta\gamma}$.

The family $\{f_{\alpha\beta\gamma}\}$ of trivializations (local S^1 valued functions) satisfies the cocycle condition

$$f_{\beta\gamma\delta} f^{-1}_{\alpha\gamma\delta} f_{\alpha\beta\delta} f^{-1}_{\alpha\beta\gamma} = 1$$

on the intersections of four open sets.

Because of the relations (3) the local curvature forms $\omega_{\alpha\beta} = \phi^*_{\alpha\beta} c$ satisfy the relations

$$[\omega_{\alpha\beta}] + [\omega_{\beta\gamma}] + [\omega_{\gamma\alpha}] = 0 \tag{4a}$$

in de Rham cohomology $H^2(\mathcal{U}_{\alpha\beta\gamma})$ on the base. Note that these equations do not hold on the level of differential forms. However, this can be corrected by adding an exact 2-form $d\theta_{\alpha\beta}$ to each of the closed forms $\omega_{\alpha\beta}$; the modified forms $\omega_{\alpha\beta}$ satisfy then

$$\omega_{\alpha\beta} + \omega_{\beta\gamma} + \omega_{\gamma\alpha} = 0. \tag{4b}$$

Actually, because of the given local trivializations $f_{\alpha\beta\gamma}$ on triple intersections we have the consistency condition

$$\nabla_{\alpha\beta\gamma} f_{\alpha\beta\gamma} = 0, \tag{5}$$

where $\nabla_{\alpha\beta\gamma}$ is the connection on the trivial bundle $\mathcal{L}_{\alpha\beta}\otimes\mathcal{L}_{\beta\gamma}\otimes\mathcal{L}_{\gamma\alpha}$ composed from the connections on the factors $\mathcal{L}_{\alpha\beta}$ with curvature forms $\omega_{\alpha\beta}$. In fact, (5) implies (4b): If $A_{\alpha\beta}$ is a local potential, $dA_{\alpha\beta} = \omega_{\alpha\beta}$, then (5) can be written as

$$df_{\alpha\beta\gamma} + (A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha}) f_{\alpha\beta\gamma} = 0,$$

and multiplying by $f^{-1}_{\alpha\beta\gamma}$ and then taking the exterior derivative gives the cocycle relations (4b) for the forms $\omega_{\alpha\beta}$.

If (ρ_α) is a partition of unity on G subordinate to the covering $(\mathcal{U}_\alpha)$ then we can produce a closed 3-form on G in the usual way. First, we have a closed 3-form ω_α on $\mathcal{U}_\alpha$,

$$\omega_\alpha = d\sum_\beta \omega_{\alpha\beta}\rho_\beta = \sum_\beta \omega_{\alpha\beta} d\rho_\beta.$$

Since $\omega_\alpha - \omega_\beta = 0$ on $U_{\alpha\beta}$ they can be pasted together to give the closed 3-form ω_3 on G. This is the de Rham form of the Dixmier-Douady (DD) class of the bundle gerbe. Of course, this description disregards all potential torsion information. However, for our purposes the differential form picture is quite sufficient since we going to show that there is already on this level an obstruction to quantization.

Unfortunately the existence of a partition of unity on the infinite-dimensional manifold G does not appear to be known. However, in order to define the DD class as an element of the dual of the 3-homology classes it is sufficient to use the partition of unity on the (singular) homology 3-cycles and to pull-back the forms $\omega_{\alpha\beta}$ down to the embedded 3-cycles and then proceed as above. An alternative solution would be to replace G by the group of unitaries differing from the identity by a Hilbert-Schmidt operator (which is a Hilbert manifold) where we would have a partition of unity.

Note however that whichever method we use to define the DD class we can normalize it so that its integral around closed 3-cycles is 2π times an integer.

In the case when the gerbe is coming from a principal U_{res} bundle $\mathcal{P}$ over G there is another method to construct the Dixmier-Douady class which gives directly the integrals of ω_3 over 3-cycles in G. The homology cycles can be generated by mappings from S^3 to G, so we shall restrict to this case.

Map the 3-disk D^3 onto the sphere S^3 such that the boundary is mapped to one point $g \in S^3$. Pulling back the bundle $\mathcal{P}$ to D^3 leads to a trivial U_{res} bundle over D^3. In this trivialization the boundary $S^2 \subset D^3$ is mapped to the fiber $\mathcal{P}_g$ over g. Selecting a base point x in the fiber we can identify $\mathcal{P}_g \simeq U_{res}$. The integral of the curvature form c over S^2 gives then the integral of ω_3 over $S^3 \subset G$. Note that the result does not depend on the choice of the base point x since a different choice x' is related by a right translation $x' = x \cdot q$ by an element $q \in U_{res}$. The cohomology class $[c]$ is invariant under right (and left) translations on the group manifold. One can check that this construction gives the same result as the one starting from the cocycle of 2-forms $\omega_{\alpha\beta}$. Namely, selecting a local trivialization of the bundle $\mathcal{P}$ at $g \in G$ such that g is mapped to the point x in the fiber, we observe that the map $S^2 \to U_{res}$ above is just the transition function from the local trivialization over D^3 to the local trivialization around g. The rest follows from Stokes' theorem, using the fact that $\omega_{\alpha\beta} = \omega_\alpha - \omega_\beta$ on the overlap and that $d\omega_\alpha = d\omega_\beta = \omega_3$.

Universal U_{res} bundle over G. Let $\mathcal{P}$ be the space of smooth paths (parametrized by $0 \leq t \leq 2\pi$) in G with initial point $1 \in G$. We also require that the derivatives of $g(t)$ vanish at the end points. To each $g \in \mathcal{P}$ there corresponds a vector potential A on the circle S^1, with values in the Lie algebra $\mathbf{g}$ of G, $A(t) = g(t)^{-1}g'(t)$. The group ΩG of based loops at $1 \in G$ acts freely from the right on $\mathcal{P}$ through gauge transformations $A^g = g^{-1}Ag + g^{-1}g'$ and the set of orbits is $\mathcal{P}/\Omega G = G$. Since clearly $\mathcal{P}$ is contractible, $\mathcal{P}$ is the universal ΩG bundle over G. The local triviality of the path fibration is obtained by the exponential mapping; locally, near the unit element, the trivialization is $(g, h) \mapsto \tilde{h}$, where $g \in G$, $h(t)$ is a based loop at 1, and $\tilde{h}(t) = \exp(tZ)h(t)$ with $\exp(Z) = g$.

By Bott periodicity, ΩG is homotopy equivalent with U_{res}. Actually, we can define a group homomorphism $j : \Omega G \to U_{res}$ which is a homotopy equivalence. The group G acts, by definition, as the unitary group of 1 + trace-class operators in a complex Hilbert space H. Let $\mathcal{H} = L^2(S^1, H)$ and let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be the polarization to nonnegative and negative Fourier components. Then each $g \in \Omega G$ acts naturally in $\mathcal{H}$, by pointwise multiplication, and this action gives the promised homomorphism to $U_{res}(\mathcal{H}_+ \oplus \mathcal{H}_-)$. The homomorphism can be used to define an associated bundle $\tilde{\mathcal{P}} = \mathcal{P} \times_{\Omega G} U_{res}$ with fiber U_{res}. This latter bundle is then a universal U_{res} bundle over G, see Proposition 2.

Remark. There is an alternative construction of the universal U_{res} bundle over G which makes its appearance in quantum field theory (QFT) more transparent. For each gauge potential $A(t) = g(t)^{-1} dg(t), g \in \mathcal{P}$ let W_A be the set of unitary operators T in $\mathcal{H}$ such that the projections onto the subspaces $T \cdot \mathcal{H}_+$ and $\mathcal{H}_+(D_A)$ differ by a Hilbert-Schmidt operator; here $\mathcal{H}_+(D_A)$ is the positive frequency subspace for the Dirac operator D_A. It is easy to see that if both $T, T' \in W_A$ then $T' = Th$ for some $h \in U_{res}$. Consequently, one can view W_A as a fiber in a principal U_{res} bundle over $\mathcal{P}$ [CMM2]. The base is contractible, thus there exists a global section $A \mapsto T_A$. Actually, we can construct explicitly a global section by setting

$$T_A = T_{f^{-1}df} = f,$$

that is, T_A is the multiplication operator by the function $f(t)$ on the interval $[0, 1]$.

We can define a U_{res} valued 1-cocycle for the natural ΩG action on $\mathcal{P}$ by

$$S(A; g) = T_A^{-1} T_{A^g}.$$

In the case of the choice $T_A = f$ above, we have simply $S(A; g) = g$. This cocycle defines the same associated U_{res} bundle $\tilde{\mathcal{P}}$ over G as above, by the equivalence relation $(A, T) \simeq (A^g, TS(A; g))$ for $g \in \Omega G$ and $T \in W_A$.

Proposition 2. *The total space of the bundle $\tilde{P}$ is contractible and thus $\tilde{P} \to G$ is the universal U_{res} bundle over G.*

Proof. By a well known theorem in homotopy theory [G] a space with the homotopy type of a CW complex is contractible if it is weakly homotopy equivalent to a point. All of the spaces we consider are Banach manifolds and have the homotopy type that of a CW complex. We want to compare the homotopy groups of $\mathcal{P}$, which are all trivial, with the homotopy groups $\pi_i(\tilde{\mathcal{P}})$. From the appendix we know that the embedding j of ΩG to U_{res} is a homotopy equivalence. This embedding extends to an embedding of principal bundles $\tilde{j} : \mathcal{P} \to \tilde{\mathcal{P}}$ by mapping $p \in \mathcal{P}$ to the equivalence class of the pair $(p, e) \in \mathcal{P} \times U_{res}$ in $\tilde{\mathcal{P}}$; e is the identity element in U_{res}. On each fibre this map reduces to j. This means we have a commutative diagram

$$\begin{array}{ccc} \pi_i(\Omega G) & \xrightarrow{j^*} & \pi_i(U_{res}) \\ \nwarrow & & \nearrow \\ & \pi_{i+1}(G) & \end{array}$$

where the up arrows represent the connecting maps for the homotopy long exact sequences for the locally trivial fibrations $\mathcal{P}$ and $\tilde{\mathcal{P}}$ respectively.

Since $\pi_i(P) = 0$ for all i, by the homotopy exact sequence

$$...\pi_{i+1}(G) \to \pi_i(\Omega G) \to \pi_i(P) \to \pi_i(G) \to ...$$

the connecting map $k : \pi_{i+1}(G) \to \pi_i(\Omega G)$ is an isomorphism (both groups are known to be 0 when i is odd and equal to $\mathbb{Z}$ when i is even (these facts rely on results of Palais, see [Q] for a discussion). From the commutative diagram above it follows that the corresponding map for the fibering $U_{res} \to \tilde{P} \to G$ is also an isomorphism and from the homotopy long exact sequence for this fibering we conclude that $\pi_i(\tilde{P}) = 0$. Moreover this shows that $\tilde{j}$ is a weak homotopy equivalence from which we deduce that $\tilde{P}$ is a contractible.

3. THE DIXMIER DOUADY CLASS

Next we want to relate the Dixmier-Douady class $\omega_3 \in H^3(G, \mathbb{Z})$ of the various gerbes over G to the natural curvature 2-form on the group ΩG. Recall that this curvature form on ΩG is the homogeneous 2-form

$$\omega_2 = \frac{1}{2\pi}\int_{S^1} \mathrm{tr}\,(g^{-1}dg)\frac{d}{dt}(g^{-1}dg). \tag{6}$$

We start from the universal ΩG bundle $\mathcal{P}$ over G and its bundle gerbe as described in Section 2.

Proposition 3. *The cohomology class $[\omega_3]$ is represented by the closed 3-form*

$$\theta_3 = \frac{1}{12\pi} tr\,(g^{-1}dg)^3. \tag{7}$$

Proof. We have to show that the integral of θ_3 over any closed 3-cycle on G agrees with the integral of ω_3. We can generate 3-cycles by 3-spheres, so we take as the 3-cycle a (differentiable) mapping χ of S^3 into G. Since G is connected, we can assume the image under χ of S^3 is such that the poles of S^3 are mapped to the unit element in G.

We cut the 3-cycle S^3 along the equator to two disks B_+ and B_-. Over these disks the pullback under χ of the principal bundle $\mathcal{P} \to G$ is trivial; choose a pair of local trivializations $\psi_\pm$ over $B_\pm$. Concretely, the local trivializations can be defined as follows. For each $x \in S^2$ on the equator we have path $t \mapsto \phi_+(x)(t) \in S^3$ by connecting the point x by a segment of a great circle to the to the 'north pole'. Similarly, we obtain a path $\phi_-(x)(t)$ by connecting x by a great circle to the 'south pole'. We choose the parametrization such that $t = 2\pi$ corresponds to the point x and $t = 0$ corresponds to either of the poles (which is mapped to the unit element in G). Setting $\psi_\pm = \chi \circ \phi_\pm$ the transition function on the equator (with values in ΩG) is then given by $\psi_+(x,t) = \psi_-(x)(t)g(x,t)$ for some $g(x,.) \in \Omega G$. On the other hand, $(x,t) \mapsto \psi_\pm(x,t)$ is a parametrization of points on $\chi(B_\pm) \subset G$ and therefore, by a simple calculation,

$$
\begin{aligned}
12\pi \int_{S^3} \theta_3 &= \sum_{\alpha=\pm} \int_{B_\pm} \operatorname{tr} ({\psi_\alpha}^{-1} d\psi_\alpha)^3 \\
&= \int_{S^2 \times S^1} \operatorname{tr} (g^{-1} dg)^3 + 3 \int_{S^2} \operatorname{tr} {\psi_-}^{-1} d\psi_- \wedge dg g^{-1}.
\end{aligned}
$$

The second term on the right vanishes since $g(x, 2\pi) = 1$ on S^2. The first term on the right is

$$
\int_{S^2 \times S^1} \operatorname{tr} (g^{-1} dg)^3 = 12\pi \int_{S^2} \omega_2. \tag{8}
$$

On the other hand, the integral on the right is equal to the integral of ω_3 over S^3; this follows, by Stokes' theorem, directly from the construction of the DD class ω_3 from the family of local 2-forms η_α such that $\eta_\alpha - \eta_\beta = \omega_{\alpha\beta}$ and $d\eta_\alpha = \omega_\alpha = \omega_3|_{U_\alpha}$; recall that the class $[\omega_{\alpha\beta}]$ is given by the pull-back of ω_2 with respect to the transition function $g : U_{\alpha\beta} \to \Omega G$.

Thus indeed

$$
\int_{S^3} \theta_3 = \int_{S^3} \omega_3 \tag{9}
$$

and so ω_3 and θ_3 represent the same cohomology class.

The DD class is an obstruction to writing the line bundles $\mathcal{L}_{\alpha\beta}$ as tensor products $\mathcal{L}_\alpha \otimes \mathcal{L}^*_\beta$ of local line bundles over the open sets $\mathcal{U}_\alpha$ on the base, that is, an obstruction to a trivialization of the gerbe $\mathcal{Q}$.

We use the following theorem, taken from [[K], Theorem 3.17] in a slightly reformulated form:

Theorem 1. *[K]. Let M be a compact space and $[M, G]$ the set of homotopy classes of maps from M to G. Then $K^1(M)$ is isomorphic with $[M, G]$. The group structure in the latter group is given by pointwise multiplication of maps.*

On the other hand, we have seen that $\mathcal{P}$ can be viewed as an universal U_{res} bundle (or, what is essentially the same, as an $\Omega\mathcal{G}$ bundle) over G. Thus we can construct U_{res} bundles over any compact space M by pulling back this universal bundle via a map from M to G.

The index theorem in [MP], Proposition 12, tells us that for a compact subset $M \subset G$ $K^1(M)$ is realised by homotopy classes of maps into the family of odd-dimensional Dirac operators parametrized by the boundary conditions M. By the above theorem we can conclude that the universal U_{res} bundle over G can be identified as a (universal) U_{res} bundle over the family of Dirac operators (identified topologically as the family G of boundary conditions).

In the case of the universal U_{res} bundle over G we already know the DD class of the bundle gerbe $\mathcal{Q}$ is given by the generator (7) (devided by 2π) of $H^3(G, \mathbb{Z})$. Thus this is also the (nontrivial) obstruction to a trivialization of the gerbe over the space of Dirac operators parametrized by the boundary conditions G.

As a conclusion we obtain our main result:

Theorem 2. *There is an obstruction to a prolongation of the U_{res} bundle $\mathcal{P}$ (as defined in the end of Section 1) over a compact submanifold $M \subset G$ of hermitean elliptic boundary conditions, that is, an obstruction to the construction of the bundle of fermionic Fock spaces for the Dirac operators parametrized by M. The Dixmier-Douady class of the obstruction is given by the restriction of the de Rham class θ_3 to M.*

In particular, the obstruction is nontrivial when $M = U(N)$ is any finite-dimensional subgroup of G with $N \geq 2$.

Remark 1. We could interpret our family of Dirac operators parametrised by G as an element of $K^1(G)$ if the latter were defined as homotopy classes of maps into the self adjoint Fredholm operators. As G is not compact this is problematic. For our purposes it is enough to work with compact subsets.

Remark 2. In the case of an odd-dimensional manifold without boundary there is a similar obstruction to *gauge invariant* quantization, related to Schwinger terms in current algebra, [CMM1, CMM2]. Recently the case of gravitational Schwinger terms was discussed in the same formalism, [EM].

APPENDIX: PROOF OF THE HOMOTOPY EQUIVALENCE j

We define a system of closed n forms ($n = 2, 4, 6, \dots$) on the Hilbert-Schmidt Grassmannian $Gr(H_+)$ for the polarization $H = H_+ \oplus H_-$, by

(A-1) $$\omega_n = a_n \mathrm{tr}\, F (dF)^n,$$

where F is the grading operator associated to $W \in Gr(H_+)$, that is, F restricted to W is $+1$ and the restriction to W^- is -1. Note that since $F^2 = 1$, the differentials dF anticommute with F. Thus for odd n the form ω_n vanishes identically; α_n is a normalization coefficient given by

$$a_n = -(\frac{1}{2\pi i})^j \frac{(j-1)!}{(2j-1)!}, \text{ with } n = 2j.$$

With this normalization the form ω_n is the generator in $H^n(Gr(H_+), \mathbb{Z})$.

Since $U_{res}/(U(H_+) \times U(H_-)) = Gr(H_+)$ and the diagonal subgroup is contractible, the natural projection $p : U_{res} \to Gr(H_+)$ can be used to pull back the generator ω_n to the generator ϕ_n in $H^n(U_{res}, \mathbb{Z})$. The projection can be written as $p(g) = g\epsilon g^{-1} = F_g$, where ϵ is the grading associated to $W = H_+$. It follows that the pull-back of dF is $[\theta, F_g]$, where $\theta = dgg^{-1}$ is the Maurer-Cartan 1-form on U_{res}, and so

(A-2) $$\phi_n = a_n \mathrm{tr}\, F_g [\theta, F_g]^n.$$

The homotopy type of both ΩG and U_{res} is known. The homotopy groups vanish in odd dimensions whereas in even dimensions the homotopy groups are all isomorphic with $\mathbb{Z}$, [C],[PrSe], [Q]. A generator x_n of the homotopy group $\pi_n(U_{res})$ when paired with the generator ϕ_n of $H^n(U_{res}, \mathbb{Z})$ gives $< x_n, \phi_n >= 1$. Thus the only thing we need to check to prove that the embedding $j : \Omega G \to U_{res}$ is a homotopy equivalence is to show that $< j(y_n), \phi_n >= 1$ for all even n, where y_n is the generator of $\pi_n(\Omega G)$.

The odd generators in $H^*(G,\mathbb{Z})$ are given by the differential forms

$$\psi_{2j-1} = a_{2j} \mathrm{tr}\, \theta^{2j-1}. \tag{A-3}$$

From this one obtains by transgression the even generators in $H^*(\Omega G, \mathbb{Z})$,

$$\psi'_{2j} = (2j+1) a_{2j+2} \int_{S^1} \mathrm{tr}\, (g'(t) g(t)^{-1}) \theta^{2j}. \tag{A-4}$$

We shall show that ψ'_n is in the same cohomology class as the restriction of ϕ_n to the subgroup $\Omega G \subset U_{res}$. Provided that this is the case, we have $1 =< y_n, \psi'_n >=< j(y_n), \phi_n >$ and thus indeed $j(y_n)$ is the generator of $\pi_n(U_{res})$, and consequently j is a homotopy equivalence.

Note that $[\theta, F_g] = g[g^{-1}dg, \epsilon]g^{-1}$ and therefore

$$\phi_n = a_n \mathrm{tr}\, \epsilon [\theta_L, \epsilon]^n, \tag{A-5}$$

where $\theta_L = g^{-1}dg$. The proof of the equivalence of ϕ_n and ψ'_n is through a standard trick using the Cartan homotopy method. We set $\theta_L = \theta_0 + \theta_1$ where θ_0 commutes with ϵ and θ_1 anticommutes with ϵ. Define $G_t = -t\theta_1^2 + (1-t)\theta_0^2$ for $0 \le t \le 1$. Then by $d\theta_L = -\theta_L^2$ we get

$$\begin{aligned} dG_t &= -\frac{1}{t}[\theta_0, G_t] \\ \frac{d}{dt} G_t &= -(\theta_0^2 + \theta_1^2) = d\theta_0. \end{aligned} \tag{A-6}$$

Thus we obtain

(A-7)

$$\mathrm{tr}\, \epsilon G_1^n - \mathrm{tr}\, \epsilon G_0^n = \int_0^1 \frac{d}{dt} G_t^n dt = -n \int_0^1 \mathrm{tr}\, \epsilon (\theta_0^2 + \theta_1^2) G_t^{n-1} dt = nd \int_0^1 \mathrm{tr}\, \epsilon \theta_0 G_t^{n-1} dt.$$

Actually, ϵG_0^n is not trace-class. However, this operator is an even wedge power of θ_0 and by the permutation properties of the wedge product it is a commutator of some zero order pseudodifferential operators on the circle. Such an operator is always conditionally trace-class: this means that the trace is evaluated by first taking the trace over matrix indices, then integrating the product symbol over the circle and finally integrating over the momentum variable from $-\Lambda$ to Λ and taking the limit $\Lambda \to \infty$. But for conditionally trace-class operators $\mathrm{tr}[A, B] \neq 0$, in general.

In the present case we can write

$$\text{(A-8)} \qquad \mathrm{tr}\, \epsilon G_0^n = \frac{1}{2} \mathrm{tr}\, [\theta_0, \epsilon \theta_0^{2n-1}]$$

where we have used the fact that θ_0 commutes with ϵ. If $a(x,p)$ and $b(x,p)$ are the symbol functions of a pair of zero order PSDO's A, B on the circle, then the conditional trace of the commutator $[A, B]$ is given by the expression

(A-9)

$$\mathrm{tr}[A, B] = \frac{-i}{2\pi} \lim_{\Lambda \to \infty} \int_{p=-\Lambda}^{\Lambda} dp \frac{\partial}{\partial p} \int_x dx\, \mathrm{tr} \left(\frac{\partial a}{\partial x}(x,p) b(x,p) - a(x,p) \frac{\partial b}{\partial x}(x,p) \right).$$

Since in the expression (A-8) above the momentum variable is contained only in $\epsilon = p/|p|$ and the derivative of this symbol is twice the delta function in momentum space, we obtain

$$\text{(A-10)} \qquad \mathrm{tr}\, \epsilon G_0^n = \frac{1}{2} \frac{-i}{2\pi} \int_x \mathrm{tr}\, \theta^{2n-1} \frac{d}{dx} \theta dx.$$

Note that in the case $n = 1$ this is just the curvature on a loop group arising from the canonical central extension of the loop algebra.

By integration in parts, we can conclude that the integral of (A-10) over any compact $2n$ manifold M_{2n} in G is equal to the integral

$$\text{(A-11)} \qquad \frac{-i}{4\pi} \int_{M_{2n}} \int_{S^1} dx\, \mathrm{tr}\, \theta^{2n} (g^{-1} \frac{d}{dx} g) = \frac{-i}{4(2n+1)\pi} \int_{S^1 \times M_{2n}} \mathrm{tr}\, \theta^{2n+1}$$

and therefore this last expression under the integral sign represents the same cohomology class as (A-10), and therefore after a multiplication by the normalization coefficient a_{2n}, through (A-7), the same class as (A-4). Note that the coefficient in front of the integral on the right-hand side of the equation is equal to the ratio a_{2n+2}/a_{2n} which is necessary to recover the correct normalization for θ^{2n+1}. This proves that $< j(y_n), \phi_n > = < j(y_n), \psi'_n > = 1$.

References

[AS] M.F. Atiyah and I.M. Singer: Index theory for skew-adjoint Fredholm operators. I.H.E.S. Publ. Math. **37,** p. 305-326 (1969)

[C] A.L. Carey: Infinite dimensional groups and quantum field theory Acta Applicandae Mathematicae, **1** (1983), 321-331.

[CMM1] A. L. Carey, J. Mickelsson, and M. Murray: Index theory, gerbes, and hamiltonian quantization. Commun. Math. Phys. **183,** 707 (1997). hep-th/9511151

[CMM2] ——: Bundle gerbes applied to field theory. hep-th/9711133. To be publ. in Reviews in Math. Phys.

[EM] C. Ekstrand and J. Mickelsson: Gravitational anomalies, gerbes, and hamiltonian quantization. hep-th/9904189

[G] B. Gray: Homotopy Theory: an introduction to algebraic topology, Academic Press, New York, 1975.

[K] M. Karoubi: *K-Theory. An Introduction.* Grundlehren der Mathematischen Wissenschaften 226. Springer-Verlag, Berlin (1978)

[MP] R.B. Melrose and P. Piazza, An index theorem for families of Dirac operators on odd-dimensional manifolds with boundary. J. Diff. Geometry **46,** p. 287-334 (1997)

[PrSe] A. Pressley and G. Segal: *Loop Groups.* Clarendon Press, Oxford (1986)

[W] E. Witten: Anti-de Sitter space and holography, Adv. Theor. Math. Phys. **2** 253 (1998); hep-th/9802150.

[Q] D. Quillen: Superconnection character forms and the Cayley transform, Topology **27** 1988, 211-238 and P de la Harpe: *Classical Banach-Lie algebras and Banach Lie groups of operators in Hilbert Space,* Springer Lecture Notes in Mathematics **285** 1972