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Abstract

We classify all homogeneous symplectic manifolds with a torsion free connection of special symplectic holonomy, i.e. a connection whose holonomy is an absolutely irreducible proper subgroup of the full symplectic group. Thereby, we obtain many new explicit descriptions of manifolds with special symplectic holonomies. We also show that manifolds with such a connection are homogeneous iff they contain no symmetric points and their symplectic scalar curvature is constant.

Contents

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1 Introduction

A connection ∇ on a manifold M provides a recipe for parallel translation of tangent vectors along curves. One of its basic invariants is the *holonomy group* of ∇ which is defined as the group of the automorphisms of the tangent space T_pM induced by parallel translation along p-based loops. Identifying the tangent space T_pM at a point $p \in M$ with a fixed vector space V of the appropriate dimension, we may regard the holonomy group as a subgroup $\mathrm{H} \subset \mathrm{Aut}(V)$ which is well-defined up to conjugacy, independent of the choice of $p \in M$.

There is a natural first-order integrability condition that can be posed on the connection, namely the vanishing of its torsion. In fact, throughout this paper we shall assume all connections to be torsion free.

Any H-invariant tensor on V induces a parallel tensor field on M in a canonical way. For example, if $\mathrm{H} \subset \mathrm{O}(n)$ then M carries a parallel Riemannian metric, whence ∇ is the Levi-Civita connection of this metric. Therefore, the holonomy groups $\mathrm{H} \subset \mathrm{O}(n)$ are called *Riemannian holonomy groups*. Moreover, if H is properly contained in $\mathrm{SO}(n)$ and acts irreducibly on V then H is called a *special Riemannian holonomy group*.

Analogously, if $\mathrm{H} \subset \mathrm{Sp}(V)$ where $\mathrm{Sp}(V)$ is the group of automorphisms of V which preserve a symplectic form, then there is a parallel symplectic form on M. Such an H is called a *symplectic holonomy group*, and we call H a *special symplectic holonomy group* if it is properly contained in $\mathrm{Sp}(V)$ and acts absolutely irreducibly on V.

While the possible special Riemannian holonomies were classified in 1955 by Berger [Be1], the existence of *special symplectic holonomy groups* was not known until the beginning of this decade. Namely, in [Br1] Bryant discovered two irreducible holonomy groups of torsion free connections in dimension four which he denoted by G_3 and H_3, respectively. While H_3 is special symplectic, its conformal extension G_3 preserves a symplectic form only up to a scale. (This is a four dimensional phenomenon; indeed, for all dimensions bigger than four, any holonomy group preserving a symplectic form up to a scale must preserve it properly.) Later, Chi, Merkulov and this author found several other examples of special symplectic holonomy groups [CS, CMS1, CMS2, MS1, MS2]. Moreover, in [MS1, S3] it was shown that these examples exhaust all possible special symplectic holonomy groups. Since Bryant had classified the holonomies which are neither Riemannian nor symplectic and showed existence of connections with these holonomies [Br2, Br3] , this finally completed the classification of irreducible holonomy groups.

A most remarkable feature of the special symplectic holonomy groups is the existence of a universal method for the construction of connections with these holonomies. This method is based on a certain quadratic deformation of a linear Poisson structure [CMS1]. As a consequence, it follows that, for a fixed special symplectic holonomy group H, the moduli space of (local) torsion free connections with holonomy contained in H is finite dimensional. This has some strong implications on the rigidity of these connections. For example, there is always a local symmetry group of positive dimension acting on M. Due to these local symmetries, there is an ambiguity when glueing together local neighborhoods of such a manifold, and this implies that there may be cohomological obstructions for the existence of maximal examples [Br1, S3].

There are several explicitly known classes of manifolds with the "conformal symplectic" holonomy group G_3. For example, moduli of rational curves in $\mathbb{CP}^2$ of fixed degree passing through a number of given points carry such a structure [Br1]; in [S2], all *homogeneous* G_3-connections are classified; finally, in [C] degenerate G_3-connections are described via the construction of their moduli space.

On the other hand, the only explicit examples of connections with the special symplectic holonomy H_3 are the space of conics passing through one fixed point [Br1], and a few more connections with a symmetry group of cohomogeneity one [S1]. In particular, no global examples of connections with any of the remaining special symplectic holonomies have been known so far.

It is one task of the present paper to provide such examples for almost all special symplectic holonomy groups. We call a triple (M, Ω, ∇) consisting of a symplectic manifold (M, Ω) and a symplectic connection ∇ *maximal* if it is not equivalent to a proper open subset of another manifold with a symplectic connection. Further, we introduce the notion of the *symplectic scalar curvature* of such a connection which is a quadratic scalar invariant of the curvature tensor. Then we obtain the following result.

Theorem 1.1 *Each of the total spaces of the flat homogeneous vector bundles over symplectic symmetric spaces $\pi : E \to G/L_0$ in Table 1 carries a G-invariant symplectic connection with special symplectic holonomy group H whose symplectic scalar curvature is constant non-zero. These connections are maximal and share the following properties.*

1. *The 0-section $E_0 \subset E$ is totally geodesic, and the restriction of the connection to $E_0 \cong G/L_0$ is equivalent to the symmetric connection on G/L_0.*

2. *All fibers $E_p = \pi^{-1}(p)$ are totally geodesic. Moreover, $\Omega|_{E_p}$ is the (unique) L_0-invariant symplectic form on E_p where Ω is the parallel symplectic form on E.*

3. *Let $\mathcal{H}$ be the horizontal distribution on E induced by the symmetric connection on G/L_0. Then $\mathcal{H}$ is Ω-orthogonal to the fibers, and $\Omega|_{\mathcal{H}} = \pi^*(\omega)$ where ω is the symplectic form on G/L_0.*

Moreover, every connection with special symplectic holonomy and with non-zero constant symplectic scalar curvature is locally equivalent to one of these connections.

To give further examples of (global) symplectic manifolds with special symplectic holonomy, we investigate connections which are (locally) homogeneous. Evidently, any homogeneous manifold must have constant symplectic scalar curvature. But the converse is true as well.

Theorem 1.2 *Let (M, Ω, ∇) be a symplectic manifold with a torsion free symplectic connection, i.e. $\nabla\Omega \equiv 0$. Moreover, assume that the holonomy group of ∇ is special symplectic. Call a point $p \in M$* symmetric *if $(\nabla R)_p = 0$, R being the curvature of ∇. Then the following are equivalent.*

1. *M is locally homogeneous, i.e. there is a locally transitive group action via local diffeomorphisms preserving Ω and ∇.*

Table 1: Flat homogeneous vector bundles $\mathbf{E} \to \mathbf{G}/\mathbf{L}_0$ with symplectic holonomy H
Notation: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$\mathrm{H} \subset \mathrm{End}(V)$	G/L_0	$\mathrm{L}_0 \subset \mathrm{Aut}(W)$ where $E = \mathrm{G} \times_{\mathrm{L}_0} W$
$\mathrm{Sp}(3, \mathbb{F})$ $V = (\Lambda^3 \mathbb{F}^6)_0$	$\mathrm{Spin}(4,3)/(\mathrm{Spin}(3,2) \cdot \mathrm{Spin}(1,1))$ or $\mathrm{Spin}(7,\mathbb{C})/(\mathrm{Spin}(5,\mathbb{C}) \cdot \mathrm{Spin}(2,\mathbb{C}))$	$\mathrm{L}_0 \cong \mathrm{Sp}(2,\mathbb{F}) \cdot \mathbb{F}^*$ $W = \mathbb{F}^4$
$\mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SO}(n+1,n)$ or $\mathrm{SL}(2,\mathbb{C}) \cdot \mathrm{SO}(2n+1,\mathbb{C})$ $V = \mathbb{F}^2 \otimes \mathbb{F}^{2n+1}$	$\mathrm{SL}(n+2,\mathbb{F})/\mathrm{S}(\mathrm{GL}(2,\mathbb{F}) \cdot \mathrm{GL}(n,\mathbb{F}))$	$W = \mathbb{F}^2 \otimes \mathbb{F}$
$\mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SO}(2p+1,2q)$ $V = \mathbb{R}^2 \otimes \mathbb{R}^{2(p+q)+1}$	$\mathrm{SU}(p+1,q+1)/\mathrm{S}(\mathrm{U}(1,1) \cdot \mathrm{U}(p,q))$	$\mathrm{L}_0 \cong \mathrm{S}(\mathrm{GL}(2,\mathbb{R}) \cdot \mathrm{U}(p,q))$ $W = \mathbb{R}^2 \otimes \mathbb{R}$

2. *M contains no symmetric points and has constant symplectic scalar curvature.*

3. *M contains no symmetric points, and there is a point $p \in M$ for which the function $scal - scal(p)$ vanishes at p of order at least three.*

Of course, the complement of the 0-section of each vector bundle in Table 1, i.e. the complement of the set of symmetric points, is G-homogeneous.

Every locally homogeneous space is modelled on a globally homogeneous space, and we completely classify these.

Theorem 1.3 *Let $M = G/L$ be a homogeneous space with a G-invariant symplectic form Ω and a G-invariant symplectic connection ∇ with special symplectic holonomy group. Then – up to coverings – M is the complement of the 0-section of one of the vector bundles in Table 1, or an entry of one of the Tables 2 or 3.*

Moreover, the homogeneous connections in Table 2 are maximal, while the homogeneous connections in Table 3 are not.

For the sake of simplicity of the presentation, we give only the Lie algebra of the symmetry group of the homogeneous spaces from Table 3. The explicit form of $\mathrm{L} \subset \mathrm{G}$ is given in section 5.2.2.

Recall that a homogeneous space $M = \mathrm{G}/\mathrm{L}$ is called *reductive* if there is a vector space decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$ where $\mathfrak{g}$ and $\mathfrak{l}$ are the Lie algebras of G and L, respectively, such that $[\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m}$.

After passing to an appropriate cover of M if necessary, we may assume that there is a momentum map $\mu : M \to \mathfrak{g}^*$ where $\mathfrak{g}$ is the Lie algebra of G. The homogeneity implies

Table 2: Homogeneous Spaces with special symplectic holonomy of type 2
Notation: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$\mathrm{H} \subset \mathrm{End}(V)$	G	$M = \mathrm{G} \cdot \eta \subset \mathfrak{g}^* \cong \mathfrak{g}$ where $\eta \in \mathfrak{g}^* \cong \mathfrak{g}$ equals:
$\mathrm{H} = \mathrm{SL}(2, \mathbb{F})$ $V = \odot^3 \mathbb{F}^2$	$\mathrm{SL}(2, \mathbb{F}) \ltimes \mathbb{F}^2$	$\eta = x + v$, where $0 \neq x \in \mathfrak{sl}(2, \mathbb{F})$ is nilpotent $0 \neq v \in ker(x) \subset \mathbb{F}^2$
$\mathrm{Sp}(3, \mathbb{F})$ $V = (\Lambda^3 \mathbb{F}^6)_0$	$\mathrm{G}_2^{4,3} \ltimes \mathbb{R}^7$ if $\mathbb{F} = \mathbb{R}$ $\mathrm{G}_2^{\mathbb{C}} \ltimes \mathbb{C}^7$ if $\mathbb{F} = \mathbb{C}$	$\eta = x_\alpha + v_\lambda$, where $0 \neq x_\alpha \in (\mathfrak{g}_2)_\alpha$, α a long root of $\mathfrak{g}_2$ $0 \neq v_\lambda \in (\mathbb{F}^7)_\lambda$, λ a weight of $\mathbb{F}^7$ $(\alpha, \lambda) = 0$

that μ is an immersion – in fact a covering map – hence we may identify M with its image $\mu(M) \subset \mathfrak{g}^*$. Recall that the coadjoint orbit of any element in $\mathfrak{g}^*$ carries a canonical symplectic structure. We determine which homogeneous spaces $\mu(M) \subset \mathfrak{g}^*$ are coadjoint orbits.

Theorem 1.4 *Let $\pi : E \to G/L_0$ be a homogeneous vector bundle from Table 1. Then the momentum map $\mu : E \backslash 0 \to \mathfrak{g}^*$ is the double cover of a coadjoint orbit and thus, $\mu : E \to \mathfrak{g}^*$ is a branched double cover of its image.*

The homogeneous spaces in Table 2 are equivalent to coadjoint orbits while the homogeneous spaces in Table 3 are not.

The two homogeneous spaces in Table 2 with holonomy $H = SL(2, \mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, are reductive; the remaining homogeneous spaces are not reductive.

Since every holonomy irreducible symmetric space must be pseudo-Riemannian, there cannot be any locally symmetric connections with special symplectic holonomy. By our classification, there are also some special symplectic holonomy groups which do not even admit any *locally homogeneous* connections or, equivalently, no connections of constant symplectic scalar curvature. We list the number of possibilities of non-isomorphic homogeneous connections for the various special symplectic holonomy groups in Table 4.

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2 Symplectic manifolds and homogeneous connections

Let V be a vector space over $\mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We call a pair $(V, \langle\ ,\ \rangle)$ a *symplectic vector space* if $\langle\ ,\ \rangle$ is a non-degenerate skew-symmetric bilinear form on the vector space V. The Lie group of symplectic automorphisms is then defined as

$$\mathrm{Sp}(V, \langle\ ,\ \rangle) = \{x \in \mathrm{Aut}(V) \mid \langle xv, xw \rangle = \langle v, w \rangle \text{ for all } v, w \in V\},$$

Table 3: Homogeneous Spaces with special symplectic holonomy of type 3

notations/conventions: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j}$.

$\mathrm{H} \subset \mathrm{End}(V)$	$\mathfrak{g} = \sum_{i=0}^{3} \mathfrak{g}_i$	restrictions
$\mathrm{H} = \mathrm{Sp}(3,\mathbb{R})$ $V = (\Lambda^3\mathbb{R}^6)_0$	$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(p,q)$ $\mathfrak{g}_1 = \mathbb{R}^2 \otimes (\odot^2\mathbb{R}^{p,q})_0$ $\mathfrak{g}_2 = \mathbb{R} \otimes (\odot^2\mathbb{R}^{p,q})_0$ $\mathfrak{g}_3 = \mathbb{R}^2 \otimes \mathbb{R}$	$(p,q) = (3,0)$ or $(p,q) = (2,1)$
$\mathrm{H} = \mathrm{Sp}(3,\mathbb{C})$ $V = (\Lambda^3\mathbb{C}^6)_0$	$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C})$ $\mathfrak{g}_1 = \mathbb{C}^2 \otimes (\odot^2\mathbb{C}^3)_0$ $\mathfrak{g}_2 = \mathbb{C} \otimes (\odot^2\mathbb{C}^3)_0$ $\mathfrak{g}_3 = \mathbb{C}^2 \otimes \mathbb{C}$	
$\mathrm{H} = \mathrm{SU}(3,3)$ $V = \{\alpha \in \Lambda^3\mathbb{C}^6 \mid *\alpha = \alpha\}$	$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(p,q)$ $\mathfrak{g}_1 = \mathbb{R}^2 \otimes \mathfrak{su}(p,q)$ $\mathfrak{g}_2 = \mathbb{R} \otimes \mathfrak{su}(p,q)$ $\mathfrak{g}_3 = \mathbb{R}^2 \otimes \mathbb{R}$	$(p,q) = (3,0)$ or $(p,q) = (2,1)$
$\mathrm{H} = \mathrm{SL}(6,\mathbb{F})$ $V = \Lambda^3\mathbb{F}^6$	$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{F}) \oplus \mathfrak{sl}(3,\mathbb{F})$ $\mathfrak{g}_1 = \mathbb{F}^2 \otimes \mathfrak{sl}(3,\mathbb{F})$ $\mathfrak{g}_2 = \mathbb{F} \otimes \mathfrak{sl}(3,\mathbb{F})$ $\mathfrak{g}_3 = \mathbb{F}^2 \otimes \mathbb{F}$	
$\mathrm{H} = \begin{cases} \mathrm{Spin}(6,6) & \text{for } \mathbb{F} = \mathbb{R}, \\ \mathrm{Spin}(12,\mathbb{C}) & \text{for } \mathbb{F} = \mathbb{C} \end{cases}$ $V = \Delta_{12} \cong \mathbb{F}^{32}$	$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{F}) \oplus \mathfrak{sp}(3,\mathbb{F})$ $\mathfrak{g}_1 = \mathbb{F}^2 \otimes (\Lambda^2\mathbb{F}^6)_0$ $\mathfrak{g}_2 = \mathbb{F} \otimes (\Lambda^2\mathbb{F}^6)_0$ $\mathfrak{g}_3 = \mathbb{F}^2 \otimes \mathbb{F}$	
$\mathrm{H} = \mathrm{Spin}(6,\mathbb{H})$ $V = \Delta_6^{\mathbb{H}} \cong \mathbb{R}^{32}$	$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sp}(p,q)$ $\mathfrak{g}_1 = \mathbb{R}^2 \otimes (\Lambda^2\mathbb{H}^3)_0$ $\mathfrak{g}_2 = \mathbb{R} \otimes (\Lambda^2\mathbb{H}^3)_0$ $\mathfrak{g}_3 = \mathbb{R}^2 \otimes \mathbb{F}$	$(p,q) = (3,0)$ or $(p,q) = (2,1)$
$\mathrm{H} = \begin{cases} \mathrm{Spin}(6,6) & \text{for } \mathbb{F} = \mathbb{R}, \\ \mathrm{Spin}(12,\mathbb{C}) & \text{for } \mathbb{F} = \mathbb{C} \end{cases}$ $V = \Delta_{12} \cong \mathbb{F}^{32}$	$\mathfrak{g}_0 = \mathfrak{sp}(3,\mathbb{F})$ $\mathfrak{g}_1 = (\Lambda^3\mathbb{F}^6)_0 \oplus \mathbb{F}^6$ $\mathfrak{g}_2 = (\Lambda^2\mathbb{F}^6)_0 \oplus \mathbb{F}^6$ $\mathfrak{g}_3 = \mathbb{F}^6$	2 non-equivalent connections for $\mathbb{F} = \mathbb{R}$
$\mathrm{H} = \begin{cases} \mathrm{E}_7^{(5)} & \text{with } \mathbb{F} = \mathbb{R}, \\ \mathrm{E}_7^{(7)} & \text{with } \mathbb{F} = \mathbb{R}, \\ \mathrm{E}_7^{\mathbb{C}} & \text{with } \mathbb{F} = \mathbb{C} \end{cases}$ $V = \mathbb{F}^{56}$	$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{F}) \oplus (\mathfrak{f}_4^{(a)})$ $\mathfrak{g}_1 = \mathbb{F}^2 \otimes \mathbb{F}^{26}$ $\mathfrak{g}_2 = \mathbb{F} \otimes \mathbb{F}^{26}$ $\mathfrak{g}_3 = \mathbb{F}^2 \otimes \mathbb{F}$	$\mathfrak{f}_4^{(a)} = \begin{cases} \mathfrak{f}_4^{(1)} & \text{for } \mathrm{H} = \mathrm{E}_7^{(5)}, \\ \mathfrak{f}_4^{(2)} & \text{for } \mathrm{H} = \mathrm{E}_7^{(7)}, \\ \mathfrak{f}_4 & \text{for } \mathrm{H} = \mathrm{E}_7^{(7)}, \\ \mathfrak{f}_4^{\mathbb{C}} & \text{for } \mathrm{H} = \mathrm{E}_7^{\mathbb{C}} \end{cases}$

Table 3: Homogeneous Spaces with special symplectic holonomy of type 3 (cont.)

$H \subset \mathrm{End}(V)$	$\mathfrak{g} = \sum_{i=0}^{3} \mathfrak{g}_i$	restrictions/remarks
$H = \begin{cases} E_7^{(5)} & \text{for } \mathbb{F} = \mathbb{R}, \\ E_7^{\mathbb{C}} & \text{for } \mathbb{F} = \mathbb{C} \end{cases}$ $V = \mathbb{F}^{56}$	$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{sp}(4,\mathbb{F}) \\ \mathfrak{g}_1 &= (\Lambda^3\mathbb{F}^8)_0 \\ \mathfrak{g}_2 &= (\Lambda^2\mathbb{F}^8)_0 \\ \mathfrak{g}_3 &= \mathbb{F}^6 \end{aligned}$	
$\mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SO}(p,q)$ $V = \mathbb{R}^2 \otimes \mathbb{R}^{p+q}$	$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{sp}(k+1,\mathbb{R}) \oplus \mathfrak{so}(p',q') \\ \mathfrak{g}_1 &= \mathbb{R}^{2(k+1)} \otimes (\mathbb{R}^{p',q'} \oplus \mathbb{R}) \\ \mathfrak{g}_2 &= \Lambda^2\mathbb{R}^{2(k+1)} \otimes \mathbb{R} \\ &\oplus \mathbb{R} \otimes (\mathbb{R}^{p',q'} \oplus \mathbb{R}) \\ \mathfrak{g}_3 &= \mathbb{R}^{2(k+1)} \otimes \mathbb{R} \end{aligned}$	$p \geq 2, \quad q \geq 1$ $p' := p - 2k - 2$ $q' := q - 2k - 1$ $0 \leq k \leq min(\frac{p-2}{2}, \frac{q-1}{2})$
$\mathrm{SL}(2,\mathbb{C}) \cdot \mathrm{SO}(n,\mathbb{C})$ $V = \mathbb{C}^2 \otimes \mathbb{C}^n$	$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{sp}(k+1,\mathbb{C}) \oplus \mathfrak{so}(m,\mathbb{C}) \\ \mathfrak{g}_1 &= \mathbb{C}^{2(k+1)} \otimes (\mathbb{C}^m \oplus \mathbb{C}) \\ \mathfrak{g}_2 &= \Lambda^2\mathbb{C}^{2(k+1)} \otimes \mathbb{C} \\ &\oplus \mathbb{C} \otimes (\mathbb{C}^m \oplus \mathbb{C}) \\ \mathfrak{g}_3 &= \mathbb{C}^{2(k+1)} \otimes \mathbb{C} \end{aligned}$	$n \geq 3$ $m := n - 4k - 3$ $0 \leq k \leq \frac{n-3}{4}$

and the Lie algebra of symplectic endomorphisms of V is

$$\mathfrak{sp}(V, \langle\ ,\ \rangle) = \{x \in \mathrm{End}(V) \mid \langle xv, w\rangle + \langle v, xw\rangle = 0 \text{ for all } v, w \in V\}.$$

We shall frequently omit the explicit reference to $\langle\ ,\ \rangle$ and thus write $\mathrm{Sp}(V)$ and $\mathfrak{sp}(V)$, respectively. It is known that $\mathfrak{sp}(V)$ is the Lie algebra of $\mathrm{Sp}(V)$, and that both are simple. Moreover, $\mathfrak{sp}(V) \cong \odot^2 V$, with an isomorphism given by

$$(vw) \cdot u := \langle v, u\rangle w + \langle w, u\rangle v. \tag{1}$$

Definition 2.1 *Let (M, Ω, ∇) be a triple consisting of a connected manifold M with a symplectic form Ω and a torsion free affine connection ∇ such that $\nabla\Omega \equiv 0$. Then ∇ is called a* symplectic connection *on M.*

A torsion free connection ∇ on a manifold M is symplectic w.r.t. some symplectic form Ω iff the holonomy group of the connection is conjugate to a subgroup of $\mathrm{Sp}(V)$. Also, since Ω^n is a parallel volume form for $n = \frac{1}{2}\dim M$, all curvature endomorphisms are trace free, i.e. $tr(R_p(v,w)) = 0$ for all $v, w \in T_pM$ and all $p \in M$. Thus, the first Bianchi identity for R_p implies that the *Ricci curvature* which is given by

$$Ric_p(v,w) := tr(R_p(v, \_)w)$$

is *symmetric*, i.e. $Ric_p(v,w) = Ric_p(w,v)$. We define the the section of the endomorphism bundle $\underline{Ric} \in \Gamma(\mathrm{End}(TM))$ by

$$Ric_p(v,w) = \Omega(\underline{Ric}_p v, w) \quad \text{for all } v, w \in T_pM,\ p \in M.$$

Table 4: Number of homogeneous connections for all special symplectic holonomies

$\mathrm{H} \subset \mathrm{End}(V)$	V	$\#$(homog. conn. with Hol = H)
$\mathrm{SL}(2,\mathbb{R})$	$\odot^3\mathbb{R}^2 \cong \mathbb{R}^4$	1
$\mathrm{SL}(2,\mathbb{C})$	$\odot^3\mathbb{C}^2 \cong \mathbb{C}^4$	1
$\mathrm{Sp}(3,\mathbb{R})$	$(\Lambda^3\mathbb{R}^6)_0 \cong \mathbb{R}^{14}$	4
$\mathrm{Sp}(3,\mathbb{C})$	$(\Lambda^3\mathbb{C}^6)_0 \cong \mathbb{C}^{14}$	3
$\mathrm{SU}(1,5)$	$\{\alpha \in \Lambda^3\mathbb{C}^6 \mid *\alpha = \alpha\} \cong \mathbb{R}^{20}$	0
$\mathrm{SU}(3,3)$	$\{\alpha \in \Lambda^3\mathbb{C}^6 \mid *\alpha = \alpha\} \cong \mathbb{R}^{20}$	2
$\mathrm{SL}(6,\mathbb{R})$	$\Lambda^3\mathbb{R}^6 \cong \mathbb{R}^{20}$	1
$\mathrm{SL}(6,\mathbb{C})$	$\Lambda^3\mathbb{C}^6 \cong \mathbb{C}^{20}$	1
$\mathrm{Spin}(2,10)$	$\Delta_{2,10} \cong \mathbb{R}^{32}$	0
$\mathrm{Spin}(6,6)$	$\Delta_{6,6} \cong \mathbb{R}^{32}$	3
$\mathrm{Spin}(6,\mathbb{H})$	$\Delta_6^{\mathbb{H}} \cong \mathbb{R}^{32}$	2
$\mathrm{Spin}(12,\mathbb{C})$	$\Delta_{12}^{\mathbb{C}} \cong \mathbb{C}^{32}$	2
$\mathrm{E}_7^{(5)}$	$\mathbb{R}^{56}$	2
$\mathrm{E}_7^{(7)}$	$\mathbb{R}^{56}$	2
$\mathrm{E}_7^{\mathbb{C}}$	$\mathbb{C}^{56}$	2
$\mathrm{Sp}(1)\cdot\mathrm{SO}(n,\mathbb{H})$	$\mathbb{H}^n \cong \mathbb{R}^{4n}$, $n \geq 2$	0
$\mathrm{SL}(2,\mathbb{R})\cdot\mathrm{SO}(p,q)$	$\mathbb{R}^2 \otimes \mathbb{R}^{p,q}$, $p \geq q$, $p+q \geq 3$	$q+\varepsilon, \quad \varepsilon = \begin{cases} 1 & \text{if } p=q \text{ and } q \text{ odd} \\ 1 & \text{if } p+q \text{ odd}, p \geq q+2 \\ 2 & \text{if } p=q+1 \\ 0 & \text{otherwise} \end{cases}$
$\mathrm{SL}(2,\mathbb{C})\cdot\mathrm{SO}(n,\mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^n$, $n \geq 3$	$[\frac{n+1}{4}]+\varepsilon, \quad \varepsilon = \begin{cases} 0 & \text{for } n \text{ even} \\ 1 & \text{for } n \text{ odd} \end{cases}$

From the symmetry of Ric_p it follows that $\underline{Ric}_p \in \mathfrak{sp}(T_pM, \Omega_p)$ and hence, $tr\underline{Ric}_p = 0$. Therefore, the definition of scalar curvature which would be analogous to the one from Riemannian geometry contains no information at all. Instead, we introduce the following notion.

Definition 2.2 *Let (M, Ω, ∇) be a symplectic manifold with a symplectic connection ∇, and define $\underline{Ric} \in \Gamma(End(TM))$ as above. Then the* symplectic scalar curvature of ∇ *is the function $scal : M \to \mathbb{F}$ given by $scal := tr(\underline{Ric}^2)$.*

If a manifold M carries a torsion free connection whose holonomy is contained in H then there is a principal H-bundle $\pi : F \to M$, called the *holonomy bundle*, and a $V \oplus \mathfrak{h}$-valued coframe $\theta + \omega$ where θ and ω are called the *tautological one-form* and the *connection one-form,* respectively [KN]. Throughout, we shall assume that H is a connected Lie group. Since $\theta + \omega : T_pF \to V \oplus \mathfrak{h}$ is an isomorphism, we call $\theta + \omega$ the *connection coframe on F.*

Definition 2.3 *Let $\pi : F \to M$ be the holonomy bundle of a torsion free connection and let $\theta + \omega$ be the connection coframe. A vector field X on F is called an* infinitesimal connection symmetry on F *if $\mathcal{L}_X(\theta + \omega) = 0$.*

A vector field X_0 on M is called an infinitesimal connection symmetry on M *if there exists an infinitesimal connection symmetry X on F with $\pi_*(X) = X_0$.*

The connection is called locally homogeneous *if the infinitesimal connection symmetries on M act locally transitive on M, i.e. if for each $v_p \in TM$ there is a connection symmetry X_0 on M such that $(X_0)_p = v_p$.*

It is evident that the infinitesimal symmetries form a Lie algebra which we shall denote by $\mathfrak{g}$. Note that $\mathcal{L}_X\theta = 0$ implies that $\pi_*(X)$ is well-defined. Also, if X_0 is an infinitesimal connection symmetry on M then the infinitesimal connection symmetry X on F satisfying $\pi_*(X) = X_0$ is unique. Thus, the Lie algebras of infinitesimal connection symmetries on M and on F are canonically isomorphic, justifying the ambiguous use of the term.

If ∇ is a symplectic connection on M, i.e. if $\mathrm{H} \subset \mathrm{Sp}(V)$, and if $\pi : F \to M$ is the holonomy bundle then the parallel symplectic form Ω on M is determined by

$$\pi^*(\Omega) = \langle \theta, \theta \rangle .$$

Thus, the action of $\mathfrak{g}$ on M is *symplectic*, i.e. $\mathfrak{L}_X\Omega = 0$ for all $X \in \mathfrak{g}$. For $p \in F$ we let

$$\mathfrak{g}_p := \{(\theta + \omega)(X_p) \mid X \text{ an infinitesimal symmetry on } F\} \subset V \oplus \mathfrak{h}.$$

Since the evaluation map map $\mathfrak{g} \to T_pF$ is injective, $\mathfrak{g}_p \cong \mathfrak{g}$ as a vector space. Moreover, we let $\mathfrak{l}_p \subset \mathfrak{g}$ be the Lie algebra $(\theta + \omega)_p^{-1}(\mathfrak{g}_p \cap \mathfrak{h})$ and $pr_p : \mathfrak{g} \cong \mathfrak{g}_p \to V$ the canonical projection.

Proposition 2.4 *Let $\pi : F \to M$, θ, ω and $\mathfrak{g}$ be as above, and let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Then for every $p \in F$, the Lie subgroup $L_p \subset G$ with Lie subalgebra $\mathfrak{l}_p \subset \mathfrak{g}$ is closed.*

Moreover, for each $p \in F$ the element $\phi_p := (\pi^*\Omega)_p|_{\mathfrak{g}} \in \Lambda^2\mathfrak{g}^*$ is a 2-cocycle, i.e. satisfies $\phi_p([x, y], z) + \phi_p([y, z], x) + \phi_p([z, x], z) = 0$ for all $x, y, z \in \mathfrak{g}$.

Proof. Fix $p \in F$ and consider $\phi_p \in \Lambda^2\mathfrak{g}^*$ from above. The flow along the infinitesimal symmetries induces a free local action of an open neighborhood U of $e \in \mathrm{G}$, and the canonical embedding $g \mapsto g \cdot p$ allows us to regard U as a subset of F such that the restriction of the infinitesimal symmetries to U constitutes all left invariant vector fields. Under this identification, we have $p \cong e$.

Since $\mathfrak{L}_X \pi^*(\Omega) = 0$ for all $X \in \mathfrak{g}$, $\pi^*(\Omega)|_U$ must be the *right invariant* 2-form extending ϕ_p. The cocycle condition for ϕ_p follows from $d\pi^*(\Omega) = 0$.

Let Φ be the right invariant 2-form on G which extends ϕ_p, hence $\Phi|_U = \pi^*(\Omega)|_U$. Since ϕ_p is a cocycle, Φ is closed and the left invariant vector fields are symplectic. Since G is simply connected, there is a *momentum map* $\mu : \mathrm{G} \to \mathfrak{g}^*$ satisfying

$$\mu(e) = 0 \quad \text{and} \quad \langle d\mu_g, X \rangle = -(X \lrcorner \Phi_g) \quad \text{for all left invariant vector fields } X \in \mathfrak{g}.$$

At $p \cong e$, we have $X_p \lrcorner \phi_p = 0$ iff $X_p \lrcorner \pi^*(\Omega)_p = 0$ iff $\pi_*(X_p) = 0$ iff $X_p \in \mathfrak{l}_p$. By the right invariance of Φ it follows that $\langle d\mu_g, X \rangle = 0$ iff X is contained in the *right invariant distribution* induced by $\mathfrak{l}_p \subset \mathfrak{g} = T_e\mathrm{G}$.

Now L_p is the maximal integral leaf of this distribution which contains $e \in \mathrm{G}$. Thus, $\mu(e) = 0$ implies $\mu(\mathrm{L}_p) = 0$ and hence, $\mu(\overline{\mathrm{L}}_p) = 0$. On the other hand, since $\mathfrak{l}_p = \ker d\mu_p$, it follows that, after shrinking U if necessary, we may assume that $U \cap \mu^{-1}(0) = U \cap \mathrm{L}_p$, hence $U \cap L_p = U \cap \overline{\mathrm{L}}_p$, and from this, $\mathrm{L}_p = \overline{\mathrm{L}}_p$ follows. ∎

In general, if $\mathfrak{g}$ is the Lie algebra of infinitesimal symmetries of (M, Ω, ∇) and if we assume that M is simply connected then again, there is a momentum map $\mu_0 : M \to \mathfrak{g}^*$ which is – up to adding a constant – uniquely determined by

$$\langle d(\mu_0)_x, X_0 \rangle = -(X_0 \lrcorner \Omega_x) \quad \text{for all infinitesimal symmetries } X_0 \in \mathfrak{g} \text{ on } M.$$

Evidently, $ker(d(\mu_0)_x) = \{v \in T_xM \mid \Omega(v, (X_0)_x) = 0 \text{ for all } X_0 \in \mathfrak{g}\}$. Thus, $d(\mu_0)_x$ is injective iff x has a locally homogeneous neighborhood, and μ_0 is an immersion iff the connection is locally homogeneous.

Proposition 2.5 *Let (M, Ω, ∇) be a simply connected manifold with a locally homogeneous symplectic connection. Let $\mathfrak{g}$ be the Lie algebra of infinitesimal symmetries on M and G be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Then there is*

1. *an affine action of G on $\mathfrak{g}^*$ whose linear part is the coadjoint representation, i.e.*

$$g \cdot (\xi) = Ad^*_{g^{-1}}(\xi) + \kappa(g) \tag{2}$$

   for all $g \in G$, $\xi \in \mathfrak{g}^*$ and a map $\kappa : G \to \mathfrak{g}^*$,

2. *a G-orbit $\Sigma \subset \mathfrak{g}^*$ of this action which carries a G-invariant symplectic connection,*

3. *a $\mathfrak{g}$-equivariant connection preserving local diffeomorphism $\mu_0 : M \to \Sigma$.*

Proof. It is well known [LM, ch.4,Th.3.2] that there is an affine action of G on $\mathfrak{g}^*$ of the asserted form such that the momentum map $\mu_0 : M \to \mathfrak{g}^*$ is $\mathfrak{g}$-equivariant. Thus, $\mu_0(M) \subset \Sigma$ for some G-orbit Σ, and since $\dim M = \dim \Sigma$, it follows that μ_0 is a local diffeomorphism. But the flow along the vector fields $X \in \mathfrak{g}$ preserves the connection, hence there is a unique G-invariant connection on Σ which makes μ_0 connection preserving. ∎

The following is a standard result in the theory of Hamiltonian actions (cf. [LM, ch.4,Prop.3.6] and the definition of the canonical symplectic form on coadjoint orbits).

Corollary 2.6 *Let (M, Ω, ∇), $\mathfrak{g}$, G be as in Proposition 2.5. Then we can choose the momentum map $\mu_0 : M \to \mathfrak{g}^*$ in such a way that $\kappa : G \to \mathfrak{g}^*$ from (2) vanishes iff the 2-cocycle $\phi_p \in \Lambda^2\mathfrak{g}^*$ from Proposition 2.4 is a coboundary, i.e. $\phi_p(x, y) = \eta([x, y])$ for some $\eta \in \mathfrak{g}^*$ and all $x, y \in \mathfrak{g}$.*

If this is the case then $\mu_0 : M \to \Sigma$ is a local connection preserving diffeomorphism with the coadjoint orbit of η, and $\Omega = \mu^*(\omega)$ where ω is the canonical symplectic form on this coadjoint orbit.

Thus, in order to decide whether a given locally homogeneous symplectic connection is locally equivalent to a coadjoint orbit, we have to decide if there is an element $\eta \in \mathfrak{g}^*$ such that

$$\eta([x, y]) = \langle \pi_*(x), \pi_*(y) \rangle \tag{3}$$

for all $x, y \in \mathfrak{g}$. The obstruction for the existence of such an element $\eta \in \mathfrak{g}^*$ is represented by the cohomology $H^1(\mathfrak{g}, \mathfrak{g}^*)$ [HS]. As we shall see, this obstruction does not vanish for all homogeneous connections with special symplectic holonomy.

Let us again suppose that there is a G-invariant connection on the homogeneous space $M = \mathrm{G/L}$. Recall that a homogeneous space $M = \mathrm{G/L}$ is called *reductive* if there is a vector space decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$ where $\mathfrak{g}$ and $\mathfrak{l}$ are the Lie algebras of G and L, respectively, such that $[\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m}$.

Proposition 2.7 *[KN] Let $M = G/L$ be a homogeneous space with G, L connected and with a torsion free G-invariant connection with holonomy H. Let $\pi : F \to M$ be the holonomy bundle, fix $p \in F$ and consider the ismorphism $\mathfrak{g} \to \mathfrak{g}_p \subset \mathfrak{h} \oplus V$ from above.*

Then M is reductive iff there is a L-equivariant map $\tau : V \to \mathfrak{h}$ such that $v + \tau(v) \in \mathfrak{g}_p \subset \mathfrak{h} \oplus V$ for all $v \in V$.

Proof. Since M is homogenous, the projection $pr_p : \mathfrak{g}_p \to V$ is surjective. Thus, linear maps $\tau : V \to \mathfrak{h}$ with $v + \tau(v) \in \mathfrak{g}_p$ for all $v \in V$ exist and are in one-to-one correspondence with vector space decompositions $\mathfrak{g} \cong \mathfrak{g}_p = \{v + \tau(v) \mid v \in V\} \oplus \mathfrak{l}$. If we denote the first summand by $\mathfrak{m}$ then clearly, $[\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m}$ iff τ is $\mathfrak{l}$-equivariant. ∎

3 Holonomy and structure equations for symplectic connections

Let V be a finite dimensional vector space and let $\mathrm{H} \subset \mathrm{Aut}(V)$ be any connected closed Lie subgroup with Lie algebra $\mathfrak{h} \subset \mathrm{End}(V)$. We define the *space of formal curvature maps* $K(\mathfrak{h})$ by the exact sequence

$$0 \longrightarrow K(\mathfrak{h}) \longrightarrow \Lambda^2 V^* \otimes \mathfrak{h} \longrightarrow \Lambda^3 V^* \otimes V$$

where the last map is given by the composition of the natural inclusion and the skew-symmetrization map, i.e. $\Lambda^2 V^* \otimes \mathfrak{h} \hookrightarrow \Lambda^2 V^* \otimes V^* \otimes V \to \Lambda^3 V^* \otimes V$.

The significance of the space $K(\mathfrak{h})$ is that the curvature R_p at $p \in M$ of a torsion free connection ∇ on M satisfies the first Bianchi identity and thus, $R_p \in K(\mathfrak{hol}_p)$ where $\mathfrak{hol}_p \subset \mathrm{End}(T_p M)$ is the Lie algebra of the holonomy group at p. In particular, if $K(\mathfrak{h}) = 0$ then every torsion free connection whose holonomy algebra is contained in $\mathfrak{h}$ must be flat. We are therefore interested in those subalgebras $\mathfrak{h}$ with $K(\mathfrak{h}) \neq 0$.

If $(V, \langle\ ,\ \rangle)$ is a symplectic vector space then we use the contraction isomorphism $\imath : V \to V^*$ given by

$$\imath(v)w := \langle v, w \rangle \quad \text{for all } v, w \in V,$$

to identify V and V^*.

We observe that $\mathfrak{sl}(2, \mathbb{F}) \cong \mathfrak{sp}(\mathbb{F}^2, \wedge)$ where $\wedge$ denotes the determinant of $\mathbb{F}^2$; therefore, we have the identification $\mathfrak{sl}(2, \mathbb{F}) \cong \odot^2 \mathbb{F}^2$ from (1).

Theorem 3.1 *Let $\mathfrak{h} \subset End(V)$ be an irreducible semi-simple Lie subalgebra where V is a finite dimensional vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $W := \mathbb{F}^2 \otimes V$ and consider the induced tensor representation of $\mathfrak{h}^+ := \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h} \subset End(W)$. Then the following are equivalent.*

1. *There is an irreducible symmetric pair $(\mathfrak{g}, \mathfrak{h}^+)$ whose isotropy representation is equivalent to the representation of $\mathfrak{h}^+$ on W.*

2. *There is a symplectic form $\langle\ ,\ \rangle$ on V such that $\mathfrak{h} \subset \mathfrak{sp}(V, \langle\ ,\ \rangle)$, and an $\mathfrak{h}$-equivariant map $\circ : \odot^2 V \to \mathfrak{h}$ which satisfies for all $u, v, w \in V$*

$$(u \circ v)w - (u \circ w)v = 2 \langle v, w \rangle u + \langle u, w \rangle v - \langle u, v \rangle w. \tag{4}$$

3. *There is a symplectic form $\langle\ ,\ \rangle$ on V such that $\mathfrak{h} \subset \mathfrak{sp}(V, \langle\ ,\ \rangle)$, and an $\mathfrak{h}$-equivariant map $\circ : \odot^2 V \to \mathfrak{h}$ such that for $\mathbf{a} \in \mathfrak{h}$ the map $R_{\mathbf{a}} : \Lambda^2 V \to \mathfrak{h}$ given by*

$$R_{\mathbf{a}}(v, w) = 2 \langle v, w \rangle\ \mathbf{a} + v \circ (\mathbf{a}w) - w \circ (\mathbf{a}v) \tag{5}$$

lies in $K(\mathfrak{h})$.

If these conditions are satisfied then the map $\mathfrak{h} \to K(\mathfrak{h})$, $\mathbf{a} \mapsto R_{\mathbf{a}}$ is injective, thus $K(\mathfrak{h}) \neq 0$.

Proof. It is well-known that the first statement is equivalent to the existence of an $\mathfrak{h}^+$-invariant element $0 \neq R \in K(\mathfrak{h}^+)$ [H, S3].

Let $R : \Lambda^2 W \to \mathfrak{h}^+ = \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h}$. Since $\Lambda^2 W = \odot^2 V \oplus \mathfrak{sl}(2, \mathbb{F}) \otimes \Lambda^2 V$ as an $\mathfrak{h}^+$-module, it follows that R is $\mathfrak{h}^+$-equivariant iff there is an $\mathfrak{h}$-invariant $\Omega \in \Lambda^2 V$ and an $\mathfrak{h}$-equivariant map $\circ : \odot^2 V \to \mathfrak{h}$ such that

$$R(e \otimes v, f \otimes w) = \Omega(v, w) ef + (e \wedge f) v \circ w$$

for all $e, f \in \mathbb{F}^2$ and $v, w \in V$, where $\wedge$ is the determinant on $\mathbb{F}^2$. Moreover, (4) is equivalent to the first Bianchi identity for R, i.e. to $R \in K(\mathfrak{h}^+)$. Thus, the first and second statement are equivalent.

The equivalence of the second and third statement follows from an easy calculation, and evidently, $R_{\mathbf{a}} = 0$ only if $\mathbf{a} = 0$. ∎

From the classification of irreducible symmetric spaces [Be2], we immediately get the following

Corollary 3.2 *The irreducible subgroups $H \subset Sp(V)$ listed in Table 5 satisfy $K(\mathfrak{h}) \neq 0$.*

Table 5: List of special symplectic holonomy groups

Group H	Representation space	Group H	Representation space
$\mathrm{SL}(2, \mathbb{R})$	$\mathbb{R}^4 \cong \odot^3 \mathbb{R}^2$	E_7^5	$\mathbb{R}^{56}$
$\mathrm{SL}(2, \mathbb{C})$	$\mathbb{C}^4 \cong \odot^3 \mathbb{C}^2$	E_7^7	$\mathbb{R}^{56}$
$\mathrm{SL}(2, \mathbb{R}) \cdot \mathrm{SO}(p, q)$	$\mathbb{R}^{2(p+q)}$, $(p+q) \geq 3$	$\mathrm{E}_7^{\mathbb{C}}$	$\mathbb{C}^{56}$
$\mathrm{SL}(2, \mathbb{C}) \cdot \mathrm{SO}(n, \mathbb{C})$	$\mathbb{C}^{2n}$, $n \geq 3$	$\mathrm{Spin}(2, 10)$	$\mathbb{R}^{32}$
$\mathrm{Sp}(1) \cdot \mathrm{SO}(n, \mathbb{H})$	$\mathbb{H}^n \cong \mathbb{R}^{4n}$, $n \geq 2$	$\mathrm{Spin}(6, 6)$	$\mathbb{R}^{32}$
$\mathrm{SL}(6, \mathbb{R})$	$\mathbb{R}^{20} \cong \Lambda^3 \mathbb{R}^6$	$\mathrm{Spin}(6, \mathbb{H})$[1]	$\mathbb{R}^{32}$
$\mathrm{SU}(1, 5)$	$\mathbb{R}^{20} \subset \Lambda^3 \mathbb{C}^6$	$\mathrm{Spin}(12, \mathbb{C})$	$\mathbb{C}^{32}$
$\mathrm{SU}(3, 3)$	$\mathbb{R}^{20} \subset \Lambda^3 \mathbb{C}^6$	$\mathrm{Sp}(3, \mathbb{R})$	$\mathbb{R}^{14} \subset \Lambda^3 \mathbb{C}^6$
$\mathrm{SL}(6, \mathbb{C})$	$\mathbb{C}^{20} \cong \Lambda^3 \mathbb{C}^6$	$\mathrm{Sp}(3, \mathbb{C})$	$\mathbb{C}^{14} \subset \Lambda^3 \mathbb{C}^6$

[1] This representation has been erroneously omitted in the reference [S3]

The following result follows then from a cumbersome calculation which we omit. For details, see [MS1, ch.4].

Proposition 3.3 *For all subalgebras listed in Table 5 we have $K(\mathfrak{h}) \cong \mathfrak{h}$, i.e. the injective map $\mathfrak{h} \to K(\mathfrak{h})$ from Theorem 3.1 is an isomorphism.*

There is a one-to-one correspondence between the subalgebras listed in Table 5, i.e. symmetric pairs $(\mathfrak{g}, \mathfrak{h}^+) = (\mathfrak{g}, \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h})$ and simple Lie algebras $\mathfrak{g}$ which contain a long root space. This can be described as follows.

Let $\mathfrak{g}$ be such a simple Lie algebra. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ and a fundamental Weyl chamber. Let $\alpha_0 \in \Delta$ be the maximal root. By assumption, there are elements $A_\pm \in \mathfrak{g}_{\pm\alpha_0} \subset \mathfrak{g}$ such that $\alpha_0([A_+, A_-]) = 2$, and we let

$$\mathfrak{s} := span(A_+, A_-, [A_+, A_-]).$$

Evidently, $\mathfrak{s} \cong \mathfrak{sl}(2, \mathbb{F})$. For each integer i, we let

$$\Delta_i := \{\alpha \in \Delta \mid \langle \alpha, \alpha_0 \rangle = i\}.$$

Since α_0 is long, we have $\Delta_{\pm 2} = \{\pm\alpha_0\}$, and $\Delta_i = \emptyset$ if $|i| \geq 3$. Moreover, we define

$$\mathfrak{h} := (\alpha_0)^- \oplus \bigoplus_{\alpha \in \Delta_0} \mathfrak{g}_\alpha,$$

which is a semi-simple subalgebra of $\mathfrak{g}$ where $(\alpha_0)^- \subset \mathfrak{t}$ is the annihilator of $\alpha_0 \in \mathfrak{t}^*$. Finally, we let

$$V_i := \bigoplus_{\alpha \in \Delta_i} \mathfrak{g}_\alpha,$$

for all i and $W := V_1 \oplus V_{-1}$. Evidently, $[V_i, V_j] \subset V_{i+j}$, whence $[\mathfrak{s} \oplus \mathfrak{h}, W] \subset W$ and $[W, W] \subset \mathfrak{s} \oplus \mathfrak{h}$. Thus, we get the direct sum decomposition

$$\mathfrak{g} = (\mathfrak{s} \oplus \mathfrak{h}) \oplus W.$$

and $(\mathfrak{g}, \mathfrak{s} \oplus \mathfrak{h})$ is a symmetric pair.

It is now elementary to verify that the representation of $\mathfrak{s} \oplus \mathfrak{h}$ on W is equivalent to the representation of $\mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h}$ on $\mathbb{F}^2 \otimes V_1$ and that this representation is irreducible iff $\mathfrak{g}$ is *simple*.

There is an easy algorithm to describe the representations of $\mathfrak{h}$ on V_1 from the Satake diagram of $\mathfrak{g}$ [H, OV] as follows. $\mathfrak{g}$ contains a long root space iff in the Satake diagram of $\mathfrak{g}$ the nodes corresponding to simple roots α_i which are *not* orthogonal to the maximal root are white. One obtains $\mathfrak{h}$ by deleting these nodes α_i. The representation of $\mathfrak{h}$ on V_1 has one irreducible summand for each deleted node α_i, and it is described by writing on all nodes α_j adjacent to α_i the Cartan number $\langle \alpha_i, \alpha_j \rangle$.

From this and a glance at the Satake diagrams one easily verifies the following:

1. The representation of $\mathfrak{h}$ on V_1 has more than one irreducible summand iff $\mathfrak{g}$ is of type A_n with $n \geq 2$, i.e. $\mathfrak{g}$ is (a real form of) $\mathfrak{sl}(n+1, \mathbb{C})$. In this case, $\mathfrak{h}$ is (a real form of) $\mathfrak{sl}(n, \mathbb{C})$, acting on $V \oplus V^*$ where $V = \mathbb{F}^n$.

2. If $\mathfrak{g}$ is of type C_n then we have $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{F})$ and $\mathfrak{h} = \mathfrak{sp}(n-1, \mathbb{F})$. The representation of $\mathfrak{h}$ on V_1 is equivalent to the standard representation of $\mathfrak{sp}(n-1, \mathbb{F})$ on $\mathbb{F}^{2(n-1)}$.

3. There is a one-to-one correspondence between the entries of Table 5 and the Satake diagrams not of type A_n or C_n for which the node of the adjoint representation is white.

Table 6: Complex simple Lie algebras and the corresponding symplectic representations

$\mathfrak{g}$	Dynkin Diagram	$\mathfrak{h}$	Representation space
$\mathfrak{so}(n+4,\mathbb{C})$	$B_{\frac{n+3}{2}}$ or $D_{\frac{n+4}{2}}$	$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(n,\mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^n$
$\mathfrak{g}_2$	G_2	$\mathfrak{sl}(2,\mathbb{C})$	$\odot^3\mathbb{C}^2$
$\mathfrak{f}_4$	F_4	$\mathfrak{sp}(3,\mathbb{C})$	$\mathbb{C}^{14} = Ker(\wedge\omega : \Lambda^3\mathbb{C}^6 \to \Lambda^5\mathbb{C}^6)$
$\mathfrak{e}_6$	E_6	$\mathfrak{sl}(6,\mathbb{C})$	$\Lambda^3\mathbb{C}^6$
$\mathfrak{e}_7$	E_7	$\mathfrak{spin}(12,\mathbb{C})$	$\Delta_{12}^{\mathbb{C}}$
$\mathfrak{e}_8$	E_8	$\mathfrak{e}_7$	$\mathbb{C}^{56}$

Table 6 lists the correspondence between *complex* simple Lie algebras $\mathfrak{g}$ which are not of type A_n or C_n and the corresponding subgroups $\mathfrak{h} \subset \mathfrak{sp}(V)$.

Let us fix the (unique) $ad_{\mathfrak{g}}$-invariant inner product $(\ ,\ )$ on $\mathfrak{g}$ which satisfies $(\alpha,\alpha) = 2$ ($(\alpha,\alpha) = 3$, respectively) for all long roots α if $\mathfrak{g} \not\cong \mathfrak{g}_2$ ($\mathfrak{g} \cong \mathfrak{g}_2$, respectively). Since $\mathfrak{h} \subset \mathfrak{g}$, this restricts to an $ad_{\mathfrak{h}}$-invariant inner product on $\mathfrak{h}$. With this, we can deduce some statements about the weights of $\mathfrak{h} \subset \mathfrak{sp}(V)$ where we denote by Δ the set of roots of $\mathfrak{h}$ and by Φ the set of weights of V.

Proposition 3.4 *Let $\mathfrak{h} \subset \mathfrak{sp}(V)$ be one of the subalgebras listed in Table 5, and let $(\ ,\ )$ be the $ad_{\mathfrak{h}}$-invariant inner product on $\mathfrak{h}$ from the preceding paragraph. Then the following hold.*

1. *All weight spaces are one-dimensional.*
2. *For $\lambda \in \Phi$, we have $(\lambda,\lambda) \in \{\frac{3}{2}, \frac{1}{2}\}$ which allows us to refer to* long *and* short *weights. Moreover, if all roots of Δ are long, then so are all weights of Φ.*
3. *For $\lambda \in \Phi$ and $\alpha \in \Delta$ we have $(\lambda,\alpha) \in \{0, \pm 1\}$, provided that λ is a long weight or α is a long root.*
4. *If $\lambda \in \Phi$ is a short weight then $2\lambda \in \Delta$ is a long root.*
5. *Let $\lambda, \mu \in \Phi$ with λ long. Then exactly one of the following holds.*
   - *(a) $(\lambda,\mu) = \pm\frac{3}{2}$ and $\lambda = \pm\mu$.*
   - *(b) $(\lambda,\mu) = \pm\frac{1}{2}$ and $\lambda = \pm(\mu + \alpha)$ for some $\alpha \in \Delta$ with $(\lambda,\alpha) = 1$.*
6. *If $rk(\mathfrak{h}) \geq 3$ then there are long roots $\alpha_i \in \Delta$ with the following properties:*
   - *(a) $2\lambda_0 = \alpha_1 + \alpha_2 + \alpha_3$, where λ_0 is the maximal weight of Φ,*
   - *(b) $(\alpha_i, \alpha_j) = 2\delta_i^j$,*
   - *(c) α_i is the maximal root in the root system $\{\alpha \in \Delta \mid (\alpha,\alpha_j) = 0, j = 1, \ldots, i-1\}$.*

Proof. To simplify notation, we shall carry out the argument only for the case $\mathfrak{g} \not\cong \mathfrak{g}_2$, leaving the straightforward alterations of the scale in the case $\mathfrak{g} \cong \mathfrak{g}_2$ to the reader.

Let α_0 be the maximal (long) root of $\mathfrak{g}$. Then $V := V_1$ is spanned by all roots $\alpha \in \Delta_{\mathfrak{g}}$ with $(\alpha, \alpha_0) = 1$. Under the representation of $\mathfrak{h}$, these become weight spaces of weights $\alpha - \frac{1}{2}\alpha_0$. That is,

$$\Phi = \left\{ \alpha - \frac{1}{2}\alpha_0 \;\middle|\; \alpha \text{ a root of } \mathfrak{g} \text{ with } (\alpha, \alpha_0) = 1 \right\}.$$

The first assertion follows since all weight spaces of V are root spaces of $\mathfrak{g}$. Next, if $\lambda = \alpha - \frac{1}{2}\alpha_0$ and $\mu = \beta - \frac{1}{2}\alpha_0$ are weights of V and $\gamma \in \Delta$ then

$$(\lambda, \mu) = (\alpha, \beta) - \frac{1}{2} \quad \text{and} \quad (\lambda, \gamma) = (\alpha, \gamma).$$

Due to the scaling of $(\ ,\ )$, we have $(\alpha, \alpha) \in \{1, 2\}$ which implies the second statement while the third follows since $\gamma \neq \pm\alpha$ and one of α or γ is long.

The fourth part follows since if $\lambda \in \Phi$ is a short weight then $\phi = \lambda + \frac{1}{2}\alpha_0 \in \Delta_{\mathfrak{g}}$ is a short root with $(\alpha_0, \phi) = 1$ and hence, $2\lambda = 2\phi - \alpha_0$ is a root in $\mathfrak{h}$.

To show the fifth assertion, let α be a long root. Then $(\alpha, \beta) \in \{2, \pm 1, 0\}$ since $\alpha + \beta \neq 0$, and $(\alpha, \beta) = 2$ iff $\alpha = \beta$. If $(\alpha, \beta) = 1$ then $\lambda - \mu = \alpha - \beta \in \Delta$ and the claim follows.

For the last part, note that if $\mathrm{rk}(\mathfrak{h}) \geq 3$ it follows that $\mathrm{rk}(\mathfrak{g}) \geq 4$ and since $\mathfrak{g}$ is simple and not of type A_n, C_n, the root system of long roots of Δ contains a subsystem isomorphic to D_4. This means that there is a root α of $\mathfrak{g}$ such that $2\alpha = \sum_{i=0}^{3} \alpha_i$ and with $(\alpha_i, \alpha_j) = 2\delta_i^j$ and α_0 as before. Thus, α_i are roots of $\mathfrak{h}$ for $i = 1, 2, 3$ and can be chosen such that they satisfy the maximality properties stated above. Moreover, $(\alpha, \alpha_0) = 1$ and hence, $\lambda = \alpha - \frac{1}{2}\alpha_0 \in \Phi$ is of the desired form. ∎

We shall now explain the significance of these representations in the context of symplectic holonomy groups by recalling some known results.

Definition 3.5 *Let $H \subset Sp(V)$ be a proper irreducible Lie subgroup. We call H a* special symplectic holonomy group *if there exists a symplectic connection (M, Ω, ∇) on some symplectic manifold M whose holonomy group is conjugate to H. The corresponding Lie subalgebra $\mathfrak{h} \subset \mathfrak{sp}(V)$ is called a* special symplectic holonomy algebra.

Theorem 3.6
1. *A proper subgroup $H \subset Sp(V)$ is a special symplectic holonomy group iff it is an entry of Table 5.*
2. *Let M be a manifold with a torsion free connection whose holonomy is (contained in) the special symplectic holonomy group $H \subset Sp(V)$, and let $\pi : F \to M$ be the holonomy bundle with connection coframe $\theta + \omega$.*

   Then there are maps $\mathbf{a} : F \to \mathfrak{h}$ and $\mathbf{b} : F \to V$ and a constant $\mathbf{c} \in \mathbb{F}$ such that the following structure equations are satisfied.

$$\begin{aligned} d\theta &= -\omega \wedge \theta \\ d\omega &= -\omega \wedge \omega - 2R_{\mathbf{a}}(\theta \wedge \theta) \\ d\mathbf{a} &= -\omega \cdot \mathbf{a} + \mathbf{b} \circ \theta \\ d\mathbf{b} &= -\omega \cdot \mathbf{b} + (2\mathbf{a}^2 + (2(\mathbf{a}, \mathbf{a}) + \mathbf{c}) Id_V) \cdot \theta \end{aligned} \tag{6}$$

In particular, there is a constant $k_0 \neq 0$ such that $\pi^*(scal) = k_0(\mathbf{a}, \mathbf{a})$.

3. *Every symplectic connection whose holonomy is contained in a special symplectic holonomy group is analytic.*

4. *Let (M, Ω, ∇) and (M', Ω', ∇') be two manifolds with special symplectic holonomy $H \subset Sp(V)$, let $\pi : F \to M$ and $\pi' : F' \to M'$ be their holonomy reductions, and let $\mathbf{a} + \mathbf{b} : F \to \mathfrak{h} \oplus V$, $\mathbf{a}' + \mathbf{b}' : F' \to \mathfrak{h} \oplus V$ and $\mathbf{c}, \mathbf{c}'$ be the maps and constants for which the structure equations (6) hold.*

   If $\mathbf{c} = \mathbf{c}'$ and if there are points $p \in F$ and $p' \in F'$ such that $(\mathbf{a}, \mathbf{b})(p) = (\mathbf{a}', \mathbf{b}')(p')$ then there are neighborhoods of $\pi(p) \in M$ and $\pi'(p') \in M'$ on which the connections are equivalent.

5. *Let $\mathfrak{g}$ be the Lie algebra of infinitesimal symmetries on F. Then for any point $p \in F$ there is an isomorphism*

$$\begin{array}{rcl} \mathfrak{g} & \longrightarrow & \mathfrak{g}_p := \{(\theta + \omega)(w) \in V \oplus \mathfrak{h} \mid w \in T_pF,\ d\rho_p(w) = 0\} \\ X & \longmapsto & (\theta + \omega)(X_p) \end{array} \tag{7}$$

   where $\rho := \mathbf{a} + \mathbf{b} : F \to \mathfrak{h} \oplus V$. In particular, $d\rho$ has constant rank.

Proof. The first assertion follows from the classification of irreducible holonomy groups [MS1, S3] while the structure equations were determined in [CMS1, Th.3.10]. The second and third assertion were demonstrated in [CMS1, Cor.3.12]. The form of the symplectic scalar curvature follows from Definition 2.2 and (5). Finally, the last statement follows from the structure equations (6) together with [S4, Prop.4.8]. ∎

4 Special symplectic holonomy algebras

4.1 Complex symplectic holonomy algebras

Throughout this section, all vector spaces and Lie algebras are assumed to be complex. Let $\mathfrak{h} \subset \mathfrak{sp}(V, \langle\ ,\ \rangle)$ be a special symplectic holonomy algebra and let Δ and Φ be the sets of roots of $\mathfrak{h}$ and weights of V, respectively. Moreover, (4) holds by Theorems 3.1 and 3.6,1. Also, we use the $ad_{\mathfrak{h}}$-invariant inner product $(\ ,\ )$ on $\mathfrak{h}$ for which Proposition 3.4 is valid.

Let $\lambda_0 \in \Phi$ be the dominant weight of a special symplectic holonomy algebra. For $r \in \{\pm 1, \pm 3\}$, we let $V_r := \bigoplus_{\{\lambda \mid (\lambda_0, \lambda) = \frac{r}{2}\}} V_\lambda$. Then, by Proposition 3.4 we have the decomposition

$$V = V_3 \oplus V_1 \oplus V_{-1} \oplus V_{-3} \tag{8}$$

and $V_{\pm 3} = V_{\pm\lambda_0}$. We then define the Lie subgroups $P, N^i \subset \mathrm{H}$ for $i \in \{0, \pm 1\}$ by

$$\begin{array}{rcl} N^i & := & \{g \in \mathrm{H} \mid gV_r \subset V_{r+2i}, r = \pm 1, \pm 3\} \\ P & := & \{g \in N^0 \mid g|_{V_3} = Id_{V_3}\}. \end{array} \tag{9}$$

For $r \in \mathbb{Z}$, we define $\Delta_r := \{\alpha \in \Delta \mid (\lambda_0, \alpha) = r\}$ and

$$\mathfrak{n}^{\pm} := \bigoplus_{\alpha \in \Delta_{\pm 1}} \mathfrak{h}_\alpha, \qquad \mathfrak{n}^0 := \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta_0} \mathfrak{h}_\alpha. \tag{10}$$

Since λ_0 is long, Proposition 3.4 implies that $\Delta = \Delta_{-1} \cup \Delta_0 \cup \Delta_1$ and thus,

$$\mathfrak{h} = \mathfrak{n}^- \oplus \mathfrak{n}^0 \oplus \mathfrak{n}^+, \qquad [\mathfrak{n}^i, \mathfrak{n}^j] \subset \mathfrak{n}^{i+j} \qquad \text{and} \qquad V_r \circ V_s \subset \mathfrak{n}^{\frac{1}{2}(r+s)} \tag{11}$$

with the convention that $\mathfrak{n}^{\pm 2} = \mathfrak{n}^{\pm 3} = 0$. We also define the following subalgebras of $\mathfrak{h}$:

$$\begin{aligned} &\mathfrak{p}_s := \textstyle\bigoplus_{\alpha \in \Delta_0} \mathfrak{h}_\alpha \oplus \langle \{[\mathfrak{h}_\alpha, \mathfrak{h}_{-\alpha}] \mid \alpha \in \Delta_0\} \rangle \subset \mathfrak{n}^0, \\ &\mathfrak{t}^0 := \mathfrak{t} \cap \mathfrak{p}_s^-, \quad \text{so that} \quad \mathfrak{n}^0 = \mathfrak{t}^0 \oplus \mathfrak{p}_s, \\ &\mathfrak{p} := \{x \in \mathfrak{n}^0 \mid x \cdot V_{\pm 3} = 0\}. \end{aligned} \tag{12}$$

Then $\mathfrak{p}_s \subset \mathfrak{p} \subset \mathfrak{n}^0$ and $\mathfrak{p}_s$ is the maximal semi-simple subalgebra stabilizing λ_0, and $\mathfrak{p}$, $\mathfrak{n}^0$ are central extensions of $\mathfrak{p}_s$. We also have the following:

$$\mathfrak{n}^i \cdot V_r \subset V_{r+2i}, \qquad [\mathfrak{p}, \mathfrak{n}^i] \subset \mathfrak{n}^i, \qquad \mathfrak{p} \cdot V_{\pm 3} = 0. \tag{13}$$

Thus, $\mathfrak{n}^i$ and $\mathfrak{p}$ are the Lie algebras of N^i and P, respectively, and we let $P_s \subset P$ be the connected (semisimple) subgroup with Lie algebra $\mathfrak{p}_s$. From Proposition 3.4, the following is now evident.

Lemma 4.1 *Fix $0 \neq v_\pm \in V_{\pm 3}$. Then the maps*

$$\mathfrak{n}^{\mp} \longrightarrow V_{\pm 1}, \quad x \longmapsto x v_\pm \quad \text{and} \quad V_{\pm 1} \longrightarrow (V_{\mp 1})^*, \quad v \longmapsto \langle v, \_ \rangle$$

are P-equivariant isomorphisms.

By (8) and (13), $\mathfrak{n}^{\pm}$ are nilpotent. Indeed, if $x \in \mathfrak{n}^{\pm}$ then $x^4 V = 0$ and $x^3 V \subset V_{\pm 3}$.

Definition 4.2 *An element $x \in \mathfrak{n}^{\pm}$ is said to be* non-degenerate *if $x^3 V \neq 0$. Otherwise, x is called* degenerate.

Lemma 4.3 *Let $v_\pm \in V_{\pm 3}$ with $\langle v_+, v_- \rangle = 1$ and $x \in \mathfrak{n}^+$ be such that $\langle x^3 v_-, v_- \rangle = c$.*

1. $v_+ \circ v_- \in \mathfrak{t}$ is determined by the equation

$$\mu(v_+ \circ v_-) = -2(\lambda_0, \mu) \quad \text{for all } \mu \in \mathfrak{t}^*. \tag{14}$$

Thus, $ad(v_+ \circ v_-)|_{\mathfrak{n}^i} = -2i\, Id_{\mathfrak{n}^i}$ for $i = 0, \pm 1$.

2.

$$\begin{aligned} [x, (x^2 v_-) \circ v_-] &= \tfrac{2}{3} c v_+ \circ v_- \\ (x^2 v_-) \circ (x v_-) &= -\tfrac{1}{3} c v_+ \circ v_- \\ (x^2 v_-) \circ (x^2 v_-) &= -\tfrac{8}{3} c x. \end{aligned} \tag{15}$$

3.

$$(\mathbf{a}, v \circ w) = -2\langle \mathbf{a}v, w\rangle \textit{ for all } \mathbf{a} \in \mathfrak{h} \textit{ and } v, w \in V. \tag{16}$$

Proof. Since $v_\pm \in V_{\pm 3} = V_{\pm\lambda_0}$, it follows that $v_+ \circ v_- \in \mathfrak{t}$. Now, $\mathfrak{t}^*$ is spanned by all elements $\mu \in \Phi_1 \cup \Phi_3$, so it suffices to show (14) for these. If we let $v_\mu \in V_\mu$ then

$$\begin{aligned}
\mu(v_+ \circ v_-)v_\mu &= (v_+ \circ v_-)v_\mu \\
&= (v_+ \circ v_\mu)v_- + 2\langle v_-, v_\mu\rangle v_+ \\
&\qquad - \langle v_+, v_-\rangle v_\mu + \langle v_+, v_\mu\rangle v_- \quad \text{by (4)} \\
&= 2\langle v_-, v_\mu\rangle v_+ - v_\mu \quad \text{since } \langle v_+, v_\mu\rangle = 0 \text{ and } v_+ \circ v_\mu = 0 \\
&\qquad\qquad \text{for } \mu \in \Delta_1 \cup \Delta_3, \text{ and } \langle v_+, v_-\rangle = 1.
\end{aligned}$$

Thus, if we let $\mu = \lambda_0$ and $v_\mu = v_+$ then this implies that $\mu(v_+ \circ v_-) = -3 = -2(\lambda_0, \mu)$. If, on the other hand, $\mu \in \Delta_1$, then $\langle v_-, v_\mu\rangle = 0$ and hence $\mu(v_+ \circ v_-) = -1 = -2(\lambda_0, \mu)$.

The next assertion follows since $\mathfrak{n}^i$ is spanned by root spaces $\mathfrak{h}_\alpha$ with $(\lambda_0, \alpha) = i$ and $[v_+ \circ v_-, x_\alpha] = \alpha(v_+ \circ v_-)x_\alpha$.

Moreover, $[x, (x^2v_-) \circ v_-] = cv_+ \circ v_- + (x^2v_-) \circ (xv_-) = cv_+ \circ v_- + \frac{1}{2}[x, (xv_-) \circ (xv_-)]$ and $(xv_-) \circ (xv_-) = [x, (xv_-) \circ v_-] - (x^2v_-) \circ v_- = -(x^2v_-) \circ v_-$ as $(xv_-) \circ v_- = \frac{1}{2}[x, v_- \circ v_-] = 0$. This implies the second part.

For the last part, note that both sides of (16) are $ad_{\mathfrak{h}}$-invariant, hence it suffices to verify this identity for $\mathbf{a} \in \mathfrak{t}$, $v = v_+$ and $w = v_-$. In this case, (16) follows immediately from (14). ∎

Proposition 4.4 *If $rk(\mathfrak{h}) \geq 3$ then every P-orbit of an element $x \in \mathfrak{n}^+$ contains an element of the form*

$$\tilde{x} = x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3}, \tag{17}$$

where $x_{\alpha_i} \in \mathfrak{h}_{\alpha_i}$ and the α_i are as in Proposition 3.4,6.

The element $\tilde{x}$ from (17) is called the *normal form* of x. Note that there are only two holonomies $\mathfrak{h}$ with $\mathrm{rk}(\mathfrak{h}) \leq 2$, namely $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{C})$ with $V = \odot^3\mathbb{C}^2$ and $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C})$ with $V = \mathbb{C}^2 \otimes \mathbb{C}^3$.

Proof. Let $\alpha \in \Delta$ be a root. Then $(\alpha, \beta) \geq -2$ for all $\beta \in \Delta$ and thus, $(ad(\mathfrak{h}_\alpha))^3 = 0$. Therefore,

$$Ad_{exp(x_\alpha)}x = x + [x_\alpha, x] + \frac{1}{2}[x_\alpha, [x_\alpha, x]] \quad \text{for all } x_\alpha \in \mathfrak{h}_\alpha, \alpha \in \Delta \text{ and } x \in \mathfrak{h}. \tag{18}$$

Let $\alpha_1 \in \Delta$ be the maximal long root, and let $x \in \mathfrak{n}^+$. We decompose x as

$$x = \sum_{\alpha \in \Delta_1} x_\alpha,$$

where $x_\alpha \in \mathfrak{h}_\alpha$. W.l.o.g. we assume that $x_{\alpha_1} \neq 0$. Moreover, we let $\Delta_1' := \{\alpha \in \Delta_1 \mid (\alpha, \alpha_1) = 0\}$.

Since $[\mathfrak{n}^+, \mathfrak{n}^+] = 0$ by (11), we have $(\beta, \gamma) \geq 0$ for all $\beta, \gamma \in \Delta_1$. Let $\beta \in \Delta_1$ be a root with $(\beta, \alpha_1) = 1$, i.e. $\beta - \alpha_1 \in \Delta_0$. We first assert that $[\mathfrak{h}_{\beta-\alpha_1}, \mathfrak{n}^+] \subset \mathfrak{h}_\beta \oplus \bigoplus_{\alpha \in {\Delta_1}'} \mathfrak{h}_\alpha$. To see this, let $\gamma \in \Delta_1, \gamma \neq \alpha_1$ be a root for which $[\mathfrak{h}_{\beta-\alpha_1}, \mathfrak{h}_\gamma] = \mathfrak{h}_{\beta-\alpha_1+\gamma} \neq 0$, i.e. $\beta - \alpha_1 + \gamma \in \Delta_1$. Then $0 \leq (\beta - \alpha_1 + \gamma, \alpha_1) = 1 - 2 + (\gamma, \alpha_1)$, and since $\gamma \neq \alpha_1$ and α_1 is long, this implies that $(\gamma, \alpha_1) = 1$, i.e. $\beta - \alpha_1 + \gamma \in {\Delta_1}'$ as claimed.

Thus, if $\beta \in \Delta_1$ is a root such that $(\beta, \alpha_1) = 1$ and $x_\beta \neq 0$, then replacing x by $x' := Ad_{exp(x_{\beta-\alpha_1})}(x)$ for a suitable element $x_{\beta-\alpha_1} \in \mathfrak{h}_{\beta-\alpha_1}$ then, using (18) and the above, we have $x'_\beta = 0$ so that the number of roots $\beta \in \Delta_1$ with $(\beta, \alpha_1) = 1$ and $x_\beta \neq 0$ can be reduced by one. Repeating this process, we conclude that the P-orbit of x contains an element of the form

$$x' = x_{\alpha_1} + \sum_{\alpha \in {\Delta_1}'} x_\alpha.$$

Next, let α_2 be the maximal root in the root system $\{\alpha \in \Delta \mid (\alpha, \alpha_1) = 0\}$. By Proposition 3.4, $(\lambda_0, \alpha_2) = 1$. Then a similar discussion as above implies that the P-orbit of x contains an element of the form

$$x'' = x_{\alpha_1} + x_{\alpha_2} + \sum_{\alpha \in {\Delta_1}''} x_\alpha,$$

where $\Delta_1'' = \{\alpha \in \Delta_1 \mid (\alpha, \alpha_i) = 0,\ i = 1, 2\}$. Let $\beta \in {\Delta_1}''$. Then by Proposition 3.4, $(\beta, \alpha_3) = (\beta, 2\lambda_0) = 2$, hence $\beta = \alpha_3$ since α_3 is long, i.e. ${\Delta_1}'' = \{\alpha_3\}$. ∎

Corollary 4.5 *Suppose $rk(\mathfrak{h}) \geq 3$. Then an element $x \in \mathfrak{n}^+$ is non-degenerate iff in its normal form (17) we have $x_{\alpha_i} \neq 0$ for $i = 1, 2, 3$.*

Proof. Let $\tilde{x}$ be the normal form of x. Since $\alpha_i^2 \alpha_j V_{-\lambda_0} = V_{-\lambda_0+2\alpha_i+\alpha_j} = 0$ by Proposition 3.4 and since $[x_{\alpha_i}, x_{\alpha_j}] \in [\mathfrak{n}^+, \mathfrak{n}^+] = 0$, it follows that $\tilde{x}^3 = 6x_{\alpha_1}x_{\alpha_2}x_{\alpha_3}$. ∎

Corollary 4.6 *Suppose $rk(\mathfrak{h}) \geq 3$. Then every $v \in V_{\pm 1}$ lies in the P-orbit of an element of the form*

$$v = v_{\pm(\lambda_0-\alpha_1)} + v_{\pm(\lambda_0-\alpha_2)} + v_{\pm(\lambda_0-\alpha_3)} \tag{19}$$

with $v_{\pm(\lambda_0-\alpha_i)} \in V_{\pm(\lambda_0-\alpha_i)}$ where $\alpha_i \in \Delta$ are as in Proposition 3.4,6.

Proof. It suffices to treat the case $v \in V_{-1}$ since $V_1 = (V_{-1})^*$. Let $0 \neq v_- \in V_{-3}$. By Lemma 4.1, we have $v = xv_-$ with $x \in \mathfrak{n}^+$. Replacing v by an element in its P-orbit, we may assume that x is of normal form (17), whence the claim follows. ∎

Corollary 4.7 *Let $v \in V_1 \oplus V_3$. Then $v \circ v \in \mathfrak{n}^+$. Moreover, if $v \circ v$ is degenerate then $(v \circ v)^2 = 0$; in this case, $v \circ v$ lies in the P-orbit of $\mathfrak{h}_\alpha$ where $\alpha \in \Delta$ is a long root.*

Thus, if $v \circ v$ is degenerate then either $v \circ v = 0$ or $rk(v \circ v) = \#\{\lambda \in \Phi \mid (\lambda, \alpha) = 1\}$.

Proof. The cases with $\mathrm{rk}(\mathfrak{h}) \leq 2$ are easily proven, thus we assume the contrary.

By (11), we have $V_3 \circ V_3 = V_3 \circ V_1 = 0$ and $V_1 \circ V_1 \subset \mathfrak{n}^+$. Thus, we may assume that $v \in V_1$ and write it in its normal form (19)

$$v = \sum_{i=1}^{3} v_{\lambda_0 - \alpha_i}.$$

Since $\lambda_0 - \alpha_i$ is a long weight, it follows that $2(\lambda_0 - \alpha_i)$ is not a root and hence $v_{\lambda_0-\alpha_i} \circ v_{\lambda_0-\alpha_i} = 0$. Thus,

$$v \circ v = 2 \sum_{i<j} v_{\lambda_0-\alpha_i} \circ v_{\lambda_0-\alpha_j}.$$

But now, $v_{\lambda_0-\alpha_i} \circ v_{\lambda_0-\alpha_j} \in \mathfrak{h}_{2\lambda_0-\alpha_i-\alpha_j} = \mathfrak{h}_{\alpha_k}$ where $\{i,j,k\} = \{1,2,3\}$. Thus, by Corollary 4.5, $v \circ v$ is non-degenerate iff $v_{\lambda_0-\alpha_i} \circ v_{\lambda_0-\alpha_j} \neq 0$ for all $i \neq j$ which happens iff $v_{\lambda_0-\alpha_i} \neq 0$ for all i. On the other hand, if, say, $v_{\lambda_0-\alpha_1} = 0$ then $v \circ v = 2v_{\lambda_0-\alpha_2} \circ v_{\lambda_0-\alpha_3} \in \mathfrak{h}_{\alpha_1}$, and the claim follows. ∎

Proposition 4.8 *Let $v \in V_1 \oplus V_3$.*

1. *If $v \notin V_3$ then its N^+-orbit contains an element in V_1.*
2. *If $v \neq 0$ then the H-orbit of v contains an element v of the form*

$$\tilde{v} = v_+ + xv_-, \tag{20}$$

for some degenerate $x \in \mathfrak{n}^+$ and where $v_\pm \in V_{\pm 3}$ are such that $\langle v_+, v_- \rangle = 1$. Moreover, if $v \circ v$ is degenerate then $v \circ v = 4x$.

Proof. Let $v := v_1 + v_3$ with $v_i \in V_i$ and suppose that $v_1 \neq 0$. By Lemma 4.1, there is an $x \in \mathfrak{n}^+$ for which $0 \neq xv_1 \in V_3$. Moreover, $xV_3 = 0$. Thus, $exp(x)v = v_1 + (v_3 + xv_1)$, and since $\dim V_3 = 1$ it follows that, after replacing x by a suitable scalar multiple, we may assume that $exp(x)v = v_1$ which shows the first assertion.

The second assertion is easily proven for the cases with $\mathrm{rk}(\mathfrak{h}) \leq 2$, thus we assume the contrary. Moreover, it is obvious if $v \in V_3$, thus by the first part, we may assume that $v \in V_1$ and that v is in its normal form (19). Since $v \neq 0$ we assume w.l.o.g. that $v_{\lambda_0-\alpha_1} \neq 0$.

Now let w be the element of the Weyl group of $\mathfrak{h}$ which corresponds to the reflection σ_{α_1}. Then $wV_{\lambda_0-\alpha_1} = V_{\lambda_0} = V_3$ and $wV_{\lambda_0-\alpha_2} = V_{\lambda_0-\alpha_1-\alpha_2} = V_{-\lambda_0+\alpha_3}$, and likewise, $wV_{\lambda_0-\alpha_3} = V_{-\lambda_0+\alpha_2}$. Thus, $\tilde{v} := w \cdot v \in V_3 \oplus \mathfrak{h}_{\alpha_2} V_{-3} \oplus \mathfrak{h}_{\alpha_3} V_{-3}$ with non-vanishing V_3-component, and this implies that $\tilde{v}$ can be written in the form (20) with $x \in \mathfrak{h}_{\alpha_2} + \mathfrak{h}_{\alpha_3} \subset \mathfrak{n}^+$ and hence, x is degenerate.

If $v \circ v$ is degenerate then as in the proof of Corollary 4.7 we may assume that $v_{\lambda_0-\alpha_3} = 0$ and hence, $x \in \mathfrak{h}_{\alpha_3}$ by the above and therefore, $x^2 = 0$. Thus,

$$\begin{aligned}
\tilde{v} \circ \tilde{v} &= v_+ \circ v_+ + 2v_+ \circ (xv_-) + (xv_-) \circ (xv_-) \\
&= 2[x, v_+ \circ v_-] + [x, v_- \circ (xv_-)] && \text{since } v_+ \circ v_+ = 0,\ xv_+ = 0 \text{ and } x^2 = 0 \\
&= 4x + \tfrac{1}{2}[x, [x, v_- \circ v_-]] && \text{by Lemma 4.3} \\
&= 4x && \text{since } v_- \circ v_- = 0.
\end{aligned}$$

■

If $p \in \mathfrak{p}$ and $x \in \mathfrak{n}^+$ then $x[p,x] = [p,x]x$ since $x, [p,x] \in \mathfrak{n}^+$. This implies by induction that for $v_- \in V_{-3}$ we have

$$px^k v_- = k[p,x]x^{k-1}v_- \quad \text{for all } k \geq 1. \tag{21}$$

Corollary 4.9 *Fix $0 \neq v_- \in V_{-3}$. For any constant $c \neq 0$, the set*

$$S_c := \{x \in \mathfrak{n}^+ \mid \langle x^3 v_-, v_- \rangle = c\} \tag{22}$$

is a single P-orbit of codimension 1. Moreover, if $x \in \mathfrak{n}^+$ is non-degenerate then every element $y \in \mathfrak{n}^+$ satisfies $y = rx + [p,x]$ for some $r \in \mathbb{C}$ and $p \in \mathfrak{p}$.

Proof. For the two holonomies with $\mathrm{rk}(\mathfrak{h}) \leq 2$ the statement is easily verified, thus we assume that $\mathrm{rk}(\mathfrak{h}) \geq 3$.

The P-invariance of S_c follows directly from the definition of P in (9). Moreover, if $\sum_{i=1}^3 x_{\alpha_i}$ lies in this set, then any other element of S_c in normal form is of the form $\sum_{i=1}^3 c_i x_{\alpha_i}$ for constants c_i with $c_1c_2c_3 = 1$. But then these elements lie in the same P-orbit, and the first statement follows from Proposition 4.4. Therefore, $[\mathfrak{p}, x]$ is a hyperplane in $\mathfrak{n}^+$ and $x \notin [\mathfrak{p}, x]$ which shows the second assertion. ■

Corollary 4.10 *If $x \in \mathfrak{n}^+$ is non-degenerate then $x : V_{-1} \to V_1$ and $ad(x)^2 : \mathfrak{n}^- \to \mathfrak{n}^+$ are isomorphisms.*

Proof. For the two holonomies with $\mathrm{rk}(\mathfrak{h}) \leq 2$ the statements are easily verified, thus we assume that $\mathrm{rk}(\mathfrak{h}) \geq 3$. Also, since $V_{\pm 1}$ and $\mathfrak{n}^\pm$ have equal dimensions, it suffices to show injectivity in each case. We fix $v_\pm \in V_{\pm 3}$ such that $\langle v_+, v_- \rangle = 1$ and define $0 \neq c \in \mathbb{C}$ by $x^3 v_- = cv_+$.

By Lemma 4.1, there is an $x_- \in \mathfrak{n}^-$ such that $x^2 v_- = x_- v_+$. From (17) it is easy to see that x_- is also non-degenerate and thus by Corollary 4.9, the dimensions of the stabilizers of x and x_- are equal. Let $p \in \mathfrak{p}$ be in the stabilizer of x. Then $[p, x_-]v_+ = px_-v_+ = px^2v_- = x^2pv_- = 0$, and hence $[p, x_-] = 0$. Thus the stabilizer of x_- contains the stabilizer of x, thus these stabilizers are equal. That is to say, $[p,x] = 0$ iff $[p, x_-] = 0$ iff $px^2v_- = 0$.

Let $v \in V_{-1}$ be such that $xv = 0$. By Corollary 4.9, $v = rxv_- + [p,x]v_-$ for some $r \in \mathbb{C}$ and $p \in \mathfrak{p}$. Thus, $0 = x^2v = rcv_+ + [p,x]x^2v_- = rcv_+ + \frac{1}{3}px^3v_- = rcv_+$ by (21). Thus, $r = 0$, i.e. $v = [p,x]v_-$. But then, $0 = xv = x[p,x]v_- = \frac{1}{2}px^2v_- = \frac{1}{2}[p, x_-]v_+$ again by (21), thus $[p, x_-] = 0$ by Lemma 4.1. By the above, this implies that $v = [p,x]v_- = 0$.

To show the second assertion, let $y \in \mathfrak{n}^-$ be such that $ad(x)^2 y = 0$. Since $yxv_- \in V_{-3}$, there is a $r \in \mathbb{C}$ such that $yxv_- = rv_-$. Then $\langle xyv_+, v_- \rangle = \langle v_+, yxv_- \rangle = r$ so that $xyv_+ = rv_+$. Also, since $ad(x)^3(\mathfrak{n}^-) = 0$ by (11), we have

$$\begin{aligned} 0 &= [x,[x,[x,y]]]v_- \\ &= (x^3y - 3x^2yx + 3xyx^2 - yx^3)v_- \\ &= 0 - 3rx^2v_- + 3xyx^2v_- - cyv_+ \end{aligned} \tag{23}$$

and

$$\begin{aligned} 0 &= [x,[x,y]]]v_- \\ &= (x^2y - 2xyx + yx^2)v_- \\ &= 0 - 2rxv_- + yx^2v_-. \end{aligned} \tag{24}$$

Multiplying (24) by x and using (23) yields

$$0 = -2rx^2v_- + xyx^2v_- = -rx^2v_- + \frac{c}{3}yv_+. \tag{25}$$

Multiplying (25) again by x yields $-\frac{2c}{3}rv_+ = 0$. Thus $r = 0$, and by (25) and Lemma 4.1 we have also $y = 0$. ∎

Let us fix once and for all a non-degenerate element $\mathbf{a}_+ \in \mathfrak{n}^+$ and let $P_0 \subset P$ and $\mathfrak{p}_0 \subset \mathfrak{p}$ be the stabilizers of $\mathbf{a}_+$. We define the symmetric bilinear form $\sigma = \sigma_{\mathbf{a}_+}$ on V_{-1} by the equation

$$\sigma(v,w) = \sigma_{\mathbf{a}_+}(v,w) := \langle \mathbf{a}_+ v, w\rangle \quad \text{for all } v,w \in V_{-1}. \tag{26}$$

Thus,

$$P_0 = P \cap \mathrm{O}(V_{-1},\sigma) \quad \text{and} \quad \mathfrak{p}_0 := \mathfrak{p} \cap \mathfrak{so}(V_{-1},\sigma). \tag{27}$$

Of course, P_0 fixes $\mathbf{a}_+ v_- \in V_{-1}$ and hence its orthogonal complement

$$W_{-1} := V_{-1} \cap \mathbf{a}_+^- = \{y \in V_{-1} \mid \sigma(\mathbf{a}_+ v_-, y) = \langle \mathbf{a}_+^2 v_-, y\rangle = 0\} \tag{28}$$

and the space

$$W_1 := \mathbf{a}_+ W_{-1} \subset V_1.$$

Evidently, $W_1 \cong W_{-1}$ as a P_0-module.

Proposition 4.11 *Let $\mathbf{a}_+ \in \mathfrak{n}^+$ be non-degenerate and let $W_{\pm 1} \subset V_{\pm 1}$ and $\mathfrak{p}_0 \subset \mathfrak{p}$ be as above. Then*

$$W_{-1} = \ker(\mathbf{a}_+^2 \cdot \_ : V_{-1} \to V_3) = \mathfrak{p}(\mathbf{a}_+ v_-) = [\mathfrak{p}, \mathbf{a}_+]v_-.$$

and

$$W_1 = \ker(\mathbf{a}_+ : V_1 \to V_3) = \mathfrak{p}(\mathbf{a}_+^2 v_-) = [\mathfrak{p}, \mathbf{a}_+^2]v_-.$$

Proof. The last two spaces of the first line are equal since $\mathfrak{p}v_- = 0$. Let $w \in W_{-1}$. Then $0 = \sigma(\mathbf{a}_+ v_-, w) = \langle \mathbf{a}_+^2 v_-, w\rangle = \langle v_-, \mathbf{a}_+^2 w\rangle$, and $\mathbf{a}_+^2 w \in V_3$ so that $\mathbf{a}_+^2 w = 0$. Also, $\mathbf{a}_+^2[\mathfrak{p}, \mathbf{a}_+]v_- = \frac{1}{3}\mathfrak{p}\mathbf{a}_+^3 v_- = 0$ by (21). Thus, all spaces of the first line are contained in the kernel of $\mathbf{a}_+^2$. Since all of these are hyperplanes in V_{-1} by Corollary 4.9, equality follows.

The equalities in the second line are shown analogously. ∎

As it turns out, $\mathfrak{p}_0$ is again semi-simple for almost all special symplectic holonomy algebras $\mathfrak{h} \subset \mathfrak{sp}(n,\mathbb{C})$. (Some exceptions occur if $\mathrm{rk}(\mathfrak{h}) \le 3$.) Also, $\mathfrak{p}_s = \mathfrak{p}$ for all entries except $\mathfrak{h} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(n,\mathbb{C})$ in which case $\mathfrak{p} = \mathfrak{p}_s \oplus \mathbb{C}$. We list the Lie algebras $\mathfrak{p}$ and their representation on V_1 as well as the representations of $\mathfrak{p}_0$ on W_1 in Table 7.

Table 7: $\mathfrak{p}$ and $\mathfrak{p}_0$ for special symplectic holonomies

$\mathfrak{h}$	V	$\mathfrak{p}$	V_1	$\mathfrak{p}_0$	$W_1 \subset V_1$
$\mathfrak{sl}(2,\mathbb{C})$	$\odot^3\mathbb{C}^2$	0	$\mathbb{C}$	0	0
$\mathfrak{sp}(3,\mathbb{C})$	$(\Lambda^3\mathbb{C}^6)_0$	$\mathfrak{sl}(3,\mathbb{C})$	$\odot^2\mathbb{C}^3$	$\mathfrak{so}(3,\mathbb{C})$	$(\odot^2\mathbb{C}^3)_0$
$\mathfrak{sl}(6,\mathbb{C})$	$\Lambda^3\mathbb{C}^6$	$\mathfrak{sl}(3,\mathbb{C}) \oplus \mathfrak{sl}(3,\mathbb{C})$	$\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{C})$
$\mathfrak{spin}(12,\mathbb{C})$	$\Delta_{12}^{\mathbb{C}}$	$\mathfrak{sl}(6,\mathbb{C})$	$\Lambda^2\mathbb{C}^6$	$\mathfrak{sp}(3,\mathbb{C})$	$(\Lambda^2\mathbb{C}^6)_0$
$\mathfrak{e}_7$	$\mathbb{C}^{56}$	$\mathfrak{e}_6$	$\mathbb{C}^{27}$	$\mathfrak{f}_4$	$\mathbb{C}^{26}$
$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(n,\mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^n$	$\mathbb{C} \oplus \mathfrak{so}(n-2,\mathbb{C})$	$\mathbb{C} \oplus \mathbb{C}^{n-2}$	$\mathfrak{so}(n-3,\mathbb{C})$	$\mathbb{C} \oplus \mathbb{C}^{n-3}$

An important observation we make is that $(\mathfrak{p}_s, \mathfrak{p}_0)$ is an *irreducible symmetric pair* in all cases. Next, we let

$$Sym(W_{-1},\sigma) := \{\phi \in \mathrm{End}(W_{-1}) \mid \sigma(\phi(v),w) = \sigma(\phi(w),v) \text{ for all } v,w \in W_{-1}\}$$

and also, let

$$\mathfrak{n}^{\pm}_{\mathbf{a}_+} := \{y \in \mathfrak{n}^{\pm} \mid yV_{\mp 3} \subset W_{\mp 1}\} = \{y \in \mathfrak{n}^{\pm} \mid \mathbf{a}_+^2 y V_{\mp 3} = 0\}, \tag{29}$$

where the equality of these sets follows from Proposition 4.11. By Lemma 4.1, $\mathfrak{n}^{\pm}_{\mathbf{a}_+} \cong W_{\pm 1}$ as a P_0-module.

Let $\pi : V_{-1} \to W_{-1}$ be the σ-orthogonal projection. It is then straightforward to verify that the maps

$$\begin{array}{cccc} \imath: & \mathfrak{n}^- & \longrightarrow & Sym(W_{-1},\sigma) \\ & y & \longmapsto & (\pi \circ y \circ \mathbf{a}_+)|_{W_{-1}} \end{array} \quad \text{and} \quad \begin{array}{cccc} \jmath: & \odot^2\mathfrak{p}_0 & \longrightarrow & Sym(W_{-1},\sigma) \\ & (x,y) & \longmapsto & \frac{1}{2}(xy+yx)|_{W_{-1}} \end{array} \tag{30}$$

are well defined and P_0-equivariant.

Proposition 4.12 *Let $\mathfrak{h}$ be a complex special symplectic holonomy group and fix a non-degenerate element $\mathbf{a}_+ \in \mathfrak{n}^+$. Then the map $\imath : \mathfrak{n}^- \to Sym(W_{-1},\sigma)$ from above is injective. Moreover, the solutions of the equation*

$$\imath(y) = \jmath(p^2) \quad \text{for } p \in \mathfrak{p}_0 \text{ and } y \in \mathfrak{n}^-_{\mathbf{a}_+} \tag{31}$$

can be classified as follows.

1. *If $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{C})$ or $\mathfrak{h} = \mathfrak{sl}(6,\mathbb{C})$ then $y = 0$ and $p = 0$.*
2. *If $\mathfrak{h} = \mathfrak{sp}(3,\mathbb{C})$ then $\imath$ is an isomorphism. Either*
   (a) *$y = 0$ and $p = 0$, or*
   (b) *p lies in the orbit of maximal root of $\mathfrak{p}_0 \cong \mathfrak{sl}(2,\mathbb{C})$ and y lies in the orbit of the maximal weight vector of $\mathfrak{n}^-_x \cong M_4$ where M_k denotes the (unique) $(k+1)$-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module.*
3. *If $\mathfrak{h} = \mathfrak{spin}(12,\mathbb{C})$ or $\mathfrak{h} = \mathfrak{e}_7$ then $y = 0$ and either*

(a) $p = 0$ or

(b) p lies in the orbit of the maximal root of $\mathfrak{p}_0$.

4. If $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(n,\mathbb{C})$ then $y = 0$ and $p \in \mathfrak{so}(n-3,\mathbb{C})$ is such that $p^2\mathbb{C}^{n-3} = 0$.

Proof. The cases with $\mathrm{rk}(\mathfrak{h}) \leq 2$ are easily proven, thus we assume the contrary.

Let $y \in \ker(\imath) \subset \mathfrak{n}^-$, i.e. $yW_1 \subset \mathbf{a}_+V_{-3}$. This is equivalent to saying that $y^+W_{-1} \subset \mathbf{a}_+^2V_{-3}$ for $y^+ := [\mathbf{a}_+, [\mathbf{a}_+, y]]$ by Proposition 4.11.

Let $z^+ \in \mathfrak{n}^+_{\mathbf{a}_+}$. Then $y^+z^+W_{-1} = z^+y^+W_{-1} \subset z^+\mathbf{a}_+^2V_{-3} = \mathbf{a}_+^2z^+V_{-3} \subset \mathbf{a}_+^2W_{-1} = 0$ by Proposition 4.11 so that $y^+(\mathfrak{n}^+_{\mathbf{a}_+})^2V_{-3} = 0$. But now, one shows that $(\mathfrak{n}^+_{\mathbf{a}_+})^2V_{-3} = V_1$ using the normal form (17) of elements of $\mathfrak{n}^+_{\mathbf{a}_+}$. This implies that $y^+V_1 = 0$ and thus, $y^+ = 0$ by Lemma 4.1. Therefore, $y = 0$ by Corollary 4.10 and hence, $\imath$ is injective.

Now let us investigate equation (31) for each holonomy group separately. Note that $Sym(W_{-1}, \sigma) \cong \odot^2 W_{-1} \cong \odot^2 W_1$ as a $\mathfrak{p}_0$-module.

1. $\mathfrak{h} \cong \mathfrak{sp}(3,\mathbb{C})$:

   In this case, $\dim(\odot^2\mathfrak{p}_0) = \dim(\odot^2 W_{\pm 1}) = 6$, so that $\imath$ is an isomorphism. Moreover, $\odot^2\mathfrak{p}_0 \cong \odot^2 M_2 \cong M_4 \oplus M_0$ by the Clebsch-Gordon formula and thus, $\mathfrak{n}^-_{\mathbf{a}_+} \cong W_{-1} \cong M_4$ as a $\mathfrak{sl}(2,\mathbb{C})$-module. The M_0-summand on $\mathfrak{p}_0$ is represented by the Killing form B of $\mathfrak{p}_0$ so that $\jmath(p^2) \in M_4$ iff $B(p,p) = 0$ iff $p = 0$ or p lies in the orbit of a root space.

2. $\mathfrak{h} \cong \mathfrak{sl}(6,\mathbb{C})$:

   We decompose $\odot^2\mathfrak{p}_0 \cong M_{max} \oplus \imath(\mathfrak{n}^-_{\mathbf{a}_+}) \oplus M'$ where M_{max} is the irreducible summand whose maximal weight is given by 2α with $\alpha \in \mathfrak{sl}(3,\mathbb{C})$ the maximal root. Then $\jmath(M_{max}) \neq 0$ since $(ad_{\mathfrak{p}_\alpha})^2 \neq 0$.

   Let $\pi : \odot^2\mathfrak{p}_0 \to M_{max}$ be the projection. Then the set $\{p \in \mathfrak{p}_0 \mid \pi(\jmath(p^2)) = 0\}$ is closed, $\mathfrak{p}_0$-invariant and does not intersect the orbit of the highest weight vector. It is well-known that for an irreducible representation, the only set with these properties is $\{0\}$, i.e. $\pi(\jmath(p^2)) = 0$ iff $p = 0$. Since $\pi(\jmath(p^2)) = \pi(\imath(y)) = 0$, the claim follows.

3. $\mathfrak{h} \cong \mathfrak{spin}(12,\mathbb{C})$:

   There is only one summand of $\odot^2\mathfrak{p}_0$ isomorphic to $\mathfrak{n}^-_{\mathbf{a}_+} \cong (\Lambda^2\mathbb{C}^6)_0$, namely the one given by the image of the map $\kappa : (\Lambda^2\mathbb{C}^6)_0 \to Sym(\Lambda^2\mathbb{C}^6)$ characterized by the equation $\kappa(\alpha)(\beta) \wedge \omega^2 = \alpha \wedge \beta \wedge \omega$ where ω is the symplectic form preserved by $\mathfrak{sp}(3,\mathbb{C})$. Thus, we must have $\jmath(p^2) = \kappa(\alpha)$ for some $\alpha \in (\Lambda^2\mathbb{C}^6)_0$.

   The set $\{\alpha \in (\Lambda^2\mathbb{C}^6)_0 \mid \kappa(\alpha) = \jmath(p^2) \text{ for some } p \in \mathfrak{p}_0\} \subset (\Lambda^2\mathbb{C}^6)_0$ is closed and $\mathfrak{p}_0$-invariant and hence is either $\{0\}$ or contains the orbit of the maximal weight vector.

   Let us suppose that there is a $p \in \mathfrak{p}_0$ such that $\jmath(p^2) = \kappa(\alpha)$ where $\alpha \in (\Lambda^2\mathbb{C}^6)_0$ is a maximal weight vector. Fixing a basis $\{e_{\pm i} \mid i = 1,2,3\}$ of $\mathbb{C}^6$ such that $\omega = \sum_i e_i \wedge e_{-i}$, we may assume that $\alpha = e_1 \wedge e_2$. Thus, $\kappa(\alpha)(e_i \wedge \mathbb{C}^6) = 0$ for $i = 1,2,\pm 3$. Now, a straightforward investigation yields that $p^2(e_i \wedge \mathbb{C}^6) = 0$ for $i = 1,2,\pm 3$ and $p \in \mathfrak{sp}(3,\mathbb{C})$ implies that $p(v) = \omega(v,u)u$ for some fixed $u \in \mathbb{C}^6$. But this means that $\jmath(p^2) = 0$ which is a contradiction.

Thus, if $\jmath(p^2) \in \imath(\mathfrak{n}^-_{\mathfrak{a}_+}) \subset \kappa(\Lambda^2\mathbb{C}^6)_0$ then we must have $\jmath(p^2) = 0$ which implies that either $p = 0$ or p lies in the orbit of the maximal root. Also, $y = 0$ by the first part.

4. $\mathfrak{h} \cong \mathfrak{f}_4$:

   The decomposition of $\odot^2(\mathfrak{p}_0)$ into its irreducible components yields that there is no summand isomorphic to $\mathfrak{n}^-_{\mathfrak{a}_+} \cong W_{-1}$. Thus, $\imath(\mathfrak{n}^-_{\mathfrak{a}_+}) \cap \jmath(\odot^2\mathfrak{p}_0) = 0$ which means that (31) implies that $\imath(y) = \jmath(p^2) = 0$, thus $y = 0$ by the first part.

   One can then show that each $\mathfrak{p} \in \mathfrak{f}_4$ satisfies $\jmath(p^2) = 0$ iff $p = 0$ or p lies in the orbit of a maximal root. We omit the details.

5. $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C})$:

   In this case, $\odot^2 W_1 \cong \odot^2\mathbb{C} \oplus (\mathbb{C} \otimes \mathbb{C}^{n-3}) \oplus (\odot^2\mathbb{C}^{n-3})_0 \oplus \mathbb{C}Id_{\mathbb{C}^{n-3}}$. On the one hand, since $\mathfrak{p}_0$ acts trivially on the $\mathbb{C}$-factor, it follows that $\jmath(\odot^2\mathfrak{p}_0) \subset (\odot^2\mathbb{C}^{n-3})_0 \oplus \mathbb{C}Id_{\mathbb{C}^{n-3}}$. On the other hand, $\mathfrak{n}^-_{\mathfrak{a}_+}$ does not contain $(\odot^2\mathbb{C}^{n-3})_0$ as a summand. Comparing these decompositions yields $\imath(\mathfrak{n}^-_{\mathfrak{a}_+}) \cap \jmath(\odot^2\mathfrak{p}) \subset \mathbb{C}Id_{\mathbb{C}^{n-3}}$.

   However, a glance at the normal form (17) reveals that $\imath(\mathfrak{n}^-) \cap \mathbb{C}Id_{\mathbb{C}^{n-3}} = 0$ so that (31) implies that $\imath(y) = 0$ and therefore, $y = 0$ by the first part. ∎

4.2 Real symplectic holonomy algebras

The facts about real forms and the notation used in this section are taken from [OV, ch.3,4]. Throughout this section, all vector spaces and Lie groups are assumed to be real unless they are indexed by $\mathbb{C}$.

Let $\mathrm{H} \subset \mathrm{Aut}(V)$ be a real special symplectic holonomy group where V is a real vector space, let $\mathfrak{h} \subset \mathrm{End}(V)$ be its Lie algebra and let $\mathrm{H}_{\mathbb{C}} \subset \mathrm{Aut}(V_{\mathbb{C}})$ and $\mathfrak{h}_{\mathbb{C}} \subset \mathrm{End}(V_{\mathbb{C}})$ be the complexifications. Thus, $\mathfrak{h} \oplus V \subset \mathfrak{h}_{\mathbb{C}} \oplus V_{\mathbb{C}}$ as a real subspace. Let $\mathcal{C}_{\mathbb{C}} \subset \mathbb{P}(V_{\mathbb{C}})$ be the $\mathrm{H}_{\mathbb{C}}$-orbit of the highest weight vector and let $\mathcal{C} := \mathcal{C}_{\mathbb{C}} \cap \mathbb{RP}(V)$.

We choose a Cartan subalgebra $\mathfrak{t}_{\mathbb{C}} \subset \mathfrak{h}_{\mathbb{C}}$ and let $\mathfrak{t} := \mathfrak{t}_{\mathbb{C}} \cap \mathfrak{h}$. Since $\mathfrak{h}_{\mathbb{C}} \subset \mathrm{End}(V_{\mathbb{C}})$ is also a special symplectic holonomy algebra, there is the decomposition $\mathfrak{h}_{\mathbb{C}} = \mathfrak{n}^-_{\mathbb{C}} \oplus \mathfrak{n}^0_{\mathbb{C}} \oplus \mathfrak{n}^+_{\mathbb{C}}$ and the subalgebra $\mathfrak{p}_{\mathbb{C}} \subset \mathfrak{n}^0_{\mathbb{C}}$ from (10) and (12).

Let $\mathfrak{a} \subset \mathfrak{t} \subset \mathfrak{h}$ be the maximal $\mathbb{R}$-diagonalizable subalgebra. Then there is a decomposition

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{h}_\lambda \quad \text{where } \mathfrak{h}_0 = \mathfrak{a} \oplus \mathfrak{m} \tag{32}$$

for some subset $\Sigma \subset \mathfrak{a}^*$ and some compact subalgebra $\mathfrak{m}$. Then Σ is a root system and has as many simple components as $\mathfrak{h}$. Moreover, the action of the Weyl group W_Σ on $\mathfrak{a}^*$ leaves $\mathfrak{m}$ invariant and is induced by conjugation by elements of H. We call an element $\lambda \in \mathfrak{t}^*$ *real* if $\lambda(\mathfrak{t}) \subset \mathbb{R}$ which occurs iff $\lambda(\mathfrak{m} \cap \mathfrak{t}) = 0$. It follows that for some real $\lambda \in \mathfrak{t}^*$, the orbit $W_\Sigma \cdot \lambda = (W \cdot \lambda) \cap \{\text{real elements}\}$, where W denotes the Weyl group of $\mathfrak{h}$.

Let us suppose that $\mathcal{C} \neq \emptyset$, and let $v_+ \in V$ be an element which determines a line in $\mathcal{C}$. Then we can choose $\mathfrak{t}_{\mathbb{C}}$ and a fundamental Weyl chamber such that v_+ lies in the dominant weight space V_{λ_0} and $\lambda_0(\mathfrak{t}) \subset \mathbb{R}$ whence λ_0 is a real weight. Conversely, if there is a long real weight then its weight space lies in $\mathcal{C}$. Thus, $\mathcal{C} \neq \emptyset$ iff there are real long weights.

If this is the case, i.e. if λ_0 is real, then the weights $\{\mu \mid (\lambda_0, \mu) = \frac{r}{2}\}$ and the roots $\{\alpha \mid (\lambda_0, \alpha) = i\}$ are invariant under conjugation so that by (8) and (10) we get the decompositions

$$\mathfrak{h} = \mathfrak{n}^+ \oplus \mathfrak{n}^0 \oplus \mathfrak{n}^-, \quad V = V_3 \oplus V_1 \oplus V_{-1} \oplus V_{-3} \tag{33}$$

where $\mathfrak{n}^i = \mathfrak{n}^i_{\mathbb{C}} \cap \mathfrak{h}$ and $V_r = (V_r)_{\mathbb{C}} \cap V$. We also define the analogues of N^i, P and $\mathfrak{p} \subset \mathfrak{n}^0$ as in (9) and (12).

In terms of the Satake diagram of $\mathfrak{h}$, the dominant weight is real iff in the description of the representation of $\mathfrak{h}$ on V via Dynkin diagrams, there are no non-zero coefficient over a black node, as these correspond to the roots of $\mathfrak{m}$. A glance at the Satake diagrams of the representations in question implies the following.

Lemma 4.13 *Let $H \subset End(V)$ be a real special symplectic holonomy group and choose a maximal $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a} \subset \mathfrak{t} \subset \mathfrak{h}$ and $\mathfrak{m}$ as above. There are real long weights in Φ iff $\mathcal{C} \neq \emptyset$. In this case, we have the decompositions (33).*

In particular, these conditions are satisfied unless $H \subset End(V)$ is one of the following real holonomy groups:

1. $H = SL(2, \mathbb{R}) \cdot SO(n) \subset End(\mathbb{R}^2 \otimes \mathbb{R}^n)$,
2. $H = Sp(1) \cdot SO(n, \mathbb{H}) \subset End(\mathbb{H}^n)$,
3. $H = SU(1, 5) \subset End(\mathbb{R}^{20})$,
4. $H = Spin(2, 10) \subset End(\Delta_{2,10})$.

Lemma 4.14 *Let $H \subset End(V)$ be a real special symplectic holonomy group. Then either $\mathcal{C} = \emptyset$ or H acts transitively on $\mathcal{C}$.*

Proof. Suppose that $\mathcal{C} \neq \emptyset$. Since all Cartan algebras are conjugate, it follows that $\mathcal{C}$ contains an element in the maximal weight space V_{λ_0} w.r.t. some fixed fundamental chamber. Thus, all that remains to be shown is that $v_+ \in V_{\lambda_0}$ lies in the same H-orbit as $-v_+$.

Let $\alpha \in \Delta$ be the maximal root, let $(\mathrm{H}_\alpha)_{\mathbb{C}} := exp(\{(\mathfrak{h}_\alpha)_{\mathbb{C}}, (\mathfrak{h}_{-\alpha})_{\mathbb{C}}\}) \subset \mathrm{H}_{\mathbb{C}}$. Then $(\mathrm{H}_\alpha)_{\mathbb{C}} \cong \mathrm{SL}(2, \mathbb{C})$ and hence, the real form $\mathrm{H}_\alpha := (\mathrm{H}_\alpha)_{\mathbb{C}} \cap \mathrm{H}$ is (up to covering) isomorphic to $\mathrm{SL}(2, \mathbb{R})$ or $\mathrm{SU}(2)$.

Let $r_0 := (\lambda_0, \alpha)$. Then by Proposition 3.4, r_0 is odd (in fact, $r_0 = 1$ if $\mathrm{rk}(\mathfrak{h}) \geq 2$ and $r_0 = 3$ if $\mathrm{rk}(\mathfrak{h}) = 1$). H_α leaves $\bigoplus_{i=0}^{r_0}((V_{\lambda_0})_{\mathbb{C}} \oplus (V_{\lambda_0 - i\alpha})_{\mathbb{C}}) \cap V$ invariant and acts on this via the irreducible (r_0+1)-dimensional representation. Since r_0 is odd, $-Id \in \mathrm{H}_\alpha$ acts as $-Id$ on this space and thus, in particular, on V_{λ_0}. ∎

Suppose now that we have a fixed decomposition of $\mathfrak{h}$ and V from (33) and elements $v_\pm \in V_{\pm 3}$ with $\langle v_+, v_- \rangle = 1$ and define for a constant $c \neq 0$ the set $S_c \subset \mathfrak{n}^+$ as in (22). We denote its complexification by $(S_c)_{\mathbb{C}} \subset (\mathfrak{n}^+)_{\mathbb{C}}$. Recall that by Corollary 4.9, $(S_c)_{\mathbb{C}}$ is a single $P_{\mathbb{C}}$-orbit.

Proposition 4.15 *Let $H \subset End(V)$ be a real special symplectic holonomy group for which $\mathcal{C} \neq \emptyset$, fix a Cartan decomposition, $v_\pm \in V_{\pm 3}$ and $c \neq 0$ as above.*

Then there is a one-to-one correspondence between P-orbits of S_c and isomorphism classes of real subalgebras $\mathfrak{p}_0 \subset \mathfrak{p}$ such that $\mathfrak{p}_0 \otimes \mathbb{C} = (\mathfrak{p}_0)_\mathbb{C}$. This correspondence is given by associating to each element $x \in S_c$ its infinitesimal stabilizer.

Proof. Evidently, the infinitesimal stabilizer $\mathfrak{p}_0$ of $x \in S_c$ must be a real form of $(\mathfrak{p}_0)_\mathbb{C}$ since $S_c = (S_c)_\mathbb{C} \cap \mathfrak{n}^+$. Also, if two elements lie in the same P-orbit then their stabilizers are conjugate and hence isomorphic.

Let $\mathfrak{p}_0 \subset \mathfrak{p}$ be a real form of $(\mathfrak{p}_0)_\mathbb{C}$. Then the representation of $\mathfrak{p}_0$ on $\mathfrak{n}^+ \cong V_1$ is a real form of the representation of $(\mathfrak{p}_0)_\mathbb{C}$ on $(V_1)_\mathbb{C}$. By Table 7, this means that there is a $\mathfrak{p}_0$-invariant subspace which intersects S_c for all $c \neq 0$. Thus, $\mathfrak{p}_0$ is the infinitesimal stabilizer of some element $x \in S_c$.

Finally, suppose that $x, x' \in S_c$ have isomorphic infinitesimal stabilizers $\mathfrak{p}_0$ and $\mathfrak{p}_0'$, respectively. A glance at Table 7 shows that $((\mathfrak{p}_s)_\mathbb{C}, (\mathfrak{p}_0)_\mathbb{C})$ is an irreducible symmetric pair in each case, hence so are $(\mathfrak{p}_s, \mathfrak{p}_0)$ and $(\mathfrak{p}_s, \mathfrak{p}_0')$. Since $\mathfrak{p}_0 \cong \mathfrak{p}_0'$, this implies that there is an automorphism $\imath : \mathfrak{p}_s \to \mathfrak{p}_s$ with $\imath(\mathfrak{p}_0') = \mathfrak{p}_0$. Moreover, one verifies that each class of the outer automorphisms of $\mathfrak{p}_s$ contains an element which leaves $\mathfrak{p}_0$ invariant, thus $\imath$ can be chosen to be inner, hence $\mathfrak{p}_0$ and $\mathfrak{p}_0'$ are conjugate. Therefore, after replacing x' by an element in its P-orbit, we may assume that $\mathfrak{p}_0 = \mathfrak{p}_0'$.

If H is simple then the representation of $\mathfrak{p}_0$ on $\mathfrak{n}^+$ has a one-dimensional invariant subspace, hence x, x' are linearly dependent. Since $x, x' \in S_c$, we conclude that $x = x'$.

If H is not simple then there is a two-dimensional $\mathfrak{p}_0$-invariant subspace W_0 of $\mathfrak{n}^+$, and one verifies directly that P acts transitively on $W_0 \cap S_c$. ∎

In Table 8, we list the real forms $\mathfrak{p}_0$ of $(\mathfrak{p}_0)_\mathbb{C}$ in each case. Since $(\mathfrak{p}_s, \mathfrak{p}_0)$ is a symmetric pair one may either calculate the possible subgroups directly, or refer to the classification of irreducible symmetric spaces in [Be2].

Proposition 4.16 *Let $H \subset End(V)$ be a real special symplectic holonomy group with $\mathcal{C} \neq \emptyset$, suppose a decomposition (33) and $\mathsf{a}_+ \in S_c \subset \mathfrak{n}^+$ for some $c \neq 0$ has been fixed, and $P_0 \subset P$ and $\mathfrak{p}_0 \subset \mathfrak{p}$ are defined as in (27). Then the number of P_0-orbits of solutions of (31) is as specified in Table 8.*

Proof. Since $\mathfrak{h} \oplus V \subset \mathfrak{h}_\mathbb{C} \oplus V_\mathbb{C}$, each solution $(y, p) \in \mathfrak{n}^-_{\mathsf{a}_+} \oplus \mathfrak{p}_0$ of (31) must also be a solution in the complexification $(\mathfrak{n}^-_{\mathsf{a}_+})_\mathbb{C} \oplus (\mathfrak{p}_0)_\mathbb{C}$. Thus, by Proposition 4.12, we have either $(y, p) = (0, 0)$, or $\mathfrak{h}$ is simple and p lies in the orbit of the maximal root of $(\mathfrak{p}_0)_\mathbb{C}$, or $\mathfrak{h}$ is not simple and $y = 0$.

If $\mathfrak{h}$ is simple and $(y, p) \neq (0, 0)$ then by Proposition 4.12 $\mathfrak{h}_\mathbb{C} \cong \mathfrak{spin}(12, \mathbb{C})$, $\mathfrak{e}_7^\mathbb{C}$ or $\mathfrak{sp}(3, \mathbb{C})$, and $p \in \mathcal{R}_\mathbb{C} \cap \mathfrak{p}_0$ where $\mathcal{R}_\mathbb{C}$ is the orbit of the maximal root of $(\mathfrak{p}_0)_\mathbb{C}$. But $\mathcal{R}_\mathbb{C} \cap \mathfrak{p}_0 \neq \emptyset$ iff in the Satake diagram of $\mathfrak{p}_0$ the nodes corresponding to the simple roots of $(\mathfrak{p}_0)_\mathbb{C}$ which are not perpendicular to the maximal root are white. Verifying this condition it follows that this is the case iff $\mathfrak{p}_0 \subset (\mathfrak{p}_0)_\mathbb{C}$, $\mathfrak{p} \subset \mathfrak{p}_\mathbb{C}$ and $\mathfrak{h} \subset \mathfrak{h}_\mathbb{C}$ are the split forms. Thus each P_0-orbit of $\mathcal{R}_\mathbb{C} \cap \mathfrak{p}_0$ intersects the maximal root space. It remains to decide if p and $-p$ lie in the same P_0-orbit.

Table 8: $\mathfrak{p}_s$ and $\mathfrak{p}_0$ for simple real special symplectic holonomies with $\mathcal{C} \neq \emptyset$

$\mathfrak{h}$	$\mathfrak{p}_s$	V_1	$\mathfrak{p}_0$	# P_0-orbits of solutions of (31)
$\mathfrak{sl}(2,\mathbb{R})$	0	$\mathbb{R}$		1
$\mathfrak{sp}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{R})$	$\odot^2\mathbb{R}^3$	$\mathfrak{so}(3)$ $\mathfrak{so}(2,1)$	2 2
$\mathfrak{sl}(6,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})$	$\mathbb{R}^3 \otimes (\mathbb{R}^3)^*$	$\mathfrak{sl}(3,\mathbb{R})$	1
$\mathfrak{su}(3,3)$	$\mathfrak{sl}(3,\mathbb{C})$	$\left\{ \begin{array}{c} A \in M_{\mathbb{C}}(3) \\ A = A^* \end{array} \right\}$	$\mathfrak{su}(3)$ $\mathfrak{su}(2,1)$	1 1
$\mathfrak{spin}(6,6)$	$\mathfrak{sl}(6,\mathbb{R})$	$\Lambda^2\mathbb{R}^6$	$\mathfrak{sp}(3,\mathbb{R})$	3
$\mathfrak{spin}(6,\mathbb{H})$	$\mathfrak{sl}(3,\mathbb{H})$	$\Lambda^2\mathbb{H}^3$	$\mathfrak{sp}(3)$ $\mathfrak{sp}(2,1)$	1 1
$\mathfrak{e}_7^{(5)}$	$\mathfrak{e}_6^{(1)}$	$\mathbb{R}^{27}$	$\mathfrak{f}_4^{(1)}$	2
$\mathfrak{e}_7^{(7)}$	$\mathfrak{e}_6^{(4)}$	$\mathbb{R}^{27}$	$\mathfrak{f}_4$ or $\mathfrak{f}_4^{(2)}$	1
$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(p,q)$ $p \geq q \geq 1,\ p \geq 2$	$\mathfrak{so}(p-1,q-1)$	$\mathbb{R}^{p-1,q-1}$	$\mathfrak{so}(p-2,q-1)$ $\mathfrak{so}(p-1,q-2)$	$[\frac{min(p,q+1)}{2}]$ $[\frac{q}{2}]$

If $\mathfrak{h} \cong \mathfrak{sp}(3,\mathbb{R})$ then we choose a basis $e_{\pm 1}, e_{\pm 2}, e_{\pm 3}$ of $\mathbb{R}^6$ such that $\mathfrak{h}$ preserves the symplectic form $\sum_i e_i \wedge e_{-i}$, and let $v_\pm := e_{\pm 1} \wedge e_{\pm 2} \wedge e_{\pm 3}$. A calculation then yields that we may choose $\mathbf{a}_+ := c_0(e_1^2 + e_2 e_3)$ for some appropriate constant $c_0 \neq 0$. Moreover, $\mathcal{R}_{\mathbb{C}} \cap \mathfrak{p}_0$ contains the element $p := 2e_1 e_{-2} - e_{-1} e_3$. However, the element of $\mathrm{H} = \mathrm{Sp}(3,\mathbb{R})$ which maps $e_{\pm i} \mapsto \varepsilon_i e_{\pm i}$ with $\varepsilon_1 = 1$, $\varepsilon_2 = \varepsilon_3 = -1$ fixes $v_\pm$ and $\mathbf{a}_+$ and hence lies in P_0 and maps p to $-p$.

If $\mathfrak{h} \cong \mathfrak{spin}(6,6)$ or $\mathfrak{e}_7^{(5)}$ then by Proposition 4.15 S_c has a single P-orbit, thus we may assume that $\mathbf{a}_+ = x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3}$ is in its normal form (17). Moreover, $p^2 = 0$ is satisfied if $p \in \mathfrak{h}_\beta$ where β is a long root perpendicular to α_i, so we may assume this.

Now, if $\mathfrak{h} \cong \mathfrak{spin}(6,6)$ then this implies that the image of $\mathbf{a}_+ \pm p \in \mathfrak{h} \cong \mathfrak{so}(6,6)$ is a positive (negative, respectively) semidefinite plane. Hence $\mathbf{a}_+ + p$ and $\mathbf{a}_+ - p$ cannot be H-equivalent, hence p and $-p$ lie in different P_0-orbits.

On the other hand, if $\mathfrak{h} = \mathfrak{e}_7^{(5)}$ then there is a long root γ with $\langle \beta, \gamma \rangle = 1$ and $\langle \alpha_i, \gamma \rangle = 0$. Letting $S_\gamma := \exp(\mathfrak{h}_\gamma, \mathfrak{h}_{-\gamma})$ then $S_\gamma \cong \mathrm{SL}(2,\mathbb{R})$, $\mathfrak{h}_{\alpha_i}$ is S_γ-invariant so that $S_\gamma \subset P_0$, and $span(\mathfrak{h}_\beta, \mathfrak{h}_{\beta-\gamma})$ is a two-dimensional S_γ-module. In particular, $-Id \in S_\gamma$ maps p to $-p$ so that these lie in the same P_0-orbit.

Finally, if $\mathfrak{h}$ is not simple then $y = 0$ and $p \in \mathfrak{so}(p-1, q-2)$ ($\mathfrak{so}(p-2, q-1)$, respectively) is an endomorphism with $p^2 = 0$. It is easy to show that two such p's are P_0-equivalent iff they have equal rank, and the rank can be any even integer $\leq min(p-1, q-2)$ ($\leq min(p-2, q-1)$, respectively). From this, the asserted numbers follow. ∎

5 Classification of degenerate pairs

Let $\mathrm{H} \subset \mathrm{Sp}(V, \langle\ ,\ \rangle)$ be a special symplectic holonomy group with Lie algebra $\mathfrak{h} \subset \mathfrak{sp}(V, \langle\ ,\ \rangle)$ and let (M, Ω, ∇) be a symplectic manifold with a symplectic connections with holonomy H. Let $\pi : F \to M$ be the holonomy bundle and $\rho := \mathbf{a} + \mathbf{b} : F \to \mathfrak{h} \oplus V$ be the curvature maps from Theorem 3.6. We wish to investigate the degenerate critical points of the symplectic scalar curvature of ∇. By Theorem 3.6, the pull-back of this function to F coincides with the function $(\mathbf{a}, \mathbf{a}) : F \to \mathbb{F}$, up to multiplication by a constant. Thus, we shall investigate the degenerate critical points of $(\mathbf{a}, \mathbf{a}) : F \to \mathbb{F}$.

Suppose that the first and second order derivatives of $(\mathbf{a}, \mathbf{a})$ vanish on some fiber $\pi^{-1}(p) \subset F$ for some $p \in M$. By (6), the vanishing of the first derivative means that on $\pi^{-1}(p)$ we have $0 = \xi_v(\mathbf{a}, \mathbf{a}) = 2(\xi_v \mathbf{a}, \mathbf{a}) = 2(\mathbf{b} \circ v, \mathbf{a}) = -4 \langle \mathbf{ab}, v \rangle$ for all $v \in V$ by (16), i.e.

$$\mathbf{ab} = 0, \tag{34}$$

while the vanishing of the second derivatives implies that the differential of (34) vanishes, which by (6) yields

$$(\mathbf{b} \circ v)\mathbf{b} + \mathbf{a}(2\mathbf{a}^2 + \overline{c}\ Id_V)v = 0 \quad \text{for all } v \in V, \tag{35}$$

where $\overline{c} = \mathbf{c} + 2(\mathbf{a}, \mathbf{a})$. Utilizing (4), we obtain

$$(\mathbf{b} \circ v)\mathbf{b} = -3 \langle \mathbf{b}, v \rangle \mathbf{b} + (\mathbf{b} \circ \mathbf{b})v,$$

so that (35) is equivalent to

$$-3 \langle \mathbf{b}, v \rangle \mathbf{b} + (\mathbf{b} \circ \mathbf{b})v + \mathbf{a}(2\mathbf{a}^2 + \overline{c} Id_V)v = 0 \quad \text{for all } v \in V. \tag{36}$$

As it turns out, it is convenient to distinguish the following types of solutions of (34) and (36).

Definition 5.1 *A pair $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ with $\mathbf{b} \neq 0$ which satisfies (34) and (36) is called* degenerate. *A degenerate pair is said to be of*

type 1 if $\overline{c} \neq 0$.

type 2 if $\overline{c} = 0$ and $\mathbf{b} \circ \mathbf{b} \neq 0$, or $rk(\mathfrak{h}) = 1$.

type 3 if $\overline{c} = 0$, $\mathbf{b} \circ \mathbf{b} = 0$ and $rk(\mathfrak{h}) \geq 2$.

Let us choose a Cartan decomposition of $\mathfrak{h}$

$$\mathfrak{h} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{h}_\alpha.$$

Then we obtain the following result.

Lemma 5.2 *Let $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ be a degenerate pair. Then we may replace $(\mathbf{a}, \mathbf{b})$ by an element in its H-orbit such that the following hold:*

1. $\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_+$ *with* $\mathbf{a}_i \in \mathfrak{n}^i$,

2. $\mathbf{b} \in V_1$ *and* $0 \neq \mathbf{b} \circ \mathbf{b} \in \mathfrak{n}^+$, *or* $\mathbf{b} \in V_3$ *and* $\mathbf{b} \circ \mathbf{b} = 0$,

3. $\mathbf{a}_0\mathbf{b} = \mathbf{a}_+\mathbf{b} = 0$,

4.
$$\mathbf{a}_0(2\mathbf{a}_0^2 + \bar{c}Id_V) = 0, \tag{37}$$

5. *if* $\bar{c} \neq 0$ *then* $[\mathbf{a}_0, \mathbf{a}_+] = 0$ *and* $\mathbf{a}_0 \in \mathfrak{t}$.

Proof. Let $\mathbf{a} = \mathbf{a}_s + \mathbf{a}_n$ be the Jordan decomposition of $\mathbf{a}$ into its semi-simple and nilpotent part such that $[\mathbf{a}_s, \mathbf{a}_n] = 0$. After replacing $\mathbf{a}$ by an element in its H-orbit, we may assume that $\mathbf{a}_s \in \mathfrak{t}$ and $\mathbf{a}_n \in \bigoplus_{\alpha\in\Delta^+} \mathfrak{h}_\alpha$.

Also, from (11) we have the decomposition $\mathbf{a} = \mathbf{a}_+ + \mathbf{a}_0 + \mathbf{a}_-$ with $\mathbf{a}_i \in \mathfrak{n}^i$. Comparing this with the Jordan decompositions yields that $\mathbf{a}_- = 0$, which is the first assertion.

Let $\mathbf{b} = \sum_r \mathbf{b}_r$ with $\mathbf{b}_r \in V_r$ be the decomposition of $\mathbf{b} \in V$.

Substitute $v = v_+ \in V_3$ into (36). Since $\mathbf{a}V_3 \subset V_3$ and $(\mathbf{b} \circ \mathbf{b})V_3 \subset V_1 \oplus V_3$, the V_{-3}-component of this equation reads $-3\langle \mathbf{b}_{-3}, v_+\rangle \mathbf{b}_{-3} = 0$, whence $\mathbf{b}_{-3} = 0$ and thus, $\langle \mathbf{b}, v_+\rangle = 0$. Thus, the vanishing of the V_1-component implies that $(\mathbf{b} \circ \mathbf{b})v_+ \subset V_3$, i.e. $\mathbf{b} \circ \mathbf{b} \in \mathfrak{n}^0 \oplus \mathfrak{n}^+$ by Lemma 4.1.

Next, substitute $v \in V_1$ into (36). By the above, both $\mathbf{a}$ and $\mathbf{b}\circ\mathbf{b}$ preserve $V_1 \oplus V_3$, whence the vanishing of the V_{-1}-component of (36) implies $-3\langle \mathbf{b}_{-1}, v\rangle \mathbf{b}_{-1} = 0$ for all $v \in V_1$, whence $\mathbf{b}_{-1} = 0$ so that $\mathbf{b} \in V_1 \oplus V_3$. The second assertion follows from Proposition 4.8 since N^+ preserves the decomposition of $\mathbf{a}$.

The third assertion follows from the second and the decomposition of $0 = \mathbf{ab} = \mathbf{a}_0\mathbf{b} + \mathbf{a}_+\mathbf{b}$ into its V_1- and V_3-components.

For $v \in V_r$, $r = 1, 3$, the vanishing of the V_r-component of (36) implies that $\mathbf{a}_0(2\mathbf{a}_0^2 + \bar{c}Id_V)v = 0$ for all $v \in V_1 \oplus V_3$, and since $\mathbf{a}_0(2\mathbf{a}_0^2 + \bar{c}Id_V)V_i \subset V_i$ for all i, this implies (37).

Finally, note that $0 = [\mathbf{a}_s, \mathbf{a}] = [\mathbf{a}_s, \mathbf{a}_0] + [\mathbf{a}_s, \mathbf{a}_+]$ and since $\mathbf{a}_s \in \mathfrak{t}$ and $[\mathfrak{t}, \mathfrak{n}^i] \subset \mathfrak{n}^i$, this implies that $[\mathbf{a}_0, \mathbf{a}_s] = 0$. If $\bar{c} \neq 0$ then $\mathbf{a}_0$ is semisimple by (37), hence so is $\mathbf{a}_0 - \mathbf{a}_s$. On the other hand, $\mathbf{a}_0 - \mathbf{a}_s = \mathbf{a}_n - \mathbf{a}_+ \in \bigoplus_{\alpha\in\Delta^+} \mathfrak{h}_\alpha$ is nilpotent. Therefore, $\mathbf{a}_0 = \mathbf{a}_s$ and $\mathbf{a}_n = \mathbf{a}_+$ which implies the last statement. ∎

5.1 Degenerate pairs of type 1

Proposition 5.3 *Let* $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ *be a degenerate pair of type 1. Then we can choose the Cartan decomposition of* $\mathfrak{h}$ *such that* $\mathbf{b} \in V_\lambda$ *where* $\lambda \in \Phi$ *is a short weight, and* $\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_+$ *with* $\mathbf{a}_0 \in \mathfrak{t}$ *and*

$$\lambda(\mathbf{a}_0) = 0, \quad \mathbf{b} \circ \mathbf{b} = 2\bar{c}\mathbf{a}_+, \quad \textit{and} \quad \langle \mathbf{b}, v\rangle \mathbf{b} = (2\mathbf{a}_0^2 + \bar{c}Id_V)\mathbf{a}_+ v \quad \textit{for all } v \in V. \tag{38}$$

In particular, degenerate pairs of type 1 exist only for those holonomies $\mathfrak{h}$ *which have short roots, i.e. for* $\mathfrak{h} \cong \mathfrak{sp}(3, \mathbb{F})$, $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(2p + 1, 2q)$ *and* $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(2n + 1, \mathbb{C})$.

Proof. By Lemma 5.2, we may assume that $\mathbf{b} \in V_1 \oplus V_3$ and $\mathbf{b} \circ \mathbf{b} \in \mathfrak{n}^+$, $\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_+$ with $\mathbf{a}_0 \in \mathfrak{t}, \mathbf{a}_+ \in \mathfrak{n}^+$ and $[\mathbf{a}_0, \mathbf{a}_+] = 0$. Using (37) we get

$$\mathbf{a}(2\mathbf{a}^2 + \bar{c} Id_V) = \mathbf{a}_+(6\mathbf{a}_0^2 + \bar{c} Id_V) + 6\mathbf{a}_0\mathbf{a}_+^2 + 2\mathbf{a}_+^3. \tag{39}$$

Let $r := \lambda_0(\mathbf{a}_0)$. Then by (37), $2r^2 \in \{0, -\bar{c}\}$. Applying (36) with $v_- \in V_{-3}$, the V_{-1}-component reads $(\mathbf{b} \circ \mathbf{b})v_- + (6r^2 + \bar{c})\mathbf{a}_+ v_- = 0$ and since $6r^2 + \bar{c} \in \{\bar{c}, -2\bar{c}\}$, Lemma 4.1 implies that

$$\mathbf{b} \circ \mathbf{b} = k\bar{c}\mathbf{a}_+, \quad \text{where } k \in \{2, -1\}. \tag{40}$$

Suppose that $\mathbf{b} \circ \mathbf{b} = 0$. Then by (40), $\mathbf{a}_+ = 0$, i.e. $\mathbf{a} = \mathbf{a}_0$ and hence by (37), $\mathbf{a}(2\mathbf{a}^2 + \bar{c} Id_V) = 0$. But then (36) reads $-3\langle \mathbf{b}, v\rangle \mathbf{b} = 0$ for all $v \in V$, implying that $\mathbf{b} = 0$ which was excluded.

Thus, $\mathbf{b} \circ \mathbf{b} \neq 0$, hence by Lemma 5.2 we may assume that $\mathbf{b} \in V_1$. Then the V_3-component of (36) with $v = v_- \in V_{-3}$ reads $2\mathbf{a}_+^3 v_- = 0$, thus $\mathbf{a}_+$ and hence $\mathbf{b} \circ \mathbf{b}$ are degenerate. In particular, Corollary 4.7 implies that $(\mathbf{b} \circ \mathbf{b})^2 = 0$ and hence, $\mathbf{a}_+^2 = 0$. Thus, (36), (39) and (40) imply

$$-3\langle \mathbf{b}, v\rangle \mathbf{b} + (6\mathbf{a}_0^2 + (k+1)\bar{c} Id_V)\mathbf{a}_+ v = 0 \quad \text{for all } v \in V, \text{ where } k \in \{2, -1\}. \tag{41}$$

Next, multiplying (41) by $\mathbf{a}_0$ and using (37) and $\mathbf{a}_0\mathbf{b} = 0$ by Lemma 5.2 we get

$$(k-2)\bar{c}\mathbf{a}_0\mathbf{a}_+ = 0.$$

If $k = -1$ this would imply $\mathbf{a}_0\mathbf{a}_+ = 0$, and hence by (41), $-3\langle \mathbf{b}, v\rangle \mathbf{b} = 0$ for all $v \in V$, thus, $\mathbf{b} = 0$ which is impossible. Thus, we have $k = 2$, and (40) and (41) yield the last two equations of (38).

Let $V = W_0 \oplus W_r \oplus W_{-r}$ be the decomposition of V into the $\mathbf{a}_0$-Eigenspaces. Since $\mathbf{a}_0$ preserves $\langle\ ,\ \rangle$, it follows that $\langle W_0, W_{\pm r}\rangle = \langle W_r, W_r\rangle = \langle W_{-r}, W_{-r}\rangle = 0$; since $[\mathbf{a}_0, \mathbf{a}_+] = 0$ it follows that $\mathbf{a}_+$ preserves this decomposition, and $\mathrm{rk}(\mathbf{a}_+|_{W_r}) = \mathrm{rk}(\mathbf{a}_+|_{W_{-r}})$. Moreover, from $\mathbf{b} \in W_0$ and (38) we obtain $\mathbf{a}_+(W_0) = \mathbf{b}$.

Thus, $\mathrm{rk}(\mathbf{a}_+) = 2\mathrm{rk}(\mathbf{a}_+|_{W_r}) + 1$ is odd, and hence, by (40), so is the rank of $\mathbf{b} \circ \mathbf{b}$. But then, since $\mathbf{b} \circ \mathbf{b}$ is degenerate, Corollary 4.7 implies that $\mathbf{b} \circ \mathbf{b}$ lies in the orbit of $\mathfrak{h}_\alpha$ where $\alpha \in \Delta$ is a long root such that the set

$$S := \{\lambda \in \Phi \mid (\lambda, \alpha) = 1\}$$

contains an odd number of elements. Note, however, that there is an involution $\sigma : S \to S$, $\sigma(\lambda) := \alpha - \lambda$ which has at most one fixed point $\lambda = \frac{1}{2}\alpha$. That is to say, S contains an odd number of elements iff $\frac{1}{2}\alpha \in \Phi$. In particular, Φ contains short weights.

Replacing $\mathbf{b}$ by an element in its P-orbit, we may assume that $\mathbf{b}$ is in its normal form (19). Since $(\mathbf{b} \circ \mathbf{b})^2 = 0$ and by Proposition 3.4,6. this means that

$$2\lambda_0 = \alpha_1 + \alpha_2 + \alpha_3 = \alpha_1 + \alpha_2 + 2\lambda.$$

and

$$\mathbf{b} = \mathbf{b}_{\lambda_0 - \alpha_1} + \mathbf{b}_{\lambda_0 - \alpha_2} = \mathbf{b}_{\lambda + \beta} + \mathbf{b}_{\lambda - \beta},$$

where $\beta = \frac{1}{2}(\alpha_1 - \alpha_2)$. Since $V_\lambda \subset V_1$ by Proposition 3.4,2, $\lambda_0 - \lambda \in \Delta_1$, and $(\lambda_0 - \lambda, \alpha_2) = 1$, whence $\beta = \lambda_0 - \lambda - \alpha_2 \in \Delta$.

Let $\mathrm{H}_\beta := \exp(\langle \mathfrak{h}_\beta, \mathfrak{h}_{-\beta} \rangle)$. Then $\mathrm{H}_\beta \cong \mathrm{SL}(2, \mathbb{C})$ and since $(\beta, \lambda_0) = 0$ it follows that $\mathrm{H}_\beta \subset P$. Moreover, since $\mathbf{a}_0 \mathbf{b} = 0$, we have $\beta(\mathbf{a}_0) = \lambda(\mathbf{a}_0) = 0$ so that H_β leaves $\mathbf{a}_0$ invariant, while it acts on $V_{\lambda+\beta} \oplus V_\lambda \oplus V_{\lambda-\beta}$ via the irreducible 3-dimensional representation. Thus, the H_β-orbit of $\mathbf{b}$ either contains an element in $V_{\lambda+\beta}$ or an element in V_λ. Since $\mathbf{b} \circ \mathbf{b} \neq 0$, the latter is the case, and this completes the proof. ∎

From this explicit description of the degenerate pairs we also get the following result by a straightforward calculation which we omit.

Corollary 5.4 *Let $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ be a degenerate pair of type 1. Then the image of $R_{\mathbf{a}} : \Lambda^2 V \to \mathfrak{h}$ generates all of $\mathfrak{h}$.*

Proposition 5.5 *Let $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ be a degenerate pair of type 1, and let $\mathbf{a} = \mathbf{a}_+ + \mathbf{a}_0$ be the Jordan decomposition of $\mathbf{a}$ where $\mathbf{a}_0$ is semi-simple and $\mathbf{a}_+$ is nilpotent. Let $\mathbf{b}_- \in V$ be such that $\mathbf{a}_0 \mathbf{b}_- = 0$ and $\langle \mathbf{b}, \mathbf{b}_- \rangle = 1$. Then the linear map $\tau : V \to \mathfrak{h}$ given by*

$$\tau(v) := \frac{2}{\overline{c}} \mathbf{b} \circ (\mathbf{a}_0 v) + \mathbf{b}_- \circ \left(\frac{\overline{c}}{2} \langle \mathbf{b}, v \rangle \mathbf{b}_- - (2\mathbf{a}_0^2 + \overline{c} Id_V) v \right)$$

satisfies the identity

$$\rho_v(\mathbf{a} + \mathbf{b}) := \xi_{v+\tau(v)}(\mathbf{a} + \mathbf{b}) = 0 \quad \textit{for all } v \in V. \tag{42}$$

Moreover, if we let $\mathfrak{l} := stab(\mathbf{a}, \mathbf{b}) \subset \mathfrak{h}$ then there is no $\mathfrak{l}$-equivariant map $\tau : V \to \mathfrak{h}$ which satisfies (42).

Proof. We may assume that $(\mathbf{a}, \mathbf{b}) = (\mathbf{a}_+ + \mathbf{a}_0, \mathbf{b})$ is the pair from Proposition 5.3 and thus satisfies (38). Then we may assume that $\mathbf{b}_- \in V_{-\lambda}$ and hence, $\mathbf{b} \circ \mathbf{b}_- \in \mathfrak{t}$. We begin by showing the identities

$$(\mathbf{b} \circ \mathbf{b}_-)\mathbf{b}_\pm = \mp \mathbf{b}_\pm \quad \text{and} \quad \mathbf{b} = \overline{c} \mathbf{a}_+ \mathbf{b}_-. \tag{43}$$

Here we use the notational convention that $\mathbf{b}_+ = \mathbf{b}$.

The second of these equations follows from the last identity in (38) with $v := \mathbf{b}_-$ since $\mathbf{a}_0 \mathbf{b}_- = 0$. The second identity in (38) and (4) implies $2\mathbf{b} = (\mathbf{b} \circ \mathbf{b})\mathbf{b}_- = (\mathbf{b} \circ \mathbf{b}_-)\mathbf{b} + 3\mathbf{b}$, whence $(\mathbf{b} \circ \mathbf{b}_-)\mathbf{b} = -\mathbf{b}$. Also, $\mathbf{b}_- \in V_{-\lambda}$ is an eigenvector of $\mathbf{b} \circ \mathbf{b}_- \in \mathfrak{t}$, and $\langle (\mathbf{b} \circ \mathbf{b}_-)\mathbf{b}_-, \mathbf{b} \rangle = \langle (\mathbf{b} \circ \mathbf{b}_-)\mathbf{b}, \mathbf{b}_- \rangle = -1$ from which (43) follows.

By (6), we obtain

$$\begin{aligned} \rho_v(\mathbf{a}) &= \mathbf{b} \circ v - [\tau(v), \mathbf{a}], \\ \rho_v(\mathbf{b}) &= (2\mathbf{a}^2 + \overline{c} Id_V)v - \tau(v)\mathbf{b} \end{aligned} \tag{44}$$

Now $[\mathbf{a}, \mathbf{b} \circ \mathbf{a}_0 v] = \mathbf{b} \circ (\mathbf{a}\mathbf{a}_0 v) = \mathbf{b} \circ (\mathbf{a}_0^2 v) + \mathbf{b} \circ (\mathbf{a}_+ \mathbf{a}_0 v)$. But the second summand equals $\frac{1}{\overline{c}}(\mathbf{a}_+ \mathbf{b}_-) \circ (\mathbf{a}_0 \mathbf{a}_+ v) = \frac{1}{2\overline{c}} [\mathbf{a}_0, [\mathbf{a}_+, [\mathbf{a}_+, \mathbf{b}_- \circ v]]]$, by (43) and $\mathbf{a}_+^2 = 0$. Since $\mathbf{a}_+ \in \mathfrak{h}_{2\lambda}$ and 2λ is a long root, $[\mathbf{a}_+, [\mathbf{a}_+, \mathfrak{h}]] \subset \mathfrak{h}_{2\lambda}$ and $[\mathbf{a}_0, \mathfrak{h}_{2\lambda}] = 0$, so that this term vanishes and we get

$$[\mathbf{a}, \mathbf{b} \circ \mathbf{a}_0 v] = \mathbf{b} \circ (\mathbf{a}_0^2 v).$$

Next, $[\mathbf{a}, \mathbf{b}_- \circ ((2\mathbf{a}_0^2 + \bar{c} Id_V)v)] = \frac{1}{\bar{c}} \mathbf{b} \circ ((2\mathbf{a}_0^2 + \bar{c} Id_V)v) + \mathbf{b}_- \circ (\langle \mathbf{b}, v \rangle \mathbf{b})$ by (37), (38) and (43). Finally, $[\mathbf{a}, \mathbf{b}_- \circ \mathbf{b}_-] = \frac{2}{\bar{c}} \mathbf{b} \circ \mathbf{b}_-$ by (43), and putting all of these together, $\rho_v(\mathbf{a}) = 0$ follows.

For the second part, we have $(\mathbf{b} \circ (\mathbf{a}_0 v))\mathbf{b} = (\mathbf{b} \circ \mathbf{b})\mathbf{a}_0 v = 2\bar{c} \mathbf{a}_+ \mathbf{a}_0 v$ by (4) since $\langle \mathbf{b}, \mathbf{a}_0 v \rangle = -\langle \mathbf{a}_0 \mathbf{b}, v \rangle = 0$ and by (38). Also, $(\mathbf{b}_- \circ \mathbf{b}_-)\mathbf{b} = (\mathbf{b}_- \circ \mathbf{b})\mathbf{b}_- - 3\mathbf{b}_- = -2\mathbf{b}_-$ by (4) and (43). Finally, again by (4), $(\mathbf{b}_- \circ (2\mathbf{a}_0^2 + \bar{c} Id_V)v)\mathbf{b} = (\mathbf{b} \circ \mathbf{b}_-)(2\mathbf{a}_0^2 + \bar{c} Id_V)v - (2\mathbf{a}_0^2 + \bar{c} Id_V)v - 2\bar{c} \langle \mathbf{b}, v \rangle \mathbf{b}_- - \bar{c} \langle \mathbf{b}_-, v \rangle \mathbf{b}$. Now,

$$\begin{aligned}
2(\mathbf{b} \circ \mathbf{b}_-)(2\mathbf{a}_0^2 + \bar{c} Id_V)v &= \bar{c}[\mathbf{a}_+, \mathbf{b}_- \circ \mathbf{b}_-](2\mathbf{a}_0^2 + \bar{c} Id_V)v && \text{by (43)} \\
&= \bar{c}(\mathbf{a}_+(2\mathbf{a}_0^2 + \bar{c} Id_V)(\mathbf{b}_- \circ \mathbf{b}_-)v \\
&\qquad -(\mathbf{b}_- \circ \mathbf{b}_-)\mathbf{a}_+(2\mathbf{a}_0^2 + \bar{c} Id_V)v) && \text{since } [\mathbf{a}_0, \mathbf{b}_- \circ \mathbf{b}_-] = 0 \\
&= \bar{c}(\langle \mathbf{b}, (\mathbf{b}_- \circ \mathbf{b}_-)v \rangle \mathbf{b} \\
&\qquad -(\mathbf{b}_- \circ \mathbf{b}_-) \langle \mathbf{b}, v \rangle \mathbf{b}) && \text{by (38)} \\
&= 2\bar{c}(\langle \mathbf{b}_-, v \rangle \mathbf{b} + \langle \mathbf{b}, v \rangle \mathbf{b}_-) && \text{since } (\mathbf{b}_- \circ \mathbf{b}_-)\mathbf{b} = -2\mathbf{b}_-
\end{aligned}$$

Putting these together, $\rho_v(\mathbf{b}) = 0$ follows.

Finally, if $\sigma : V \to \mathfrak{h}$ is another map satisfying (42) then $\alpha := \sigma - \tau$ has values in $\mathfrak{l}$. Then $(\mathbf{a}_+ \sigma)(\mathbf{b}) = [\mathbf{a}_+, \sigma(\mathbf{b})] - \sigma(\mathbf{a}_+ \mathbf{b}) = [\mathbf{a}_+, -\bar{c} \mathbf{b} \circ \mathbf{b}_- + \alpha(\mathbf{b})] - 0 = -\mathbf{b} \circ \mathbf{b} + [\mathbf{a}_+, \alpha(\mathbf{b})]$. But since $[\mathbf{a}_+, \mathfrak{l}] = 0$, we get $(\mathbf{a}_+ \sigma)(\mathbf{b}) = -\mathbf{b} \circ \mathbf{b}$, i.e. $\mathbf{a}_+ \sigma \neq 0$. ∎

Theorem 5.6 *Let* $(\mathbf{a}, \mathbf{b}) \subset \mathfrak{h} \oplus V$ *be a degenerate pair of type 1. Then there is a G-invariant symplectic connection on the total space of the homogeneous vector bundle* $E \to G/L_0$ *from Table 1 corresponding to holonomy group* H *whose curvature at any point* $p \in E \backslash 0$ *is represented by* $(\mathbf{a}, \mathbf{b})$*. Moreover, the homogeneous space* $E \backslash 0 = G/L$ *is not reductive, and the momentum map* $\mu : G/L \to \mathfrak{g}^*$ *is a double covering of a coadjoint orbit.*

Proof. Using (7) from Theorem 3.6, it follows that the Lie algebra $\mathfrak{g}$ of infinitesimal symmetries is given by

$$\mathfrak{g} := \left\{ \rho_v := \xi_{v + \tau(v)} \mid v \in V \right\} \oplus \mathfrak{l}, \tag{45}$$

where $\mathfrak{l} := \{ \rho_x := \xi_x \mid x \in \mathfrak{h}, x \cdot (\mathbf{a}, \mathbf{b}) = 0 \}$ is the stabilizer of $(\mathbf{a}, \mathbf{b})$, and there is a G-invariant symplectic connection on G/L whose curvature is represented by $(\mathbf{a}, \mathbf{b})$. Thus, Corollary 5.4 and the *Ambrose-Singer Holonomy theorem* [AS] imply that the holonomy of this connection equals *all of H*, and Propositions 2.7 and 5.5 imply that this space is not reductive.

In order to identify $\mathfrak{g}$ and $\mathfrak{l}$, it suffices to consider the complex case, as the investigation of the corresponding real forms is standard. As before, we assume that $(\mathbf{a}, \mathbf{b})$ is the degenerate pair given in Proposition 5.3. We are given the explicit formulae for the vector fields ρ_w in Proposition 5.5, and their Lie brackets in (6). First, we compute that there is a Lie algebra isomorphism between $\mathfrak{sl}(2, \mathbb{C}) \cong \odot^2 \mathbb{C}^2$ and $\mathfrak{s}_1 := span(\rho_{\mathbf{b}}, \bar{c} \rho_{\mathbf{b}_-} + 2\mathbf{a}_0, \mathbf{a}_+)$, given by

$$\frac{1}{\bar{c}} \rho_{\mathbf{b}} \longleftrightarrow e_1 e_2, \quad \rho_{\mathbf{b}_-} + \frac{2}{\bar{c}} \mathbf{a}_0 \longleftrightarrow e_1^2, \quad -4\mathbf{a}_+ \longleftrightarrow e_2^2,$$

with a basis $e_1, e_2 \in \mathbb{C}^2$ with $\det(e_1, e_2) = 1$.

Let $\mu \in \Phi$ be a weight with $(\mu, \lambda) \geq 0$, $\mu \neq \lambda$ and $v_\mu \in V_\mu$. Then one calculates that

$$[\rho_{\mathbf{b}}, \rho_{v_\mu}] = -\overline{c}(\mu, \lambda)\rho_{v_\mu},$$

so that these vectors lie in the -1-eigenspace or the 0-eigenspace of $ad(\frac{1}{\overline{c}}\rho_{\mathbf{b}})$.

If $\mathfrak{h} \cong \mathfrak{sp}(3, \mathbb{C})$ then we let $\mathfrak{s}_2$ be the semisimple part of $\mathfrak{l}$. One calculates that $\mathfrak{s}_2 \cong \mathfrak{sl}(2, \mathbb{C})$. Moreover, there are two short weights $\pm\mu \in \Phi$ with $(\mu, \lambda) = 0$ and $r := \mu(\mathbf{a}_0) \neq 0$ where $2r^2 + \overline{c} = 0$. For these, we have $\mathfrak{s}_2 \cdot V_{\pm\mu} = 0$. A calculation then shows that $\mathfrak{s}_3 := span(\mathbf{a}_0, V_{\pm\mu}) \cong \mathfrak{sl}(2, \mathbb{C})$ with an explicit isomorphism given by

$$\frac{2}{r}\mathbf{a}_0 \longleftrightarrow e_1 e_2, \quad v_{-\mu} \longleftrightarrow e_1^2, \quad v_\mu \longleftrightarrow e_2^2,$$

where $v_{\pm\mu} \in V_{\pm\mu}$ are such that $(\mathbf{a}_0, v_\mu \circ v_{-\mu}) = 1$. Also, $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$ for $i \neq j$. Finally, for all long weights $\nu \in \Phi$ with $(\nu, \lambda) > 0$, V_ν, lies in the -1-eigenspace of $\frac{1}{\overline{c}}\rho_{\mathbf{b}}$, in the ± 1-eigenspace of the diagonal element of $\mathfrak{s}_2$ and in the ± 2-eigenspace of $\frac{2}{r}\mathbf{a}_0$. Thus, we have the decomposition

$$\mathfrak{g} \cong \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \mathfrak{s}_3 \oplus V_{1,1,2} \cong \mathfrak{so}(4, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C}) \oplus (\mathbb{C}^4 \otimes \mathbb{C}^3).$$

For weight reasons, $[V_{1,1,2}, V_{1,1,2}] \subset \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \mathfrak{s}_3$ and it is easy to verify that $[V_{1,1,2}, V_{1,1,2}] \neq 0$. Thus, this is the symmetric pair corresponding to the Grassmannian $\mathrm{SO}(7, \mathbb{C})/(\mathrm{SO}(4, \mathbb{C}) \cdot \mathrm{SO}(3, \mathbb{C}))$ which shows that $\mathfrak{g} \cong \mathfrak{so}(7, \mathbb{C})$. Also, we have the explicit description of $\mathfrak{l} \subset \mathfrak{g}$ which yields the assertion.

In the case $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(2n+1, \mathbb{C})$, the semisimple component of $\mathfrak{l}$ is $\mathfrak{sl}(n, \mathbb{C})$ and every weight $\mu \in \Phi$ with $\mu \neq \lambda$ satisfies $(\lambda, \mu) \neq 0$. On the weight vectors v_μ with $(\lambda, \mu) = 1$, $\mathfrak{sl}(n, \mathbb{C})$ acts as on $V \oplus V^*$ with $V \cong \mathbb{C}^n$. But $\frac{1}{r}\mathbf{a}_0$ commutes with $\mathfrak{s}_1 \oplus \mathfrak{sl}(n, \mathbb{C})$ and has V and V^* as its ± 1-eigenspaces.

Therefore, we get the decomposition

$$\mathfrak{g} \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C}) \oplus (\mathbb{C}^2 \otimes (V \oplus V^*)).$$

Again, for representation theoretic reasons, $[\mathbb{C}^2 \otimes (V \oplus V^*), \mathbb{C}^2 \otimes (V \oplus V^*)] \subset \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C})$ and a calculation shows that this bracket does not vanish. Therefore, we obtain the symmetric pair corresponding to the "Grassmannian" $\mathrm{SL}(n+2, \mathbb{C})/(\mathrm{S}(\mathrm{GL}(2, \mathbb{C}) \cdot \mathrm{GL}(n, \mathbb{C}))$ which shows that $\mathfrak{g} \cong \mathfrak{sl}(n+2, \mathbb{C})$.

From the explicit calculations, we find that $\mathfrak{l} \subset \mathfrak{l}_0$ is the stabilizer of the maximal weight of the representation of L_0 on W where $\mathfrak{l}_0$ is the Lie algebra of L_0 and $E = \mathrm{G} \times_{\mathrm{L}_0} W \to \mathrm{G}/\mathrm{L}_0$ is the homogeneous vector bundle corresponding to holonomy group H from Table 1. Whence, G/L is equivalent to $E \backslash 0$. That $\mu : \mathrm{G}/\mathrm{L} \to \mathfrak{g}^*$ is a double covering of a coadjoint orbit follows from another explicit calculation which we omit. ∎

Of course, Corollary 2.6 already implies that $\mu(\mathrm{G}/\mathrm{L}) \subset \mathfrak{g}^*$ must be a coadjoint orbit since $H^1(\mathfrak{g}, \mathfrak{g}^*) = 0$ for simple $\mathfrak{g}$ by the *Whitehead Lemma* [HS].

5.2 Degenerate pairs of type 2 or 3

Lemma 5.7 *If* $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ *is a degenerate pair of type 2 or 3 for some special symplectic holonomy algebra* $\mathfrak{h}$ *then* $\mathbf{b} \circ \mathbf{b}$ *is degenerate.*

Proof. Suppose there is a solution of (34) and (36) with $\bar{c} = 0$. Using the decomposition of $\mathbf{a}$ from Lemma 5.2, we may assume that $\mathbf{b} \in V_1 \oplus V_3$ and $\mathbf{b} \circ \mathbf{b} \in \mathfrak{n}^+$ and that $\mathbf{a}_0$ is nilpotent. Since $V_{\pm 3}$ are Eigenspaces for $\mathbf{a}_0$, it follows that $\mathbf{a}_0 V_{\pm 3} = 0$. Then the V_{-1}-component of (36) with $v = v_- \in V_{-3}$ yields

$$0 = (\mathbf{b} \circ \mathbf{b}) v_- + 2\mathbf{a}_0^2 \mathbf{a}_+ v_- = ((\mathbf{b} \circ \mathbf{b}) + 2[\mathbf{a}_0, [\mathbf{a}_0, \mathbf{a}_+]]) v_-,$$

and hence by Lemma 4.1, $\mathbf{b} \circ \mathbf{b} = -2[\mathbf{a}_0, [\mathbf{a}_0, \mathbf{a}_+]] = -2(\mathbf{a}_0^2 \mathbf{a}_+ - 2\mathbf{a}_0 \mathbf{a}_+ \mathbf{a}_0 + \mathbf{a}_+ \mathbf{a}_0^2)$. But now, expanding $(\mathbf{b} \circ \mathbf{b})^3$ and using $\mathbf{a}_0^3 = 0$ from (37) and $\mathbf{a}_0 V_{\pm 3} = 0$, it follows that $(\mathbf{b} \circ \mathbf{b})^3 v_- = 0$, i.e. $\mathbf{b} \circ \mathbf{b}$ is degenerate as claimed. ∎

Proposition 5.8 *Let* $\mathfrak{h} \subset End(V)$ *be a complex symplectic holonomy algebra and fix a decomposition of* V *and* $\mathfrak{h}$ *as in (8) and (11). Let* $v_\pm \in V_{\pm 3}$ *be such that* $\langle v_+, v_- \rangle = 1$ *and define* $S_c \subset \mathfrak{n}^+$ *as in (22). Fix an element* $\mathbf{a}_+ \in S_{\frac{3}{2}} \subset \mathfrak{n}^+$*, i.e.*

$$\mathbf{a}_+^3 v_- = \frac{3}{2} v_+ \tag{46}$$

and let $P_0 \subset P$, $\mathfrak{p}_0 \subset \mathfrak{p}$, $\mathfrak{n}^-_{\mathbf{a}_+} \subset \mathfrak{n}^-$ *and* $W_{-1} \subset V_{-1}$ *as defined in (27), (28) and (29).*

Then $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ *is a degenerate pair of type 2 or 3 iff the H-orbit of* $(\mathbf{a}, \mathbf{b})$ *contains a pair of the form* $(\mathbf{a}_+ + \mathbf{a}_0 + \mathbf{a}_-, v_+ - 2\mathbf{a}_- \mathbf{a}_+^2 v_-)$ *where* $\mathbf{a}_0 \in \mathfrak{p}_0$ *and* $\mathbf{a}_- \in \mathfrak{n}^-_{\mathbf{a}_+}$ *are such that*

$$\jmath(\mathbf{a}_0^2) = -3\imath(\mathbf{a}_-) \tag{47}$$

with $\imath, \jmath$ *from (30).*

Proof. Suppose that $(\mathbf{a}, \mathbf{b})$ satisfies (34) and (36) with $\bar{c} = 0$. By Lemma 5.7 and Propositions 4.8 we may change $(\mathbf{a}, \mathbf{b})$ to an element in its H-orbit such that

$$\mathbf{b} = v_+ + \frac{1}{4} (\mathbf{b} \circ \mathbf{b}) v_-.$$

Also, we decompose $\mathbf{a}$ according to (11) as

$$\mathbf{a} = \tilde{\mathbf{a}}_+ + \mathbf{a}_0 + \mathbf{a}_- \quad \text{with} \quad \tilde{\mathbf{a}}_+ \in \mathfrak{n}^+, \mathbf{a}_i \in \mathfrak{n}^i.$$

Now, the V_3-component of (36) with $v = v_-$ reads

$$-3v_+ + 2\tilde{\mathbf{a}}_+^3 v_- = 0,$$

and thus, by (46) and Corollary 4.9, we may replace $(\mathbf{a}, \mathbf{b})$ by an element in its P-orbit and assume that

$$\tilde{\mathbf{a}}_+ = \mathbf{a}_+.$$

Then one verifies easily that (34) implies $\mathbf{a}_0 V_3 = 0$ so that $\mathbf{a}_0 \in \mathfrak{p}$ and $\mathbf{a}_- v^+ + \frac{1}{4}\mathbf{a}_+(\mathbf{b} \circ \mathbf{b})v_- = 0$. Moreover, $0 = [\mathbf{a}, \mathbf{b} \circ \mathbf{b}] = [\mathbf{a}_+, \mathbf{b} \circ \mathbf{b}] + [\mathbf{a}_0, \mathbf{b} \circ \mathbf{b}] + [\mathbf{a}_-, \mathbf{b} \circ \mathbf{b}]$. But by (11), $[\mathbf{a}_+, \mathbf{b} \circ \mathbf{b}] \in [\mathfrak{n}^+, \mathfrak{n}^+] = 0$, $[\mathbf{a}_-, \mathbf{b} \circ \mathbf{b}] \in [\mathfrak{n}^-, \mathfrak{n}^+] \subset \mathfrak{n}_0$ and $[\mathbf{a}_0, \mathbf{b} \circ \mathbf{b}] \in [\mathfrak{n}_0, \mathfrak{n}^+] \subset \mathfrak{n}^+$, so that we obtain

$$\mathbf{a}_0 \in \mathfrak{p}, \quad [\mathbf{a}_\pm, \mathbf{b} \circ \mathbf{b}] = [\mathbf{a}_0, \mathbf{b} \circ \mathbf{b}] = 0, \quad \mathbf{a}_- v^+ + \frac{1}{4}\mathbf{a}_+(\mathbf{b} \circ \mathbf{b})v_- = 0. \tag{48}$$

Since $\mathbf{a}_+\mathbf{a}_- V_3 \subset V_3$, there is an $r \in \mathbb{C}$ with

$$\mathbf{a}_+\mathbf{a}_- v_+ = r v_+ \quad \text{and} \quad \mathbf{a}_-\mathbf{a}_+ v_- = r v_-, \tag{49}$$

because $\langle \mathbf{a}_-\mathbf{a}_+ v_-, v_+ \rangle = \langle v_-, \mathbf{a}_+\mathbf{a}_- v_+ \rangle$. Now, decomposing (36) with $v = v_-$ into its V_r-components and using that $\mathbf{a}_0 v_- = \mathbf{a}_- v_- = 0$ and (46) yields:

$$\begin{array}{rcl} 0 & = & 0, \\ 2(\mathbf{a}_+\mathbf{a}_0\mathbf{a}_+ + \mathbf{a}_0\mathbf{a}_+^2)v_- & = & 0, \\ (\frac{1}{4}(\mathbf{b} \circ \mathbf{b}) + 2(r\mathbf{a}_+ + \mathbf{a}_-\mathbf{a}_+^2 + \mathbf{a}_0^2\mathbf{a}_+))v_- & = & 0, \\ 2\mathbf{a}_-\mathbf{a}_0\mathbf{a}_+ v_- & = & 0. \end{array} \tag{50}$$

Since $\mathbf{a}_0 \in \mathfrak{p}$ we have $\mathbf{a}_0\mathbf{a}_+ v_- = [\mathbf{a}_0, \mathbf{a}_+]v_-$ and $\mathbf{a}_0\mathbf{a}_+^2 v_- = 2\mathbf{a}_+[\mathbf{a}_0, \mathbf{a}_+]v_-$ by (21) so that the second equation in (50) implies

$$\mathbf{a}_+[\mathbf{a}_0, \mathbf{a}_+]v_- = 0.$$

But now, since $\mathbf{a}_+$ is non-degenerate, Corollary 4.10 implies that $[\mathbf{a}_0, \mathbf{a}_+]v_-$, and thus, by Lemma 4.1, $[\mathbf{a}_0, \mathbf{a}_+] = 0$, i.e. $\mathbf{a}_0 \in \mathfrak{p}_0$. But then, (48) implies that $\mathbf{a}_0\mathbf{a}_- v_+ = -\frac{1}{4}\mathbf{a}_0\mathbf{a}_+(\mathbf{b} \circ \mathbf{b})v_- = -\frac{1}{4}\mathbf{a}_+(\mathbf{b} \circ \mathbf{b})\mathbf{a}_0 v_- = 0$, hence $[\mathbf{a}_0, \mathbf{a}_-]v_+ = 0$, so that, by Lemma 4.1,

$$[\mathbf{a}_0, \mathbf{a}_\pm] = 0, \quad \mathbf{a}_0 \in \mathfrak{p}_0. \tag{51}$$

With this, (50) yields

$$\mathbf{a}_-\mathbf{a}_+^2 v_- = -\left(\frac{1}{8}(\mathbf{b} \circ \mathbf{b}) + r\mathbf{a}_+\right) v_-. \tag{52}$$

Next, note that $(ad(\mathfrak{n}^+))^3(\mathfrak{n}^-) = 0$ by (11), so that

$$0 = [\mathbf{a}_+, [\mathbf{a}_+, [\mathbf{a}_+, \mathbf{a}_-]]] = \mathbf{a}_+^3\mathbf{a}_- - 3\mathbf{a}_+^2\mathbf{a}_-\mathbf{a}_+ + 3\mathbf{a}_+\mathbf{a}_-\mathbf{a}_+^2 - \mathbf{a}_-\mathbf{a}_+^3. \tag{53}$$

Thus, we get

$$\begin{array}{rcll} 0 & = & (\mathbf{a}_+^3\mathbf{a}_- - 3\mathbf{a}_+^2\mathbf{a}_-\mathbf{a}_+ + 3\mathbf{a}_+\mathbf{a}_-\mathbf{a}_+^2 - \mathbf{a}_-\mathbf{a}_+^3)v_- & \\ & = & 0 - 3r\mathbf{a}_+^2 v_- - 3\mathbf{a}_+\left(\frac{1}{8}(\mathbf{b} \circ \mathbf{b}) + r\mathbf{a}_+\right)v_- - \frac{3}{2}\mathbf{a}_- v_+ & \text{by (46), (49) and (52)} \\ & = & -6r\mathbf{a}_+^2 v_- & \text{by (48).} \end{array}$$

This means that $r = 0$, and with (29), this means that

$$\mathbf{a}_+\mathbf{a}_- V_{\pm 3} = \mathbf{a}_-\mathbf{a}_+ V_{\pm 3} = 0 \quad \text{and} \quad \mathbf{a}_- \in \mathfrak{n}_{\mathbf{a}_+}^-. \tag{54}$$

Moreover, $8[\mathbf{a}_+, [\mathbf{a}_+, \mathbf{a}_-]]v_- = 8(\mathbf{a}_+^2\mathbf{a}_- - 2\mathbf{a}_+\mathbf{a}_-\mathbf{a}_+ + \mathbf{a}_-\mathbf{a}_+^2)v_- = -(\mathbf{b} \circ \mathbf{b})v_-$ by (52) and (54), and since $[\mathbf{a}_+, [\mathbf{a}_+, \mathbf{a}_-]], \mathbf{b} \circ \mathbf{b} \in \mathfrak{n}^+$, Lemma 4.1 implies that

$$\mathbf{b} \circ \mathbf{b} = -8[\mathbf{a}_+, [\mathbf{a}_+, \mathbf{a}_-]], \quad \mathbf{b} = v_+ - 2\mathbf{a}_-\mathbf{a}_+^2 v_-. \tag{55}$$

Using (55) and decomposing (36) with $v \in V_{-1}$ into its V_r-components yields

$$\begin{aligned} -6(\mathbf{a}_+^2\mathbf{a}_- - 3\mathbf{a}_+\mathbf{a}_-\mathbf{a}_+ + \mathbf{a}_-\mathbf{a}_+^2)v + 6\mathbf{a}_0^2\mathbf{a}_+v &= 0 \\ 2(\mathbf{a}_0^3 + 3\mathbf{a}_0\mathbf{a}_-\mathbf{a}_+)v &= 0 \\ 2\mathbf{a}_-^2\mathbf{a}_+v &= 0. \end{aligned} \tag{56}$$

The last equation together with Corollary 4.10 implies that $\mathbf{a}_-^2V_1 = 0$ and hence, $\langle \mathbf{a}_-v_+, V_1\rangle = \langle v_+, \mathbf{a}_-^2V_1\rangle = 0$, i.e. $\mathbf{a}_-^2 = 0$.

On the other hand, $[\mathbf{a}_-, [\mathbf{a}_-, \mathbf{a}_+]]v_+ = (\mathbf{a}_-^2\mathbf{a}_+ - 2\mathbf{a}_-\mathbf{a}_+\mathbf{a}_- + \mathbf{a}_+\mathbf{a}_-^2)v_+ = 0$ by the above and (54) and hence by Lemma 4.1, $[\mathbf{a}_-, [\mathbf{a}_-, \mathbf{a}_+]] = 0$, i.e.

$$\mathbf{a}_-^2 = \mathbf{a}_-\mathbf{a}_+\mathbf{a}_- = 0. \tag{57}$$

Since $\mathbf{a}_+^2v \in V_3$, it follows that $\mathbf{a}_+^2v = \langle \mathbf{a}_+^2v, v_-\rangle v_+ = -\sigma(v, \mathbf{a}_+v_-)v_+$ with $\sigma = \sigma_{\mathbf{a}_+}$ from (26) and thus, $\mathbf{a}_-\mathbf{a}_+^2v = -\sigma(v, \mathbf{a}_+v_-)\mathbf{a}_-v_+ = -2\sigma(v, \mathbf{a}_+v_-)\mathbf{a}_+\mathbf{a}_-\mathbf{a}_+^2v_-$ by (48) and (55). Substituting this into the first equation of (56) yields

$$-6\mathbf{a}_+((\mathbf{a}_+\mathbf{a}_- - 3\mathbf{a}_-\mathbf{a}_+ - \mathbf{a}_0^2)v - 2\sigma(v, \mathbf{a}_+v_-)\mathbf{a}_-\mathbf{a}_+^2v_-) = 0$$

for all $v \in V_{-1}$, and since $\mathbf{a}_+$ is non-degenerate, Corollary 4.10 yields

$$\mathbf{a}_0^2v = (\mathbf{a}_+\mathbf{a}_- - 3\mathbf{a}_-\mathbf{a}_+)v - 2\sigma(v, \mathbf{a}_+v_-)\mathbf{a}_-\mathbf{a}_+^2v_- \quad \text{for all } v \in V_{-1}. \tag{58}$$

Note that the σ-orthogonal projection $\pi : V_{-1} \to W_{-1}$ is given by $\pi(v) = v + \frac{2}{3}\sigma(v, \mathbf{a}_+v_-)\mathbf{a}_+v_-$, and from there it is straightforward to verify that (58) is equivalent to (47).

Conversely, suppose the orbit of $(\mathbf{a}, \mathbf{b})$ contains an element of the form $(\mathbf{a}_+ + \mathbf{a}_0 + \mathbf{a}_-, v_+ - 2\mathbf{a}_-\mathbf{a}_+^2v_-)$ with $\mathbf{a}_0 \in \mathfrak{p}_0$ and $\mathbf{a}_- \in \mathfrak{n}_{\mathbf{a}_+}^-$ such that $\jmath(\mathbf{a}_0^2) = -3\imath(\mathbf{a}_-)$. By (29) we have also (55).

If $\mathbf{a}_- \neq 0$ then $\mathfrak{h} \cong \mathfrak{sp}(3, \mathbb{C})$ and $\mathfrak{p}_0 \cong \mathfrak{sl}(2, \mathbb{C})$ by Proposition 4.12, and $\mathbf{a}_- \in \mathfrak{n}_{\mathbf{a}_+}^-$ lies in the orbit of the maximal weight vector. From there, one sees easily that $\mathbf{a}_-^2 = 0$ and $[\mathbf{a}_0, \mathbf{a}_-] = 0$. Thus, we may assume the latter equations for any $\mathfrak{h}$.

By (46), (53) and (54) we get $0 = (\mathbf{a}_+^3\mathbf{a}_- - 3\mathbf{a}_+^2\mathbf{a}_-\mathbf{a}_+ + 3\mathbf{a}_+\mathbf{a}_-\mathbf{a}_+^2 - \mathbf{a}_-\mathbf{a}_+^3)v_- = 3\mathbf{a}_+\mathbf{a}_-\mathbf{a}_+^2v_- - \frac{3}{2}\mathbf{a}_-v_+$, and from this, $\mathbf{a}\mathbf{b} = 0$ follows.

A straightforward calculation then shows that (36) follows from (46), (55), the identities $\mathbf{a}_-^2 = 0$ and $[\mathbf{a}_0, \mathbf{a}_-] = 0$, and (47) which is equivalent to (58). ∎

Therefore, Propositions 4.12 and 5.8 immediately yield the following

Theorem 5.9 *Let $H \subset Aut(V)$ be a complex symplectic holonomy group with holonomy algebra $\mathfrak{h}$. Choose $v_\pm \in V_{\pm 3}$ with $\langle v_+, v_-\rangle = 1$ and a non-degenerate $\mathbf{a}_+ \in \mathfrak{n}^+$ with $\mathbf{a}_+^3v_- = \frac{3}{2}v_+$ and define $\mathfrak{p}_0 \subset \mathfrak{p}$ as in (27). Then $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ is a degenerate pair of type 3 iff one of the following holds.*

1. *$\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C})$ and the H-orbit of $(\mathbf{a}, \mathbf{b})$ contains a pair $(\mathbf{a}_+ + \mathbf{a}_0, v_+)$ where $\mathbf{a}_0 \in \mathfrak{p}_0$ is such that $\mathbf{a}_0^2 = 0$. The $P_0 \cong SO(n-3, \mathbb{C})$-orbit of $\mathbf{a}_0 \in \mathfrak{p}_0$ with $\mathbf{a}_0^2 = 0$ is determined by the rank of $\mathbf{a}_0$ which can be any even integer $\leq \frac{n-3}{2}$. Thus, there are exactly $[\frac{n+1}{4}]$ such orbits.*

2. $\mathfrak{h} \cong \mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{h} \cong \mathfrak{sp}(3,\mathbb{C})$ *or* $\mathfrak{h} \cong \mathfrak{sl}(6,\mathbb{C})$ *and* $(\mathbf{a},\mathbf{b})$ *lies in the H-orbit of the pair* $(\mathbf{a}_+, v_+)$.

3. $\mathfrak{h} \cong \mathfrak{spin}(12,\mathbb{C})$ *or* $\mathfrak{h} \cong \mathfrak{e}_7^{\mathbb{C}}$ *and* $(\mathbf{a},\mathbf{b})$ *lies in the H-orbit of the pair* $(\mathbf{a}_+ + \mathbf{a}_0, v_+)$, *where either* $\mathbf{a}_0 = 0$ *or* $\mathbf{a}_0$ *is a long root vector of* $\mathfrak{p}_0$.

Moreover, $(\mathbf{a},\mathbf{b}) \in \mathfrak{h} \oplus V$ *is a degenerate pair of type 2 iff* $\mathfrak{h} \cong \mathfrak{sl}(2,\mathbb{C})$ *and* $(\mathbf{a},\mathbf{b})$ *lies in the H-orbit of the pair* $(\mathbf{a}_+, v_+)$, *or* $\mathfrak{h} \cong \mathfrak{sp}(3,\mathbb{C})$ *and* $(\mathbf{a},\mathbf{b})$ *lies in the H-orbit of the pair* $(\mathbf{a}_+ + \mathbf{a}_0 + \mathbf{a}_-, v_+ - 2\mathbf{a}_-\mathbf{a}_+^2 v_-)$ *where* $\mathbf{a}_0 \in \mathfrak{p}_0 \cong \mathfrak{sl}(2,\mathbb{C})$ *is a root vector and* $\mathbf{a}_-$ *is uniquely determined by (47).*

Corollary 5.10 *Let* $H \subset Aut(V)$ *be a real symplectic holonomy algebra.*

1. *There are no degenerate pairs of type 2 or type 3 if* $\mathcal{C} = \emptyset$, *i.e. for those holonomy groups listed in Lemma 4.13.*

2. *If H satisfies* $\mathcal{C} \neq \emptyset$ *then there is a one-to-one correspondence between H-orbits of degenerate pairs of type 2 and type 3, and* P_0*-orbits of solutions of (31) for a fixed* $v_+ \in \mathcal{C}$ *and* $\mathbf{a}_+ \in S_{\frac{3}{2}}$. *In particular, the number of such orbits is the one specified in Table 8 on page 29. Moreover, for both* $H = SL(2,\mathbb{R})$ *and* $H = Sp(3,\mathbb{R})$, *there is exactly one H-orbit of degenerate pairs of type 2.*

Proof. In the proof, we use the notational conventions of section 4.2.

If $(\mathbf{a},\mathbf{b}) \in \mathfrak{h} \oplus V \subset \mathfrak{h}_{\mathbb{C}} \oplus V_{\mathbb{C}}$ is a degenerate pair of type 3 then $[\mathbf{b}] \in \mathcal{C}$, hence the first part follows from Lemma 4.13. Moreover, by Lemma 4.14, we may assume that $\mathbf{b} = v_+ \in V_3$ for some fixed decomposition of V and $\mathfrak{h}$ as in (33), and $\mathbf{a} = \mathbf{a}_+ + \mathbf{a}_0$ with $\mathbf{a}_+ \in S_{\frac{3}{2}}$ and $\jmath(\mathbf{a}_0^2) = 0$. Any two such pairs $(\mathbf{a}, v_+)$ lie in the same H-orbit iff they lie in the same P-orbit, and the statement then follows from Proposition 4.16.

If $(\mathbf{a},\mathbf{b}) \in \mathfrak{h} \oplus V \subset \mathfrak{h}_{\mathbb{C}} \oplus V_{\mathbb{C}}$ is a degenerate pair of type 2 then, by Theorem 5.9 we must have H = Sp$(3,\mathbb{R})$, hence $\mathcal{C} \neq \emptyset$. Again Lemma 4.14 and Proposition 4.16 apply. ∎

Proposition 5.11 *Let* $H \subset Aut(V)$ *be a (real or complex) special symplectic holonomy group. If* $(\mathbf{a},\mathbf{b}) \in \mathfrak{h} \oplus V$ *is a symmetric pair of type 2 or type 3 then the image of* $R_{\mathbf{a}} : \Lambda^2 V \to \mathfrak{h}$ *generates all of* $\mathfrak{h}$.

Proof. Clearly, it suffices to show the assertion in the case where H is complex. By Theorem 5.9 we may assume that $\mathbf{a} = \mathbf{a}_+ + \mathbf{a}_0 + \mathbf{a}_-$ with $\mathbf{a}_i \in \mathfrak{n}^i$, $\mathbf{a}_+$ non-degenerate and $[\mathbf{a}_\pm, \mathbf{a}_0] = 0$. Again, we fix elements $v_\pm \in V_{\pm 3}$ such that $\langle v_+, v_- \rangle = 1$.

We let $\underline{\mathfrak{h}} \subset \mathfrak{h}$ be the Lie algebra generated by the image of $R_{\mathbf{a}}$. Let $v \in V_{-1}$. Then $v \circ v_- = 0$ by (11), thus, $0 = [\mathbf{a}_+, v \circ v_-] = (\mathbf{a}_+ v) \circ v_- + v \circ (\mathbf{a}_+ v_-)$. Moreover, $\langle v, v_- \rangle = 0$. Therefore, $R_{\mathbf{a}}(v, v_-) = v \circ (\mathbf{a}_+ v_-) - v_- \circ (\mathbf{a}_+ v) = -2 v_- \circ \mathbf{a}_+ v$, again by (11). By Corollary 4.10, this means that $\underline{\mathfrak{h}} \supset V_1 \circ V_{-3}$. But each $w \in V_1$ can be written as $w = x v_+$ for some $x \in \mathfrak{n}^-$, so that $w \circ v_- = [x, v_+ \circ v_-] = -2x$ by (14), so that $\mathfrak{n}^- \subset \underline{\mathfrak{h}}$.

Now, $R_{\mathbf{a}}(v_+, v_-) = 2\mathbf{a} + v_+ \circ (\mathbf{a}_+ v_-) - v_- \circ (\mathbf{a}_- v_+) = 2\mathbf{a} + [\mathbf{a}_+, v_+ \circ v_-] - [\mathbf{a}_-, v_+ \circ v_-] = 2\mathbf{a} + 2\mathbf{a}_+ + 2\mathbf{a}_-$ by (14), i.e. $2\mathbf{a}_+ + \mathbf{a}_0 \in \underline{\mathfrak{h}}$. But then, $[2\mathbf{a}_+ + \mathbf{a}_0, \mathfrak{n}^-] = [\mathbf{a}_+, \mathfrak{n}^-] \bmod \mathfrak{n}^-$ so that $[\mathbf{a}_+, \mathfrak{n}^-] \subset \underline{\mathfrak{h}}$. Also, $[2\mathbf{a}_+ + \mathbf{a}_0, [\mathbf{a}_+, \mathfrak{n}^-]] = 2[\mathbf{a}_+, [\mathbf{a}_+, \mathfrak{n}^-]] + [\mathbf{a}_+, [\mathbf{a}_0, \mathfrak{n}^-]]$ and since the second summand lies in $[\mathbf{a}_+, \mathfrak{n}^-] \subset \underline{\mathfrak{h}}$, it follows that $[\mathbf{a}_+, [\mathbf{a}_+, \mathfrak{n}^-]] \subset \underline{\mathfrak{h}}$, hence $\mathfrak{n}^+ \subset \underline{\mathfrak{h}}$ by Corollary 4.10.

It remains to show that $\mathfrak{n}^+$ and $\mathfrak{n}^-$ generate all of $\mathfrak{h}$. Let $\alpha \in \Delta_0$, and let $\lambda \in \Phi$ be a weight for which $(\lambda, \alpha) \neq 0$. By Proposition 3.4, $\lambda = \lambda_0 - \alpha_1$ for some $\alpha_1 \in \Delta_1$, and since $(\lambda_0, \alpha) = 0$ by definition of Δ_0, it follows that $(\alpha, \alpha_1) \neq 0$. Choose $\varepsilon = \pm 1$ such that $\beta_\varepsilon := \alpha + \varepsilon\alpha_1$ is a root. Moreover, we let $\beta_{-\varepsilon} := -\varepsilon\alpha_1$. Evidently, $\beta_{\pm 1} \in \Delta_{\pm 1}$ and $\alpha = \beta_1 + \beta_{-1}$. Therefore, $\mathfrak{h}_\alpha = [\mathfrak{h}_{\beta_1}, \mathfrak{h}_{\beta_{-1}}] \subset [\mathfrak{n}^+, \mathfrak{n}^-] \subset \underline{\mathfrak{h}}$, so that $\underline{\mathfrak{h}}$ contains all root spaces $\mathfrak{h}_\alpha$ and hence, $\underline{\mathfrak{h}} = \mathfrak{h}$ as claimed. ∎

5.2.1 Degenerate pairs of type 2

If H is a special symplectic holonomy group then by Theorem 5.9, degenerate pairs of type 2 may exist only if (the complexification of) H equals $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{Sp}(3,\mathbb{C})$, i.e. $\mathrm{H} = \mathrm{SL}(2,\mathbb{F})$ or $\mathrm{H} = \mathrm{Sp}(3,\mathbb{F})$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $V = \odot^3\mathbb{F}^2$ or $V = (\Lambda^3\mathbb{F}^6)_0$, respectively.

The case where $\mathrm{H} = \mathrm{SL}(2,\mathbb{F})$ has been treated in [Br1, S1]. It follows that there is exactly one degenerate pair of type 2 which represents the curvature of a homogeneous symplectic connection on the coadjoint orbit specified in Table 2. This homogeneous space is known to be reductive [S1].

Thus, we shall concentrate on the case where $\mathrm{H} = \mathrm{Sp}(3,\mathbb{F})$. We fix a basis $e_{\pm i}$, $i = 1,2,3$ of $\mathbb{F}^6$ such that the symplectic form $\omega = \sum_i e_i \wedge e_{-i}$ is H-invariant, and the symplectic form $\langle\ ,\ \rangle$ on V is determined by the equation

$$-6\ \alpha \wedge \beta = \langle \alpha, \beta\rangle \omega^3 \quad \text{for all } \alpha, \beta \in V.$$

To convenience our notation, we let $\alpha_{ijk} := e_i \wedge e_j \wedge e_k$ and $\beta_i := e_i \wedge (e_j \wedge e_{-j} - e_k \wedge e_{-k})$ where $(|i|, j, k)$ is an even permutation of $(1,2,3)$. Evidently, $\beta_i \in V$ spans the weight space of weight θ_i while α_{ijk} spans the weight space of weight $\theta_i + \theta_j + \theta_k$ if $\{|i|, |j|, |k|\} = \{1,2,3\}$.

We let $v_\pm := \alpha_{\pm 1, \pm 2, \pm 3}$ so that $\langle v_+, v_-\rangle = 1$, and set $V_{\pm 3} = span\{v_\pm\}$ and $V_{\pm 1} := span\{\alpha_{\pm i, \pm j, \mp k} \mid i,j,k = 1,2,3\} \cap V$. Then (8), (10), (11) and (12) are satisfied if we let $\mathfrak{n}^\pm = span\{e_{\pm i} e_{\pm j} \mid i,j = 1,2,3\}$ and $\mathfrak{n}^0 = span\{e_i e_{-j} \mid i,j = 1,2,3\}$, viewing $\mathfrak{n}^i \subset \odot^2(V) \cong \mathrm{H}$. Evidently, $\mathfrak{n}^0 \cong \mathfrak{gl}(3,\mathbb{F})$ so that $\mathfrak{p} = \mathfrak{p}_s \cong \mathfrak{sl}(3,\mathbb{F})$.

By Theorem 5.9, the (unique) H-orbit of a degenerate pair of type 2 contains the element $(\mathbf{a}_+ + \mathbf{a}_0 + \mathbf{a}_-, v_+ - 2\mathbf{a}_-\mathbf{a}_+^2 v_-)$ with $\mathbf{a}_+ = c_0(e_1^2 + e_2 e_3)$ for some constant c_0 and $\mathbf{a}_0 = 2e_1 e_{-2} - e_{-1}e_3$. Moreover, (47) implies that then $\mathbf{a}_- = -2e_{-2}^2$, and $\mathbf{a}_+ \in S_{\frac{3}{2}}$ iff $c_0 = -\frac{1}{2}$.

Consider the transformation $\phi \in \mathrm{End}(\mathbb{F}^6)$ with $\phi(e_i) = \tilde{e}_i$ where

$$\begin{array}{rclrclrcl}
\tilde{e}_1 &=& -\frac{1}{2}(e_1 - 2e_{-2}) & \tilde{e}_2 &=& (e_2 - 2e_{-1}) & \tilde{e}_3 &=& e_3 \\
\tilde{e}_{-1} &=& -\frac{1}{2}(e_2 + 2e_{-1}) & \tilde{e}_{-2} &=& \frac{1}{4}(e_1 + 2e_{-2}) & \tilde{e}_{-3} &=& e_{-3}
\end{array}$$

One verifies that $\phi \in \mathrm{H}$ and hence the H-orbit of any degenerate pair $(\mathbf{a}, \mathbf{b})$ of type 2 contains the element

$$\phi^{-1}(\mathbf{a}, \mathbf{b}) = (-2e_1^2 + e_{-1}e_3,\ \ 2\beta_3). \tag{59}$$

The infinitesimal stabilizer of this pair is the Lie subalgebra

$$\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1,$$

where

$$\begin{aligned} \mathfrak{l}_0 &:= span\{e_{\pm 2}^2, e_2 e_{-2}\} \cong \mathfrak{sl}(2, \mathbb{F}), \\ \mathfrak{l}_1 &:= span\{e_{\pm 2} e_3, e_3^2, \phi^{-1}(\mathbf{a})\}. \end{aligned}$$

Evidently, $[\mathfrak{l}_i, \mathfrak{l}_j] \subset \mathfrak{l}_{i+j}$. Also, one calculates that the linear map $\tau : V \to \mathfrak{h}$ given by

$$\begin{aligned} \tau(\alpha_{\pm 1, \pm 2, 3}) &:= 0 \\ \tau(\alpha_{-1, \pm 2, -3}) &:= -4 e_{\pm 2} e_{-3} \\ \tau(\alpha_{1, \pm 2, -3}) &:= e_{-1} e_{\pm 2} \end{aligned} \qquad \begin{aligned} \tau(\beta_1) &:= -2e_1^2, \quad \tau(\beta_{-1}) := 4 e_3 e_{-3} \\ \tau(\beta_3) &:= 4 e_1 e_3, \quad \tau(\beta_{-3}) := 4 e_1 e_{-3} + \tfrac{1}{2} e_{-1}^2 \\ \tau(\beta_{\pm 2}) &:= -4 e_1 e_{\pm 2} \end{aligned}$$

satisfies the identity

$$\rho_v(\mathbf{a} + \mathbf{b}) := \xi_{v+\tau(v)}(\mathbf{a} + \mathbf{b}) = 0 \quad \text{for all } v \in V. \tag{60}$$

If there was a $\mathfrak{l}$-equivariant map $\sigma : V \to \mathfrak{h}$ satisfying (60) then $\sigma = \tau + \delta$ with $\delta : V \to \mathfrak{l}$. But then, since $\mathbf{a}\beta_3 = 0$, we would have $0 = [\mathbf{a}, \sigma(\beta_3)] = [\mathbf{a}, 4e_1 e_3 + \delta(\beta_3)] = -4e_3^2 + [\mathbf{a}, \delta(\beta_3)]$, and since $[\mathbf{a}, \delta(\beta_3)] \in [\mathbf{a}, \mathfrak{l}] = 0$ this yields a contradiction. Thus, there cannot be a $\mathfrak{l}$-equivariant map $\tau : V \to \mathfrak{h}$ satisfying (60), i.e. the corresponding homogeneous space is *not* reductive by Proposition 2.7.

Next, one verifies that the map

$$\rho_{\beta_{-1}} \longleftrightarrow 4 e_1 e_2, \quad \rho_{\beta_1 + 3\mathbf{a}} \longleftrightarrow e_2^2, \quad \rho_{\beta_{-3}} \longleftrightarrow e_1^2$$

yields a Lie algebra morphism between $\mathfrak{s} := span\{\rho_{\beta_{-1}}, \rho_{\beta_1 + 3\mathbf{a}}, \rho_{\beta_{-3}}\}$ and $\mathfrak{sl}(2, \mathbb{F}) \cong \odot^2 \mathbb{F}^2$. Decomposing the symmetry algebra $\mathfrak{g} = \mathfrak{l} \oplus \{\rho_v \mid v \in V\}$ as a $\mathfrak{s} \oplus \mathfrak{l}_0 \cong \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{sl}(2, \mathbb{F})$-module, one obtains

$$\mathfrak{g} \cong \mathfrak{s} \oplus \mathfrak{l}_0 \oplus V_{3,1} \oplus V_{1,1} \oplus V_{2,0}.$$

Moreover, $0 \neq [V_{3,1}, V_{3,1}] \subset \mathfrak{s} \oplus \mathfrak{l}_0$, and from there it follows that $\mathfrak{s} \oplus \mathfrak{l}_0 \oplus V_{3,1} \cong \mathfrak{g}_2$, and $V_{1,1} \oplus V_{2,0}$ is the (unique) non-trivial 7-dimensional $\mathfrak{g}_2$-module. Finally, one verifies that $V_{1,1} \oplus V_{2,0}$ is abelian, so that in conclusion we have

$$\mathfrak{g} \cong \begin{cases} \mathfrak{g}_2^{4,3} \rtimes \mathbb{R}^7 & \text{if } \mathbb{F} = \mathbb{R} \\ \mathfrak{g}_2^{\mathbb{C}} \rtimes \mathbb{C}^7 & \text{if } \mathbb{F} = \mathbb{C}, \end{cases}$$

where $\mathfrak{g}_2^{4,3}$ stands for the (unique) non-compact real form of the exceptional Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$.

Another calculation then yields that the element $\eta \in \mathfrak{g}^*$ determined by

$$\eta(\phi^{-1}(\mathbf{a})) = -\eta(\rho_{\beta_1}) = \frac{1}{8}, \quad \eta(\rho_\lambda) = 0, \text{ all weights } \lambda \neq \beta_1, \quad \eta([\mathfrak{l}, \mathfrak{l}]) = 0$$

satisfies the identity (3). Therefore, the homogeneous space G/L is the coadjoint G-orbit of $\mathrm{G} \cdot \eta \subset \mathfrak{g}^*$ with its canonical symplectic form by Corollary 2.6, and its holonomy is all of H by Corollary 5.11. In conclusion, we have the following

Theorem 5.12 *Let $(\mathbf{a}, \mathbf{b}) \subset \mathfrak{h} \oplus V$ be a degenerate pair of type 2. Then there exists a Lie group G and a coadjoint orbit $\Sigma := G \cdot \eta \subset \mathfrak{g}^*$ for some $\eta \in \mathfrak{g}^*$ with its canonical symplectic structure and a G-invariant symplectic connection on Σ whose holonomy is conjugate to H and whose curvature is represented by $(\mathbf{a}, \mathbf{b})$. This homogeneous space is reductive if $rk(H) = 1$ and not reductive otherwise. The possible choices for $\eta \in \mathfrak{g}^*$ are listed in Table 2.*

5.2.2 Degenerate pairs of type 3

By Theorem 5.9 and Corollary 5.10, we may assume that a degenerate pair of type 3 is of the form $(\mathbf{a}, \mathbf{b}) = (\mathbf{a}_+ + \mathbf{a}_0, v_+)$ with $v_+ \in V_3$, $\mathbf{a}_+ \in S_{\frac{3}{2}} \subset \mathfrak{n}^+$ and $\mathbf{a}_0 \in \mathfrak{p}$ satisfying $\mathbf{a}_0^2 V = 0$. First of all, we determine the stabilizer algebra $\mathfrak{l} := \{x \in \mathfrak{h} \mid [x, \mathbf{a}] = x \cdot \mathbf{b} = 0\}$.

Lemma 5.13 *Let $(\mathbf{a}, \mathbf{b}) = (\mathbf{a}_+ + \mathbf{a}_0, v_+) \in \mathfrak{h} \oplus V$ be the degenerate pair of type 3 from Theorem 5.9. Then $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_0$ where*

$$\begin{aligned} \mathfrak{l}_1 &:= l(\mathfrak{n}^-), \\ \mathfrak{l}_0 &:= \mathfrak{l} \cap \mathfrak{p}_0 = \{n_0 \in \mathfrak{p}_0 \mid [n_0, \mathbf{a}_0] = 0\}, \end{aligned}$$

and with the map

$$l(n_-) := [\mathbf{a}_+, [\mathbf{a}_+ - \mathbf{a}_0, n_-]] \quad \text{for all } n_- \in \mathfrak{n}^-. \tag{61}$$

Moreover, $[\mathfrak{l}_0, \mathfrak{l}_i] \subset \mathfrak{l}_i$ for $i \in \{0, 1\}$, and $l : \mathfrak{n}^- \to \mathfrak{l}_1$ is an $\mathfrak{l}_0$-equivariant isomorphism.

Proof. It is straightforward to verify that $\mathfrak{l}_0, \mathfrak{l}_1 \subset \mathfrak{l}$, $[\mathfrak{l}_0, \mathfrak{l}_i] \subset \mathfrak{l}_i$ and l is $\mathfrak{l}_0$-equivariant by direct verification.

By Corollary 4.10 it follows that l is injective, thus an isomorphism onto its image $\mathfrak{l}_1$, and that $\mathfrak{l}_0 \cap \mathfrak{l}_1 = 0$. Finally, let $x \in \mathfrak{l}$. Since $xv_+ = 0$, we have $x = x_+ + x_0$ with $x_+ \in \mathfrak{n}^+$ and $x_0 \in \mathfrak{p}$, and by Corollary 4.10, $x_+ = [\mathbf{a}_+, [\mathbf{a}_+, x_-]]$ for some $x_- \in \mathfrak{n}^-$. Thus, $x - l(x_-) \in \mathfrak{l} \cap \mathfrak{p} = \mathfrak{l}_0$ which shows that $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1$. ∎

For $n_-, m_- \in \mathfrak{n}^-_{\mathbf{a}_+}$, we define $\phi_1(n_-, m_-) \in \mathfrak{n}^-_{\mathbf{a}_+}$ and $\sigma_0(n_-, m_-) \in \mathbb{C}$ by the equation

$$n_- m_- v_+ = \phi_1(n_-, m_-)\mathbf{a}_+^2 v_- + \sigma_0(n_-, m_-)\mathbf{a}_+ v_-. \tag{62}$$

Clearly, $\phi_1 : \odot^2 \mathfrak{n}^-_{\mathbf{a}_+} \to \mathfrak{n}^-_{\mathbf{a}_+}$ and $\sigma_0 : \odot^2 \mathfrak{n}^-_{\mathbf{a}_+} \to \mathbb{C}$ are linear and $\mathfrak{p}_0$-equivariant. Indeed, the relation of σ_0 and $\sigma = \sigma_{\mathbf{a}_+}$ from (26) is determined by $\sigma_0(n_-, m_-) = -\frac{2}{3}\sigma(n_- m_- v_+, \mathbf{a}_+ v_-) = -\frac{2}{3}\left\langle \mathbf{a}_+^2 v_-, n_- m_- v_+ \right\rangle$.

Furthermore, we shall also need the following maps.

Lemma 5.14 *The following identities yield well defined $\mathfrak{p}_0$-equivariant linear maps.*

$$\begin{aligned} \phi_2 : \ & \odot^2 \mathfrak{n}^-_{\mathbf{a}_+} \longrightarrow \mathfrak{l}_0, \\ & \phi_2(n_-, m_-) := [[\mathbf{a}_+, n_-], [\mathbf{a}_0, [\mathbf{a}_+, m_-]]] + [[\mathbf{a}_+, m_-], [\mathbf{a}_0, [\mathbf{a}_+, n_-]]] \end{aligned}$$

$$\begin{aligned} \phi_3 : \ & \Lambda^2 \mathfrak{n}^-_{\mathbf{a}_+} \longrightarrow \mathfrak{l}_0, \\ & \phi_3(n_-, m_-) := [[\mathbf{a}_+, [\mathbf{a}_0, n_-]], [\mathbf{a}_+, [\mathbf{a}_0, m_-]]]. \end{aligned}$$

Proof. It is evident that $\phi_i(n_-, m_-) \in \mathfrak{n}^0$ and that ϕ_i is $\mathfrak{p}_0$-equivariant for $i = 2, 3$. To see that $\phi_i(n_-, m_-) \in \mathfrak{l}_0$, note that $[\mathbf{a}_+, \mathfrak{n}^-_{\mathbf{a}_+}]v_+ = \mathbf{a}_0 v_+ = 0$, thus $\phi_i(n_-, m_-)v_+ = 0$.

Also, $[\mathbf{a}_0, [\mathbf{a}_0, [\mathbf{a}_+, \mathfrak{n}^-_{\mathbf{a}_+}]]] = [\mathbf{a}_+, [\mathbf{a}_0, [\mathbf{a}_0, \mathfrak{n}^-_{\mathbf{a}_+}]]] = 0$ since $[\mathbf{a}_0, [\mathbf{a}_0, \mathfrak{n}^-_{\mathbf{a}_+}]] = 0$ which implies that $[\mathbf{a}_0, \phi_i(n_-, m_-)] = 0$.

Finally, one calculates that $\phi_i(n_-, m_-)\mathbf{a}_+ v_- = 0$ which implies that $[\mathbf{a}_+, \phi_i(n_-, m_-)] = 0$ by Lemma 4.1. ∎

Proposition 5.15 *Let $(\mathbf{a}, \mathbf{b}) = (\mathbf{a}_+ + \mathbf{a}_0, v_+) \in \mathfrak{h} \oplus V$ be a degenerate pair of type 3. Then the linear map $\tau : V \to \mathfrak{h}$ given by*

$$\begin{array}{llll} \tau(v_+) & := & 0 & \\ \tau(n_- v_+) & := & 0 & \textit{for all } n_- \in \mathfrak{n}^- \\ \tau(n_+ v_-) & := & -[n_+, (\mathbf{a}_+^2 v_-) \circ v_-] + 2[\mathbf{a}_0, n_-] & \textit{where } n_\pm \in \mathfrak{n}^\pm,\ n_+ = ad(\mathbf{a}_+)^2 n_- \\ \tau(v_-) & := & -(\mathbf{a}_+^2 v_-) \circ v_-. & \end{array}$$

satisfies the identity

$$\rho_v(\mathbf{a} + \mathbf{b}) := \xi_{v+\tau(v)}(\mathbf{a} + \mathbf{b}) = 0 \quad \textit{for all } v \in V. \tag{63}$$

Moreover, if we let $\mathfrak{l} := stab(\mathbf{a}, \mathbf{b}) \subset \mathfrak{h}$ then there is no $\mathfrak{l}$-equivariant map $\tau : V \to \mathfrak{h}$ which satisfies (63).

Proof. Again, we have to show that (44) holds. We have $\mathbf{b} = v_+$ and thus, $v_+ \circ (V_1 \oplus V_3) = 0$ and $\mathbf{a}^2(V_1 \oplus V_3) = 0$ which implies (44) for all $v \in V_1 \oplus V_3$.

For $v = v_-$, we calculate

$$\begin{aligned} [\mathbf{a}, \tau(v_-)] &= -[\mathbf{a}, (\mathbf{a}_+^2 v_-) \circ v_-] \\ &= -[\mathbf{a}_+, (\mathbf{a}_+^2 v_-) \circ v_-] \\ &= -v_+ \circ v_- = -\mathbf{b} \circ v_- \quad \text{by (15)}, \end{aligned}$$

and $((\mathbf{a}_+^2 v_-) \circ v_-)v_+ = ((\mathbf{a}_+^2 v_-) \circ v_+)v_- - 2\mathbf{a}_+^2 v_- = -2\mathbf{a}_+^2 v_- = -2\mathbf{a}^2 v_-$ by (4) and since $(\mathbf{a}_+^2 v_-) \circ v_+ \in V_1 \circ V_3 = 0$. This shows (44) for $v \in V_{-3}$.

Next, for $v \in V_{-1}$, i.e. for $v = n_+ v_-$, some $n_+ \in \mathfrak{n}^+$, we calculate – using (14) repeatedly

$$\begin{aligned} [\mathbf{a}, \tau(v)] &= [\mathbf{a}, [n_+, \tau(v_-)] + 2[\mathbf{a}_0, n_-]] \\ &= [n_+, [\mathbf{a}, \tau(v_-)]] + [[\mathbf{a}, n_+], \tau(v_-)] + 2[\mathbf{a}, [\mathbf{a}_0, n_-]] \\ &= -[n_+, v_+ \circ v_-] + [[\mathbf{a}_0, n_+], \tau(v_-)] + 2[\mathbf{a}_+, [\mathbf{a}_0, n_-]] \\ &= -v_+ \circ (n_+ v_-) + [\mathbf{a}_0, [n_+, \tau(v_-)]] + 2[\mathbf{a}_0, [\mathbf{a}_+, n_-]] \quad \text{since } [\mathbf{a}_0, \tau(v_-)] = 0 \\ &= -\mathbf{b} \circ v + [\mathbf{a}_0, [[\mathbf{a}_+, [\mathbf{a}_+, n_-]], \tau(v_-)] + 2[\mathbf{a}_+, n_-]], \end{aligned}$$

and

$$\begin{aligned} [[\mathbf{a}_+, [\mathbf{a}_+, n_-]], \tau(v_-)] &= [\mathbf{a}_+, [[\mathbf{a}_+, n_-], \tau(v_-)]] - [[\mathbf{a}_+, n_-], [\mathbf{a}_+, \tau(v_-)]] \\ &= [\mathbf{a}_+, [[\mathbf{a}_+, n_-], \tau(v_-)]] + [[\mathbf{a}_+, n_-], v_+ \circ v_-] \\ &= [\mathbf{a}_+, [[\mathbf{a}_+, \tau(v_-)], n_-]] \quad \text{since } [\mathbf{a}_+, n_-] \in \mathfrak{n}^0 \\ &= -[\mathbf{a}_+, [v_+ \circ v_-, n_-]] \\ &= -2[\mathbf{a}_+, n_-] \end{aligned}$$

so that $[\mathbf{a}, \tau(v)] = -\mathbf{b} \circ v$ as claimed. Finally, $\tau(n_+ v_-)\mathbf{b} = ([n_+, \tau(v_-)] + 2[\mathbf{a}_0, n_-])v_+ = 2n_+\mathbf{a}_+^2 v_- + 2\mathbf{a}_0 n_- v_+ = 2\mathbf{a}_+^2 n_+ v_- + 4\mathbf{a}_0\mathbf{a}_+ n_+ v_- = 2\mathbf{a}^2 n_+ v_-$.

We shall demonstrate the last assertion about the non-existence of an $\mathfrak{l}$-equivariant map $\sigma : V \to \mathfrak{h}$ satisfying (63) only for the case where $\mathbf{a}_0 = 0$ as the general case works analogously but with further calculations. Suppose therefore that such an $\mathfrak{l}$-equivariant map σ exists, so that $\delta : \tau - \sigma$ takes values in $\mathfrak{l} \subset \mathfrak{n}^0 \oplus \mathfrak{n}^+$.

Since $\tau(l(n_-)v_-) = [l(n_-), \tau(v_-)]$ it follows that $\delta(l(n_-)v_-) = [l(n_-), \delta(v_-)] \in \mathfrak{n}^+$, hence $\delta(V_{-1}) \subset \mathfrak{n}^+$. Likewise, $\delta(\mathbf{a}_+ l(n_-)v_-) = [\mathbf{a}_+, \tau(l(n_-)v_-)] + [\mathbf{a}_+, \delta(l(n_-)v_-)] = -2l(n_-)$ since $[\mathbf{a}_+, \delta(l(n_-)v_-)] \in [\mathfrak{n}^+, \mathfrak{n}^+] = 0$, i.e.

$$\sigma(\mathbf{a}_+ n_+ v_-) = -2n_+ \quad \text{for all } n_+ \in \mathfrak{n}^+. \tag{64}$$

Now let $n_-, m_- \in \mathfrak{n}^-_{\mathbf{a}_+}$, set $(n_+, m_+) := ad(\mathbf{a}_+)^2(n_-, m_-)$. Define $c \in \mathbb{F}$ and $x \in \mathfrak{n}^+_{\mathbf{a}_+}$ by the identity $n_+ m_+ v_- = \mathbf{a}_+(c\mathbf{a}_+ + x)v_-$. Then

$$\begin{aligned}
[n_+, \sigma(m_+ v_-)] &= [n_+, \tau(m_+ v_-)] && \text{since } [n_+, \delta(m_+ v_-)] \in [\mathfrak{n}^+, \mathfrak{n}^+] = 0 \\
&= -(\mathbf{a}_+^2 v_-) \circ (n_+ m_+ v_-) && \text{since } n_+, m_+ \in \mathfrak{n}^+_{\mathbf{a}_+} \\
&= -(\mathbf{a}_+^2 v_-) \circ (c\mathbf{a}_+^2 v_- + x\mathbf{a}_+ v_-) && \\
&= -c(\mathbf{a}_+^2 v_-) \circ (\mathbf{a}_+^2 v_-) && \\
&\quad -[x, (\mathbf{a}_+^2 v_-) \circ (\mathbf{a}_+ v_-)] && \\
&= 4c\mathbf{a}_+ + x && \text{by (14) and (15),}
\end{aligned}$$

while $\sigma(n_+ m_+ v_-) = -2(c\mathbf{a}_+ + x)$ by (64). Thus, $(n_+\sigma)(m_+ v_-) = 6c\mathbf{a}_+ + 3x = 0$ iff $n_+ m_+ v_- = 0$. Hence, we must have $(\mathfrak{n}^+_{\mathbf{a}_+})^2 v_- = 0$ which is a contradiction. ∎

Now let us determine the structure of the symmetry algebra $\mathfrak{g}$ from (45) for these examples.

Theorem 5.16 *Let $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ be the degenerate pair of type 3 from Theorem 5.9, let $\mathfrak{g}$ be the symmetry algebra from (45), G be a Lie group with Lie algebra $\mathfrak{g}$ and $L \subset G$ be the Lie subgroup corresponding to the Lie subalgebra $\mathfrak{l} \subset \mathfrak{g}$.*

Then there is a G-invariant connection with special symplectic holonomy H on the homogeneous space G/L whose curvature at any point is represented by $(\mathbf{a}, \mathbf{b})$.

Moreover, we have

$$\mathfrak{g} \cong \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{l}_0 \oplus V_{\nu\otimes 1} \oplus V_{\nu\otimes\lambda} \oplus V_{1\otimes\lambda}, \tag{65}$$

where the last three summands are modules of the first two, and where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Here, 1 denotes the trivial representation of either summand, ν denotes the standard representation of $\mathfrak{sl}(2, \mathbb{F})$ and λ the representation of $\mathfrak{l}_0$ on $\mathfrak{n}^-_{\mathbf{a}_+}$.

The bracket relations of these modules is given as follows:

$$\begin{aligned}
[e, f] &= 0, \\
[e, n_-] &= 0, \\
[e, f \otimes n_-] &= -36\langle e, f\rangle [\mathbf{a}_0, n_-], \\
[e \otimes n_-, f \otimes m_-] &= 18(6\sigma_0([\mathbf{a}_0, n_-], m_-)ef - \\
&\qquad \langle e, f\rangle (\phi_1(n_-, m_-) + 2\sigma_0(n_-, m_-)\mathbf{a}_0 + 4\phi_2(n_-, m_-))), \\
[e \otimes n_-, m_-] &= -3\sigma_0(n_-, m_-)e + e \otimes (\phi_1([\mathbf{a}_0, n_-], m_-) - \tfrac{1}{2}\phi_1(n_-, [\mathbf{a}_0, m_-])), \\
[n_-, m_-] &= -\phi_1([\mathbf{a}_0, n_-], m_-) + \phi_1([\mathbf{a}_0, m_-], n_-) \\
&\qquad -8\sigma_0([\mathbf{a}_0, n_-], m_-)\mathbf{a}_0 + 4\phi_3(n_-, m_-),
\end{aligned} \tag{66}$$

where, $e, f \in \mathbb{F}^2$, $n_-, m_- \in \mathfrak{n}^-_{\mathbf{a}_+}$ and where $\langle\ ,\ \rangle$ denotes the determinant of $\mathbb{F}^2$.

Proof. We proceed similarly as in the proof of Theorem 5.6. Namely, we are given the explicit formulae for the vector fields ρ_w in Proposition 5.15, and their Lie brackets in (6). First, we calculate that there is a Lie algebra isomorphism between $\mathfrak{sl}(2,\mathbb{F}) \cong \odot^2\mathbb{F}^2$ and $\mathfrak{s}_1 := span(\rho_{\mathbf{a}_+v_-}, \rho_{\mathbf{a}_+^2v_-} + 5\mathbf{a}_+ + \mathbf{a}_0, \rho_{v_-})$, given by

$$\rho_{\mathbf{a}_+v_-} \longleftrightarrow 3e_1e_2, \quad 2\rho_{v_-} \longleftrightarrow 9e_1^2, \quad 2(\rho_{\mathbf{a}_+^2v_-} - 5\mathbf{a}_+ - \mathbf{a}_0) \longleftrightarrow 3e_2^2,$$

with a basis $e_1, e_2 \in \mathbb{F}^2$ with $\langle e_1, e_2\rangle = 1$. Obviously, $[\mathfrak{s}_1, \mathfrak{l}_0] = 0$ so that $\mathfrak{g}$ contains $\mathfrak{s}_1 \oplus \mathfrak{l}_0$ as a subalgebra.

Next, one verifies that the following yield $\mathfrak{s}_1 \oplus \mathfrak{l}_0$-equivariant embeddings of $V_{\nu\otimes 1} \oplus V_{\nu\otimes\lambda} \oplus V_{1\otimes\lambda}$ into $\mathfrak{g}$:

$$\begin{array}{rcl} 2(\rho_{\mathbf{a}_+^2v_-} + 4\mathbf{a}_+ + 2\mathbf{a}_0) & \longleftrightarrow & e_1 \\ 9\rho_{v_+} & \longleftrightarrow & e_2 \end{array} \qquad \begin{array}{rcl} 2\rho_{n_-\mathbf{a}_+^2v_-} & \longleftrightarrow & e_1 \otimes n_- \\ 3(\rho_{n_-v_+} + \frac{2}{3}\rho_{[\mathbf{a}_0,n_-]\mathbf{a}_+^2v_-} - 4l(n_-)) & \longleftrightarrow & e_2 \otimes n_- \\ \rho_{n_-v_+} - \frac{2}{3}\rho_{[\mathbf{a}_0,n_-]\mathbf{a}_+^2v_-} + 2l(n_-) & \longleftrightarrow & n_- \end{array}$$

The asserted bracket relations are then shown by a cumbersome but straightforward calculation. ∎

From the description of the Lie algebra $\mathfrak{g}$ in Theorem 5.16 one can now give an explicit description of $\mathfrak{g}$ by a case-by-case investigation. To get a flavour of the arguments involved, we discuss the case $\mathfrak{h} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(n,\mathbb{C})$ in some more detail below.

If $\mathfrak{h} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(n,\mathbb{C})$ and $\mathrm{rk}(\mathbf{a}_0) = 2k \leq \frac{n-3}{2}$, then one verifies easily that $\mathfrak{l}_0 = \mathfrak{sp}(k,\mathbb{C}) \oplus \mathfrak{so}(n-4k-3) \oplus V_{\nu_k\otimes\mu} \oplus V_{\Lambda^2\otimes 1} \oplus V_1$ and $\lambda = V_{1\otimes\mu} \oplus 2V_{\nu_k\otimes 1} \oplus V_1$ where ν_k, μ denote the standard representations of $\mathfrak{sp}(k,\mathbb{C})$ and $\mathfrak{so}(n-4k-3,\mathbb{C})$, respectively, and Λ_2 denotes the second fundamental representation of $\mathfrak{sp}(k,\mathbb{C})$. Since $\phi_1 : \odot^2\mathfrak{n}^-_{\mathbf{a}_+} \to \mathfrak{n}^-_{\mathbf{a}_+}$ and $\phi_3 : \odot^2\mathfrak{n}^-_{\mathbf{a}_+} \to \mathfrak{l}_0$ are $\mathfrak{l}_0$-equivariant, this decomposition implies that $\phi_i(2V_{\nu_1\otimes\nu_k\otimes 1}) = 0$ and thus, by (66), $0 \neq [2V_{\nu_1\otimes\nu_k\otimes 1}, 2V_{\nu_1\otimes\nu_k\otimes 1}] \subset \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sp}(k,\mathbb{C})$. Thus, $(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sp}(k,\mathbb{C}), V_{\nu_1\otimes\nu_k\otimes 1})$ is a symmetric pair, whence $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sp}(k,\mathbb{C}) \oplus V_{\nu_1\otimes\nu_k\otimes 1} \cong \mathfrak{sp}(k+1,\mathbb{C})$.

From (65) we then get the decomposition $\mathfrak{g} \cong \mathfrak{sp}(k+1,\mathbb{C}) \oplus \mathfrak{so}(n-4k-3,\mathbb{C}) \oplus V_{\nu_{k+1}\otimes\mu} \oplus V_{1\otimes\mu} \oplus V_{\Lambda^2\otimes 1} \oplus 2V_{\nu_{k+1}\otimes 1} \oplus 2V_1$, and from (66) we get the bracket relations described in Table 3.

Proposition 5.17 *The homogeneous spaces G/L from Theorem 5.16 are not reductive and are not equivalent to a coadjoint orbit in $\mathfrak{g}^*$, i.e. there is no $\eta \in \mathfrak{g}^*$ which satisfies (3).*

Proof. The non-reductivity follows from Propositions 2.7 and 5.15. Next, from (66) one calculates that $\eta \in \mathfrak{g}^*$ satisfies (3) iff η satisfies the following conditions:

$$\begin{array}{lll} \text{on } \mathfrak{sl}(2,\mathbb{F}): & \eta(e_2^2) = \frac{1}{6}, & \eta(e_1^2) = \eta(e_1e_2) = 0 \\ \text{on } \mathfrak{l}_0: & \eta(\mathbf{a}_0) = \frac{1}{4}, & \eta(\phi_2(\odot^2\mathfrak{n}^-_{\mathbf{a}_+})) = \eta([\mathfrak{l}_0,\mathfrak{l}_0]) = 0 \\ \text{on } V_{\nu,1}: & \eta(e_1) = -1, & \eta(e_2) = 0 \\ \eta(V_{\nu\otimes\lambda} \oplus V_{1\otimes\lambda}) = 0. & & \end{array}$$

Thus, such an $\eta \in \mathfrak{g}^*$ exists iff $\mathbf{a}_0 \notin \phi_2(\odot^2 \mathfrak{n}^-_{\mathbf{a}_+}) + [\mathfrak{l}_0, \mathfrak{l}_0]$.

In particular, $\mathbf{a}_0 \neq 0$ and thus, by Theorem 5.9, $\mathfrak{h}_{\mathbb{C}}$ must be $\mathfrak{spin}(12, \mathbb{C})$, $\mathfrak{e}_7^{\mathbb{C}}$ or $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C})$.

In the first two cases, $\mathbf{a}_0$ lies in the orbit of a long root in $\mathfrak{p}_0$, say α. Let β be a root of $\mathfrak{p}_0$ with $(\alpha, \beta) = 1$. Then the root spaces $(\mathfrak{p}_0)_\beta, (\mathfrak{p}_0)_{\alpha-\beta} \subset \mathfrak{l}_0$, thus, $\mathbf{a}_0 \in (\mathfrak{p}_0)_\alpha = [(\mathfrak{p}_0)_\beta, (\mathfrak{p}_0)_{\alpha-\beta}] \subset [\mathfrak{l}_0, \mathfrak{l}_0]$, so in these cases, there cannot exists $\eta \in \mathfrak{g}^*$ satisfying (3).

Next, let us suppose that $\mathfrak{h}_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C})$ and $\mathrm{rk}(\mathbf{a}_0) = 2k > 0$, so that $\mathfrak{n}^-_{\mathbf{a}_+} = V_{1\otimes\mu} \oplus 2V_{\nu_k \otimes 1} \oplus V_1$ as a $\mathfrak{sp}(k) \oplus \mathfrak{so}(n - 4k - 3)$-module. Using the definitions it is then easy to calculate that $\phi_2(2V_{\nu_k\otimes 1}, 2V_{\nu_k \otimes 1}) = \mathfrak{sp}(k) \oplus V_{\Lambda^2 \otimes 1} \oplus V_1$ and $\mathbf{a}_0 \in V_1$. So in this case, $\mathbf{a}_0 \in \phi_2(\odot^2 \mathfrak{n}^-_{a_+})$ and there cannot be an $\eta \in \mathfrak{g}^*$ satisfying (3) either. ∎

6 Proofs of the main results

As before, we let (M, Ω, ∇) be a symplectic manifold with a connection of special symplectic holonomy and let $\pi : F \to M$ be its holonomy bundle and $\rho := \mathbf{a} + \mathbf{b} : F \to \mathfrak{h} \oplus V$ be the map from Theorem 3.6. Throughout this section, we define $F_0 \subset F$ and $M_0 \subset M$ by

$$F_0 := \{p \in F \mid \mathbf{b}(p) = 0\}, \qquad M_0 := \pi(F_0).$$

Since F_0 is H-invariant, it follows that the restriction $\pi : F_0 \to M_0$ is again a principal H-bundle, and M_0 is the set of *symmetric points*, i.e. of those $p \in M$ with $(\nabla R)_p = 0$.

Suppose that $M_0 = M$, i.e. $\mathbf{b} \equiv 0$. Then it follows that $\nabla R \equiv 0$, i.e. ∇ is locally symmetric. Thus, the curvature tensor R_p must be invariant under the holonomy group at $p \in M$. But Proposition 3.3 implies that there are no non-trivial H-invariant curvature tensors which means that the holonomy must be a *proper subgroup* of H which is excluded.

Thus, $M_0 \subset M$ and $F_0 \subset F$ are proper closed subsets.

Proof of Theorem 1.1. The constancy of the symplectic scalar curvature and Definition 5.1 implies that $\rho(F \backslash F_0)$ consists of degenerate pairs. Note that for degenerate pairs $(\mathbf{a}, \mathbf{b})$ of type 2 or 3, $\mathbf{a}$ is nilpotent and hence, $(\mathbf{a}, \mathbf{a}) = 0$. Thus, by the non-vanishing of the symplectic scalar curvature, $\rho(F \backslash F_0)$ must consist of degenerate pairs of type 1.

Define $\rho_s : F \to \mathfrak{h}$ by $\rho_s := \mathbf{a} - \frac{1}{2\bar{c}} \mathbf{b} \circ \mathbf{b}$. Since ρ_s is analytic by Theorem 3.6,3, (37) and (38) imply that $\rho_s(2\rho_s^2 + \bar{c} Id_V) = 0$ and $\rho_s(p)$ is the semisimple part of $\mathbf{a}(p)$ for all $p \in F$. In particular, the image of ρ_s consists of a single H-orbit of a semi-simple element of $\mathfrak{h}$.

Fix $\mathbf{a}_s \in \rho_s(F)$, let $\mathrm{L}_0 \subset \mathrm{H}$ be the stabilizer of $\mathbf{a}_s$ and $\mathfrak{l}_0 \subset \mathfrak{h}$ be the Lie algebra of L_0. Let

$$F_s := \rho_s^{-1}(\mathbf{a}_s) \subset F, \quad \text{whence} \quad F = F_s \times_{\mathrm{L}_0} \mathrm{H}. \tag{67}$$

Then the H-equivariance of ρ_s implies that $\pi : F_s \to M$ is a principal L_0-bundle. Moreover, by Lemma 5.2,3 the restriction $\mathbf{b}|_{F_s}$ takes values in $W := \ker(\mathbf{a}_s) \subset V$. For $v \in W$ we have $\xi_v(\mathbf{b}) = \bar{c}v$ by (6) where $\xi_v \in \mathfrak{X}(F)$ is the vector field determined by $\theta(\xi_v) = v$, $\omega(\xi_v) = 0$. Thus, $\mathbf{b} : F_s \to W$ is an L_0-equivariant local submersion.

Let us assume that F_s is simply connected. For all $p \in F_s$, $\ker(d(\mathbf{b}|F_s)_p) = \ker(d\rho_p)$ by (38), whence by Theorem 3.6,5 there is a $(\mathfrak{g} \oplus \mathfrak{l}_0)$-equivariant immersion $\imath : F_s \to \mathrm{G} \times W$

such that $\mathbf{b} = pr_2 \circ \imath$ with the projection $pr_2 : \mathrm{G} \times W \to W$. By (67), $\imath$ extends to an $(\mathfrak{g} \oplus \mathfrak{l}_0)$-equivariant immersion

$$\jmath : F \to \mathrm{G} \times_{\mathrm{L}_0} (W \times \mathrm{H}) =: F_{max}.$$

The $(\mathfrak{g} \oplus \mathfrak{l}_0)$-equivariance of $\imath$ implies that the connection coframe and the curvature map extend to F_{max}, i.e. there is a G-invariant $\mathfrak{h} \oplus V$-valued coframe $\hat{\theta} + \hat{\omega}$ on F_{max} such that $\jmath^*(\hat{\theta} + \hat{\omega}) = \theta + \omega$ and a $\mathfrak{g}$-invariant function $\hat{\rho} : F_{max} \to \mathfrak{h} \oplus V$ such that $\jmath^*(\hat{\rho}) = \rho$. Thus, $(\hat{\theta}, \hat{\omega})$ and $\hat{\rho} = \hat{\mathbf{a}} + \hat{\mathbf{b}}$ also satisfy (6).

All of this implies now that the principal H-bundle $\hat{\pi} : F_{max} \to \mathrm{G} \times_{\mathrm{L}_0} W =: E$ is an H-structure with a symplectic connection of special symmetric holonomy H with constant non-zero symplectic scalar curvature, and there is a connection preserving G-equivariant immersion $\underline{\jmath} : M \to \mathrm{G} \times_{\mathrm{L}_0} W$. Thus, every manifold with a symplectic connection of non-zero symplectic scalar curvature is locally equivalent to the G-invariant connection on E.

We shall now show the asserted properties of the connection on E and omit the superscripts $\hat{}$. Thus, $F = \mathrm{G} \times_{\mathrm{L}_0} (W \times \mathrm{H})$ and $F_s = \mathrm{G} \times W$, $\mathbf{b} : F_s \to W$ is the canonical projection and $\mathbf{a} = \mathbf{a}_s + 2\overline{c}\mathbf{b} \circ \mathbf{b}$. Moreover, $F_0 = \mathrm{G} \times_{\mathrm{L}_0} (\{0\} \times \mathrm{H})$ and $E_0 \subset E$ is the 0-section.

Let $v \in \mathbf{a}_s(V)$. By (6), $\xi_{\mathbf{a}_s v}(\rho_s) = 0$, thus the vectors $\xi_{\mathbf{a}_s v}$ are tangent to $F_0 \cap F_s = \mathrm{G} \times \{0\} \cong \mathrm{G}$. In fact, $E_0 = \pi(F_0 \cap F_s)$ and $\pi_*(\xi_{\mathbf{a}_s V}) = TE_0$, whence E_0 is totally geodesic and the connection restricts to the symmetric connection on $E_0 = \mathrm{G}/\mathrm{L}_0$.

Moreover, for $v \in W = \ker \mathbf{a}_s$ we have $\xi_v(\rho_s) = 0$ by (6), whence ξ_v is tangent to F_s for all $v \in W$. We claim that $\mathfrak{l}_0 \oplus \{\xi_v \mid v \in W\}$ spans the tangent spaces of $\mathrm{L}_0 \times W \subset F_s$. This is evident on the set $\mathrm{L}_0 \times \{0\}$, while on its complement, it follows from Proposition 5.5, since $\tau(v) \in \mathfrak{l}_0$ for all $v \in W$.

Thus, the fibers of $E \to \mathrm{G}/\mathrm{L}_0$ are totally geodesic, and since the horizontal distribution $\mathcal{H}$ on E induced by the symmetric connection on G/L_0 is spanned by $\{\pi_*(\xi_{\mathbf{a}_s v}) \mid v \in V\}$, it follows that $\mathcal{H}$ is symplectically orthogonal to the fibers as claimed. ∎

Proof of Theorem 1.2. (1) $\Longrightarrow$ (2): Since $M_0 \subset M$ is invariant under the local symmetry group, the local homogeneity implies that $M_0 \subset M$ and $F_0 \subset F$ is open. Since M_0 is also closed and M_0 is a *proper* subset of M, we must have $M_0 = \emptyset$, i.e. M contains no symmetric points. Since the symplectic scalar curvature is preserved by the symmetry group, it must be constant.

(2) $\Longrightarrow$ (3): Trivial

(3) $\Longrightarrow$ (1): Let $p \in F \backslash F_0$ such that $scal - scal(\pi(p))$ vanishes of order at least 3. Thus, the first and second order derivatives of $(\mathbf{a}, \mathbf{a})$ vanish at p whence $\rho(p)$ is degenerate by Definition 5.1. Thus, the local homogeneity of ∇ follows from Theorem 3.6,4. together with Theorems 5.6, 5.12 and 5.16. ∎

Proof of Theorem 1.3. Let (M, Ω, ∇) be a manifold with a symplectic connection with vanishing symplectic scalar curvature and let $\pi : F \to M$ as before. Suppose that $F_0 \neq \emptyset$ and fix $p \in F_0$. Then by (36) we have $\rho(p) = (\mathbf{a}, 0)$ with $\mathbf{a}^3 = 0$. If we have $\mathbf{a}^2 = 0$ then by

(6), $\mathfrak{g}_p = V \oplus \mathfrak{l}_0$ whence the connection is locally homogeneous and thus, $\mathbf{b} \equiv 0$. However, this was already excluded. Thus, we must have $\mathbf{a}^2 \neq 0$.

If $\mathrm{rk}(\mathfrak{h}) = 1$ then any nilpotent element of $\mathbf{a} \in \mathfrak{h}$ satisfies $\mathbf{a}^3 \neq 0$ or $\mathbf{a} = 0$. Thus, if $\mathrm{rk}(\mathfrak{h}) = 1$ then M cannot contain any symmetric points, hence must be homogeneous by Theorem 1.2. Therefore, the homogeneous examples in Table 2 for $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{F})$ are maximal.

We assume from now on that $\mathrm{rk}(\mathfrak{h}) \geq 2$. Let $v \in \mathcal{C}$, i.e. $v \in V$ with $v \circ v = 0$, and let ξ_v be the vector field on F with $\theta(\xi_v) = v$, $\omega(\xi_v) = 0$. We assert that the trajectory γ of ξ_v with $\gamma(0) = p$ satisfies

$$\rho(\gamma(t)) = (\mathbf{a} + t^2(\mathbf{a}^2 v) \circ v,\ 2t\mathbf{a}^2 v) =: (\mathbf{a}(t), \mathbf{b}(t)) \tag{68}$$

for all t. Namely, $\langle \mathbf{a}v, v \rangle = -\frac{1}{2}(\mathbf{a}, v \circ v) = 0$ by (16) and $v \circ v = 0$; also, $\langle \mathbf{a}^2 v, v \rangle = -\langle \mathbf{a}v, \mathbf{a}v \rangle = 0$, $\langle \mathbf{a}^2 v, \mathbf{a}v \rangle = -\langle \mathbf{a}^3 v, v \rangle = 0$. All of this and (4) implies that $((\mathbf{a}^2 v) \circ v)v = (v \circ v)\mathbf{a}^2 v = 0$ and $((\mathbf{a}^2 v) \circ v)\mathbf{a}v = (\mathbf{a}v \circ v)\mathbf{a}^2 v = \frac{1}{2}[\mathbf{a}, v \circ v]\mathbf{a}^2 v = 0$. Thus, it follows that $\mathbf{a}(t)^2 v = \mathbf{a}^2 v$ for all t, whence $\rho(\gamma(t))' = (\mathbf{b}(t) \circ v, 2\mathbf{a}(t)^2 v)$ which by (6) implies that $\rho(\gamma(t))' = d\rho(\xi_v)_{\gamma(t)}$ as claimed.

Next, we have $0 = [\mathbf{a}, v \circ v] = 2(\mathbf{a}v) \circ v$, whence $0 = [\mathbf{a}, (\mathbf{a}v) \circ v] = (\mathbf{a}^2 v) \circ v + (\mathbf{a}v) \circ (\mathbf{a}v)$. Therefore, since $\mathbf{a}^3 = 0$, $(\mathbf{a}^2 v) \circ (\mathbf{a}^2 v) = [\mathbf{a}, [\mathbf{a}, (\mathbf{a}^2 v) \circ v]] = -[\mathbf{a}, [\mathbf{a}, (\mathbf{a}v) \circ (\mathbf{a}v)]] = -2(\mathbf{a}^2 v) \circ (\mathbf{a}^2 v)$, whence $(\mathbf{a}^2 v) \circ (\mathbf{a}^2 v) = 0$.

Since ∇ has constant symplectic scalar curvature, we must have $\mathcal{C} \neq \emptyset$ by Corollary 5.10,1. Thus, we can choose $v \in \mathcal{C}$ such that $\mathbf{a}^2 v \neq 0$. Therefore, the trajectory $(\mathbf{a}(t), \mathbf{b}(t))$ from (68) satisfies $0 \neq \mathbf{b}(t) \in \mathcal{C}$ for all $t \neq 0$. Since ∇ has constant symplectic scalar curvature, $(\mathbf{a}(t), \mathbf{b}(t))$ must be degenerate for all $t \neq 0$. Thus, $\mathbf{b}(t) \in \mathcal{C}$ implies that $(\mathbf{a}(t), \mathbf{b}(t))$ is of type 3. Therefore, if ∇ has constant scalar symplectic curvature and $\rho(F)$ contains a degenerate pair of type 2 then M cannot contain any symmetric points and thus must be locally homogeneous by Theorem 1.2. In particular, the homogeneous connections in Table 2 are maximal.

Finally, if $(\mathbf{a}_+ + \mathbf{a}_0, v_+)$ is a degenerate pair of type 3 then one can show that there is a decomposition $\mathbf{a}_+ = \mathbf{a}_1 + \mathbf{a}_2$ with $\mathbf{a}_i \in \mathfrak{n}^+$, $[\mathbf{a}_0, \mathbf{a}_i] = 0$, $\mathbf{a}_1^2 = 0$ and $\mathbf{a}_2^3 = 0$. For the complex holonomies and the split forms, this follows from the normal form (17), while in the remaining cases one has to perform a direct investigation.

Let $v := \mathbf{a}_1 v_- \in V_1$. Then $v \circ v = [\mathbf{a}_1, [\mathbf{a}_1, v_- \circ v_-]] = 0$, whence $v \in \mathcal{C}$. Moreover, since $[\mathbf{a}_1, \mathbf{a}_2] \in [\mathfrak{n}^+, \mathfrak{n}^+] = 0$, we have $\frac{3}{2}\mathbf{a}_+^3 v_- = 3\mathbf{a}_1 \mathbf{a}_2^2 v_- = 3\mathbf{a}_2^2 v$. Thus, the trajectory γ of ξ_v passing through a point $p \in F$ with $\rho(p) = (\mathbf{a}_2 + \mathbf{a}_0, 0)$ satisfies $\rho(\gamma(1)) = (\mathbf{a}_+ + \mathbf{a}_0, v_+)$ by (68). In other words, any maximal manifold with a symplectic connection for which $\rho(F)$ contains a degenerate pair of type 3 must also contain symmetric points and, in particular, cannot be locally homogeneous by Theorem 1.2. Thus, the homogeneous spaces in Table 3 are not maximal. ∎

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