Group Theoretical Quantization and the Example of a Phase Space $S^1 \times \mathbb{R}^+$

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Abstract

The group theoretical quantization scheme is reconsidered by means of elementary systems. Already the quantization of a particle on a circle shows that the standard procedure has to be supplemented by an additional condition on the admissibility of group actions. A systematic strategy for finding admissible group actions for particular subbundles of cotangent spaces is developed, two–dimensional prototypes of which are $T^*\mathbb{R}^+$ and $\mathcal{S} = S^1 \times \mathbb{R}^+$ (interpreted as restrictions of $T^*\mathbb{R}$ and T^*S^1 to positive coordinate and momentum, respectively). In this framework (and under an additional, natural condition) an $SO^{\uparrow}(1,2)$–action on $\mathcal{S}$ results as the *unique* admissible group action.

For symplectic manifolds which are (specific) parts of phase spaces with known quantum theory a simple "projection method" of quantization is formulated. For $T^*\mathbb{R}^+$ and $\mathcal{S}$ equivalent results to those of more established (but more involved) quantization schemes are obtained. The approach may be of interest, e.g., in attempts to quantize gravity theories where demanding nondegenerate metrics of a fixed signature imposes similar constraints.

1 Introduction

To quantize a classical phase space there are techniques generalizing the standard quantization method which is only applicable to simple cotangent bundles. Most prominent are geometric [1] and group theoretical quantization [2, 3]. But since each method of quantization has its own advantages and disadvantages and gives rise to certain types of ambiguities, none of them can be regarded as a final and unique route to a quantum theory. Usually, such a scheme is developed on simple examples so as to reproduce standard results. Studying more complicated systems then can lead to the necessity of further specifications which are necessary to exclude unphysical results.

The present paper is divided into two main parts. In the first one, Sec. 2, we review the group theoretical quantization (following Ref. [2]) focussing on some points which,

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in our opinion, deserve further study. In this scheme one studies the irreducible unitary representations of a group which has a transitive, almost effective Hamiltonian action with a momentum map on the phase space (keeping only those representations which are physically acceptable). The main ingredients of this method and its application to the simplest examples (phase space $T^*\mathbb{R}$, $T^*\mathbb{R}^+$, and T^*S^1) are recalled here. In the course of this review we will see the necessity of a global generating principle for phase space functions as opposed to a local one, which is already implied by transitivity of the group action. Therefore, we have to supplement the rules of group theoretical quantization as outlined in Ref. [2] by a further condition on the allowed group actions in order to recover the results of standard quantizations of T^*S^1.

Thereafter a general strategy for finding an appropriate group action on particular subbundles of cotangent bundles is discussed. This extends considerations presented in Ref. [2] for actions on cotangent bundles.

Sec. 2 is concluded with a proposal ("projection method") for the quantization of certain submanifolds $\mathcal{P}$ of phase spaces $\widetilde{\mathcal{P}}$, where quantum realizations of $\widetilde{\mathcal{P}}$ are known.

$T^*\mathbb{R}^+$ may be used as illustrating example. As a restriction of $T^*\mathbb{R}$ it fits into the framework of both of the final two subsections of Sec. 2. Standard results are reproduced in this case.

In Sec. 3 we will consider the phase space $\mathcal{S} := S^1 \times \mathbb{R}^+ := T^*S^1|_{p>0}$ defined as the restriction of the cotangent bundle of S^1 to positive momenta, which is maybe the simplest example for a phase space which is not symplectomorphic to a cotangent bundle. It will be found that $SO^\uparrow(1,2)$, the identity component of $SO(1,2)$, provides an appropriate action on $\mathcal{S}$ for applying group theoretical quantization. To find this group action we exploit the fact that $\mathcal{S}$ is a subbundle of T^*S^1 for which we can apply the methods of Sec. 2 described above. It will be seen that the canonical lift of an $SO^\uparrow(1,2)$ subgroup of the diffeomorphism group of S^1 to T^*S^1 provides a *transitive* and effective Hamiltonian action on the *half* cylinder $\mathcal{S} = S^1 \times \mathbb{R}^+$. Under a further condition we will be able to show that any such subgroup of the lift of the diffeomorphisms is necessarily isomorphic to a covering group of $SO^\uparrow(1,2)$. All effective actions of proper covering groups of $SO^\uparrow(1,2)$, which would be allowed according to the commonly used rules of group theoretical quantization, will be seen to be excluded by the additional condition mentioned above.

Applying standard knowledge on the unitary irreducible representations (IRREPs) of $SO^\uparrow(1,2)$ and its covering groups leads to possible quantum realizations of the system under consideration. Actually, as any representation of a group is also a representation of a covering group of that group, it is sufficient to analyze the unitary IRREPs of the universal covering group $\widetilde{SO}^\uparrow(1,2)$ of $SO^\uparrow(1,2)$ to obtain the most general possible quantum theory. This in turn amounts to an analysis of the unitary IRREPs of the Lie algebra of $so(1,2) \sim su(1,1) \sim sl(2,\mathbb{R})$. Selecting appropriate representations which fulfill the relation $p > 0$ at the quantum level will complete the group theoretical quantization of $\mathcal{S}$, leading to a one–parameter family of inequivalent quantizations given by the positive discrete series D^k of $SO^\uparrow(1,2)$–representations, which is labeled by a parameter $k \in \mathbb{R}^+$.

These considerations are supplementary to those in Ref. [4], where the group theoretical

quantization of $\mathcal{S}$ has been carried out already. Other recent related work is Ref. [5, 6, 7], where the quantization of a system is discussed the reduced phase space of which (or, rather, its regular part) turns out to be the four–fold copy of our phase space $\mathcal{S}$.

By definition, $\mathcal{S}$ is the restriction of the cotangent bundle T^*S^1 to positive momenta. The above mentioned projection method may therefore be used as an alternative (and simple) route to the quantization of $\mathcal{S}$. Equivalence of this quantization with the group theoretical one restricts the parameter k to the interval $0 < k \leq 1$ (due to a maximality condition in the projection method). This restricted range for k coincides also with what one expects on general grounds [1] for a phase space with fundamental group $\pi_1 = \mathbb{Z}$ (θ-angle). In the group theoretical quantization, however, all positive values of k, labeling the inequivalent $so(1,2)$–representations of the positive discrete series, come out on an equal footing. (Here, k can be restricted to the interval $0 < k \leq 1$ by regarding representations with $k > 1$ as "unphysical" — it is not unusual that not all possible unitary IRREPs are physically acceptable and thus taken into account. However, here this exclusion cannot be done "intrinsically" such as, e.g., in terms of an operator condition.) On the other hand, relaxing the maximality condition in the projection method, all the values of $k \in \mathbb{R}^+$ can be realized also there. This is seen to lead to an apparently novel realization of the positive discrete series in terms of functions over S^1.

In the context of the projection method, our phase space can also be viewed as a toy model for imposing similar constraints, e.g., the constraint $\det e > 0$ in a dreibein formulation of general relativity. This analogy, and some of its limitations, are discussed briefly in Subsec. 2.5.

A further application of our considerations to a gravitational problem can be found in Ref. [4], where it is shown that a suitable periodic identification of the reduced phase space of Schwarzschild black holes in an arbitrary spacetime dimension yields the phase space $\mathcal{S}$.

2   General remarks

In this section we briefly recapitulate the group theoretical quantization scheme by means of some elementary systems. To reobtain the standard results for the quantization of T^*S^1, we will find it necessary to reconsider the generating principle: Already this simple example illustrates the necessity to require that *any* function on phase space can be generated (globally) by means of the fundamental observables obtained from the momentum map of the group action ("strong generating principle"). As shown in Subsec. 2.3, in many cases (such as, e.g., when the group $\mathcal{G}$ under discussion is semisimple) it turns out to be sufficient to simply consider the center of the group $\mathcal{G}$ (or the center of a group G closely related to $\mathcal{G}$, cf Lemma 1 below) as to the effect of excluding a candidate $\mathcal{G}$–action (instead of explicitly checking the strong generating principle for the respective set of fundamental observables).

In Subsec. 2.4 we collect some of the remarks of Ref. [2] on group actions on general cotangent bundles (related to lifts of the diffeomorphism group of the base manifold). The situation will be found to simplify considerably when certain subbundles are considered,

which, in the two–dimensional case, are nothing but $T^*\mathbb{R}^+$, $\mathcal{S}$, or disjoint unions of these two. This leads to a general strategy of finding admissible group actions on such subbundles, which is then subsequently illustrated for both of the two–dimensional cases.

Finally, a projection method is introduced in Subsec. 2.5 which is also applicable to these two–dimensional examples. For higher–dimensional phase spaces it is not applicable to the subbundles considered in Subsec. 2.4, while, on the other hand, its range of applicability is much wider.

2.1 Review of the group theoretical quantization scheme

The Heisenberg commutation relations

$$[q^i, p_j] = i\hbar \delta^i_j \qquad \text{(all other commutators vanishing)} \tag{1}$$

are at the heart of many introductory textbooks on quantum mechanics. Mathematically they are, however, not an adequate *starting point* for quantization. Firstly, these relations can certainly be valid only on a dense subspace of the full Hilbert space. But even worse, there exist many inequivalent (and unphysical) representations of these relations on a dense subspace (cf Ref. [8], p. 88 for a simple example), which in part is connected to the fact that the commutation relations take into account only local information about the phase space (cf the example in Ref. [2], p. 1131). The "exponentiated" Heisenberg relations, defining the Weyl algebra, on the other hand, *are* a good starting point for an algebraic approach to quantization. With $U(a) := \exp(-ia^j p_j)$ and $V(b) := \exp(-ib_j q^j)$, where $a, b \in \mathbb{R}^n$ (up to respective units), the Weyl algebra has the form

$$U(a)U(a') = U(a + a'), \; V(b)V(b') = V(b + b'), \; U(a)V(b) = V(b)U(a)e^{i\hbar a^j b_j} \,. \tag{2}$$

It is a mathematical fact that (for finite n and for fixed $\hbar$) the irreducible, strongly continuous representations of the Weyl algebra are *unique* (up to unitary equivalence) and equivalent to the standard representation of quantum mechanics in a Hilbert space $L^2(\mathbb{R}^n, d^n x)$ with q^i being the multiplication operator x^i and $p_i = -i\hbar d/dx^i$. (Cf e.g. Refs. [9, 8, 2] for further details).

If the configuration space is no more an $\mathbb{R}^n$ or the phase space even no more a cotangent bundle, quantization is no more that unique and different generalizations or alterations of the above approach come into question. E.g. on a configuration space T^n (n-torus) the relations (2) are required to hold for $b_i \in \mathbb{Z}$ only and the space of unitarily inequivalent representations becomes $U(1)^n$ (corresponding to the different possibilities of (mutually commuting, cf Ref. [10]) self–adjoint extensions of the operators $p_i = -i\hbar d/dx^i$ on $[0, 2\pi]^n$). And on a configuration space $\mathbb{R}^+$ it is even no more adequate to consider unitary representations of $\exp(i\lambda p)$ (among the other elements in the Weyl algebra); the hermitean operator $p = -i\hbar d/dx$ has no self–adjoint extensions on $\mathbb{R}^+$, $\exp(i\lambda p)$ is a translation operator that does not map $\mathbb{R}^+$ into itself for all values of λ.

Geometric quantization is one of the most prominent attempts for a quantization procedure applicable to more or less arbitrary phase spaces (cf, e.g., Ref. [1] for an introduction).

Another method is the group theoretical quantization, which is inspired in part by geometric quantization as well as by work of Mackey [11]; cf Ref. [2] for a review. Since this approach requires the phase space to be some coset space, it has the drawback that even in some cases of finite dimensional phase spaces one may be forced to use infinite dimensional groups and their representation theory. However, in many finite dimensional examples of physical interest (cf e.g. Ref. [2]), including the phase space $\mathcal{S}$ studied in detail in Sec. 3, this is not the case.

In the context of a standard configuration space $\mathbb{R}^n$, the group theoretical approach arises as follows: Reinterpreting the phase factor in the Weyl algebra (2) as a central element, the relations (2) may be understood as the multiplication law for a $2n+1$–dimensional Lie group. Its Lie algebra is given by the Heisenberg relations (1) with generators q^i, p_j, and 1, where the need for the latter generator results from the right-hand side of the commutators, 1 denoting a central element of the full Lie algebra. The study of the irreducible unitary representations of this group (or its universal covering group, the Heisenberg group) yields our standard quantum theory.

Let us analyse this situation more carefully so that it allows a generalization to more general phase spaces: The operators U and V in Eq. (2) are translation operators in the configuration space and momentum space, respectively. *Each* of these n–dimensional translation groups has an analogue in the classical phase space $\mathcal{P} \equiv T^* \mathbb{R}^n$, $q \to q + a$ and $p \to p + b$. Put together, these two transformation groups form the $2n$–dimensional *abelian* group[1] $\mathcal{G} = (\mathbb{R}^{2n}, +)$, which acts transitively and effectively on $\mathcal{P}$ and leaves the symplectic form $\omega = dq^i \wedge dp_i$ invariant, i.e. it is a group of canonical transformations. (*Transitivity* means that for any two points in $\mathcal{P}$ there is a group element such that its application to one of the points yields the other one, *effectiveness* implies that only the identity of $\mathcal{G}$ acts trivially on $\mathcal{P}$). The action of $\mathcal{G}$ is Hamiltonian, moreover, i.e. there exist (globally defined) functions F_A on $\mathcal{P}$, such that the vector fields $V_A \equiv \{\cdot, F_A\}$ generate the group action (the invariance of ω guarantees only the *local* existence of functions F_A); in the present case with group $(\mathbb{R}^{2n}, +)$ the respective Hamiltonians (or observables) are $q^i, p_i, i = 1, \ldots, n$ (up to an addition of constants, which drop out from the generating vector fields V_A).

In the general case of an (effective) action of a group $\mathcal{G}$ on a phase space $\mathcal{P}$, the Lie bracket of the generating vector fields V_A always mimics the Lie algebra $\mathcal{L}(\mathcal{G})$ of the group $\mathcal{G}$: $[V_A, V_B] = f_{AB}^C V_C$, where f_{AB}^C are the structure constants of $\mathcal{L}(\mathcal{G})$ (in the above case of an abelian group G, $f_{AB}^C \equiv 0$). If the action is Hamiltonian, one may conclude from this in general only that $\{F_A, F_B\} = -f_{AB}^C F_C + \kappa(A, B)$, where κ is a constant on $\mathcal{P}$. As a function on the Lie algebra, κ is a two–cocycle (as a consequence of the Jacobi identity for the Poisson bracket), which changes by a two–coboundary upon redefining the functions F_A by a constant; thus $\kappa \in H^2(\mathcal{L}(\mathcal{G}), \mathbb{R})$ (cf e.g. Ref. [1] for more details on this and related aspects). In many cases the constants κ can be made to vanish upon an appropriate choice of functions F_A in which case the Hamiltonian action is said to allow a *momentum map*. E.g., this is the case when the group $\mathcal{G}$ is semisimple, because then $H^2(\mathcal{L}(\mathcal{G}), \mathbb{R})$ is

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[1]This is in contrast to putting together the operators U and V, as seen by the last relation in Eq. (2); we will shortly come back to this difference.

trivial. However, also any Hamiltonian action of a Lie group on a *compact* phase space has a momentum map (independently of the second cohomology of the respective group) or, similarly, any subgroup of the lift of the diffeomorphism group of an arbitrary configuration space.

It is obvious from the Poisson brackets $\{q^i, p_j\} = \delta^i_j$ that the action of $(\mathbb{R}^{2n}, +)$ on $T^*\mathbb{R}^n$ does not allow a momentum map (clearly the righthand side of these relations cannot be removed by shifting q^i and p_j by constants). Although $T^*\mathbb{R}^n$ is the simplest choice of a phase space, from the point of view of group theoretical quantization it is rather an involved example (due the absence of a momentum map). Instead of $\mathcal{G} = (\mathbb{R}^{2n}, +)$ one is then lead to focus on a central extension $\mathcal{E}$ of this group, the Lie algebra of which may be spanned by F_A and a central element 1, with the Lie bracket provided by the Poisson bracket between the corresponding functions on $\mathcal{P}$. The unique simply connected choice for this group $\mathcal{E}$ is the Heisenberg group. Note that $(\mathbb{R}^{2n}, +)$ is *not* a subgroup of $\mathcal{E}$, any abelian subgroup having at most dimension $n + 1$; only the factor group $\mathcal{E}/\mathbb{R}$ with respect to the central subgroup $\mathcal{N} = \mathbb{R}$ yields $(\mathbb{R}^{2n}, +)$. Also, in contrast to the latter group, $\mathcal{E}$ does not act effectively on $\mathcal{P}$ anymore (as $\mathcal{N}$ acts trivially on $\mathcal{P}$), while it certainly still is transitive.

There is a one–parameter family of weakly continuous unitary[2] IRREPs of the Heisenberg group $\mathcal{E}$. This parameter stems from the unitary representation of the central subgroup $\mathcal{N} = (\mathbb{R}, +)$. Following Isham, the freedom in this parameter is fixed by nature's value of $\hbar$. This brings us back to the first paragraph of this section with its unique quantum theory.

Preliminarily, we state the following general strategy in the group theoretical approach to a quantum theory for a given phase space $\mathcal{P}$ (cf Ref. [2] for further motivation and details): First find a Hamilonian, transitive, and almost effective[3] action of a group $\mathcal{G}$ on $\mathcal{P}$. *If* this action allows a momentum map, the next and final step is to study the weakly continuous, unitary IRREPs of $\mathcal{G}$ (discarding possibly physically unacceptable representations). If, on the other hand, there is no momentum map for the action of $\mathcal{G}$, one again considers the one–parameter central extension $\mathcal{E}$ of $\mathcal{G}$ and then studies the weakly continuous, unitary IRREPs of $\mathcal{E}$.

In general there may be different admissible groups acting on the phase space and each of these groups may have different, inequivalent actions. Moreover, for any group there may be various admissible unitary representations. Some of the latter may be excluded upon physical considerations (such as, e.g., by positivity of a classically positive Hamiltonian), the possible ambiguity in the remaining IRREPs being interpreted as part of the ambiguity in the transition from a classical system to its quantum version.

Note that clearly *any* function on $T^*\mathbb{R}^n$ is a function of the elementary observables q^i and p_i. As will be found below, a similar requirement on the fundamental observables

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[2] As a consequence of unitarity, weak continuity implies strong continuity. Alternatively, we may require strong continuity and then find, although only for specific cases such as for the Heisenberg group, that all the representations are unitary.

[3] I.e., there is only a *discrete* set of elements of $\mathcal{G}$ (which necessarily is an invariant subgroup) acting trivially on all points of $\mathcal{P}$.

F_A of the group $\mathcal{G}$ *has* to be asked for also in the general case, leading to an *additional* constraint on the admissibility of group actions. This will be taken up in the following two subsections, after illustrating the above considerations by means of the elementary systems T^*S^1 and $T^*\mathbb{R}^+$. Thereafter we will add some remarks on the quantization of general cotangent bundles T^*Q and subbundles including our example system $\mathcal{S}$.

2.2   Application to $T^*\mathbb{R}^+$ and T^*S^1

To illustrate the quantization scheme reviewed in the previous subsection, we present the examples $T^*\mathbb{R}^+$ and T^*S^1. The second of these examples will lead us to discuss the issue of generating phase space functions in more detail.

As T^*S^1 and $T^*\mathbb{R}^+$ may be quantized by various established quantization schemes, we present the standard results (from several perspectives) first before turning to their group theoretical quantization.

2.2.1   Standard results

As remarked already in the preceding subsection, there is a one–parameter family of differ-ent, but physically acceptable quantum theories of T^*S^1. This parameter may be viewed as a consequence of the multiple connectedness of the phase space (cf, e.g., the general statements on the quantization of multiply connected phase spaces in geometric quantiza-tion in Ref. [1] and our discussion below; cf also Ref. [10]). The resulting Hilbert space may be spanned by the wave functions $\exp i(n + \theta)\varphi$, $n \in \mathbb{Z}$, where φ is a coordinate on the interval $[0, 2\pi]$ and $\theta \in [0, 1]$ (where $\theta = 0$ is to be identified with $\theta = 1$) is the fixed parameter mentioned above. Thus the wave functions may be regarded as functions on the interval $[0, 2\pi]$ with quasi–periodic boundary conditions (having periodic probability den-sities). The momentum operator $p = (\hbar/i)\, d/d\varphi$ is self–adjoint and its spectrum obviously is of the form $\{\hbar(n+\theta), n \in \mathbb{Z}\}$. (The level spacing $\hbar$ of the spectrum is fixed by the choice $[0, 2\pi]$ for the fundamental interval of the angle variable φ. In physical applications, it may possibly be rescaled depending on the realization of φ.) The spectra of p differ for different values of θ and thus the respective quantum theories cannot be unitarily equivalent. (Note also that although θ provides only an overall shift in the spectrum of p, already for a free Hamiltonian of the form $H = p^2/2$, energy differences *are* affected by that parameter.)

¿From the point of view of geometric quantization (cf, e.g., [1]) wave functions are sections in a line bundle over S^1 or better $T^*S^1 = S^1 \times \mathbb{R}$. This line bundle is necessarily trivial. There are, however, several inequivalent connections $\hbar^{-1}\Theta$ with the same curvature $\hbar^{-1}\omega$ (ω being the symplectic form on T^*S^1; its Chern class is trivial and so is the bun-dle). Up to gauge transformations Θ may be brought into the form $\Theta = p\,d\varphi - \theta\,d\varphi$ where $\theta \sim \theta + 1$ as a consequence of the $U(1)$ gauge transformations $\exp(-i\varphi/\hbar)$. The difference between two connections with fixed curvature is a flat connection; up to gauge transfor-mations this difference is an element of $H^1(M, \mathbb{R})/H^1(M, \mathbb{Z}) \sim H^1(M, U(1))$ (M being the phase space under consideration, here $M = T^*S^1$), different choices correspond to different parallel transporters around nontrivial loops (so the ambiguity may be associated also to

elements of $Hom(\pi_1(M), U(1))$. In the above trivialization of the bundle, (polarized) wave functions correspond to ordinary functions over S^1 and the θ–angle enters in the mometum operator $\hat{p}$: $\hat{p} = -i\hbar\nabla_{X_p} + p = -i\hbar\frac{d}{d\varphi} + \theta$ (here $X_p = \frac{d}{d\varphi}$ is the Hamiltonian vector field corresponding to p and ∇ denotes the covariant derivative). We may, however, also use a nontrivial transition function at $\varphi = 0$ to remove θ in the expression for the momentum operator, transferring it simultaneoulsy into the wave functions (so that effectively they are quasiperiodic as above).

The quantum system corresponding to $T^*\mathbb{R}^+$, on the other hand, may be traced back to the one for $T^*\mathbb{R}$: Let $q > 0$ and $p \in \mathbb{R}$ parameterize $(T^*\mathbb{R}^+, dq \wedge dp)$. Then in the new chart $(\widetilde{q} := \ln q, \widetilde{p} := q\,p)$ this symplectic manifold *becomes* just $(T^*\mathbb{R}, d\widetilde{q} \wedge d\widetilde{p})$. The quantization of the latter is standard, yielding wave functions $\widetilde{\psi}(\widetilde{q})$ with measure $\int d\widetilde{q}$ if simultaneously $\widetilde{p} = (\hbar/i)\,d/d\widetilde{q}$.

The resulting quantum system may now be represented also in the original coordinates $q = \exp(\widetilde{q})$: For the wave functions $\psi(q) = \widetilde{\psi}(\ln q)$ the measure in the inner product becomes $\int_{\mathbb{R}^+} dq/q$ and $\widetilde{p} = (\hbar/i)\,q\,d/dq$. Note that, as a consequence of the nontrivial measure in q, $p = (\hbar/i)\,\sqrt{q}\,(d/dq)(1/\sqrt{q})$.

We can also get rid of the nontrivial measure by rescaling the wavefunctions: $\widehat{\psi}(q) := \psi(q)/\sqrt{q}$. Then the measure becomes $\int_{\mathbb{R}^+} dq$, while $p = (\hbar/i)\,d/dq$ and now $\widetilde{p}$ is seen to become $\widetilde{p} = (qp + pq)/2$.

We remark that while $\widetilde{p}$ is (in the above manner by construction) a self–adjoint operator, p is only hermitean, having no self–adjoint extensions (cf Ref. [8] and Ref. [2] for further details on p).

2.2.2 $T^*\mathbb{R}^+$

We next turn to the group theoretical quantization of $T^*\mathbb{R}^+$. As remarked already above, from the point of view of symplectic manifolds $T^*\mathbb{R}^+ = T^*\mathbb{R}$. The *observables* $q > 0$ and p on $T^*\mathbb{R}^+$, however, are certainly different from the observables q and p on $T^*\mathbb{R}$, which, for means of clarity, we again denote by $\widetilde{q}$ and $\widetilde{p}$ as in Subsec. 2.2.1 above. The precise correspondence between these observables (viewed as observables on *one and the same* phase space) has been provided already there, too.

In the group theoretical approach there are thus at least two admissible groups which may be used to quantize $T^*\mathbb{R}^+$ (or, likewise, to quantize $T^*\mathbb{R}$). First, we may just take the abelian group generated by the Hamiltonian vector fields corresponding to $\widetilde{q} \equiv \ln q$ and $\widetilde{p} \equiv qp$. In one–to–one correspondence with the quantization of $T^*\mathbb{R}$, this action on $T^*\mathbb{R}^+$ has no momentum map, and the canonical group $\mathcal{C}$ becomes the three–dimensional Heisenberg group. As is evident from the discussion of $T^*\mathbb{R}^+$ in the preceding Subsec. 2.2.1, in this way the correct quantum theory of $T^*\mathbb{R}^+$ is reproduced. It is identical to the quantum theory of $T^*\mathbb{R}$; one just has to take into account the nontrivial correspondence of observables.

Second, in the framework of group theoretical quantization, we may also use the group generated by q and $\widetilde{p} \equiv qp$. This is easily seen to provide an effective and transitive action on $T^*\mathbb{R}^+$ of the two–dimensional, nonabelian affine group $\mathcal{G} = \mathbb{R} \rtimes \mathbb{R}^+$. Since it obviously

has a momentum map, for the quantization of $T^*\mathbb{R}^+$ (or, likewise, also of $T^*\mathbb{R}$!) one may study, as an alternative to the Heisenberg group $\mathcal{C}$, the unitary IRREPs of the affine group $\mathcal{G}$.

There are *three* unitarily inequivalent IRREPs of $\mathcal{G}$ (again here we refer to Ref. [2] for further details). In one of them, the operator q has a strictly negative spectrum; clearly, this representation has to be excluded on physical grounds, as classically q is strictly positive. Furthermore, one of the representations uses a *one*–dimensional Hilbert space and thus does not come into question as quantum theory of $T^*\mathbb{R}^+$, too. The single remaining representation has the Hilbert space $L^2(\mathbb{R}^+, dq/q)$ with the generator $\widetilde{p}$ being represented by $(\hbar/i)\, q\, (d/dq)$. This is in coincidence with what we found above (cf Subsec. 2.2.1). In this case the parameter $\hbar$ enters on reasons of correct physical dimensions: $\widetilde{p}$ has the dimension of an action and the Poisson bracket relation $\{q, \widetilde{p}\} = q$ thus has to turn into the commutator $[q, \widetilde{p}] = i\hbar q$ in the quantum theory.

Certainly $\{\cdot, q\} = -d/dp$ and $\{\cdot, p\} = d/dq$ do *not* generate a group on $T^*\mathbb{R}^+$; d/dq generates translations of q, which may leave the positive real axis. This fits well to the previous observation that p cannot become a self–adjoint operator.

2.2.3 $\quad T^*S^1$

In the group theoretical approach to quantizing T^*S^1 one first looks for a transitive, almost effective Hamiltonian action of a group $\mathcal{G}$ on that space. Such a group is provided by the three–dimensional Euclidean group (in two dimensions) $E_2 = \mathbb{R}^2 \rtimes SO(2)$. If $\varphi \in [0, 2\pi]$ denotes the configuration space variable on S^1 and p its conjugate momentum, Hamiltonian generators of this action are provided by $\{\cdot, p\}$, generating rotations along the S^1, as well as by $\{\cdot, \sin\varphi\}$ and $\{\cdot, \cos\varphi\}$, which generate transformations along the fibers $\varphi = \text{const.}$ Since the Poisson brackets between the respective Hamiltonian functions clearly close, the action has a momentum map. The action of E_2 is easily seen to be effective and transitive on T^*S^1, moreover. The representation theory of E_2 shows that there is a one–parameter family of unitary IRREPs (cf Ref. [2] for details). The corresponding parameter $\lambda \in \mathbb{R}^+$ is, however, *not* the θ–angle, as we might have expected from our previous consideration of this example. Instead, upon working with dimensionful quantities, it may be seen that this parameter has to be identified with $\hbar$ again.

In the present quantization scheme the θ–angle arises only when considering *another* group action on T^*S^1. Clearly $\pi_1(E_2) = \pi_1(SO(2)) = \mathbb{Z}$. Thus instead of E_2 we may consider as well the action of its universal covering group, $\widetilde{E}_2$. This action is no more effective, but still almost effective[4] (and the Lie algebra isomorphism between the Poisson algebra of the generating Hamiltonians and the elements of $\mathcal{L}(\mathcal{G})$ is certainly not affected by this change of $\mathcal{G}$). The unravelling of the subgroup $SO(2)$ of E_2 to $\mathbb{R} \subset \widetilde{E}_2$ leads to an additional continuous parameter in the unitary representations. This parameter lives on a circle and may be identified readily with the angle θ. So, when using $\mathcal{G} = \widetilde{E}_2$, the

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[4]The elements which act trivially on T^*S^1 are then just the center $\mathbb{Z}$ of $\widetilde{E}_2$ in the kernel of the projection from $\widetilde{E}_2$ to E_2.

group theoretical quantization scheme reproduces the general results of other established approaches. (The representations obtained from the choice $\mathcal{G} = E_2$ correspond to the special, but still legitimate, quantum realization with periodic wave functions, $\theta = 0$, on the other hand).

2.2.4 Summary

The two examples discussed above nicely illustrate that there may be several admissible group actions on one and the same phase space $\mathcal{P}$ (which will still be the case also after imposing our additional condition on admissible group actions below).

As any covering group $\widehat{G}$ of an (almost) effectively acting group G acts almost effectively on the phase space, too, and unitary representations of G are also unitary representations of $\widehat{G}$ (but not necessarily vice versa), we will always choose the (unique) simply connected universal covering group $\widetilde{G}$ as the group $\mathcal{G}$.

We learn from the group theoretical quantization of T^*S^1 that only then we may expect to obtain the most general quantum realization of the theory with classical phase space $\mathcal{P}$.

The example $T^*\mathbb{R}^+$ (or $T^*\mathbb{R}$) demonstrates that different admissible groups need not be just coverings of one another. Moreover, this example illustrates that not all weakly continuous, unitary IRREPs of $\mathcal{G}$ need to make sense physically. In part this was concluded from a comparison of the range of values of a physically important classical observable (namely q) with its quantum spectrum.

The situation in quantizing the phase space $\mathcal{S} = S^1 \times \mathbb{R}^+$, discussed in detail in Sec. 3, will be quite analogous to the one in quantizing T^*S^1. The allowed effectively acting group will be $SO^\uparrow(1,2)$. Only by studying the IRREPs of the respective universal covering group a θ–angle, to be expected due to $\pi_1(\mathcal{S}) = \mathbb{Z}$, will be obtained. In analogy to the example $T^*\mathbb{R}^+$, on the other hand, not all unitary IRREPs will be seen to make sense "physically" as quantum realizations of $\mathcal{S}$.

2.2.5 Other group actions on T^*S^1

Up to now the discussion was in agreement with Ref. [2]. However, in the example of T^*S^1 there are much more group actions which fulfill the conditions of transitivity, effectiveness, and of being Hamiltonian with momentum map: The Lie algebra of E_2 is not only provided by the Hamiltonian generators $\{\cdot, p\}$, $\{\cdot, \sin\varphi\}$, and $\{\cdot, \cos\varphi\}$ on T^*S^1, but also by the countably infinite family $\{\cdot, l^{-1}p\}$, $\{\cdot, \sin l\varphi\}$, and $\{\cdot, \cos l\varphi\}$, $l \in \mathbb{N}$.[5] For fixed l these vector fields generate an *effective* action of the *l–fold covering* group of E_2: The vector field $l^{-1}\{\cdot, p\}$ generates the translations $\varphi \mapsto \varphi + tl^{-1}$, $t \in \mathbb{R}$, which is the identity transformation for $t = 2\pi l$, but not already for $t = 2\pi j$, $j < l$.

If we repeat the quantization described in Subsection 2.2.3 for $l \neq 1$, we have to use the same representation theory because in any case we use the universal covering group $\widetilde{E}_2$. However, now we have $l^{-1}p$ in place of p, and *this* phase space function is quantized to

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[5]This example, for which we are grateful to H. Kastrup, provides a simplified version of what will be found for the phase space $\mathcal{S}$ by the systematic procedure employed in Sec. 3 (cf Eqs. (10) and (7) below).

the same operator as p above with discrete spectrum $\hbar(\mathbb{Z} + \theta)$. Thus p will be quantized to an operator with spectrum $\hbar l(\mathbb{Z} + \theta)$. Note that the interval $\varphi \in [0, 2\pi]$ has not changed and, therefore, the obtained spectrum is not acceptable. A rescaling of $\hbar$ to absorb l, furthermore, is not possible because locally we have to preserve canonical conjugacy of p and φ.

We are thus in the need of excluding the group actions on T^*S^1 with $l \neq 1$! To extract a general strategy from this example, we now will focus on the question of what kind of phase space functions may be generated by the fundamental observables of the group action.

2.3   Generation by fundamental observables

In Ref. [2] two different principles for what phase space functions can be generated by the fundamental observables $F_1, \ldots, F_n \in C^\infty(\mathcal{P}, \mathbb{R})$ generating the group action on the phase space were presented:

Strong Generating Principle (SGP): *For any phase space function $f \in C^\infty(\mathcal{P}, \mathbb{R})$ there is a function $\Phi_f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that $f = \Phi_f(F_1, \ldots, F_n)$.*

Local Generating Principle (LGP): *Any $s \in \mathcal{P}$ has a neighborhood $\mathcal{U}_s \subset \mathcal{P}$ such that the condition of the SGP is met on $\mathcal{U}_s$.*

As noted in Ref. [2], the LGP is fulfilled if the group action is transitive. However, transitivity is not sufficient for SGP. E.g., in the example of T^*S^1 above we found an infinite family of transitive group actions parameterized by the label l. The SGP is fulfilled only for $l = 1$: For $l > 1$ the functions $\sin l\varphi$ and $\cos l\varphi$ are not sufficient to generate an arbitrary (smooth) function on the interval $0 \leq \varphi < 2\pi$, because any generated function is $2\pi l^{-1}$–periodic (globally we cannot take the l–th root). Thus, demanding SGP singles out the only group action which reproduces the results of standard quantizations in this example.

In Ref. [2] only the need for the LGP was recognized, and incorporated by means of transitivity of the group action. (Consequently it then was concluded [2], p. 1149: "... in this group theory oriented quantization scheme, we cannot always maintain the strong generating principle.") As the above example shows, however, the validity of the SGP is an essential part of group theoretical quantization and must not be ignored.

The SGP is a condition on the Hamiltonians of a given group action. For practical applications it may be worthwhile to reformulate it in terms of a property of the group action (analogously to trading in transitivity of the action for the LGP) or even the canonical group $\mathcal{G}$ itself. We did not succeed in this attempt in full generality. However, we will now present a *necessary* condition for the validity of the SGP for a rather large class of group actions.

Let $\mathcal{G}$ have an almost effective, transitive Hamiltonian action on $\mathcal{P}$ and let us, for the above purpose, assume that this action admits a momentum map (and thus there is no need for a central extension). An almost effective action can always be reduced to an effective action by factoring out a discrete subgroup: If $\mathcal{G}$ acts almost effectively, then $G := \mathcal{G}/N$, where N is the maximal invariant subgroup of $\mathcal{G}$ acting trivially on $\mathcal{P}$, acts

effectively (with all other properties of the action unchanged). The necessary condition mentioned above may now be formulated as a condition on the remaining center $Z(G)$ of G.

Lemma 1 *Let $\mathcal{G}$ be a group acting almost effectively, transitively, and Hamiltonian with a momentum map on the phase space $\mathcal{P}$ and G be the corresponding effectively acting group. If $\mathcal{G}$ is semisimple and the center $Z(G)$ of G is nontrivial, then the strong generating principle is violated. It is also violated (for a general group $\mathcal{G}$), if $Z(G)$ is nontrivial but finite.*

Proof: Let $s \in \mathcal{P}$, $g \in G$, and denote the group action of g by $L_g : s \mapsto gs$. By means of this action to each $X \in \mathcal{L}G$ a vector field $\widetilde{X}$ on $\mathcal{P}$ is associated, whose flow we denote as $\exp tX := \Phi_t(\widetilde{X})$. Its pushforward with L_g acting on a function f on $\mathcal{P}$ is

$$L_{g*}\widetilde{X}(f) = \left.\frac{d}{dt}\right|_{t=0} f \circ L_g \circ \exp(tX) = \left.\frac{d}{dt}\right|_{t=0} f \circ \exp(tAd_gX) \circ L_g. \tag{3}$$

If $\widetilde{X} = \widetilde{X}_H = \{\cdot, H\}$ is a Hamiltonian vector field, then we have, furthermore, $L_{g*}\widetilde{X}_H = \widetilde{X}_{H \circ L_g^{-1}}$ because the group action is Hamiltonian. For $g = z \in Z(G)$ in the center of G we have $Ad_gX = X$ and these equations imply $\widetilde{X}_H = \widetilde{X}_{H \circ L_z^{-1}}$. The generating function H thus has to fulfill

$$H \circ L_z^{-1} = H + c_z(H) \quad \text{for any } z \in Z(G). \tag{4}$$

Here $c_z : \mathcal{L}G \to \mathbb{R}$ is a linear map from the Lie algebra of G, which we identify using the momentum map with its isomorphic Lie algebra of generating functions of the group action on $\mathcal{P}$, to $\mathbb{R}$. This map is in fact a 1–cocycle in the cohomology of this Lie algebra: $\{H \circ L_z^{-1}, G \circ L_z^{-1}\} = \{H, G\} \circ L_z^{-1}$ implies $\{H + c_z(H), G + c_z(G)\} = \{H, G\} = \{H, G\} + c_z(\{H, G\})$ which leads to $c_z(\{H, G\}) = 0$. This observation already proves our first assertion: If G is semisimple, we have $[\mathcal{L}G, \mathcal{L}G] = \mathcal{L}G$ and $c_z(\mathcal{L}G) = c_z([\mathcal{L}G, \mathcal{L}G]) = 0$; c_z vanishes for any $z \in Z(G)$. This means that each of the generating functions, and therefore any generated function, is invariant with respect to the action of $Z(G)$. But the center of G acts nontrivially, because the group action of G is effective, and not any phase space function, which is in general not invariant, can be generated.

For groups with $[\mathcal{L}G, \mathcal{L}G] \neq \mathcal{L}G$ (nonperfect groups, cf the remark following this proof) the above argument cannot be used. However, if $Z(G)$ is finite, there is for each $z \in Z(G)$ a $k \in \mathbb{N}$ with $z^k = 1$. Due to $H = H \circ L_{z^k}^{-1} = H + kc_z(H)$ (This follows from Eq. (4) and $c_z(H \circ L_{z'}^{-1}) = c_z(H)$ for all $z, z' \in Z(G)$, which in turn is a consequence of $c_z(H+c) = c_z(H)$ for any constant function c on the phase space.) we again have $c_z(H) = 0$ for any $z \in Z(G)$ and $H \in \mathcal{L}G$. $\qquad\square$

In the above lemma, we could also relax the conditions replacing "semisimple" by "perfect".[6] The defining property of a perfect Lie group $\mathcal{G}$ is $[\mathcal{L}\mathcal{G}, \mathcal{L}\mathcal{G}] = \mathcal{L}\mathcal{G}$. A prominent example for a nonsemisimple but perfect Lie group is the Poincaré group.

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[6] We are grateful to D. Giulini for this remark.

Note that the center of semisimple Lie groups is discrete, while for perfect Lie groups per se this is not necessarily the case. However, in the present context a continuous center $Z(\mathcal{G})$ is excluded in any case due to the (almost) effectiveness of the $\mathcal{G}$–action and the existence of a momentum map: The phase space function generating the action of the center would have vanishing Poisson brackets with all other generating functions. Therefore, it would be constant, and the center would act trivially.

Thus, the only case of a nontrivial center not covered by the lemma is that of a discrete but infinite center of a nonperfect group.

A simple example for this case where, however, the SGP is still violated, may be provided on $T^*\mathbb{R}$. Such an action on $T^*\mathbb{R}$ fulfilling Isham's axioms can be constructed as a limit $l \to \infty$ of the action of the l–fold covering group of E_2 on T^*S^1: After the symplectic transformation $(\varphi, p) \mapsto (l\varphi, l^{-1}p)$ we can take the limit $l \to \infty$ for the action of the l–fold covering group of E_2. The generating functions p, $\sin \varphi$, and $\cos \varphi$ are now l–independent, but the l–fold covering group acts on a phase space with φ–interval $0 \leq \varphi < 2\pi l$. For $l \to \infty$ this phase space unwinds to $T^*\mathbb{R}$ and the action becomes an effective and transitive action of $\widetilde{E}_2$, which is neither perfect nor has finite center. The lemma does not apply, but nevertheless the group action has to be rejected because only 2π–periodic functions can be generated.

This example shows that the lemma is not sufficient to decide in all cases whether a group action is allowed, and it demonstrates even more drastically the necessity of the SGP: Trusting this group action of $\widetilde{E}_2$ would lead us to a discrete spectrum for p in a quantization of $T^*\mathbb{R}$! (This discreteness comes in because the fundamental observables are periodic, which is a global property and cannot be detected by the LGP. A further failure of this group action is that the coordinate q in $T^*\mathbb{R}$ could not be promoted to an operator, because it cannot be generated by the fundamental observables.)

Note that the lemma does not provide any statement about the validity or failure of the SGP for the case that $Z(G)$ is trivial. We are, however, not aware of an example with trivial $Z(G)$ where the SGP is violated.

In the paragraph preceding the lemma we made use of the fact that a trivially acting subgroup of $\mathcal{G}$ can always be factored out to arrive at the effectively acting group G. If the center of the latter group, $Z(G)$, is nontrivial, it can be factored out only at the cost of factoring the phase space, too. This does not change its dimensionality due to discreteness of the center. (To do so, we have to suppose that the action on the phase space of the center is properly discontinuous, which is, e.g., fulfilled if the center is finite.) If the action of $\mathcal{G}$ on this factored phase space is still Hamiltonian, the conclusion of the lemma can be evaded by regarding $\mathcal{G}$ as canonical group for this smaller phase space $\mathcal{P}' \equiv \mathcal{P}/Z(G)$ (i.e. although the SGP is violated on $\mathcal{P}$ it is not necessarily so on $\mathcal{P}'$).

In the light of this consideration we can understand the wrong p–spectrum obtained when using the action of the l–fold ($l > 1$) covering group of E_2 on T^*S^1. In this case the center is the cyclic group of order l generated by the translation in $\varphi \in [0, 2\pi]$ by $2\pi l^{-1}$. If we want to factor out the center, we have to identify the points φ and $\varphi + 2\pi l^{-1}$ to obtain an action of E_2 (or an almost effective action of the l–fold covering). This identification

effects a reduction of the configuration space to the interval $[0, 2\pi l^{-1}]$, which explains the multiplication of the p–spectrum by l.

2.4   Quantizing cotangent bundles and certain subbundles

We proceed with some general remarks [2] on the group theoretical approach when applied to phase spaces which are cotangent bundles, $\mathcal{P} = T^*Q$. As discussed in the next section, the phase space $\mathcal{S}$, on the other hand, is definitely not a cotangent bundle. However, it will turn out to be a certain subbundle of a cotangent bundle (specified below). Many of the facts applicable to cotangent bundles will be seen to be applicable to those subbundles, too. In a sense, the situation even simplifies there.

2.4.1   A general strategy for determining group actions

On T^*Q, the infinite dimensional group $\mathcal{D} := (C^\infty(Q, \mathbb{R})/\mathbb{R}) \rtimes \mathrm{Diff}(Q)$, which is a subgroup of the full group of canonical transformations, always acts transitively and effectively.

Here $\mathrm{Diff}(Q)$ is the canonical lift of the diffeomorphism group of the configuration space Q. If $q^i \to \tilde{q}^i(q)$ denotes the diffeomorphism on Q, this is lifted canonically to a symplectomorphism on T^*Q (a so–called "point transformation") when it is accompanied by $p_i \to p_j\, \partial q^j/\partial\tilde{q}^i$ (where $q(\tilde{q})$ denotes the inverse of the function $\tilde{q}(q)$). The action of $\mathrm{Diff}(Q)$ is also Hamiltonian and allows a momentum map: If $X^i(q)\, d/dq^i$ is the generating vector field of a diffeomorphism of Q (connected to the identity), then $X^i(q)\, p_i$ is a Hamiltonian of its canonical lift, and it is obvious that the Poisson algebra of these functions on T^*Q is closed without a central extension.

Although infinite–dimensional, $\mathrm{Diff}(Q)$ by itself does not act transitively on T^*Q, as the ($\dim(Q)$–dimensional) subspace $p_i = 0$ is mapped into itself. However, when enhanced by $C^\infty(Q, \mathbb{R})/\mathbb{R}$ ("diffeomorphisms up the fibers"), the action becomes transitive on T^*Q; here $C^\infty(Q, \mathbb{R})/\mathbb{R}$ consists of those canonical transformations that are generated by Hamiltonian vector fields of the form $\{\cdot, f(q)\}$, where $f \in C^\infty(Q, \mathbb{R})/\mathbb{R}$ ($\mathbb{R}$ corresponding to the constants that act trivially and which are thus removed so as to obtain an effective action).

Quantizing (finite–dimensional) cotangent bundles, one thus may look for *finite*–dimensional subgroups $\mathcal{G} = W \rtimes G$ of $\mathcal{D} \equiv (C^\infty(Q, \mathbb{R})/\mathbb{R}) \rtimes \mathrm{Diff}(Q)$ which still act transitively. As a subgroup of $\mathcal{D}$ this action is then guaranteed to be Hamiltonian and to act effectively. As seen above, moreover, separately, each of the groups $\mathrm{Diff}(Q)$ and $C^\infty(Q, \mathbb{R})/\mathbb{R}$ allows a momentum map (and thus this follows also for any of their subgroups G and W, respectively). However, the full (combined) group $\mathcal{D}$, and thus also $\mathcal{G} = W \rtimes G$, *may* have an obstruction for a momentum map (cf the example $Q = \mathbb{R}^n$ reexamined below).

2.4.2   The examples revisited

In the examples discussed above we always used subgroups of $\mathcal{D}$ (or their covering groups). For $T^*\mathbb{R} \sim T^*\mathbb{R}^+$ this was $G = \mathbb{R}, W = \mathbb{R}$ (which is more natural when viewing the phase space as $T^*\mathbb{R}$, the generating observables being q and p in the corresponding chart) *or*

$G = \mathbb{R}^+$, $W = \mathbb{R}$ (more natural when viewing the phase space as $T^*\mathbb{R}^+$, the generating observables being $q > 0$ and qp in this other chart). In the former case there is an obstruction to a momentum map and one is lead to the three–dimensinonal Heisenberg group (which is a subgroup of $C^\infty(Q, \mathbb{R}) \rtimes \mathrm{Diff}(Q)$), in the latter case there was no obstruction to a momentum map for $\mathcal{G} = W \rtimes G$. For T^*S^1, on the other hand, $W = \mathbb{R}^2$ and $G = SO(2)$ (the rotations along the S^1) or, better, the universal covering group of the latter, $G = \mathbb{R}$.

2.4.3 Subbundles

We noted above that the subspace

$$\mathcal{P}_0 = \{(p,q) \in T^*Q \,|\, p_i = 0 \quad \forall i = 1, \ldots, \dim(Q)\} \tag{5}$$

of T^*Q is left invariant by the action of $\mathrm{Diff}(Q)$. On the (connected components of the) complement $\mathcal{P}_*$ of $\mathcal{P}_0$ in T^*Q the action is, however, also transitive. More precisely, for $\dim(Q) = 1$ $\mathcal{P}_*$ has two connected components, which we will denote by $\mathcal{P}_+$ and $\mathcal{P}_-$ for $p > 0$ and $p < 0$, respectively. (The phase space $\mathcal{S}$ will be found to be of this type with $Q = S^1$ in the following section.) For $\dim(Q) > 1$, on the other hand, $\mathcal{P}_*$ is already connected and we have the following small lemma:

Lemma 2 *For $\dim(Q) > 1$ ($\dim (Q) = 1$) the canonical lift of $\mathrm{Diff}(Q)$ ($\mathrm{Diff}_+(Q)$, the component of $\mathrm{Diff}(Q)$ connected to the identity) has a transitive and effective action on (the connected components of) $\mathcal{P}_* = T^*Q \backslash \mathcal{P}_0$ with a momentum map.*

Proof: According to the invariance of $\mathcal{P}_0$ with respect to $\mathrm{Diff}(Q)$, the action of $\mathrm{Diff}(Q)$ does not lead out of the subbundle $\mathcal{P}_*$. For $\dim(Q) = 1$ each of the components of $\mathcal{P}_*$ is invariant only with respect to orientation preserving diffeomorphisms (and we thus restrict to $\mathrm{Diff}_+(Q)$ in this case).

The momentum map of the action has been provided already above, furthermore, and its effectiveness on $\mathcal{P}_*$ is obvious. Transitivity on $\mathcal{P}_*$ ($\mathcal{P}_\pm$ for $\dim(Q) = 1$) follows as $\mathrm{Diff}(Q)$ ($\mathrm{Diff}_+(Q)$) acts fiber transitively on $\mathcal{P}_* \subset T^*Q$ (i.e., it acts transitively on the space of fibers), while on the fiber of $\mathcal{P}_*$ ($\mathcal{P}_\pm$) over the origin $q^i = 0$ of some particular local coordinate system of Q the vector fields $\{\cdot, q^i p_j\}$ act transitively. $\qquad\square$

Note that when dealing with $\mathcal{P}_*$ ($\mathcal{P}_\pm$), it is not only not necessary to add the above group $C^\infty(Q, \mathbb{R})/\mathbb{R}$ (or *any* of its subgroups) to obtain a transitive action, this is even not possible: Already any one–dimensional subgroup of $C^\infty(Q, \mathbb{R})/\mathbb{R}$ moves points in a fiber of T^*Q into its origin $p_i = 0$, so that no subgroup of $C^\infty(Q, \mathbb{R})/\mathbb{R}$ yields a group action on $\mathcal{P}_*$.

Thus, if we are to quantize a phase space $\mathcal{P}_*$ (or one of its connected components), we may first search for finite–dimensional, transitively acting subgroups G of $\mathrm{Diff}(Q)$. Such an action of G then automatically acts effectively *and* now it also has a momentum map, as this is the case for $\mathrm{Diff}(Q)$. The quantum realizations of the phase space $\mathcal{P}_*$ are then to

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be found among the unitary IRREPs of $\mathcal{G} = \widetilde{G}$, where $\widetilde{G}$ is the universal covering group of G. This sets the strategy for what follows in the next section.

There certainly is no guarantee that such a finite–dimensional group G exists for a given (finite–dimensional) phase space $\mathcal{P}_*$ as likewise there need not exist a finite–dimensional, transitively acting subgroup of $\mathcal{D}$ on a cotangent bundle T^*Q. In both of these cases there still could be some other finite–dimensional subgroup $\mathcal{G}$ of the *full* group of canonical transformations on the phase space acting transitively and effectively. Moreover, certainly not any finite–dimensional cotangent bundle (and likewise not any of its subbundles $\mathcal{P}_*$) can be quantized by the group theoretical approach (using *fininte–dimensional* groups), even if it is quantizable e.g. in the sense of geometric quantization. In particular, the mere existence of a transitive, almost effective action of $\mathcal{G}$ on a phase space $\mathcal{P}$ implies that (topologically) $\mathcal{P} \cong \mathcal{G}/\mathcal{H}$, where $\mathcal{H}$ is a subgroup of $\mathcal{G}$ (the stabilizer group of some point in $\mathcal{P}$); clearly not any phase space $\mathcal{P}$ (or also cotangent bundle T^*Q or its subbundles $\mathcal{P}_*$) has the topology of some coset space of finite dimensional groups.[7] Still, the group theoretical quantization scheme, and in particular the above strategy for quantizing T^*Q and $\mathcal{P}_*$, is general enough to be applicable to a number of physical systems, and, among others, this will apply also to the phase space $\mathcal{S}$.

2.4.4 $T^*\mathbb{R}^+$ and $T^*(\mathbb{R}^2\backslash\{(0,0)\})$ as subbundles

The phase space $T^*\mathbb{R}^+$ can, after interchanging q and p, be seen as a subbundle $\mathcal{P}_+$ of $T^*\mathbb{R}$. A transitive action on $\mathbb{R}^+$ is generated by the phase space function q, whereas the proof of Lemma 2 suggests to use in addition the generating function qp to obtain a transitive action on the phase space. This brings us back to the group $\mathcal{G} = \mathbb{R} \rtimes \mathbb{R}^+$ of Subsection 2.2.2. The quantum theory obtained there was defined on the Hilbert space $L^2(\mathbb{R}^+, dq/q)$ with qp acting as $(\hbar/i)\, q\, d/dq$, which realizes the unique representation of $\mathcal{G}$ having positive spectrum for q.

An example for a phase space $\mathcal{P}_*$ is, again after interchanging coordinates and momenta, the phase space $T^*(\mathbb{R}^2\backslash\{(0,0)\})$. Such a phase space is of relevance in the context of the Aharanov Bohm effect.

The smallest transitively acting subgroup of the diffeomorphism group of $\mathbb{R}^2$ (the fibers of this phase space) is the two–dimensional abelian group of translations generated by the coordinates x and y of $\mathbb{R}^2\backslash\{(0,0)\}$. According to the proof of Lemma 2 we obtain a transitive action on the phase space if we add the functions xp_x, xp_y, yp_x and yp_y as generators. However, already the span $\langle x, y, xp_x + yp_y, xp_y - yp_x \rangle$ (the latter two functions are $xp_x + yp_y = rp_r$ and $xp_y - yp_x = p_\varphi$ in polar coordinates) is closed under Poisson brackets forming a Lie algebra isomorphic to $\mathbb{R}^2 \rtimes \mathbb{R}^2$, and we will see that it generates a transitive action on $T^*(\mathbb{R}^2\backslash\{(0,0)\})$.

The Hamiltonian vector fields are easily seen to generate an action of the group $G = (\mathbb{R}^2 \rtimes SO(2)) \rtimes \mathbb{R}^+$, the semidirect product of the group of motions of $\mathbb{R}^2$ with the group $\mathbb{R}^+$ of dilatations with composition $(\vec{v}_1, R_1, \lambda_1)(\vec{v}_2, R_2, \lambda_2) = (\vec{v}_1 + \lambda_1 R_1 \vec{v}_2, R_1 R_2, \lambda_1 \lambda_2)$. (Here

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[7]We are grateful to D. Giulini for pointing out to us that *any* manifold can be obtained as the coset space of appropriate, generically *infinite dimensional* groups.

$\vec{v}$ denotes the translation vector and R the two–by–two rotation matrix.) This group is isomorphic to $G \cong \mathbb{R}^2 \rtimes (SO(2) \times \mathbb{R}^+) \cong \mathbb{C} \rtimes \mathbb{C}^*$ with composition $(\alpha_1, \beta_1)(\alpha_2, \beta_2) = (\alpha_1 + \beta_1 \alpha_2, \beta_1 \beta_2)$. Using the latter form, the action on $T^*(\mathbb{R}^2 \backslash \{(0,0)\})$ can most compactly be written in terms of the complex coordinates $z := x + iy$, $p := p_x + ip_y$ as $(\alpha, \beta) : (z, p) \mapsto (\beta z, \beta^{-1} p + \alpha)$. (The group G can also be viewed as a subgroup of $\mathcal{D}$ in Subsec. 2.4.1, where $\mathbb{R}^2$ is a subgroup of $C^\infty(Q, \mathbb{R})/\mathbb{R}$ and $SO(2) \times \mathbb{R}^+$ a subgroup of $\mathrm{Diff}(Q)$ with $Q = \mathbb{R}^2 \backslash \{(0,0)\}$. From this point of view one still would have to check the existence of a momentum map, which is immediate from the present perspective of G (cf Lemma 2).)

Analogously to the example $T^* S^1$ we can also find effective actions of any covering group of G, but again they are excluded by the SGP. The quantum theory, however, will be most generally provided by unitary representations of the universal covering $\mathcal{G} = \widetilde{G}$.

Using Mackey theory [11], one finds that the inequivalent (nontrivial) unitary representations of this (universal covering) group may be presented on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^+ \times S^1, r\, dr d\varphi)$ according to the unitary action $(U(\vec{v}, t, \lambda)\psi)(r, \varphi) = \lambda \exp(i\theta t + ir(v_1 \cos\varphi + v_2 \sin\varphi))\psi(\lambda r, \varphi + t)$ of $\mathcal{G}$, where $t \in \mathbb{R}$ is a parameter in $\widetilde{G}$ covering the $SO(2)$–angle of G. Here $\theta \in (0, 1]$ is the θ–angle expected due to $\pi_1(T^*(\mathbb{R}^2 \backslash \{(0,0)\})) = \mathbb{Z}$, which, in the group theoretical context, may be understood to arise from the unitary representations of $\pi_1(G) = \mathbb{Z}$, the center of $\widetilde{G}$. The spectra of the fundamental observables $\widehat{x}$, $\widehat{y}$, $\widehat{rp_r}$, and $\widehat{p_\varphi}$ are $\mathbb{R}$, $\mathbb{R}$, $\mathbb{R}$, and $\mathbb{Z} + \theta$, respectively. Note that here $(0,0)$ is in the spectrum of $(\widehat{x}, \widehat{y})$, although classically this point is removed from the configuration space.

2.5 The projection method

By imposing the restriction to a subbundle at the quantum level we can arrive at the quantum theory of $T^* \mathbb{R}^+$ also in a different way: Starting from the standard quantization of $T^* \mathbb{R}$ on the Hilbert space $\widetilde{\mathcal{H}} = L^2(\mathbb{R}, dq)$ we restrict it, in a second step, to the maximal subspace $\mathcal{H}$ on which q is quantized to a positive operator (implementation of the restriction $q > 0$ at the quantum level as an operator inequality), i.e., we define $\mathcal{H}$ through completion of the maximal subspace $F \subset \mathcal{D}(\hat{q})$ on which $\int_{\mathbb{R}} \overline{f} q f\, dq \geq 0$ for all $f \in F$ (where $C D(\hat{q})$ is the domain of definition of the multiplication operator $\hat{q}$). This subspace is easily seen to be $\mathcal{H} = L^2(\mathbb{R}^+, dq)$.

Clearly there is a (unique) projector $\pi : \widetilde{\mathcal{H}} \to \mathcal{H}$, which may be used to also transport operators defined in $\widetilde{\mathcal{H}}$ to operators on $\mathcal{H}$. (The uniqueness of the projector is a result of the maximality condition required for the subspace $\mathcal{H}$ on which $\widehat{q} > 0$. This condition is necessary to reproduce standard results; in a way, it serves to capture the phase space, here $T^* \mathbb{R}^+$, globally.)

We now propose a more general setting in which the above "projection method" should be applicable.

2.5.1 Restricted phase spaces and their Hilbert spaces

A phase space $\mathcal{P}$ which can be treated using the projection method has to obey the following properties: First, $\mathcal{P}$ can be characterized as a submanifold of a phase space $\widetilde{\mathcal{P}}$ via restriction

by means of inequalities $f_i > 0$ for a set of functions $\{f_i\}$ on $\widetilde{\mathcal{P}}$ with mutually vanishing Poisson brackets. We furthermore demand that, for each i, the set on which the opposite inequality, $f_i < 0$, is fulfilled is nonempty. (This condition is necessary to exclude, e.g., cases like the restriction of $T^*\mathbb{R}^2$ to $T^*(\mathbb{R}^2\backslash\{(0,0)\})$ by means of $x^2 + y^2 > 0$, which cannot be treated by the method of the present subsection; see the remarks below.) Second, a quantum realization of $\widetilde{\mathcal{P}}$ is known in which the functions $\{f_i\}$ may be promoted to self–adjoint, simultaneously diagonalizable operators $\{\hat{f}_i\}$.[8]

For simplicity we assume that the space $\mathcal{P}$, where all conditions $f_i > 0$ are fulfilled, is connected. Otherwise, we have to quantize each connected component separately and to take eventually the direct sum of the resulting Hilbert spaces as common Hilbert space for the quantization of $\mathcal{P}$.

The general strategy of the projection method to obtain a quantum realization of $\mathcal{P}$ is then as follows: Starting with the Hilbert space $\widetilde{\mathcal{H}}$ which quantizes $\widetilde{\mathcal{P}}$ we have the self–adjoint operators $\hat{f}_i$. Their spectral families can be used to define the projectors $P_i := \Theta(\hat{f}_i)$, where $\Theta : \mathbb{R} \rightarrow \mathbb{R}$ is the step function which is zero for $x < 0$ and one for $x \geq 0$. Because the operators $\hat{f}_i$ are assumed to be simultaneously diagonalizable, their spectral families commute and the common projector $P := \prod_i P_i \colon \widetilde{\mathcal{H}} \rightarrow \widetilde{\mathcal{H}}$ can be defined unambiguously. Using this projector, the restricted Hilbert space $\mathcal{H}$ is defined as a subspace of $\widetilde{\mathcal{H}}$ according to $\mathcal{H} := P(\widetilde{\mathcal{H}})$. As a Hilbert space of its own, $\mathcal{H}$ is regarded as the Hilbert space of $\mathcal{P}$.

Restricting the image of P to $\mathcal{H}$ we obtain a map $\pi \colon \widetilde{\mathcal{H}} \rightarrow \mathcal{H}$ with adjoint being the inclusion $\iota \colon \mathcal{H} \hookrightarrow \widetilde{\mathcal{H}}$ of $\mathcal{H}$ (which is defined as a subspace of $\widetilde{\mathcal{H}}$) in $\widetilde{\mathcal{H}}$. Both these maps are partial isometries (i.e., they map closed subspaces — $\mathcal{H}$ in both cases — isometrically to their images and annihilate their orthogonal complements). Composing the two maps we obtain $\pi \circ \iota = \mathbb{I}_\mathcal{H}$ (the identity on $\mathcal{H}$) and $\iota \circ \pi = P$ (the projector on $\widetilde{\mathcal{H}}$), respectively.

In the preceding subsection we presented a general strategy for finding a group action on certain subbundles of cotangent bundles appropriate for the group theoretical quantization. For a one–dimensional configuration space the subbundle is defined by an inequality of the form $f > 0$ where f is the coordinate or its canonical momentum. All the above conditions of the projection method are fulfilled in this case and it can be used to obtain a quantum realization of this subbundle as demonstrated by the example $T^*\mathbb{R}^+$ above.

For phase spaces of dimension greater than two the situation is different. Here, the subbundles $\mathcal{P}_*$ of the previous subsection are defined by inequalities $f_i \neq 0$ removing a *lower*–dimensional submanifold from the phase space and the functions f_i do not meet all the conditions required above. They still Poisson commute and for a known quantum realization of $\mathcal{P}$ they correspond to simultaneously diagonalizable operators. Thus, one still can construct the projector, of course. However, the restriction method may fail: If zero is not contained in the discrete part of the spectrum of all the $\hat{f}_i$, then the projector is the identity on $\widetilde{\mathcal{H}}$, not leading to any restriction (an example for this case is the phase space $T^*(\mathbb{R}^2\backslash\{(0,0)\})$). If, on the other hand, zero is contained in the discrete part of the

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[8]We remark that in the case of unbounded self–adjoint operators commutativity on a dense domain is not sufficient for their simultaneous diagonalizability, needed below.

spectrum for at least one of the $\hat{f}_i$, then the projection leads to a restriction, but the point zero can be excluded from the spectra of all the $\hat{f}_i$ only if it is an isolated point.

Although the projection method is not applicable to higher–dimensional phase spaces of the form $\mathcal{P}_*$ in general, the conditions for its applicability as formulated above are fulfilled by a much wider class of systems than those considered in Subsec. 2.4.3. Given a phase space $\mathcal{P}$ one merely has to find an appropriate embedding of $\mathcal{P}$ within a phase space with known quantum realization.

2.5.2 Observables

To complete the quantum theory of $\mathcal{P}$ we have to promote a certain class of observables to densely defined operators on $\mathcal{H}$. In the quantization of $\widetilde{\mathcal{P}}$ we already have such operators $\widetilde{\mathcal{O}}$ acting on $\widetilde{\mathcal{H}}$ as quantizations of observables. These can be used to define operators on $\mathcal{H}$ by mapping $\widetilde{\mathcal{O}}\colon\widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ to $\mathcal{O}\colon\mathcal{H} \to \mathcal{H}$ by means of $\mathcal{O} := \pi \circ \widetilde{\mathcal{O}} \circ \iota$. If $\widetilde{\mathcal{O}}$ is densely defined with domain $\mathcal{D}(\widetilde{\mathcal{O}})$, then $\mathcal{O}$ is also densely defined with domain $\mathcal{D}(\mathcal{O}) = \pi(\mathcal{D}(\widetilde{\mathcal{O}}))$.

Specific properties of $\widetilde{\mathcal{O}}$ are, however, not necessarily inherited by $\mathcal{O}$. E.g., a densely defined, self–adjoint operator $\widetilde{\mathcal{O}}$ leads, in general, only to a hermitean operator $\mathcal{O}$: The product of adjoints of two densely defined operators $A\colon F \to G$ and $B\colon G \to H$ between Hilbert spaces satisfies $A^*B^* \subset (BA)^*$, and equality can be concluded, without further information on A and B, only if B is bounded (defined on all of the Hilbert space G and not just on a dense subset). This condition is fulfilled for the maps π and ι in the definition of $\mathcal{O}$, such that we obtain as its adjoint

$$\mathcal{O}^* = (\pi(\widetilde{\mathcal{O}}\iota))^* = (\widetilde{\mathcal{O}}\iota)^*\pi^* \supset \iota^*\widetilde{\mathcal{O}}^*\pi^* = \pi\widetilde{\mathcal{O}}^*\iota \ .$$

If $\widetilde{\mathcal{O}}$ is self–adjoint, $\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}^*$, then $\mathcal{O}$ is in general only hermitean: $\mathcal{O} \subset \mathcal{O}^*$. (Cf also the example of the momentum operator of $T^*\mathbb{R}^+$ below.)

Similarly, for a unitary operator $\widetilde{\mathcal{O}}$ the operator $\mathcal{O}$ is isometric ($\pi\widetilde{\mathcal{O}}\iota$ clearly preserves the norm on $\mathcal{H}$), but not necessarily also unitary: Its adjoint is given by $\mathcal{O}^* = \pi\widetilde{\mathcal{O}}^*\iota$ due to the fact that $\widetilde{\mathcal{O}}$, being unitary, is a bounded operator. Only if $\widetilde{\mathcal{O}}$ commutes with P (i.e. if $\widetilde{\mathcal{O}}$ preserves the subspace $\mathcal{H} = P(\widetilde{\mathcal{H}})$ as well as its orthogonal complement), we may in general simplify $\mathcal{O}\mathcal{O}^* = \pi\widetilde{\mathcal{O}}P\widetilde{\mathcal{O}}^*\iota$ to $\mathcal{O}\mathcal{O}^* = \pi P\iota = \mathbb{1}_{\mathcal{H}}$ and likewise conclude $\mathcal{O}^*\mathcal{O} = \mathbb{1}_{\mathcal{H}}$. An example for a unitary operator with only isometric projection will appear in Subsec. 3.2.2.

If possible, observables of the classical theory are promoted to self–adjoint operators. (The momentum operator on $T^*\mathbb{R}^+$ provides an example where this is not possible.) The operator $\mathcal{O}$ obtained from some self–adjoint operator $\widetilde{\mathcal{O}}$ in the above manner is, in general, only hermitean; this is typically the case because the conditions $f_i > 0$ introduce a boundary on the phase space $\mathcal{P}$. An operator $\mathcal{O}$ projected as above is then defined on a dense domain including a specification of boundary conditions. If this operator has self–adjoint extensions, each of them can be used as quantization of an observable (possibly introducing an additional ambiguity in defining the quantum theory of $\mathcal{P}$).

The latter scenario may be illustrated by means of a particle on a line of bounded extension. This system may be obtained as a submanifold of $T^*\mathbb{R}$ by means of $f_1 = q - a$ and $f_2 = b - q$ for some $a, b \in \mathbb{R}$ with $b > a$. The domain of definition of the momentum operator projected from the one of $T^*\mathbb{R}$ is given by absolutely continuous functions on $[a, b]$ which vanish at the boundary. So defined, it is only hermitean. However, it has a family of self–adjoint extensions, parameterized again by a θ–angle, defined on absolutely continuous functions ψ satisfying $\psi(b) = \exp(i\theta)\psi(a)$. Each of these extensions may now be chosen as a possible quantum observable corresponding to the canonical momentum on $T^*([a, b])$ (cf, e.g., Ref. [8]).

Best candidates for operators which project to a self–adjoint one on $\mathcal{H}$ correspond to phase space functions adapted to the boundary. This is similar to the situation in group theoretical quantization, where the condition that the fundamental observables generate an action on $\mathcal{P}$ forces the generating vector fields to be tangential to the boundary.

We finally illustrate these considerations by means of the quantization of $T^*\mathbb{R}^+$. The Hilbert space $\mathcal{H} = L^2(\mathbb{R}^+, dq)$ was derived at the beginning of this subsection by the projection method. Here the projector π and the inclusion ι are defined by $\pi\widetilde{\psi} = \widetilde{\psi}|_{\mathbb{R}^+}$ for $\widetilde{\psi} \in L^2(\mathbb{R}, dq)$ and $(\iota\phi)(q) = \phi(q)$ for $q > 0$ while $(\iota\phi)(q) = 0$ otherwise for $\phi \in \mathcal{H}$. The operator $\hat{q}$, whose spectral family was used to restrict the Hilbert space, remains a self–adjoint multiplication operator on $\mathcal{H}$. But the momentum operator $\widetilde{\mathcal{O}} = \hat{p} = -i\hbar d/dq$, commonly used as the other fundamental observable on $\widetilde{\mathcal{H}}$, projects down to a derivative $\mathcal{O} = -i\hbar d/dq$ on $\mathcal{H}$, which is no longer self–adjoint: The domain of definition of $\mathcal{O}$ defined by the projection is $\mathcal{D}(\mathcal{O}) = \mathcal{D}(\widetilde{\mathcal{O}}) \cap \mathcal{H} = \{\psi \in \mathcal{H} : \psi \text{ absolutely continuous}, \psi' \in \mathcal{H} \text{ and } \psi(0) = 0\}$. Its adjoint has, however, the larger domain of definition $\mathcal{D}(\mathcal{O}^*) = \{\psi \in \mathcal{H} : \psi \text{ absolutely continuous and } \psi' \in \mathcal{H}\}$, whereas $\mathcal{O}^{**} = \mathcal{O}$. This shows that $\mathcal{O}$ is not essentially self–adjoint, and, even worse, it has no self–adjoint extensions (cf, e.g., Ref. [8]).

Being a consequence of the boundary, the latter problem can easily be cured by using the self–adjoint operator $\frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) = -i\hbar(q\,d/dq + 1/2)$ (as quantization of qp) instead of $\hat{p}$. Due to the presence of $\hat{q}$, its projection to $\mathcal{H}$ no longer needs additional boundary conditions to be hermitean, and it can easily be shown to be self–adjoint (it generates the unitary transformation $\psi(q) \mapsto \sqrt{t}\psi(tq)$). This is related to the fact that the flow of qp is tangential to the boundary. We are again lead to the same fundamental observables as when using the group theoretical quantization and we obtain unitarily equivalent quantum theories.

2.5.3 Outlook on possible applications

The main advantage of the projection method (within its limited domain of applicability) as opposed to other quantization schemes is the fact that it makes use of the quantization of the embedding phase space $\widetilde{\mathcal{P}}$. So, parts of the steps in the transition from the classical to the quantum system are taken from the auxiliary system $\widetilde{\mathcal{P}}$ and need not be repeated for $\mathcal{P}$. This will become particularly transparent at the example in the subsequent section.

At the level of symplectic manifolds there is a related method, known as "symplectic cutting" in the mathematical literature (cf e.g. Ref. [12]). In this approach one is given a

torus action with momentum map on a phase space $\widetilde{\mathcal{P}}$. By means of this momentum map $\widetilde{\mathcal{P}}$ can be cut into pieces one of which is determined by setting the Hamiltonians of the torus action greater than zero yielding a certain compactification of the subspace $\mathcal{P}$ defined above. Note that due to the abelian character of the torus $U(1)^n$ these Hamiltonians always Poisson commute.

Far more complicated examples for the projection method as those provided in this paper, for which it can be relevant (when extended appropriately to deal with constrained systems), are given by gravitational theories. There, the (symmetric) matrix $g_{\mu\nu}$ of the coefficients of the metric g in some local chart is required to satisfy $\det(g) \neq 0$ (for all points of spacetime) or, more precisely, $\pm\det(g) > 0$, the sign depending on the signature of the metric g. E.g., in a dreibein formulation of Hamiltonian general relativity one has to require $\det e > 0$ for the dreibein components in order to extract the nondegenerate sector. In the context of lattice quantum gravity the implementation of this condition at the quantum level, as compatible with the general projection method above, has been investigated in Ref. [13].

We conclude these considerations with a cautionary remark: In the context of gravity theories — but also, more generally, of constrained Hamiltonian systems with (additional) "disallowed regions" in phase space — further care is needed when considering the projection method (in addition to the standard problems of the quantization of constrained systems). This becomes obvious already classically: First removing disallowed regions from phase space (degenerate sectors in gravity theories) and then performing the symplectic reduction is in general only equivalent to first reducing and then singling out the disallowed equivalence classes (or the equivalence classes without an allowed representative) if the flow of the constraints does not connect allowed with disallowed regions.

This condition is violated in several popular formulations of gravity theories in spacetime dimensions four (Ashtekar formulation), three (Chern–Simons formulation), and two (BF– or, more generally, Poisson Sigma formulation). In all of these cases, equivalence with the original, metrical formulation can be established only on the *nondegenerate* sector of phase space and (in contrast to the original diffeomorphism constraints) the flow of the constraints in the new formulation does indeed enter the degenerate sector.

To show that this can be of relevance, we provide a simple example (cf also Ref. [14] for a similar illustration): Consider a particle in $\mathbb{R}^3$ with the (original, first class) constraint $C = x\left[(x+2)^2 - (p_x)^2 - 1\right] \approx 0$, declaring the subspace with $x \leq 0$ to be "disallowed" ("degenerate sector"). Clearly, the flow of C does not leave (or enter) the forbidden region in phase space. Thus, removing the disallowed subspace and performing the symplectic reduction commute, leading to a reduced phase space (RPS) which is a *two–fold* covering of $T^*\mathbb{R}^2$. On the other hand, within the allowed region of the original phase space the constraint C may be replaced equivalently by $\widetilde{C} \equiv (x+2)^2 - (p_x)^2 - 1 \approx 0$. However, the above condition on the flow of the constraint is no more satisfied in this case. Indeed, while certainly one obtains the same RPS as before when one first removes the disallowed region and only then performs the symplectic reduction (which requires knowledge about the global topology of the orbits), the (simpler) symplectic reduction of the original theory

$T^*\mathbb{R}^3$ with respect to $\widetilde{C}$ leads to only a *single* copy of $T^*\mathbb{R}^2$ as RPS (each point of which contains allowed representatives).

Accordingly, given a procedure for solving the constraint of the original system (defined in $T^*\mathbb{R}^3$) at the quantum level, it will yield inequivalent results when performed with respect to the constraints C and $\widetilde{C}$, even if in a second step the projection method (adapted appropriately to the context) is applied to take care of $x > 0$.

Explicit examples of gravity theories in two [14] and three [15] spacetime dimensions showed that the above mechanism can indeed produce inequivalent factor spaces and, accordingly, also quantum theories. — Note, however, that in this context the failure of the projection method does not result from its insufficiency as a quantization scheme; rather, the deficiency is evident already on the classical level and results from the reformulation of the constraints, equivalence in nondegenerate sectors being, in this context, insufficient for full equivalence.

3   The phase space $\mathcal{S} = S^1 \times \mathbb{R}^+$

In this section we present the quantization of the phase space $\mathcal{S}$ which is the restriction of the cotangent bundle $T^*S^1 \sim S^1 \times \mathbb{R}$ with canonical symplectic form $\omega = d\varphi \wedge dp$ to positive values of the momentum variable p. We denote this restriction by $S^1 \times \mathbb{R}^+$, in analogy to $T^*\mathbb{R}^+ \sim \mathbb{R}^+ \times \mathbb{R}$.

As stressed already in the previous section, $T^*\mathbb{R}^+$ is symplectomorphic to $T^*\mathbb{R}$; as a symplectic manifold there is *no* difference between the phase spaces $T^*\mathbb{R}^+$ and $T^*\mathbb{R}$ (there is only a difference between what we call the *physical* momentum and position). *Topologically* we certainly also have $\mathcal{S} \sim S^1 \times \mathbb{R}$. So we may ask if possibly $\mathcal{S}$ is also symplectomorphic to T^*S^1. If this were the case, the quantization of $\mathcal{S}$ would be immediate, as then we could use the quantum theory of T^*S^1, recapitulated in the previous section.

In contrast to $T^*\mathbb{R}$ and $T^*\mathbb{R}^+$, $S^1 \times \mathbb{R} = T^*S^1$ and $S^1 \times \mathbb{R}^+ = \mathcal{S}$ are in fact *not* symplectomorphic. This may be proved by the following simple consideration: Suppose they were symplectomorphic. Then the diffeomorphism between the two phase spaces has to map a noncontractible, nonselfintersecting loop on $\mathcal{S}$ to a likewise loop on T^*S^1. Each of these loops separates the respective phase space into two disconnected parts. On $\mathcal{S} = S^1 \times \mathbb{R}^+$ *one* of these two parts has a *finite* symplectic volume. Its image on T^*S^1 under the diffeomorphism has an infinite symplectic volume, on the other hand. This is in contradiction with a symplectomorphism, which leaves symplectic volumes unchanged.

3.1 $SO^\uparrow(1,2)$ and its action

Thus $\mathcal{S}$ cannot be a cotangent bundle. However, $\mathcal{S}$ is the restriction of a cotangent bundle over S^1 to positive values of the canonical momentum. Such spaces were considered in Subsec. 2.4 (called $\mathcal{P}_+$ there). We thus may apply those considerations to construct a transitive, almost effective, and canonical group action on $\mathcal{S}$. In particular, as a consequence of Lemma 2, it is only necessary to find a (finite–dimensional) subgroup of $\mathrm{Diff}(S^1)$ with

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a lift acting transitively on $\mathcal{S}$. Its action will then be also effective and have a momentum map.

3.1.1 Finite-dimensional subgroups of $\text{Diff}(S^1)$ with transitive action on $\mathcal{S} \subset T^*S^1$

The Lie algebra $\text{diff}(S^1)$ of $\text{Diff}(S^1)$ may be represented by vector fields of the form $v = f(\varphi)\, d/d\varphi$, where f is a 2π–periodic function. Thus a dense subalgebra of $\text{diff}(S^1)$ is spanned by

$$T = \frac{d}{d\varphi} \quad , \quad S_k = \sin(k\varphi)\frac{d}{d\varphi} \text{ and } C_k = \cos(k\varphi)\frac{d}{d\varphi} \text{ with } k \in \mathbb{N} \equiv \{1, 2, \ldots\}, \quad (6)$$

and we will denote it as

$$\text{diff}_0(S^1) := \{b_0 T + \sum_{k>0}(b_k C_k + b_{-k}S_k)|b_k \in \mathbb{R} \text{ and } b_k = 0 \text{ for almost all } k\}.$$

As already mentioned, we are interested in *finite–dimensional* subgroups of the diffeomorphism group. They can have an arbitrary dimension as the following construction shows: To any $n \in \mathbb{N}$ we can choose n vector fields on the circle which have disjoint compact supports. They generate the n–dimensional abelian subgroup $\mathbb{R}^n$. Clearly, these subgroups have fixed points and thus do not act transitively on S^1 (and neither do their lifts to $\mathcal{S}$).

To eliminate these and similar subgroups from our consideration we will, in the following, constrain ourselves to the (still infinite–dimensional) subalgebra $\text{diff}_0(S^1)$ of $\text{diff}(S^1)$ generated by finite linear combinations of T, S_k and C_k in some chart of S^1. As subalgebras of $\text{diff}(S^1)$ they depend on the coordinate on S^1: Subalgebras corresponding to different coordinates are not identical; however, they are conjugate to one another and are thus isomorphic. The restriction to $\text{diff}_0(S^1)$ will allow us to draw much stronger conclusions, namely we will find that *all* finite–dimensional subgroups of $\text{Diff}_+(S^1)$ (the component of $\text{Diff}(S^1)$ connected to the identity) with Lie algebra lying in $\text{diff}_0(S^1)$ and with transitively acting lift to $\mathcal{S}$ are covering groups of $SO^\uparrow(1,2)$:

Theorem 3 *Each finite–dimensional subgroup of $\text{Diff}_+(S^1)$ which is generated by finite linear combinations of T, S_k and C_k in some chart of S^1 and which has a transitively acting lift to $\mathcal{S} \subset T^*S^1$ is isomorphic to a covering group of $SO^\uparrow(1,2)$ (the l–fold covering being generated by $l^{-1}T$, $l^{-1}S_l$, and $l^{-1}C_l$).*

3.1.2 Finite–dimensional subalgebras of the Witt algebra

To prove the theorem we first consider finite–dimensional subalgebras of the complexification of $\text{diff}_0(S^1)$, which is known as the Witt algebra

$$\mathcal{W} := \{\sum_{k \in \mathbb{Z}} a_k L_k | a_k \in \mathbb{C} \text{ and } a_k = 0 \text{ for almost all } k\}$$

with generators $L_k = -i \exp(ik\varphi)d/d\varphi$, $k \in \mathbb{Z}$ and relations $[L_j, L_k] = (k - j)L_{k+j}$.

Lemma 4 *The finite–dimensional subalgebras of $\mathcal{W}$ are at most (complex) three–dimensional, in which case they are isomorphic to $sl(2, \mathbb{C})$.*

Proof: Let A be a finite–dimensional subalgebra of $\mathcal{W}$ with at least three generators. Without any restriction these generators can be assumed to be of the form $g = L_- + cL_0 + L_+$ with $L_- = \sum_{k=1}^{M_-} a_k L_{-k}$, $L_+ = \sum_{k=1}^{M_+} b_k L_k$, $c, a_i, b_i \in \mathbb{C}$ and $a_{M_-} \neq 0 \neq b_{M_+}$. Otherwise they can be brought into this form by appropriate linear combinations. Let $g_i = L_-^{(i)} + c^{(i)} L_0 + L_+^{(i)}$, $i = 1, 2$, be two of the generators and $M_{+/-}^{(i)}$ as defined above. Then we can reveal the following conditions for A to be finite–dimensional:

(i) $M_+^{(1)} = M_+^{(2)}$, and analogously $M_-^{(1)} = M_-^{(2)}$: Otherwise $[g_1, g_2]$ would contain a contribution of $L_{M_+^{(1)} + M_+^{(2)}}$ with nonzero coefficient. By induction, repeated commutators would contain contributions from $L_{mM_+^{(1)} + nM_+^{(2)}}$ with arbitrary $m, n \in \mathbb{N}$. Therefore, the subalgebra could not be finite–dimensional.

(ii) $L_+^{(1)} \propto L_+^{(2)}$, and analogously $L_-^{(1)} \propto L_-^{(2)}$: Otherwise by appropriate linear combinations we could trade the generators for two new generators not fulfilling condition (i).

We conclude that all generators are of the form $g_i = a_i L_- + c_i L_0 + b_i L_+$, i.e., there are only three linearly independent generators $\{L_-, L_0, L_+\}$. This proves that a finite–dimensional subalgebra is at most three–dimensional.

We can now determine the form of these subalgebras $\langle L_-, L_0, L_+ \rangle$: From the commutation relations of the L_k it follows that $[L_0, L_+]$ has to be proportional to L_+ in order for $\langle L_-, L_0, L_+ \rangle$ to be closed under commutation. This can only be the case if there is an $l \in \mathbb{N}$ such that $L_+ \propto L_l$. Analogously there must be a $j \in \mathbb{N}$ such that $L_- \propto L_{-j}$. Now we must have $l = j$ because otherwise $\langle L_-, L_0, L_+ \rangle$ would not be closed. The only three–dimensional subalgebras of $\mathcal{W}$ are, therefore, given by $\langle L_{-l}, L_0, L_l \rangle$ for $l \in \mathbb{N}$, which are easily seen to be isomorphic to $sl(2, \mathbb{C})$. $\square$

Thus we know all three–dimensional subalgebras of the Witt algebra. We will see now that they also include all two–dimensional subalgebras:

Lemma 5 *Each (complex) two–dimensional subalgebra of $\mathcal{W}$ is a subalgebra of one of the $sl(2, \mathbb{C})$ subalgebras found in the preceding lemma.*

Proof: A two–dimensional Lie algebra generated by g_1 and g_2 can, without restriction, be assumed to be of the form $[g_1, g_2] = 0$ or $[g_1, g_2] = g_2$, respectively. In close analogy to the proof of the preceding lemma, one may show that the former case implies $g_1 \propto g_2$ (in contradiction to the linear independence of g_1 and g_2) and that the latter case is possible only if $g_1 \propto L_0$ and $g_2 \propto L_l$ for an $l \in \mathbb{Z}$. $\square$

To use the information about $\mathcal{W}$ contained in the preceding two lemmas we have to translate it to the real form $\mathrm{diff}_0(S^1)$. The statements on (now real) dimensionality in Lemma 4 and Lemma 5 remain true because otherwise we could construct contradictions to these lemmas by complexification.

3.1.3 $SO^{\uparrow}(1,2)$ and its covering groups

The $sl(2,\mathbb{C})$–subalgebras $\langle L_{-l}, L_0, L_l \rangle < \mathcal{W}$ have the real forms $\langle l^{-1}T, l^{-1}S_l, l^{-1}C_l \rangle$ as sub-algebras of $\mathrm{diff}_0(S^1)$. For any l this is an $so(1,2)$–algebra shown by the isomorphism

$$\frac{T}{l} \leftrightarrow T_0 \;, \quad \frac{S_l}{l} \leftrightarrow T_1 \;, \quad \frac{C_l}{l} \leftrightarrow T_2 \;. \tag{7}$$

Here the T_i, $i = 0, 1, 2$ are generators of $so(1,2)$, satisfying the standard relations $[T_i, T_j] = \varepsilon_{ij}{}^k T_k$, where $\varepsilon_{012} = 1$ and indices are raised by means of $\mathrm{diag}(-1,1,1) = \kappa/2$, κ being the Killing metric. As real forms of $sl(2,\mathbb{C})$ the above subalgebras are unique by demanding them to be real subalgebras of the real form $\mathrm{diff}_0(S^1)$ of $\mathcal{W}$.

For later use it is worthwhile to exploit the Lie algebra isomorphisms of $so(1,2)$ to $sl(2,\mathbb{R})$ and $su(1,1)$. An isomorphism between the former two in terms of their generators T_i and σ_+, σ_-, $\sigma_3/2$, respectively, is given by

$$T_0 \leftrightarrow \frac{1}{2}(\sigma_+ - \sigma_-) \quad , \qquad T_1 \leftrightarrow \frac{1}{2}(\sigma_+ + \sigma_-) \quad , \qquad T_2 \leftrightarrow \frac{1}{2}\sigma_3, \tag{8}$$

where $2\,\sigma_\pm = \sigma_1 \pm i\sigma_2$ and σ_j, $j = 1,2,3$ denote the standard Pauli matrices. An isomorphism between $so(1,2)$ and $su(1,1)$ is provided by

$$T_0 \leftrightarrow -\frac{i}{2}\sigma_3 \;, \quad T_1 \leftrightarrow \frac{1}{2}\sigma_1 \;, \quad T_2 \leftrightarrow \frac{1}{2}\sigma_2 \;. \tag{9}$$

The subgroup of $\mathrm{Diff}_+(S^1)$ generated by $l^{-1}T$, $l^{-1}S_l$, and $l^{-1}C_l$ is the l–fold *covering* group of $SO^{\uparrow}(1,2)$. This is the case because $(\exp(2\pi l^{-1}T))^j = \exp(2\pi j l^{-1}d/d\varphi) \neq 1$ for $0 < j < l$ and $(\exp(2\pi l^{-1}T))^l = 1$. In the language of $SO^{\uparrow}(1,2)$ ($l = 1$), T generates rotations in the (x_1, x_2)–plane of the $(2+1)$–dimensional Minkowski space, and S_1 and C_1 generate boosts along the x_1– and x_2–direction, respectively.

We thus arrived at $SO^{\uparrow}(1,2)$ and its covering groups as maximal finite–dimensional sub-groups of $\mathrm{Diff}_+(S^1)$ with Lie algebra in $\mathrm{diff}_0(S^1)$. They are maximal finite–dimensional sub-groups of $\mathrm{Diff}_+(S^1)$ in the sense that there is no finite–dimensional subgroup of $\mathrm{Diff}_+(S^1)$ which has one of these groups as a subgroup. This follows easily from the fact that their complexified Lie algebras contain the element L_0.

For Theorem 3 to hold the restriction to $\mathrm{diff}_0(S^1)$ is essential: As already noted at the beginning of this subsection, $\mathrm{Diff}_+(S^1)$ contains finite–dimensional subgroups of arbitrary dimension. The examples provided there were of no interest in our context, however; for physical applications, moreover, it seems natural to restrict oneself to finite linear combinations of trigonometric functions as for the *fundamental* observables (certainly this does not imply that all the observables are restricted in the same manner, since for them one still is allowed to take infinite linear combinations, cf also Subsec. 2.3 above).

Finally, to prove Theorem 3 we are left to study their possible subgroups.

Lemma 6 *For any covering group of $SO^{\uparrow}(1,2)$ there are two conjugacy classes of two-dimensional subgroups (both of which are isomorphic to $\mathbb{R} \rtimes \mathbb{R}$). The Lie algebras of respective representatives are spanned by T_2 and $T_0 \pm T_1$.*

Proof: As abelian subalgebras of $sl(2, \mathbb{R})$ are at most one–dimensional ($sl(2, \mathbb{R})$ has rank one), any two–dimensional subalgebra may be spanned by generators τ_+ and τ_3 satisfying $[\tau_3, \tau_+] = \tau_+$. In the complexified Lie algebra $sl(2, \mathbb{C})$ they span a Borel subalgebra, which is a maximally solvable subalgebra and unique up to conjugation. Thus we know that for any two–dimensional subalgebra of $sl(2, \mathbb{R})$ there is, in the fundamental representation of the algebra, a *complex* two–by–two matrix M of unit determinant such that $\tau_3 = M\,(\sigma_3/2)\,M^{-1}$ and $\tau_+ = M\,\sigma_+\,M^{-1}$ (and, up to a sign, M is unique). Reality of the matrices τ_3 and τ_+ implies that M is either real or purely imaginary. In the former case, $M \in SL(2, \mathbb{R})$ and the conjugation is compatible with the reality condition leading from $sl(2, \mathbb{C})$ to $sl(2, \mathbb{R})$. In the latter case, $M = \widetilde{M} i\sigma_1$ where $\widetilde{M} \in SL(2, \mathbb{R})$. Conjugation with the imaginary piece $i\sigma_1$ maps (σ_3, σ_+) into $(-\sigma_3, \sigma_-)$. The assertion of the lemma then follows upon the isomorphism (8) and exponentiation to group level. $\qquad\square$

To discuss transitivity of group actions on $\mathcal{S} \subset T^\star S^1$, we finally need the lifts of the diffeomorphisms generated by T, S_l, and C_l. According to Eq. (5) and the remarks in Subsec. 2.4.1, they are generated by the Hamiltonian vector fields

$$T \to \{\cdot, p\}\,, \quad S_l \to \{\cdot, p\,\sin l\varphi\}\,, \quad C_l \to \{\cdot, p\,\cos l\varphi\}, \tag{10}$$

respectively. This also provides a momentum map for the action of $SO^\uparrow(1,2)$ on $\mathcal{S}$.

We are now in the position to prove our theorem:

Proof (Of Theorem 3): According to Lemma 4 the finite–dimensional subgroups of $\mathrm{Diff}_+(S^1)$ which are generated by elements of $\mathrm{diff}_0(S^1)$ can be at most three–dimensional because the finite-dimensional subalgebras of $\mathcal{W}$, which is the complexification of $\mathrm{diff}_0(S^1)$, are at most three-dimensional.

The three-dimensional of these subgroups are isomorphic to l–fold covering groups of $SO^\uparrow(1,2)$ spanned by $l^{-1}T$, $l^{-1}S_l$, and $l^{-1}C_l$. All the two–dimensional subgroups are subgroups of these three–dimensional ones, moreover. Finally, there are the one–dimensional subgroups of $\mathrm{Diff}_+(S^1)$ which are generated by exponentiation of an arbitrary element of $\mathrm{diff}_0(S^1)$. We now investigate the action of these subgroups when lifted to $\mathcal{S} \subset T^*S^1$.

One–dimensional groups cannot have orbits filling all of the two–dimensional half–cylinder. So they cannot act transitively.

According to Lemma 6 and Eq. (7), all two–dimensional subgroups are in one of the two conjugacy classes, representatives of which are generated by the vector fields C_l and $T \pm S_l$. Their lifts $\{\cdot, p\cos l\varphi\}$ and $\{\cdot, p(1 \pm \sin l\varphi)\}$ to $\mathcal{S}$ fix the fiber over $\varphi = \mp\pi/(2l)$ and therefore the groups cannot act transitively. (The other two–dimensional subgroups, being conjugate to one of these two groups, can just as less act transitively.)

The only candidates with transitively acting lift are now the covering groups of $SO^\uparrow(1,2)$. That they act indeed transitively can be seen from the following consideration: The lift of the action of an l–fold covering group of $SO^\uparrow(1,2)$ is generated by the two vector fields given in the previous paragraph together with the vector field $\{\cdot, p\}$. The former two act transitively in some fibers and the latter one acts fiber transitively. Thus their joint action is transitive on $\mathcal{S}$. $\qquad\square$

3.1.4 Integrating the group actions

In this subsection we will derive the finite action on $\mathcal{S}$ generated by T, S_l, and C_l. There is a well known $SO^\uparrow(1,2)$–action on S^1 given by (see, e.g., Ref. [16])

$$z \mapsto \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}} \quad , \qquad z = \exp i\varphi \in S^1 \quad , \qquad A \equiv \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(1,1), \qquad (11)$$

with $|\alpha|^2 - |\beta|^2 = 1$. This action on S^1 has been written down in terms of $SU(1,1)$, which is a two–fold covering of $SO^\uparrow(1,2)$ (note that $A \in SU(1,1)$ and $-A$ have the same action).

(The relation between these two groups can be made explicit by means of the action $X \mapsto AXA^\dagger$ of $A \in SU(1,1)$ on matrices $X = X^\dagger$ satisfying $\mathrm{tr}(\sigma_3 X) = 0$, where $\dagger$ denotes transposition of the complex conjugate matrix. This transformation preserves the determinant of X, which, in the parametrization

$$X = \begin{pmatrix} x_0 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 \end{pmatrix} \quad , \qquad x_0, x_1, x_2 \in \mathbb{R} \,, \qquad (12)$$

is nothing but the bilinear form $x_0^2 - x_1^2 - x_2^2$; in this way A is seen to generate a (proper) Lorentz (or $SO^\uparrow(1,2)$) transformation on the (2+1)–dimensional Minkowski space spanned by (x_0, x_1, x_2).)

It is straightforward to verify that the infinitesimal form of Eq. (11) coincides with the action generated by the vector fields T, S_1, and C_1 (cf Eqs. (5) and (6)). In this way we may also determine the lift of the (finite) action (11) to $\mathcal{S}$ (cf Subsec. 2.4.1), yielding $p \mapsto p|\alpha e^{i\varphi} + \beta|^2$ in this case.

For $l > 1$ the action (11) on S^1 can be generalized by substituting $\exp il\varphi$ for $z = \exp i\varphi$. Infinitesimally, this action is readily seen to coincide with the one generated by the vector fields T, S_l, and C_l. However, taking the l–th root in a continuous manner to arrive at an action on $\varphi \in \mathbb{R} \mod 2\pi$ is nontrivial; in particular, for $l > 2$ it does *not* lead to an action on S^1 of the group $SU(1,1)$ itself, but of appropriate covering groups only (contrary to what is claimed, e.g., in Ref. [16]).

Actually, from the discussion preceding Lemma 6, we already know that the group generated by T, S_l, and C_l is an l–fold covering group of $SO^\uparrow(1,2)$. Thus, we can see that it is not possible to express the action in terms of $SU(1,1)$ for $l > 2$. Introducing the parameters $\gamma := \alpha^{-1}\beta$, $|\gamma| < 1$ and $0 \leq \omega < 2\pi$ by $\alpha = |\alpha|\exp i\omega$ of $SU(1,1)$ (see, e.g., Ref. [17]), which make explicit the topology of $SU(1,1)$, the n–fold covering group of $SO^\uparrow(1,2)$ can be parameterized by these parameters taking, however, ω in the range $0 \leq \omega < n\pi$. The action (11) on S^1 with φ replaced by $l\varphi$ now takes the form

$$\exp il\varphi \mapsto \exp(2i\omega)\frac{\gamma + \exp il\varphi}{\bar{\gamma}\exp il\varphi + 1} \,. \qquad (13)$$

Acting on $\exp il\varphi$, $0 \leq \varphi < 2\pi l^{-1}$, this action on S^1 is an almost effective action of the n–fold covering group of $SO^\uparrow(1,2)$, and n does not need to be identical to l. However, to

obtain an action on S^1, φ has to take values in $[0, 2\pi)$ where φ and $\varphi + 2\pi l^{-1}$ are not to be identified.

This observation will fix the covering group which acts effectively on $\mathbb{R}$ mod 2π if we take the l–th root. To this end it suffices to consider the action for $\gamma = 0$ because γ takes values in a simply connected domain. The action reduces to

$$\exp il\varphi \mapsto \exp(2i\omega)\exp il\varphi$$

which leads to $\varphi \mapsto \varphi + 2l^{-1}\omega$ if we use continuity and the fact that we have to obtain the identity transformation for $\omega = 0$. The last two conditions fix the branch of the l–th root uniquely. Now $\omega = n\pi$ must give the same result as $\omega = 0$ because we consider the action of the n–fold covering group of $SO^{\uparrow}(1,2)$. This is possible only if n is an integer multiple of l and an effective action is obtained for $n = l$. Thus we see that Eq. (13) determines an effective action of the l–fold covering group of $SO^{\uparrow}(1,2)$ on S^1 (and an almost effective action of the lm–fold covering for any $m \in \mathbb{N}$), but (for $l > 2$) not an $SU(1,1)$–action.

So, following the strategy for finding a group action on $\mathcal{S}$ as formulated in Subsec. 2.4, we thus arrive at the following action of finite dimensional groups on $\mathcal{S} \subset T^*S^1$:

$$(\exp il\varphi, p) \mapsto \left(\frac{\alpha \exp il\varphi + \beta}{\overline{\beta} \exp il\varphi + \overline{\alpha}}, p|\alpha e^{il\varphi} + \beta|^2 \right) . \tag{14}$$

This is the lift of the action of the l–fold covering group of $SO^{\uparrow}(1,2)$ presented above. (We were searching for subgroups of $\mathrm{Diff}(S^1)$ which all act effectively; therefore, these subgroups are l–fold coverings and not lm–fold ones ($m > 1$).) By construction, for any $l \in \mathbb{N}$, Eq. (14) provides a transitive, effective, and Hamiltonian action on $\mathcal{S}$ with momentum map.

Two more remarks: First, by means of the above action for $l = 2$ we may identify the phase space $\mathcal{S}$ with the coset space $SU(1,1)/N$. Here N denotes the nilpotent subgroup appearing in the Iwasawa decomposition of $SU(1,1)$ (obtained by exponentiating $T_2 - T_0$ in Eq. (8), cf also Ref. [4] for details). N is the stabilizer group of any point $(\varphi = 0, p) \in \mathcal{S}$, since obviously its generator $\{\cdot, p(\cos 2\varphi - 1)\}$ vanishes identically on the fiber over $\varphi = 0$.

Second, the (finite) action (14) on $\mathcal{S}$ may be also obtained as the (effective) action of the l–fold covering of $SO^{\uparrow}(1,2)$ on the l–fold covering of the future light cone $\mathcal{C}^+$ in $(2+1)$–Minkowski space (x_0, x_1, x_2). For $l = 1$ this is just the fundamental (defining) action of the (proper) Lorentz group which clearly maps the future light cone $\mathcal{C}^+ : x_0^2 - x_1^2 - x_2^2 = 0$, $x_0 > 0$, onto itself so that its action on Minkowski space can be restricted to an action on $\mathcal{C}^+$. The action (14) with $l = 1$ is then obtained from the $SO^{\uparrow}(1,2)$–action on $\mathcal{C}^+$ upon identifying x_0 with p and the polar angle of the light cone with φ. For $l > 1$ this generalizes to

$$(x_0, x_1 + ix_2) \leftrightarrow (p, pe^{-il\varphi}) \quad , \quad 0 \leq \varphi < 2\pi, p > 0 , \tag{15}$$

identifying the phase space $\mathcal{S}$ with an l–fold covering of $\mathcal{C}^+$. To verify the equivalence of the actions one only needs to check the infinitesimal correspondence (7) and (10) (with $T_i \in so(1,2)$ interpreted as the generators on (the l–fold covering of) $\mathcal{C}^+$). Formula (14) may now be obtained also by this approach via $X \mapsto AXA^{\dagger}$, where $A \in SU(1,1)$ as above and X results from combining Eqs. (12) and (15).

3.1.5 Admissible group actions on $\mathcal{S}$

We have now determined all the lifts of actions of the subgroups of $\mathrm{Diff}_+(S^1)$ found in Theorem 3. The possible effectively acting groups are the l–fold covering groups of $SO^\uparrow(1,2)$. We are now left only with checking the validity of the SGP for these group actions.

As is obvious from Eq. (10), the SGP is violated for $l \neq 1$. Alternatively we may also apply Lemma 1 due to the semisimplicity of $SO^\uparrow(1,2)$ and its covering groups: In the case of Eq. (14) G is identified with the l–fold covering group of $SO^\uparrow(1,2)$, which has a trivial center only for $l = 1$.

In the present case the use of the Lemma was not essential. However, let us remark that this may change drastically when more complicated phase spaces and group actions are considered (and in particular for infinite–dimensional phase spaces).

This fixes the parameter l in the countable family of (effective) group actions to be $l = 1$ so that we end up with a unique effective action of the group $SO^\uparrow(1,2)$.

Any covering group of $SO^\uparrow(1,2)$ is, however, allowed as *almost* effectively acting group provided its action projects down to the $SO^\uparrow(1,2)$–action (Eq. (14) with $l = 1$). The most general almost effective action is provided by the universal covering group $\widetilde{SO}^\uparrow(1,2)$ of $SO^\uparrow(1,2)$. According to the considerations of Sec. 2 we will thus examine the unitary representations of $\widetilde{SO}^\uparrow(1,2)$ for possible quantum realizations of $\mathcal{S}$ in the following subsection (using the momentum map (10) with $l = 1$ only).

3.2 The quantum theory

In the present subsection we will apply two methods to quantize $\mathcal{S}$. The first one completes the group theoretical quantization by using the group action derived in the preceding subsection. The second approach employs the projection method of Subsec. 2.5 making use of the fact that $\mathcal{S}$ is the restriction of T^*S^1 to positive momentum. Quantizing this phase space $\mathcal{S}$ thereby provides another example for the application of this method with, in contrast to $T^*\mathbb{R}^+$, a discrete spectrum of the observable $\hat{p}$ used to project down to the restricted Hilbert space. We will find that both quantization procedures are compatible, and that demanding equivalence constrains the quantum realizations obtained within group theoretical quantization.

3.2.1 Group theoretical quantization of $\mathcal{S}$

According to the results of the previous subsection, when applying the group theoretical quantization scheme to $\mathcal{S}$, we are to analyse the (weakly continuous) unitary IRREPs of the universal covering group $\widetilde{SO}^\uparrow(1,2)$ of $SO^\uparrow(1,2)$.

Thus we first have to look for unitary representations of $so(1,2)$. Its generators T_0, T_1, and T_2 obey the relations $[T_0, T_1] = T_2$, $[T_0, T_2] = -T_1$ and $[T_1, T_2] = -T_0$. This rank one algebra has the Casimir operator $C := T_0^2 - T_1^2 - T_2^2$. As maximal set of commuting algebra elements we choose $\{-iT_0, C\}$, which will be promoted to the maximal set $\{H, C\}$ of commuting operators on a representation space.

The states in irreducible representations can be classified by the eigenvalues λ and q of H and C, respectively. In each irreducible representation, $T_+ := T_1 - iT_2$ and $T_- := -T_1 - iT_2$ act as raising and lowering operators, respectively, which can be read off from the relations

$$[H, T_+] = T_+ \quad , \quad [H, T_-] = -T_- \quad , \quad [T_+, T_-] = -2H.$$

On an orthonormal basis $\{\phi_\lambda^q\}_{\lambda \in \Lambda}$ of a representation characterized by the eigenvalue q of C and indexed by the eigenvalues of H, the action of H, T_+, and T_- is given by

$$H\phi_\lambda^q = \lambda\phi_\lambda^q \quad , \quad T_+\phi_\lambda^q = \omega_{\lambda+1}\sqrt{q + \lambda(\lambda+1)}\phi_{\lambda+1}^q \quad , \quad T_-\phi_\lambda^q = \overline{\omega}_\lambda\sqrt{q + \lambda(\lambda-1)}\phi_{\lambda-1}^q$$
$$(16)$$

with arbitrary phase factors ω_λ, which can be chosen to be 1 by a unitary change of the basis. One can see that the spectra of H in all irreducible representations are equidistantly spaced by 1.

A more detailed analysis [17, 18, 19] of the irreducible unitary representations of $\widetilde{SO}^\uparrow(1,2)$ reveals that there are — besides the trivial representation — the following three families:

	irred. rep.	Λ	C
continuous series	$C'^{k,q}, 0 \leq k < 1, q > k(1-k)$	$k + \mathbb{Z}$	q
discrete series	$D^{-k}, k \in \mathbb{R}^+$	$-k - \mathbb{N}_0$	$k(1-k)$
	$D^k, k \in \mathbb{R}^+$	$k + \mathbb{N}_0$	$k(1-k)$

We now have to select the appropriate representations from the mathematically possible ones in accordance with the general principles outlined in Sec. 2. This will be done by checking the classical property $p > 0$ (in complete analogy with $q > 0$ for $T^*\mathbb{R}^+$, cf Subsec. 2.2.2). According to Eqs. (7) and (10) this enforces the spectrum Λ of H to be purely positive.

In the continuous series the spectrum Λ is unbounded from both sides so that these representations are to be disregarded. The same applies to the negative discrete series, where the spectrum is purely negative. The condition of positive spectrum of H is thus fulfilled only in the positive discrete series (for arbitrary parameter $k \in \mathbb{R}^+$). In this case there is a ground state $\phi_{\lambda_0}^q$, $\lambda_0 = k$, $q = k(1-k)$, which is annihilated by T_-.

The choice D^k now already determines the quantum theory of $\mathcal{S}$ in the group theoretical framework. In the following we provide one possible realization of this Hilbert space by means of antiholomorphic functions on the unit disc[9].

For $k > \frac{1}{2}$ a representation on the Hilbert space $\mathcal{H}_k(\mathcal{D})$ of antiholomorphic functions on the unit disc $\mathcal{D} := \{z \in \mathbb{C} | z\overline{z} < 1\}$ with inner product

$$(f, g)_k = \frac{2k-1}{2\pi i}\int_\mathcal{D}\overline{f(z)}g(z)(1 - z\overline{z})^{2k-2}dzd\overline{z} \tag{17}$$

is given by [19]

$$(D^k(\gamma, \omega)f)(\overline{z}) = \exp(2ik\omega)(1 - |\gamma|^2)^k(\overline{\gamma}z + \exp(2i\omega))^{-2k}f\left(\frac{\overline{z} + \gamma\exp(2i\omega)}{\overline{\gamma}z + \exp(2i\omega)}\right), \tag{18}$$

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[9] For further realizations we refer to Ref. [4] and to Subsec. 3.2.5 below

where (γ, ω) parameterize the universal covering of $SU(1,1)$ (see Subsec. 3.1.4). The factor $\exp(2ik\omega)$ determines for which values of k the representation can be projected to a representation of $SU(1,1)$ or $SO^\uparrow(1,2)$.

An orthonormal basis of $\mathcal{H}_k(\mathcal{D})$ which diagonalizes $H = -iT_0$ is given by the functions

$$g_{k,n}(z) = \sqrt{\frac{\Gamma(2k+n)}{\Gamma(2k)\Gamma(n+1)}}\, \overline{z}^n \quad , \quad n \in \mathbb{N}_0. \tag{19}$$

For $0 < k \leq \frac{1}{2}$ the Hilbert space $\mathcal{H}_k(\mathcal{D})$ can be defined by completing the span of the orthonormal basis $\{g_{k,n}\}_{n\geq 0}$.

By differentiating and using Eq. (9), we get the representations

$$\begin{aligned} H &= k + \overline{z}\frac{d}{d\overline{z}} \\ T_+ &= -2k\overline{z} - \overline{z}^2\frac{d}{d\overline{z}} \\ T_- &= -\frac{d}{d\overline{z}} \end{aligned} \tag{20}$$

of the generators $T_0 = iH$, $T_1 = (T_+ - T_-)/2$, and $T_2 = i(T_+ + T_-)/2$ of $SU(1,1)$. On the elements $g_{k,n}$ of the orthonormal basis (19) they act as

$$\begin{aligned} Hg_{k,n} &= (k+n)g_{k,n} \\ T_+g_{k,n} &= -\sqrt{(2k+n)(n+1)}g_{k,n+1} \\ T_-g_{k,n} &= -\sqrt{n(2k+n-1)}g_{k,n-1} \end{aligned} \tag{21}$$

which is identical to Eqs. (16) if we use the relations $\lambda = k+n$ and $q = k(1-k)$, choosing the phases ω_{k+n} to be -1.

According to Eq. (7) the spectrum of p in a quantization of $\mathcal{S}$ using the $SO^\uparrow(1,2)$ action is given by the spectrum of H. Reintroducing Planck's constant (cf the discussion in Sec. 2), we get the following quantization map

$$\hat{p} = \frac{\hbar}{i}T_0 = \hbar H \quad , \quad \widehat{(p\sin\varphi)} = \frac{\hbar}{i}T_1 \quad , \quad \widehat{(p\cos\varphi)} = \frac{\hbar}{i}T_2. \tag{22}$$

Thus we obtain a one–parameter family of inequivalent quantizations with spectra $\hbar(k+\mathbb{N})$, $k \in \mathbb{R}^+$, of $\hat{p}$. On the other hand, from the point of view of geometric quantization [1], the ambiguity in different quantum realizations should be parameterized by a parameter living on a circle (θ–angle) (cf our discussion in Sec. 2.2.1). Similarly, application of the alternative quantization scheme using the projection method, which is presented in the subsequent subsection, will be seen to yield $k \in (0,1]$ or, better, $k \in S^1$.

From this we conclude that representations characterized by values of k larger than one should be regarded as "unphysical" in the group theoretical quantization — similar to discarding the continuous or negative discrete series of representations. Note, however, that

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within the scheme of group theoretical quantization this cannot be obtained by a natural condition such as $p > 0$, because all values of k are obtained here on an equal footing. (Restriction to representations of effectively acting admissible groups, on the other hand, leads to $k \in \mathbb{N}$ only; this merely excludes the θ–parameter (obtained from permitting also almost effective group actions) and still leaves k unbounded.)

3.2.2 Quantum realization via restriction of a Hilbert space

By definition, our phase space $\mathcal{S}$ is the restriction of T^*S^1 to positive values of the canonical momentum p such that it can be treated by using the projection method. Quantizing $\mathcal{S} \subset T^*S^1$, we thus proceed as follows: We *first* quantize T^*S^1, which is standard and which we reviewed in Sec. 2 (from various perspectives). Thereafter, in a second step, we implement the condition $p > 0$ using the projector to the positive part of the spectrum of $\hat{p}$. (Cf Subsec. 2.5 for the strategy in general context.)

More precisely, in Sec. 2 we observed that the spectrum of $\hat{p}$ in the Hilbert space $\widetilde{\mathcal{H}}_\theta$ spanned by quasi–periodic functions on S^1 characterized by θ with inner product $(f, g) = (2\pi)^{-1} \int_{S^1} \overline{f} g \, d\varphi$ is $\{\hbar(m + \theta), m \in \mathbb{Z}\}$. The respective eigenstates $f_{\theta,m} := \exp(i(m + \theta)\varphi)$, $m \in \mathbb{Z}$, form an orthonormal basis of $\widetilde{\mathcal{H}}_\theta$. The condition $\hat{p} > 0$ is met on any subspace $\mathcal{H}_{\theta+m_{min}}$ of $\widetilde{\mathcal{H}}_\theta$ which is spanned by the vectors $f_{\theta,m}$ with $m \geq m_{min} \in \mathbb{N}_0$.

According to the general strategy of the projection method in Subsec. 2.5 we have to demand here $m_{min} = 0$ to obtain the maximal Hilbert subspace on which $\hat{p} > 0$ is fulfilled. As Hilbert spaces of $\mathcal{S}$ we will only regard $\mathcal{H}_\theta$, i.e., those with $m_{min} = 0$. In the case of $T^*\mathbb{R}^+$ the requirement of maximality was necessary so as to reproduce standard results on the quantization of this phase space (including those of group theoretical quantization). To achieve maximality also in the case of $\mathcal{S}$, on the other hand, forces us to restrict the outcome of the group theoretical quantization by declaring representations with $k > 1$ as "unphysical". For mathematical reasons it is, however, instructive in some contexts to leave m_{min} unspecified and discuss observables on all spaces $\mathcal{H}_{\theta+m_{min}}$; we will do so in Subsecs. 3.2.3 and 3.2.5 below.

All infinite–dimensional, separable Hilbert spaces are isomorphic to one another; additional structures arise only through the representation of some elementary set of observables in $\mathcal{H}_{\theta+m_{min}}$, which is induced by the respective representation in $\widetilde{\mathcal{H}}_\theta$.

We choose p and $U := \exp i\varphi$ as such a set of elementary functions. Their action on the basis $\{f_{\theta,m}, m \geq m_{min}\}$ of $\mathcal{H}_{\theta+m_{min}}$ is provided by $\hat{p} f_{\theta,m} = \hbar(m+\theta) f_{\theta,m}$ and $\hat{U} f_{\theta,m} = f_{\theta,m+1}$, where $\hat{U}$ is the obvious multiplication operator and $\hat{p} = -i\hbar \, (d/d\varphi)$. The Poisson algebra $\{U, p\} = iU$ is turned correctly into the commutation relations $[\hat{U}, \hat{p}] = -\hbar \hat{U}$.

Classically $p > 0$ and $\overline{U} U \equiv U \overline{U} = 1$. By construction of $\mathcal{H}_{\theta+m_{min}}$, $\hat{p}$ becomes positive also as an operator, and it remains self–adjoint. On the other hand, $\hat{U}$ although unitary in $\widetilde{\mathcal{H}}_\theta$, is only isometric in $\mathcal{H}_{\theta+m_{min}}$: one still finds $\hat{U}^* \hat{U} = \mathbb{1}$, $\hat{U}^*$ denoting the adjoint of $\hat{U}$, but now, due to the existence of a lowest lying state $f_{\theta,m_{min}}$ in $\mathcal{H}$ (which can be interpreted as corresponding classically to the boundary $p = 0$ of $\mathcal{S}$), $\hat{U}\hat{U}^*$ is equal only to the projector $\mathbb{1} - P_{m_{min}} \neq \mathbb{1}$ (where $P_{m_{min}}$ is the projector on the state $f_{\theta,m_{min}}$). Such a feature has been

observed already in the general context in Subsec. 2.5, and one can ask for a substitute of $\hat{U}$ with improved properties. However, as will be found in the next subsection, the operator corresponding to $\exp i\varphi$ cannot be made unitary in the group theoretical approach as well. Since unitarity of $\hat{U}$, generating translations in p as a consequence of the commutation relations, is incompatible with the restriction of the phase space, isometry is the most that can be achieved for $\hat{U}$ in a quantum theory of $\mathcal{S}$.

We finally remark that over the complex numbers the Poisson algebra of U and p is a two–dimensional affine Lie algebra and indeed $\mathcal{H}_{\theta+m_{min}}$ provides an irreducible representation of it. However, this representation is *not* unitary (even in $\widetilde{\mathcal{H}}_\theta$) and it cannot be so as a consequence of the complex structure constants appearing already in the *classical* Poisson algebra.

The classical (U,p)–algebra closes over the real numbers only when taking the real and imaginary part of U, $\cos\varphi$ and $\sin\varphi$, as separate generators. Together with p they then provide the Lie algebra of E_2 and this was precisely the algebra that yielded $\widetilde{\mathcal{H}}_\theta$, the quantum theory for T^*S^1, and not the present quantum realization in $\mathcal{H}_{\theta+m_{min}}$. This mirrors the fact that $\{\cdot,\cos\varphi\}$ and $\{\cdot,\sin\varphi\}$ cannot be used as generating vector fields on $\mathcal{S} \subset T^*S^1$ (being transversal to the boundary $p = 0$), so that they do not exponentiate to the action of a group on $\mathcal{S}$. To apply the group theoretical approach we, therefore, needed to discuss the more involved group actions provided in the previous subsection.

3.2.3 Equivalence of the two approaches

If we compare the spectra of $\hat{p}$ obtained in the approaches above, we see that they are compatible: With the identification $\theta + m_{min} = k$ of the respective parameters labeling the Hilbert spaces, the operators $\hat{p}$ of the two quantizations can be identified. We are thus lead to the following Hilbert space isomorphism between $\mathcal{H}_{\theta+m_{min}}$ and $\mathcal{H}_k(\mathcal{D})$: $f_{\theta,n+m_{min}} \mapsto g_{k,n}$, $n \in \mathbb{N}_0$.

The identification of the creation operator $\hat{U}$ of Subsec. 3.2.2 with the appropriate operator in Subsec. 3.2.1 is somewhat more involved. Classically, $U = \cos\varphi + i\sin\varphi$. Thus a first ansatz, ignoring factor ordering problems, for defining the operator $\hat{U}$ in Subsec. 3.2.1 could be of the form T_+H^{-1}, which has the correct classical limit $\cos\varphi + i\sin\varphi$ (using Eq. (22) and the definition of T_+). Again this is a creation operator. However, it cannot be identified with $\hat{U}$ of Subsec. 3.2.2 as the latter operator respects the norm — being isometric — while T_+ (or likewise T_+H^{-1}) does not.

The deficiency of this ansatz can be traced back to the fact that $T_-T_+ \neq H^2$, although the classical limit of this relation yields an equality, namely $p^2\sin^2\varphi + p^2\cos^2\varphi = p^2$. This is very similar to the difficulties of maintaining the relation $\cos^2\varphi + \sin^2\varphi = 1$ in a quantum theory of T^*S^1 discussed in Ref. [2] and we now apply a similar strategy as the one of Isham to cure our problems here. Related issues for $\mathcal{S}$ will be discussed in detail also in the next subsection.

Classically, there are certainly various possibilities to express the function $U = \exp(i\varphi)$

on phase space $\mathcal{S}$. One such a possibility is provided by

$$U = \frac{p\cos\varphi + ip\sin\varphi}{\sqrt{(p\sin\varphi)^2 + (p\cos\varphi)^2}} \,. \tag{23}$$

This function on $\mathcal{S}$ is readily translated into the operator $T_+(T_-T_+)^{-\frac{1}{2}}$ (again using Eq. (22)). Note that T_-T_+ is a positive, essentially self–adjoint operator having eigenvalues $q+\lambda(\lambda+1)$ on the states ϕ_λ^q so that this expression is a well–defined operator. (The minimal of these eigenvalues is given by $2k$ ($q = k(1-k)$ and $\lambda \geq k$ for the representation D^k in the positive discrete series). Thus, the operator $T_+(T_-T_+)^{-\frac{1}{2}}$ is well–defined only for $k > 0$, which is consistent with the fact that only under this condition the representation D^k is *unitary*.) Using the Hilbert space isomorphism between $\mathcal{H}_{\theta+m_{min}}$ and $\mathcal{H}_k(\mathcal{D})$ it is then easily verified (using Eq. (16)) that $\hat{U}$ and $T_+(T_-T_+)^{-\frac{1}{2}}$ act identically on the Hilbert space and thus may be identified. (There are factor ordering problems in defining $T_+(T_-T_+)^{-\frac{1}{2}}$ as a quantization of the classical expression (23). They are, however, fixed by asking for a quantization which acts isometrically to make possible an identification with $\hat{U}$.)

In particular, now the adjoint of $T_+(T_-T_+)^{-\frac{1}{2}}$ is $(T_-T_+)^{-\frac{1}{2}}T_-$, and we have the relations

$$\hat{U}^*\hat{U}\phi_\lambda^q = (T_-T_+)^{-\frac{1}{2}}T_-T_+(T_-T_+)^{-\frac{1}{2}}\phi_\lambda^q = \phi_\lambda^q$$

for all λ, and

$$\hat{U}\hat{U}^*\phi_\lambda^q = T_+(T_-T_+)^{-\frac{1}{2}}(T_-T_+)^{-\frac{1}{2}}T_-\phi_\lambda^q = (1 - \delta_{\lambda\lambda_0})\phi_\lambda^q.$$

This demonstrates the already known isometry and nonunitarity of $T_+(T_-T_+)^{-\frac{1}{2}} = \hat{U}$.

Up to now we expressed the operator $\hat{U}$ obtained in Subsection 3.2.2 in terms of T_+ and T_-. Conversely, we can express T_+ and T_- in terms of $\hat{p}$ and $\hat{U}$ (cf Eq. (21)):

$$T_+ = -\hbar^{-1}\sqrt{(\hat{p} + (k-1)\hbar)\,(\hat{p} - k\hbar)}\,\hat{U}\,, \tag{24}$$

while $T_- = T_+^*$.

The constructions of the present subsection provide appropriate identifications of the operators obtained in the two quantization schemes. These results hold true also for values $k > 1$, if we relax the maximality condition when using the projection method (then m_{min} is not necessarily zero), which will be necessary in Subsect. 3.2.5 to obtain realizations of the complete positive discrete series.

When quantizing $\mathcal{S}$ using the projection method we have, however, to demand $m_{min} = 0$. If we restrict k to lie in $(0,1]$ in the group theoretical quantization, the identifications of this subsection prove equivalence of the two approaches. In Subsec. 3.2.5, we will make this more explicit by studying the isomorphism of the respective Hilbert spaces in terms of function spaces.

3.2.4 Ambiguities connected with the parameter k

By comparing two quantizations, namely the group theoretical one and a quantization using the projection method, we arrived in the preceding subsections at a one–parameter

family of inequivalent quantum theories labeled by the parameter $k \in (0, 1]$. Such an ambiguity has to be expected because of $\pi_1(\mathcal{S}) = \mathbb{Z}$ (cf our discussion in Sec. 2.2.1).

Nevertheless, one could be tempted (as, e.g., the authors of Ref. [6] and Ref. [7]) to restrict this arbitrariness further by demanding that the Casimir operator $C = T_0^2 - T_1^2 - T_2^2$, whose eigenvalue $q = k(1 - k)$ determines a particular representation of the positive discrete series, should be zero (yielding $k = 1$; note that $k > 0$ for unitary representations). The apparently best argument for this step would be provided by the fact that the classical limit of C, $p^2 - p^2(\sin^2 \varphi + \cos^2 \varphi)$, vanishes identically. However, this reasoning is not compelling: Using the group theoretical quantization, we know the quantum operators corresponding to the generators p, $p \sin \varphi$, and $p \cos \varphi$, but we cannot unambiguously determine the quantization of, e.g., $\sin \varphi$ or $\cos \varphi$ (we have to divide by $\hat{p}$ in some appropriate sense), the sum of whose squares was used as one in the above conclusion. Because of factor ordering ambiguities we have to distinguish between the operators $\widehat{p} \, \widehat{\sin \varphi}$ and $\widehat{(p \sin \varphi)}$, for instance, whereas in the classical expression we can simply factor out p.

Imposing $C = 0$ to exclude representations with $k \neq 1$ is basically the argument provided in Ref. [6] (leading to Eq. (3.14) of Ref. [6]). Also in the algebraic quantization, mainly used in that paper, noninteger values of k excluded there arise when factor ordering ambiguities are taken into account. The argumentation in Sec. 3.2 of Ref. [7], on the other hand, would even lead to the trivial representation (all Ts vanishing) as the only quantum realization of $\mathcal{S}$ (not only to $k = 1$ as concluded there).

As discussed above, the condition $C = 0$ just imposes the relation $\widehat{(p \sin \varphi)}^2 + \widehat{(p \cos \varphi)}^2 = \hat{p}^2$. However, because of factor ordering ambiguities, this says nothing about the quantum version of $\sin^2 \varphi + \cos^2 \varphi = 1$, which would be used as an argument for imposing it. We thus do not find it convincing to impose $C = 0$ as a condition for singling out the value $k = 1$.

To round off the above discussion, we provide natural quantizations of $\sin \varphi$ and $\cos \varphi$ inspired by the quantization of $U = \exp i\varphi$ in the preceding subsections. As demonstrated there, it is possible to restrict the freedom in defining $\widehat{\sin \varphi}$ and $\widehat{\cos \varphi}$ by demanding that the quantization of $U \equiv \cos \varphi + i \sin \varphi$ acts isometrically. This leads to[10]

$$\widehat{\sin \varphi} := -\frac{i}{2} \left(\hat{U} - \hat{U}^* \right) = -\frac{i}{2} \left(T_+ (T_- T_+)^{-\frac{1}{2}} - (T_- T_+)^{-\frac{1}{2}} T_- \right), \tag{25}$$

$$\widehat{\cos \varphi} := \frac{1}{2} \left(\hat{U} + \hat{U}^* \right) = \frac{1}{2} \left(T_+ (T_- T_+)^{-\frac{1}{2}} + (T_- T_+)^{-\frac{1}{2}} T_- \right), \tag{26}$$

which are self–adjoint operators with the correct classical limits. Although these expressions may appear rather complicated (as compared to T_1 and T_2 for $p \sin \varphi$ and $p \cos \varphi$), they are seen to come as close to the classical properties of $\sin \varphi$ and $\cos \varphi$ as possible in

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[10]After completing this work we became aware of the fact that a similar strategy has been followed in Ref. [20] (for $m_{min} = \theta = 0$ in our notation, i.e., for parameters where the identification with the group theoretical quantization breaks down) in the context of quantum optics, where the phase space $\mathcal{S}$ plays a major role (cf Ref. [21]).

the present context: First, they satisfy

$$\widehat{(\sin\varphi)}^2 + \widehat{(\cos\varphi)}^2 = 1 - \frac{1}{2}P_{\lambda_0},$$

violating $\sin^2\varphi + \cos^2\varphi = 1$ only in the ground state characterized by $\lambda = \lambda_0$ (P_{λ_0} denotes the projector on that state). Second, also the commutator

$$[\widehat{\sin\varphi}, \widehat{\cos\varphi}] = \frac{i}{2}P_{\lambda_0}$$

is nonvanishing only in the lowest state, whereas the commutators

$$[H, \widehat{\sin\varphi}] = -i\,\widehat{\cos\varphi} \quad , \qquad [H, \widehat{\cos\varphi}] = i\,\widehat{\sin\varphi} \tag{27}$$

represent the classical Poisson relations exactly.

These are only minor violations of the classical identities, which are, moreover, independent of the value of $k \in (0,1]$. Note also that there can be *no* self–adjoint and commuting operators s and c with $[H,s] = -ic$, $[H,c] = is$ which also satisfy $s^2 + c^2 = 1$ in a quantum theory of $\mathcal{S}$. Otherwise, the operator $c + is$ would be a quantization of $U = \exp i\varphi$ as a *unitary* operator generating translations, which is a contradiction according to the discussion in Subsec. 3.2.2.

3.2.5 Different realizations of the positive discrete series on function spaces over S^1

By choosing the φ–representation of the Hilbert space $\mathcal{H}_{\theta+m_{min}}$ in Subsections 3.2.2 and 3.2.3, we are implicitly provided with a realization of the representation D^k ($k \equiv \theta + m_{min}$) on a space of sections of a (trivial) bundle over S^1 with a connection characterized by θ. On the other hand, by restricting the elements of the representation space $\mathcal{H}_k(\mathcal{D})$ of Subsection 3.2.1 to its boundary values, we obtain a realization of D^k on a space of functions on S^1, too. (Similar transitions between different Hilbert spaces have been discussed in more detail in Ref. [4].) Now we want to compare these two different realizations. In order to cover all the inequivalent representations in the positive discrete series ($k \in \mathbb{R}^+$), we drop here the condition $k \in (0,1]$ ("physical" representations in the group theoretical quantization or, respectively, $m_{min} = 0$ (maximality in the projection method)).

To allow a comparison, we first transform the space $\mathcal{H}_{\theta+m_{min}}$ into a function space over S^1 as well (more precisely, we trivialize the bundle, transferring the θ–dependence of the transition function into the momentum operator, cf our discussion in Sec. refStandard). This is done most easily by multiplying the elements $f_{\theta,m}$ by $\exp(-ik\varphi)$, yielding $\exp(in\varphi)$, $n \equiv m - m_{min} \in \mathbb{N}_0$ as the new orthonormal basis elements of a Hilbert space, which is denoted by H_+^2: It is the Hardy space of the unit circle (cf Ref. [23] for further details on this space). Note that the inner product is unaltered by the above transition and still provided by $(2\pi)^{-1} \int d\varphi\, \overline{\psi_1(\varphi)}\, \psi_2(\varphi) =: (\psi_1, \psi_2)_+$.

On H_+^2 the $so(1,2)$–generators are then easily seen to take the form (cf Eq. (24)):

$$T_0 = \frac{d}{d\varphi} + ik \ , \quad T_+ = -\exp(i\varphi)\sqrt{\left(2k - i\frac{d}{d\varphi}\right)\left(1 - i\frac{d}{d\varphi}\right)} \ , \quad T_- = T_+^* \ . \qquad (28)$$

By exponentiation this provides a unitary (irreducible) representation of the universal covering group of $SO^\uparrow(1,2)$, being a concrete realization of D^k on H_+^2.

In the following this realization shall be compared to the one obtained by restricting elements of $\mathcal{H}_k(\mathcal{D})$ to their boundary values on $S^1 = \partial\mathcal{D}$, which leads to a Hilbert space $\mathcal{H}_k(S^1)$. Because an antiholomorphic function on $\mathcal{D}$ is already determined by its boundary values, the inner product of $\mathcal{H}_k(S^1)$ is defined by anti–analytically continuing two given functions on S^1 into $\mathcal{D}$ and using the inner product $(\cdot,\cdot)_k$ of $\mathcal{H}_k(\mathcal{D})$ defined in Subsection 3.2.1. In this way an orthonormal basis in $\mathcal{H}_k(S^1)$ is seen to be provided by (cf Eq. (19))

$$\widetilde{g}_{k,n}(\varphi) = \sqrt{\frac{\Gamma(2k + n)}{\Gamma(2k)\Gamma(n + 1)}}\, e^{in\varphi} \quad , \quad n \in \mathbb{N}_0 \ . \qquad (29)$$

Here we used the coordinate φ on S^1 in reversed orientation as compared to the standard definition. This leads to $\overline{z} = \exp(i\varphi)$, slightly simplifying the following relations.

Note that except for $k = 1/2$ there is *no* representation of the inner product $(\cdot,\cdot)_k$ of $\mathcal{H}_k(S^1)$ in terms of an integral over S^1 for some measure $\mu(\varphi)$, i.e. there is no function $\mu(\varphi)$ such that $(\psi_1,\psi_2)_k = \int d\varphi\, \mu(\varphi)\,\overline{\psi_1(\varphi)}\,\psi_2(\varphi)$ except for $k = 1/2$ (in which case $\mu \equiv (2\pi)^{-1}$). This is seen most easily by inserting the orthonormal set of wave functions (29) into such an ansatz. So (for $k > 1/2$) the continuation into the disc is an essential ingredient in the definition of the inner product of $\mathcal{H}_k(S^1)$ in terms of an integral.

Alternatively, the inner product of $\mathcal{H}_k(S^1)$ may be represented as an ordinary $L^2(S^1)$–inner product with an *operator–valued* metric A_k (cf Ref. [22] for further details): $(\psi_1,\psi_2)_k = (\psi_1, A_k\,\psi_2)_+$. (This observation shows that the Hilbert spaces $\mathcal{H}_k(S^1)$ used here are identical to the Hilbert spaces $H_{A_k}^2$ of Sec. 5.3 in Ref. [4].)

In both cases H_+^2 and $\mathcal{H}_k(S^1)$ we are regarding wave functions of the form $\psi(\varphi) = \sum_{n\geq 0} a_n \exp(in\varphi)$. However, because the Hilbert spaces are completions in different inner products the function spaces are different: the Hardy space H_+^2 consists of all functions $\psi(\varphi)$ with $\sum_{n\geq 0}|a_n|^2 < \infty$, whereas in $\mathcal{H}_k(S^1)$ the functions have to obey $\sum_{n\geq 0}|a_n|^2\Gamma(2k+n)^{-1}\Gamma(2k)\Gamma(n+1) < \infty$. It follows immediately that *as function spaces* $\mathcal{H}_k(S^1) \subset H_+^2$ for $k < 1/2$, $\mathcal{H}_k(S^1) = H_+^2$ for $k = 1/2$, and $H_+^2 \subset \mathcal{H}_k(S^1)$ for $k > 1/2$. While H_+^2 is a subspace (and thus also a subset) of $L^2(S^1, d\varphi)$, $\mathcal{H}_k(S^1)$ is a subset (but not a subspace for $k < 1/2$) of $L^2(S^1, d\varphi)$ for $k \leq 1/2$ (and only for $k \leq 1/2$).

The action of the $so(1,2)$–generators in $\mathcal{H}_k(S^1)$ is derived from Eq. (20) using $\overline{z} = \exp(i\varphi)$ and $\overline{z}\,d/d\overline{z} = -i\,d/d\varphi$:

$$T_0 = \frac{d}{d\varphi} + ik \ , \quad T_+ = \exp(i\varphi)\left(-2k + i\frac{d}{d\varphi}\right) \ , \quad T_- = T_+^* \ , \qquad (30)$$

where now the adjoint is to be taken with respect to the inner product in $\mathcal{H}_k(S^1)$, certainly.

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Clearly, this presentation of the $so(1,2)$–generators as operators on wave functions over S^1 is *different* from the one obtained before in Eq. (28), *except* for $k = 1/2$ where also $\mathcal{H}_k(S^1) = H_+^2$. Eq. (28) constitutes, to the best of our knowledge, a novel realization of the positive discrete series on a space of wave functions over S^1 (namely the Hardy space). In the standard realization on wave functions over S^1, the operators have a rather simple action (provided by Eq. (30)); however, the corresponding, k–dependent Hilbert space $\mathcal{H}_k(S^1)$ carries a rather complicated and k–dependent inner product (cf the discussion above). In contrast, in the other realization the Hilbert space is simply a Hardy space with standard L^2–inner product, independently of the value of k. The price to be paid for this simplification of the Hilbert space is the appearance of roots of differential operators in the representation of the $so(1,2)$–generators (cf Eq. (28)).

Note that despite the (k–dependent) subset relations between H_+^2 and $\mathcal{H}_k(S^1)$, the $so(1,2)$–representation is certainly still irreducible in each of the respective Hilbert spaces (as is obvious from the Hilbert space isomorphism of Subsection 3.2.3); the difference in the spaces is compensated by the different action of the group generators.

4   Discussion

In Sec. 2 of this article we first motivated and recalled the basic rules for group theoretical quantization as outlined in Ref. [2]. At the example of T^*S^1 it became obvious that the strong generating principle (SGP) is an essential property to be fulfilled by the fundamental observables of the group action. Otherwise apparently admissible group actions can be provided which, however, were seen to yield an unacceptable spectrum of the momentum operator.

Checking the SGP requires the study of completeness properties of the fundamental observables generating the group action, which may be a cumbersome task for more involved phase spaces. Here Lemma 1 may be of assistance: Triviality of the center of the effectively acting projection of the canonical group was found as a necessary condition for the validity of the SGP in a wide range of cases.

We then pointed out that the lift of the diffeomorphism group of a manifold Q to $\mathcal{P} = T^*Q$ has a transitive action on (the connected parts of) $\mathcal{P}_*$ which results from $\mathcal{P}$ upon removal of the points of vanishing canonical momenta. Since, by construction, this action is also effective and Hamiltonian with momentum map (cf Lemma 2), finite–dimensional subgroups of $\text{Diff}(Q)$ are good candidates for the use in a group theoretical quantization of such subbundles. This strategy was applied in Sec. 3 to construct the $SO^\uparrow(1,2)$–action on $\mathcal{S} = T^*S^1|_{p>0}$ as the lift of the respective diffeomorphism group of S^1. Other effective actions of covering groups of $SO^\uparrow(1,2)$, found in this way as well, could be excluded by the SGP (cf also Lemma 1). In an appropriate sense (cf Theorem 3) the $SO^\uparrow(1,2)$–action on $\mathcal{S}$ was found to be the unique admissible group action for quantization of the phase space $\mathcal{S}$.

In Subsec. 2.5 we proposed a projection method for quantizing phase spaces which are appropriate submanifolds of phase spaces with known quantum realization. Examples for

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such submanifolds are $T^*\mathbb{R}^+$ and $\mathcal{S}$: The quantum theory for the phase space $T^*\mathbb{R}^+$ $(\mathcal{S})$ is obtained from the standard quantum theory of $T^*\mathbb{R}$ (T^*S^1) with its Hilbert space $\widetilde{\mathcal{H}}$ upon restriction to the maximal subspace $\mathcal{H}$ on which the operator inequality $\hat{q} > 0$ $(\hat{p} > 0)$ is satisfied. The corresponding (unique) projection operator from $\widetilde{\mathcal{H}}$ to $\mathcal{H}$ may then be used also to obtain operators defined originally only within $\widetilde{\mathcal{H}}$.

We outlined some of the prerequisites for the applicability of the projection method of quantization as well as its basic rules. It may well be that the study of further examples will lead to an adjustment and refinement of these ideas. Interesting in this context would be also a comparison of the projection method to the one of symplectic cuts [12] on their joint range of applicability.

As a possible arena for its application we discussed issues in quantum gravity, where nondegeneracy of the metric has to be imposed. In this context we remarked that the presence of constraints may lead to subtleties (in addition to the well–known problems of the quantization of constrained systems).

For $T^*\mathbb{R}^+$ the quantum theory resulting from the projection method is equivalent to the standard one for this phase space. For $\mathcal{S}$ it coincides with the quantum theory obtained from group theoretical quantization, if in addition to the negative discrete and the continuous series of the $so(1,2)$–representations (cf. Subsec. 3.1 and Ref. [4]) also representations of the positive discrete series D^k with $k > 1$ are discarded. *Within* the (present–day) scheme of group theoretical quantization this may be justified only by declaring them to be unphysical representations. In lack of a truely physical realization of the phase space $\mathcal{S}$, the above "unphysical" is not to be taken too literally. However, the resulting restriction, leaving only D^k for $0 < k \leq 1$ as possible Hilbert spaces for $\mathcal{S}$, agrees also with what one would expect on general grounds in the context of geometric quantization. The projection method yields a one–to–one relation between the θ–angle of the quantum theories of $\mathcal{S}$ and T^*S^1. Thus, agreement with other approaches to the quantization of $\mathcal{S}$ forces us to truncate the range of allowed values of k in the group theoretical one.

Within this paper we always tried to keep track of possible ambiguities in the transition from the classical to the quantum system. In the group theoretical approach this lead us to always consider representations of the universal covering of the group with admissible action. Note, however, that in all the examples studied the fundamental group of the phase space was at most $\mathbb{Z}$. The situation may become more involved for the case of nonabelian fundamental groups [10].

Acknowledgement

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