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Free Entropy Dimension ≤ 1 for Some Generators of Property T Factors of Type II_1

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Abstract: The modified free entropy dimension of certain n-tuples of self-adjoint operators, satisfying sequential commutation, is shown to be ≤ 1. In particular the von Neumann algebras of type II_1 of the groups $SL(n, \mathbb{Z})$, $n \geq 3$ have generators with free entropy dimension ≤ 1.

The free entropy $\chi(X_1, \dots, X_n)$ and the modified free entropy dimension $\delta_0(X_1, \dots, X_n)$ of an n-tuple of selfadjoint elements in a II_1-factor ([6],[7],[8]) have been the key to the solution of some old problems on von Neumann algebras ([7],[3],[4],[5]).

The natural generator of a free group factor $L(F(n))$ has δ_0 equal n, while various conditions on the II_1-factor (property Γ, existence of Cartan subalgebras) imply $\delta_0 \leq 1$ for any generator.

Here we prove $\delta_0 \leq 1$ for n-tuples of selfadjoint elements satisfying certain sequential commutation conditions and as a corollary we get that the von Neumann algebras of type II_1 $L(SL(n; \mathbb{Z}))$ $(n \geq 3)$ have generators with $\delta_0 \leq 1$.

We would like to point out a certain (not yet explained) similarity to an ergodic theory result in [2], which inspired this note.

At the end of the paper we formulate, what seems to us, a natural question about generators of $L(SL(n;\mathbb{Z}))$ in view of the free entropy dimension result.

Throughout, (M,τ) will denote a von Neumann algebra with a faithful normal trace-state and we refer the reader for the definitions of χ, δ_0 and of the sets of matricial approximants Γ to [6] and [7]. In particular,

$$\delta_0(X_1,\ldots,X_n) = n + \limsup_{\varepsilon\to 0} \frac{\chi(X_1+\varepsilon S_1,\ldots,X_n+\varepsilon S_n : S_1,\ldots,S_n)}{|\log\varepsilon|}$$

where $S_1,\ldots,S_n$ are (0,1)-semicircular and $\{X_1,\ldots,X_n\}$, $\{S_1\},\ldots,\{S_n\}$ are free.

Theorem 1 *Let $X_j = X_j^* \in M$, $1 \leq j \leq n$. Assume $[X_k, X_{k+1}] = 0$, $1 \leq k < n$ and assume that the spectral measure of X_k, $1 \leq k < n$ has no atoms. Then*

$$\delta_0(X_1,\ldots,X_n) \leq 1 \ .$$

The proof is based on a few lemmas. By $\mathcal{M}_{||}$ we will denote the $k \times k$ complex matrices and by $|\ \ |_2$ the normalized Hilbert-Schmidt norm $|A|_2^2 = k^{-1}\mathrm{Tr}(A^*A)$. Further, $(e_{pq})_{1\leq p,q\leq k}$ will be the canonical matrix-units in $\mathcal{M}_{||}$. The real subspace of selfadjoint elements will be denoted by $\mathcal{M}_{||}^{\int\dashv}$ and vol will denote the volume corresponding to the unnormalized Hilbert-Schmidt norm (i.e., $k^{\frac{1}{2}}|\cdot|_2$). We shall also use vol to denote the corresponding volume on $(\mathcal{M}_{||}^{\int\dashv})^{\backslash}$.

Lemma 1 *Let $A = \sum_{1\leq j\leq k}\lambda_j e_{jj} \in \mathcal{M}_{||}^{\int\dashv}$ where $\lambda_1 \leq \cdots \leq \lambda_k$. Assume moreover that $|\lambda_s - \lambda_t| \geq \delta$ if $|s-t| \geq k\alpha$ where $0 < \alpha < 1$. Let*

$$\Omega(A;\varepsilon) = \{B \in \mathcal{M}_{||}^{\int\dashv} \mid |\mathcal{B}|_{\in} \leq \infty,\ |[\mathcal{B},\mathcal{A}]|_{\in} \leq \varepsilon\}$$

and let $\Omega(A;\varepsilon,r)$ be the r-neighborhood $\{C \in \mathcal{M}_{||}^{\int\dashv} \mid \exists \mathcal{B} \in \Omega(\mathcal{A};\varepsilon),\ |\mathcal{B}-\mathcal{C}|_{\in} < \nabla\}$. Then if $\varepsilon\delta^{-1} + r \leq 1 + r$, we have

$$\mathrm{vol}(\Omega(A;\varepsilon,r)) \leq c((2[k\alpha]+1)k)(1+r)^{k(2[k\alpha]+1)}\cdot c((k-2[k\alpha]-1)k)(\varepsilon\delta^{-1}+r)^{k(k-2[k\alpha]-1)}k^{k^2/2}$$

where $c(m) = \pi^{m/2}(\Gamma(1+m/2))^{-1}$.

Proof. Let $\mathcal{M}_{||}^{\int\dashv} = \mathcal{V}_\infty \oplus \mathcal{V}_\in$ where $V_1 = \mathrm{span}\{e_{st} \mid |s-t| < k\alpha\}$ and $V_2 = \mathrm{span}\{e_{st} \mid |s-t| \geq k\alpha\}$. V_1, V_2 are invariant subspaces of ad A and $|\mathrm{ad}\ A(h)|_2 \geq \delta|h|_2$

if $h \in V_2$. Thus, if $h = h_1 \oplus h_2 \in \Omega(A;\varepsilon)$ then $|h_1|_2 \leq 1$ and $|h_2|_2 \leq \varepsilon\delta^{-1}$. Hence, if $h = h_1 \oplus h_2 \in \Omega(A;\varepsilon, r)$, then $|h_1|_2 \leq 1 + r$, $|h_2|_2 \leq \varepsilon\delta^{-1} + r$. We remark further that

$$\begin{aligned} \dim V_1 &\leq k(2[k\alpha] + 1) \\ \dim V_2 &\leq k(k - 2[k\alpha] - 1)\ . \end{aligned}$$

If $\varepsilon\delta^{-1} + r \ \leq 1 + r$, the volume of $\Omega(A;\varepsilon, r)$ can be majorized by the product of the volumes of a ball of dimension $k(2[k\alpha] + 1)$ and radius $(1 + r)$ and of a ball of dimension $k(k - 2[k\alpha] - 1)$ and radius $\varepsilon\delta^{-1} + r$. □

Lemma 2 *Assume the spectral projections of some* $A \in \mathcal{M}_{||}^{\int\dashv}$ *satisfy*

$$\text{Tr } E(A;(t, t + \delta)) < k\alpha$$

for all $t \in \mathbb{R}$. *Then the conclusion of Lemma 1 about* $vol(\Omega(A;\varepsilon, r))$ *holds.*

Proof. Indeed, A is unitarily equivalent to a diagonal matrix satisfying the assumptions of Lemma 1. □

We also record as the next lemma an immediate consequence of Fubini's theorem.

Lemma 3 *Let* μ_j *be measures on locally compact spaces* X_j $(1 \leq j \leq n)$ *and* $\mu = \mu_1 \otimes \cdots \otimes \mu_n$. *Let further* $Y \subset X_1 \times \ldots \times X_n$ *be an open set, such that for all* $1 \leq j \leq n$ *and* $\xi \in pr_j Y$ *we have*

$$\mu_{j+1}(pr_{j+1}(pr_j^{-1}(\xi))) \leq \rho\ .$$

Then

$$\mu(Y) \leq \mu_1(pr_1 Y) \cdot \rho^{n-1}\ .$$

Next, we define some sets which will appear in the next lemma.

By $\gamma(n;k,\varepsilon,\delta,\alpha)$ we denote the set of n-tuples $(A_1,\ldots,A_n) \in (\mathcal{M}_{||}^{\int\dashv})^{\backslash}$ such that: $|A_s|_2 < \ 1$ $(1 \leq s \leq n)$, $|[A_j, A_{j+1}]|_2 < \varepsilon$ and Tr $E(A_j;[p\delta,(p + 1)\delta]) < 2^{-1}k\alpha$ for all $p \in \mathbb{Z}$ and $1 \leq j < n$.

If $K \subset (\mathcal{M}_{||}^{\int\dashv})^{\backslash}$ we shall denote by $D(K;r)$ the set of n-tuples $(B_1,\ldots,B_n) \in (\mathcal{M}_{||}^{\int\dashv})^{\backslash}$ such that $|B_j - A_j|_2 < r$ $(1 \leq j \leq n)$ for some $(A_1,\ldots,A_n) \in K$.

Lemma 4 *Assume* X_j $(1 \leq j \leq n)$ *satisfy the assumptions of Theorem 1 and assume moreover that* $\|X_j\| < 1/2$ $(1 \leq j \leq n)$*. Let further* S_j *be* $(0,1)$*-semicircular, so that* $\{X_1, \ldots, X_n\}$, $\{S_1\}, \ldots, \{S_n\}$ *are free. Then, given* $\alpha > 0$, $\beta > 0$, *there are* $\varepsilon > 0$, $m \in \mathbb{N}$, $\varepsilon_1 > 0$, $\delta > 0$ *such that* $\varepsilon\delta^{-1} < \beta$ *and*

$$\Gamma_2(X_1 + \theta S_1, \ldots, X_n + \theta S_n : S_1, \ldots, S_n; m, k, \varepsilon_1) \subset D(\gamma(n; k, \varepsilon, \delta, \alpha); 2\theta)$$

for all $0 < \theta < 1/2$.

Proof. Given $\varepsilon_2 > 0$, for $\varepsilon_1 > 0$ small enough

$$\Gamma_2(X_1 + \theta S_1, \ldots, X_n + \theta S_n : S_1, \ldots, S_n; m, k, \varepsilon_1) \subset D(\Gamma_4(X_1, \ldots, X_n; m, k, \varepsilon_2); 2\theta) \ .$$

Thus it will suffice to prove the lemma with the inclusion at the end, replaced by

$$\Gamma_4(X_1, \ldots, X_n; m, k, \varepsilon_1) \subset \gamma(n; k, \varepsilon, \delta\alpha) \ .$$

Given $\alpha > 0$, the X_j's $(1 \leq j < n)$ having continuous spectral measures, there is $1 > \delta > 0$ such that

$$\tau(E(X_j; ((p-1)\delta, (p+2)\delta)) < 4^{-1}\alpha$$

for all $p \in \mathbb{Z}$. We then choose $\varepsilon > 0$, such that $\varepsilon\delta^{-1} < \beta$. Clearly, if $\varepsilon_1 > 0$ is small enough and $m \geq 4$ the conditions $|[A_j, A_{j+1}]|_2 < \varepsilon$ $(1 \leq j < n)$ and $|A_s|_2 < 1$ $(1 \leq s \leq n)$ will be satisfied if $(A_1, \ldots, A_n) \in \Gamma_4(X_1, \ldots, X_n; m, k, \varepsilon_1)$.

If $p \in \mathbb{Z}$ and $[(p-1)\delta, (p+2)\delta] \subset [-5, 5]$, let $P_p(t)$ be a real polynomial which is ≥ 0 for all $g \in \mathbb{R}$ and so that $P_p(t) \geq 4/5$ if $t \in [p\delta, (p+1)\delta]$ and $\tau(P_p(X_j)) < 4^{-1}\alpha$, $(1 \leq j < n)$. Then choosing $\varepsilon_1 > 0$ small enough and $m > \deg P_p$ for the finite set of p considered, we will have

$$k^{-1}\mathrm{Tr}(P_p(A_j)) < 3^{-1}\alpha \ .$$

This in turn will insure that

$$\tfrac{4}{5}k^{-1}\mathrm{Tr}(E(A_j; [p\delta, (p+1)\delta]) < 3^{-1}\alpha$$

which implies

$$k^{-1}\mathrm{Tr}(E(A_j; [p\delta, (p+1)\delta])) < 2^{-1}\alpha \ .$$

$\square$

PROOF OF THEOREM 1. There is no loss to prove the theorem under the additional assumption $\|X_s\| < 1/2$ $(1 \le s \le n)$. We shall use Lemma 4 to estimate

$$\chi(X_1 + \theta S_1, \ldots, X_n + \theta S_n : S_1, \ldots, S_n) .$$

Using 1.3 in [7] this is the same as

$$\chi_2(X_1 + \theta S_1, \ldots, X_n + \theta S_n : S_1, \ldots, S_n) .$$

By Lemma 4 we will have to estimate $\mathrm{vol}(D(\gamma(n; k, \varepsilon, \delta, \alpha); 2\theta)$. Lemma 3 gives

$$\mathrm{vol}(D(\gamma(n; k, \varepsilon, \delta, \alpha); 2\theta) \le c(k^2)(1 + 2\theta)^{k^2} k^{k^2/2} \rho^{n-1}$$

if ρ is an upper bound for $\mathrm{vol}(\Omega(A; \varepsilon, 2\theta))$ when $\mathrm{Tr}(E(A; (t, t+\delta)) < k\alpha$ for all $t \in \mathbb{R}$ (indeed this last condition follows from

$$\mathrm{Tr}(E(A_j; [p\delta, (p+1)\delta])) < 2^{-1} k\alpha \quad \text{for} \quad p \in \mathbb{Z}).$$

Further, by Stirling's formula we have

$$n^{-1} \log c(n) = 2^{-1} \log(2\pi e n^{-1}) + O(n^{-1} \log n) .$$

By Lemma 1 we can choose ρ so that

$$k^{-2} \log \rho \le 2^{-1} \log(2\pi e k^{-1}) + 2\alpha \log(1 + 2\theta) + (1 - 2\alpha) \log(\beta + 2\theta) + O(k^{-1} \log k) .$$

Thus we get

$$\begin{aligned} &\chi_2(X_1 + \theta S_1, \ldots, X_n + \theta S_n : S_1, \ldots, S_n) \\ &\le \tfrac{n}{2} \log 2\pi e + (2(n-1)\alpha + 1) \log(1+2\theta) + (1 - 2\alpha)(n-1)(\log(\beta + 2\theta)) . \end{aligned}$$

Since $\alpha > 0$, $\beta > 0$ are arbitrary we have

$$\chi_2(X_1+\theta S_1, \ldots, X_n+\theta S_n : S_1, \ldots, S_n) \le \frac{n}{2} \log 2\pi e + (n-1) \log 2\theta + \log(1 + 2\theta) .$$

Hence

$$\delta_0(X_1, \ldots, X_n) \le n + (n-1) \lim_{\theta \downarrow 0} \frac{\log 2\theta}{|\log \theta|} = 1 .$$

□

For a discrete group G let $L(G)$ denote the von Neumann algebra generated by the left regular representation λ of $L(G)$ on $\ell^2(G)$ and τ the von Neumann trace.

If G is $SL(n,\mathbb{Z})$, $(n \geq 3)$, then G is generated by the $g_{ij} = I + e_{ij} \in \mathcal{M}_\backslash$, $i \neq j$ and since g_{ij} and $g_{k\ell}$ commute if $i = k$ or $j = \ell$, we can form a generator $g_{i_1j_1}, \ldots, g_{i_pj_p}$ of $SL(n,\mathbb{Z})$ such that consecutive elements commute (repetitions are allowed) ([2]). Since $g_{i_kj_k}$ has infinite order, τ applied to the spectral measure of $\lambda(g_{i_kj_k})$ is Haar measure on the unit circle. Let log denote the Borel function on the unit circle log $\exp(2\pi i\theta) = 2\pi i\theta$ if $0 \leq \theta < 1$ and let $h_k = (2\pi i)^{-1} \log \lambda(g_{i_kj_k})$. Then $h_1, \ldots, h_p$ generate $L(SL(n,\mathbb{Z}))$, are selfadjoint, have Lebesgue absolutely continuous spectral measure, and consecutive h_k commute. Using Theorem 1 we have proved the following corollary.

Corollary 1 *$L(SL(n,\mathbb{Z}))$, $n \geq 3$, has a generator of selfadjoint elements $h_1 \ldots, h_p$ such that $\delta_0(h_1 \ldots, h_p) \leq 1$.*

Since results about δ_0 seem to have corresponding results about the number of elements of a generator, we formulate below the natural problem suggested by the preceding corollary.

Problem. Let $n \geq 3$. Does $L(SL(n,\mathbb{Z}))$ have for every $\varepsilon > 0$ a generator $\{X_1, X_2\}$ such that $X_j = X_j^*$ $(1 \leq j \leq 2)$ and $\tau(E(X_2;\{0\})) > 1 - \varepsilon\Gamma$

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