

Entropy and Asymptotic Geometry of Non-Symmetric Convex Bodies

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Abstract

We extend to the general, not necessary centrally symmetric setting a number of basic results of Local Theory which were known before for centrally symmetric bodies and were using very essentially the symmetry in their proofs. Some of these extensions look surprising. The main additional tool is a study of volume behavior around the centroid of the body.

1. Introduction

During last decade results of Geometric Functional Analysis of Finite Dimensional Normed Spaces, so called Local Theory, or also Asymptotic Theory, were regularly applied for the study of the global properties of convex (centrally-symmetric) bodies in \mathbb{R}^n . It was not the original goal of the Theory which traditionally for Functional Analysis studied the structure of subspaces, quotient spaces, operators and so on. But the level of sophistication of methods and facts allowed to deduce from the Local structure also general properties of the whole spaces and convex bodies.

However, the Theory was built for needs of the study of normed spaces and strongly used the central symmetry of convex bodies. This was not anymore a natural condition when the subject of study became a general convex body. Let us note here that it is not necessary obvious (or true) that the same facts as we know them for centrally symmetric bodies will be (essentially) true for non-symmetric ones. Indeed, let us recall an old and well known example :

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let P be a polytope in \mathbb{R}^n with $f(P)$ faces and $v(P)$ vertices, then from [FLM], for some numerical constant $c > 0$, $\log f(P) \cdot \log v(P) \geq cn$ if P is centrally symmetric, but this is completely wrong for non-symmetric as for example for the simplex.

The authors first planned to investigate these questions long ago when it became clear that Convexity theory and Computational Convexity shall need and may use many of results developed in Local Theory. The attempt was, in a sense, trivial: to repeat the construction of the whole Theory.

But the standard checking through the methods immediately revealed one difficulty : one of the most important technical tools - the Rademacher projection - estimate of Pisier used symmetry very essentially. So, we decided to go to the “final outcome” of the Theory and, using our knowledge on symmetric case to build similar not necessary symmetric theory. Of course, many more straightforward results where symmetry was essentially not used already in “symmetric” proofs are meantime written and easily available.

In section 2, we investigate the role of the barycenter, in particular, we show that if 0 is the centroid of K , then the volume of $K \cap (-K)$ is not much smaller than the volume of K . This fact plays an important role in the following study of entropy in Section 3. To understand how different is the technique of estimating entropy in non-symmetric case, consider for example, the problem of estimating $N(2K, K)$. We prove a non-symmetric version of the duality of entropy result of König-Milman [KM] and built ground for deeper study of entropy connected with M-ellipsoid that we pursue in Section 5.

In section 4, we show that “random projection” of a convex body have “bounded volume ratio” (Theorem 8) and prove the so-called (almost) Euclidean quotient spaces of subspaces result for non-symmetric body (Theorem 9), extending the corresponding result from symmetric case [M4]. This brings us to our main subject, random projection for bodies in special positions and construction of M-ellipsoid for arbitrary (not necessary symmetric) bodies. We prove in Section 5 (Theorem 11 and preceeding remarks) that for any convex body K centered at its centroid (and let $|K| = |B_2^n|$ where B_2^n is the Euclidean unit ball) there is a “position”, that is, T belonging to SL_n , such that, setting $C = TK$,

$$N(C, B_2^n) \cdot N(B_2^n, C^\circ) \cdot N(C^\circ, B_2^n) \cdot N(B_2^n, C^\circ) < e^{cn}$$

for some universal constant c . We show also how to apply this information

for global regularization of K (Theorem 13).

Through the paper we consider the space \mathbb{R}^n being equipped with its canonical Euclidean scalar product (\cdot, \cdot) and the corresponding norm $|\cdot|$. Its unit ball is denoted by B_2^n . The volume of a measurable subset A of \mathbb{R}^n is denoted by $|A|$. We denote by v_n the volume of B_2^n . The standard Gaussian measure on \mathbb{R}^n with density $e^{-|x|^2/2}/(2\pi)^{n/2}$ is denoted by γ_n .

We denote by $\text{co}(A)$ the convex hull of a subset A in \mathbb{R}^n . For any two subsets A and B of \mathbb{R}^n , and any scalars $\lambda, \mu \in \mathbb{R}$,

$$\lambda A + \mu B = \{\lambda x + \mu y; x \in A, y \in B\}$$

denotes the Minkowski sum.

Let K be convex body in \mathbb{R}^n with 0 in its interior, its polar K° is defined as usual by

$$K^\circ = \{x \in \mathbb{R}^n : (x, y) \leq 1 \text{ for every } y \in K\}.$$

2. The role of the barycenter

We say that a non-negative function ψ on \mathbb{R}^n is log-concave if $\{\psi > 0\}$ is convex and $\log \psi$ is concave on $\{\psi > 0\}$.

Lemma 1 *Let μ be a probability on \mathbb{R}^n and $\psi \in L^1(\mu)$ be a non-negative log-concave function with $\int \psi d\mu > 0$. Then*

$$\int \psi d\mu \leq \psi \left(\int x \frac{\psi(x)}{\int \psi d\mu} d\mu(x) \right).$$

Proof. Jensen's inequality applied to the convex function $t \rightarrow t \log t$ on \mathbb{R}^+ gives

$$\int \psi(x) \log \psi(x) d\mu(x) \geq \int \psi(x) d\mu(x) \log \int \psi(x) d\mu(x).$$

This can be written

$$\int \log \psi(x) \frac{\psi(x)}{\int \psi d\mu(x)} d\mu(x) \geq \log \int \psi(x) d\mu(x).$$

The concavity of $\log \psi$ on $\{\psi > 0\}$ implies that

$$\log \psi \left(\int x \frac{\psi(x)}{\int \psi d\mu} d\mu(x) \right) \geq \int \log \psi(x) \frac{\psi(x)}{\int \psi d\mu} d\mu(x) \geq \log \int \psi(x) d\mu(x),$$

which proves the lemma. \square

Theorem 2 *Let $\alpha, \beta \geq 0$ and ν be the measure on \mathbb{R}^n with density $\alpha \exp(-\beta|x|^2)$. Let $\theta \in]0, \pi/2[$ and K, L be convex compact subsets of \mathbb{R}^n . Set*

$$z = \frac{\sin \theta}{\nu(K)} \int_K x d\nu(x) - \frac{\cos \theta}{\nu(L)} \int_L y d\nu(y)$$

and

$$C(z) = \left(\frac{1}{\cos \theta} K - \frac{\sin \theta}{\cos \theta} z \right) \cap \left(\frac{1}{\sin \theta} L + \frac{\cos \theta}{\sin \theta} z \right).$$

Then we have

$$\nu(K)\nu(L) \leq \nu((\sin \theta) K + \cos \theta (-L))\nu(C(z)).$$

Proof. Let $f \in L^1(\nu)$ be a ν -integrable function, possibly vector valued and set

$$I(f) = \int_{K \times L} f(x \sin \theta - y \cos \theta) d\nu(x) d\nu(y).$$

For any $Y \in \mathbb{R}^n$, define

$$C(Y) = \left(\frac{1}{\cos \theta} K - \frac{\sin \theta}{\cos \theta} Y \right) \cap \left(\frac{1}{\sin \theta} L + \frac{\cos \theta}{\sin \theta} Y \right).$$

By the change of variable $x = X \cos \theta + Y \sin \theta$ and $y = X \sin \theta - Y \cos \theta$, $X, Y \in \mathbb{R}^n$ and because of the rotation invariant property of the measure $\nu \otimes \nu$, we get

$$I(f) = \int f(Y) \nu(C(Y)) d\nu(Y).$$

Let $M = M(\theta) = (\sin \theta) K + \cos \theta (-L)$. The function defined by $\psi(Y) = \nu(C(Y))$ for $Y \in \mathbb{R}^n$ is supported by M . Applying the above relation to the function $f(t) = t$ for $t \in M$ and 0 elsewhere, we get

$$z = \frac{1}{\nu(K)\nu(L)} \int Y \nu(C(Y)) d\nu(Y).$$

The same relation applied with $f = 1$, also gives

$$\frac{\nu(K)\nu(L)}{\nu(M)} = \int_M \nu(C(Y)) \frac{1}{\nu(M)} d\nu(Y).$$

Since ν has a log-concave density, it follows from [Bo] that the measure is log-concave; this means that for every compact subsets A and B of \mathbb{R}^n and every $\omega \in [0, 1]$, we have

$$\nu((1 - \omega)A + \omega B) \geq \nu(A)^{1-\omega} \nu(B)^\omega.$$

We deduce that the function ψ is log-concave. Applying Lemma 1 with the probability $\mu = 1_M \frac{1}{\nu(M)} \nu$ and to the log-concave function ψ , the equation

$$z = \frac{\nu(M)}{\nu(K)\nu(L)} \int Y \psi(Y) d\mu(Y) = \int Y \psi(Y) d\mu(Y) / \int \psi d\mu$$

yields to

$$\frac{\nu(K)\nu(L)}{\nu(M)} \leq \psi(z) = \nu(C(z))$$

and concludes the proof. \square

The previous result applies when the measure is the standard Gaussian probability or in the homogenous case for the Lebesgue measure.

Corollary 3 *Let K and L be two convex compact subsets of \mathbb{R}^n .*

1) If K and L have the same barycenter with respect to the measure γ_n , then

$$\gamma_n(K) \times \gamma_n(L) \leq \gamma_n\left(\frac{K+L}{\sqrt{2}}\right) \times \gamma_n(\sqrt{2}(K \cap (-L))).$$

In particular if 0 is the Gaussian barycenter of K , then

$$\gamma_n(K)^2 \leq \gamma_n(\sqrt{2}K) \times \gamma_n(\sqrt{2}(K \cap (-K))).$$

2) If K and L have the same centroid, then

$$|K| \times |L| \leq |K+L| \times |K \cap (-L)|. \quad (1)$$

In particular if 0 is the centroid of K , then

$$|K \cap (-K)| \geq 2^{-n} |K|, \quad (2)$$

and

$$\frac{|K - K|}{|K \cap (-K)|} \leq 8^n. \quad (3)$$

Proof. We apply Theorem 2 with $\theta = \pi/4$. The last inequality combines the following result of Rogers and Shephard [RS1]:

$$|K - K| \leq \binom{2n}{n} |K| \leq 4^n |K|. \quad (4)$$

□

Remarks. 1. Inequality (1) is well known for symmetric convex bodies (see [RS1]). There is a reverse form in [RS1] which states that for any convex compact subsets K and L of \mathbb{R}^n , we have

$$\binom{2n}{n} |K| \times |L| \geq |K + L| \times |K \cap (-L)|. \quad (5)$$

For our purpose, there will be no essential difference in considering $K - K$ or $\text{co}(K \cup (-K))$ when $0 \in K$. Indeed, we have $\text{co}(K \cup (-K)) \subset K - K \subset 2 \text{co}(K \cup (-K))$. An inequality similar to (4) is proved in [RS2], for the convex hull: let K be a convex compact subset of \mathbb{R}^n containing 0 , then

$$|\text{co}(K \cup (-K))| \leq 2^n |K|. \quad (6)$$

2. Let $E \oplus F = \mathbb{R}^n$ be an orthogonal decomposition of \mathbb{R}^n and denote by $P_F(K)$ the orthogonal projection of K onto F . We have

$$\frac{|K|}{|P_F(K)|} = \int_{P_F(K)} |K \cap (x + E)| \frac{1}{|P_F(K)|} dx.$$

Let $d\mu(x) = 1_{P_F(K)} \frac{1}{|P_F(K)|} dx$ and $\psi(x) = |K \cap (x + E)|$. From Brunn-Minkowski theorem, ψ is log-concave. Applying lemma 1 we obtain that

$$|K| \leq |P_F(K)| \times |K \cap (x_0 + E)|$$

where $x_0 = \int_{P_F(K)} x |K \cap (x + E)| \frac{1}{|K|} dx = P_F(\frac{1}{|K|} \int_K z dz)$. In particular if 0 is the centroid of K , we get the following result of Spingarn [Sp]:

$$|K| \leq |P_F(K)| \times |K \cap E|.$$

3. The convolution relation

$$\int_K |(2x - K) \cap K| \frac{dx}{|K|} = 2^{-n} |K|$$

gives $\max_{x \in K} |(2x - K) \cap K|/|K| \geq 2^{-n}$ (see [Gr] where this ratio is called Kovner-Besicovitch measure of symmetry). It seems reasonable to think that the maximum ratio is obtain for the simplex. Concerning the inequality (2), the best previously known estimate for $|K \cap (-K)|/|K|$ which is referred in [Gr] when 0 is the centroid of K , was of the order of $1/n^n$.

4. Bourgain-Milman inequality [BM] states that for any convex body C with 0 in its interior,

$$c_0^n v_n^2 \leq |C| \times |C^\circ| \quad (7)$$

where $c_0 > 0$ is a universal constant.

Let K be a convex body with 0 in its interior and such that $|K| \times |K^\circ| \leq c^n v_n^2$ for some constant c . So we have: $c_0^n v_n^2 \leq |K \cap -K| \times |\text{co}(K^\circ \cup -K^\circ)|$. Using (6) we arrive at

$$c_0^n v_n^2 \leq 2^n |K \cap -K| \times |K^\circ| \leq 2^n \frac{|K \cap -K|}{|K|} c^n v_n^2,$$

so that

$$2^{-n} (c_0/c)^n \leq \frac{|K \cap -K|}{|K|}.$$

Unlike inequality (7), Santaló inequality, $|K| \times |K^\circ| \leq v_n^2$ is not valid for any position with respect to 0. However, as proved by Santaló [S], there is always a shift of K such that it is true. It is satisfied if 0 is the so-called Santaló point of K ; the point that minimizes the volume product $|K| \times |K^\circ|$ when it is taken as origin for polarity [S]. This point is characterized implicitly by the fact that it is the centroid of K° . So if 0 is Santaló point of K , then $2^{-n} c_0^n \leq \frac{|K \cap -K|}{|K|}$. We were informed by Rudelson that he also received this inequality.

5. Still using Lebesgue measure but with L as a dilation of K , Theorem 2 implies that for any convex body K in \mathbb{R}^n with 0 as centroid, then for every $t \geq 0$,

$$\frac{|K \cap (-tK)|}{|K|} \geq \left(\frac{t}{1+t}\right)^n.$$

6. Using the method of Theorem 2, inequality (1) may be generalized to more than 2 bodies. For instance, we get that if K_1, K_2, K_3 are convex bodies with the same centroid, then

$$|K_1| \cdot |K_2| \cdot |K_3| \leq 3^{-n} |K_1 \cap K_2 \cap K_3| \cdot |2K_1 - K_2 - K_3| \cdot |2K_2 - K_1 - K_3|.$$

3. Duality of entropy

Let A and B be two subsets of \mathbb{R}^n , the covering number $N(A, B)$ is defined as usual as

$$N(A, B) = \min \left\{ \sharp \Lambda : \Lambda \subset \mathbb{R}^n, A \subset \Lambda + B \right\}.$$

The packing number $P(A, B)$ is defined by

$$P(A, B) = \max \left\{ \sharp \Lambda : \Lambda \subset A, \forall x, y \in \Lambda, x \neq y, (x + \text{int}(B)) \cap (y + \text{int}(B)) = \emptyset \right\}$$

where $\text{int}(B)$ denotes the interior of B .

There is an important difference between these numbers especially when B is not centrally symmetric. The following lemma bring together some easy facts.

Lemma 4 *Let A and B be convex compact subsets of \mathbb{R}^n with 0 is in their interior. Define α by $\alpha^n = \frac{|A-A|}{|A \cap (-A)|}$ then*

$$N(A, B - B) \leq P(A, B) \leq \frac{|A + B|}{|B|}, \quad (8)$$

and

$$N(A - A, A \cap (-A)) \leq 3^n \alpha^n. \quad (9)$$

Proof. To prove the first inequality, let $\{x_1, \dots, x_P\}$ be a subset of P points of A with $P = P(A, B)$ and satisfying $(x_i + \text{int}(B)) \cap (x_j + \text{int}(B)) = \emptyset$ for all $1 \leq i < j \leq P$. Then for every $x \in A$, there exists $1 \leq j \leq P$, such that $(x + \text{int}(B)) \cap (x_j + \text{int}(B)) \neq \emptyset$. Then $x - x_j \in B - B$ or $x \in x_j + (B - B)$, which means that $N(A, B - B) \leq P$. Since $\cup_{1 \leq i \leq P} (x_i + B) \subset A + B$, we have $P|B| \leq |A + B|$.

The second inequality (9) is a consequence of the following well known fact: if C and D are centrally symmetric then $N(C, D) \leq 3^n |C|/|C \cap D|$. \square

Remark. The covering number $N(2A, A)$ depends strongly on the choice of 0. Estimating this number is trivial in the centrally symmetric case. If 0 is the centroid of an n -dimensional convex body A , inequality (1) yields

$$N(2A, A) \leq N(2A, A \cap -A) \leq 10^n.$$

It is a long standing and fascinating problem to understand as precisely as possible the duality of covering numbers (or, shortly, entropy). In the next statement we extend to a non-symmetric setting the result by König and Milman [KM].

Theorem 5 *There exist a constant $c_1 > 0$ such that for any integer $n \geq 1$ and any convex compact subsets A and B of \mathbb{R}^n with θ as centroid, we have*

$$\frac{1}{c_1} N(B^\circ, A^\circ)^{1/n} \leq N(A, B)^{1/n} \leq c_1 N(B^\circ, A^\circ)^{1/n}.$$

Proof. Define α and β by $\alpha^n = \frac{|A - A|}{|A \cap (-A)|}$ and $\beta^n = \frac{|B - B|}{|B \cap (-B)|}$, then using (9) we have

$$\begin{aligned} N(\text{co}(A \cup -A), B \cap (-B)) &\leq N(A - A, B \cap (-B)) \\ &\leq N(A - A, A) N(A, B) N(B, B \cap (-B)) \\ &\leq N(A - A, A \cap (-A)) N(A, B) N(B - B, B \cap (-B)) \\ &\leq (9\alpha\beta)^n N(A, B). \end{aligned}$$

For centrally symmetric convex bodies, the result on duality from [KM], gives, for some universal constant c_2 , that

$$N(\text{co}(B^\circ \cup -B^\circ), A^\circ \cap (-A^\circ)) \leq c_2^n N(\text{co}(A \cup -A), B \cap (-B)).$$

Thus we get

$$N(B^\circ, A^\circ) \leq N(\text{co}(B^\circ \cup -B^\circ), A^\circ \cap (-A^\circ)) \leq (9\alpha\beta c_2)^n N(A, B).$$

Since 0 is the common centroid of A and B , inequality (3) gives $\alpha, \beta \leq 8$. This concludes the left-hand inequality.

The reverse inequality is proved in the same way. We estimate the numbers $|A^\circ - A^\circ|/|A^\circ \cap (-A^\circ)|$ and $|B^\circ - B^\circ|/|B^\circ \cap (-B^\circ)|$ using Santaló and Bourgain-Milman inequalities;

$$\frac{|A^\circ - A^\circ|}{|A^\circ \cap (-A^\circ)|} \leq 2^n \frac{|\text{co}(A^\circ \cup -A^\circ)|}{|A^\circ \cap (-A^\circ)|} \leq 2^n \frac{|\text{co}(A \cup -A)|}{c_0^n |A \cap (-A)|} \leq \frac{2^n \alpha^n}{c_0^n}$$

where c_0 is the numerical constant involved in (7). \square

Remark The result of the theorem remains valid if one replace the centroid by the Santaló point. Indeed what is needed here are upper estimates of $|A - A|/|A \cap (-A)|$ and $|B - B|/|B \cap (-B)|$ and concerning the polarity, we need a Santaló type inequality for A and B : $|A| \cdot |A^\circ| \leq c^n v_n^2$.

4. Random projections

The study of random quotients, subspaces and subspaces of quotient spaces of a given space is the central scheme in the Asymptotic Theory of Normed Spaces. The culmination point of this study was the QS-Theorem (of [M4]) stating that for any normed space, in a correctly chosen Euclidean structure, random quotient of subspaces of the space of proportional dimension are close to Euclidean spaces. This was the bridge between Local results (of Functional Analysis nature) and the Global Asymptotic properties of centrally symmetric convex sets. We extend in this section this QS-Theorem to the general convex setting. Surprisingly the Theory of Normed Spaces is not needed and results are true for arbitrary convex bodies.

We denote by $\mathbb{P}\text{rob}$ the rotation invariant probability measure on the orthogonal group \mathcal{O}_n . We will use the same notation to denote the rotation invariant probability measure on the Grassmann manifold $\mathcal{G}_{n,k}$, thought as the set of orthogonal projection of rank k .

Lemma 6 *Let $1 \leq k \leq n$ and $\xi \in [0, 1[$. Let $x \in \mathbb{R}^n$ and P be an orthogonal projection on \mathbb{R}^n of rank k . Then we have,*

$$I(n, k, \xi) = \mathbb{P}\text{rob}[T \in \mathcal{O}_n : |PTx| > \xi|x|] < \left(e(1 - \xi^2) \cdot \frac{n}{n - k} \right)^{\frac{n-k}{2}}. \quad (10)$$

Proof: Because of the rotation invariant property, the estimate can be reduced to the case when P is the canonical projection onto the space of the first k coordinates. For x in the sphere the mapping $T \rightarrow Tx$ maps the Haar measure of \mathcal{O}_n onto the rotation invariant measure on the sphere. By homogeneity, we transform this measure into a Gaussian measure. Thus we get

$$I(n, k, \xi) = \frac{1}{(2\pi)^{n/2}} \int_{\sum_1^k x_i^2 > \xi^2 \sum_1^n x_i^2} e^{-|x|^2/2} dx.$$

Integrating by polar coordinates yields

$$I(n, k, \xi) = \frac{kv_k(n-k)v_{n-k}}{(2\pi)^{n/2}} \int_{\rho^2 > (\xi^2/(1-\xi^2))r^2} \rho^{k-1} r^{n-k-1} e^{-\rho^2/2} e^{-r^2/2} d\rho dr.$$

The change of variables $r = \sqrt{2st}$, $\rho = \sqrt{2s(1-t)}$ gives

$$I(n, k, \xi) = \frac{kv_k(n-k)v_{n-k}}{(2\pi)^{n/2}} \int_0^\infty \left(\int_0^{1-\xi^2} 2^{\frac{n}{2}-2} s^{\frac{n}{2}-1} e^{-s} t^{\frac{n-k}{2}-1} (1-t)^{\frac{k}{2}-1} dt \right) ds.$$

Thus

$$I(n, k, \xi) = \frac{kv_k(n-k)v_{n-k}}{(2\pi)^{n/2}} 2^{\frac{n}{2}-2} \Gamma\left(\frac{n}{2}\right) \int_0^{1-\xi^2} t^{\frac{n-k}{2}-1} (1-t)^{\frac{k}{2}-1} dt,$$

with $v_k = \frac{\pi^{k/2}}{\Gamma(\frac{k}{2}+1)}$. Finally we arrive at

$$I(n, k, \xi) = B\left(\frac{n-k}{2}, \frac{k}{2}\right)^{-1} \int_0^{1-\xi^2} t^{\frac{n-k}{2}-1} (1-t)^{\frac{k}{2}-1} dt,$$

where B denotes the Beta function. Therefore

$$I(n, k, \xi) \leq B\left(\frac{n-k}{2}, \frac{k}{2}\right)^{-1} \frac{2}{n-k} (1-\xi^2)^{\frac{n-k}{2}},$$

and for any $\delta \in]0, 1[$,

$$B\left(\frac{n-k}{2}, \frac{k}{2}\right) \geq \frac{2}{n-k} \delta^{\frac{n-k}{2}} (1-\delta)^{\frac{k}{2}-1}.$$

We get

$$I(n, k, \xi) \leq (1 - \xi^2)^{\frac{n-k}{2}} / \delta^{\frac{n-k}{2}} (1 - \delta)^{\frac{k}{2}-1}$$

and we conclude by choosing $\delta = \frac{n-k}{n}$. \square

The deviation inequality (10) allows to construct “large projection” in the spirit of [LMP].

Lemma 7 *Let $1 \leq k < n$ and $\lambda = k/n$. Let $\sigma > 0$ and K a convex compact subset of \mathbb{R}^n such that there exists a subset $\Lambda \subset B_2^n$ satisfying $B_2^n \subset \Lambda + K$ and $\sharp \Lambda \leq e^{\sigma n}$, then there exists an orthogonal projection P on \mathbb{R}^n of rank k such that*

$$c(\sigma, \lambda)P(B_2^n) \subset P(K),$$

where $c(\sigma, \lambda) = \frac{1}{2e}(1 - \lambda)e^{-2(\sigma+1)/(1-\lambda)}$. Moreover, the subset of $\mathcal{G}_{n,k}$ of projections satisfying the above inclusion, has probability larger than $1 - e^{-n}$.

Proof : Define ξ by $\xi^2 = 1 - \frac{1-\lambda}{e}e^{-\frac{2(\sigma+1)}{1-\lambda}}$. From the deviation inequality (10), we have

$$I(n, k, \xi)e^{\sigma n} < \left(e(1 - \xi^2) \cdot \frac{n}{n-k} \right)^{\frac{n-k}{2}} e^{\sigma n} = e^{-n}.$$

Thus there exists an orthogonal projection P of rank k such that for every $x \in \Lambda$, one has $|Px| \leq \xi|x| \leq \xi$. Moreover the set of rank k orthogonal projections satisfying this property has a probability measure larger than $1 - e^{-n}$.

Now we have $P(B_2^n) \subset P(\Lambda) + P(K) \subset \xi P(B_2^n) + P(K)$. Since $0 \leq \xi < 1$ this clearly implies that $(1 - \xi)P(B_2^n) \subset P(K)$. Indeed, convex sets are ordered as their support function and the support functions are additive and positively homogeneous (see lemma 3 in [LMP]). We conclude by observing that $1 - \xi \geq (1 - \xi^2)/2 = \frac{1}{2e}(1 - \lambda)e^{-2(\sigma+1)/(1-\lambda)}$. \square

Fix $c \in]0, 1[$. We say that a property in \mathbb{R}^n is satisfied for “random orthogonal projection” of rank k , if the set of rank k projections satisfying the property has a probability larger than $1 - c^n$ in $\mathcal{G}_{n,k}$. We now show that random projection of a convex set have “bounded volume ratio”.

Theorem 8 *Let $1 \leq k < n$ and $\lambda = k/n$. Let C be a convex compact subset of \mathbb{R}^n with non-empty interior. There exists an affine transformation T such that in the position where $K = T(C)$, K has θ as centroid and random rank k orthogonal projections satisfy*

$$P(B_2^n) \subset P(K \cap -K) \subset P(K) \text{ and } \left(\frac{|P(K)|}{|P(B_2^n)|} \right)^{1/k} \leq r(\lambda) = e^{\frac{c_4}{1-\lambda}}$$

where c_4 is numerical constant.

Proof : According to a result of [M1] applied to the centrally symmetric convex body $C - C$, there exists an ellipsoid \mathcal{E}_0 such that

$$|C - C| = |\mathcal{E}_0| \text{ and } \frac{|(C - C) \cap \mathcal{E}_0|}{|\mathcal{E}_0|} \geq c_3^n$$

where c_3 is a universal constant. Without changing notation, we translate C so that θ is its centroid. Let $\alpha = (|C - C|/|C \cap -C|)^{1/n}$, then using (3) and (9), we get

$$\begin{aligned} N(\mathcal{E}_0, C \cap -C) &\leq N(\mathcal{E}_0, C - C) N(C - C, C \cap -C) \\ &\leq (3\alpha)^n N(\mathcal{E}_0, C - C) \leq (3 \times 8)^n N(\mathcal{E}_0, (C - C) \cap \mathcal{E}_0) \\ &\leq (24 \times 3)^n \frac{|\mathcal{E}_0|}{|(C - C) \cap \mathcal{E}_0|} \leq (72/c_3)^n. \end{aligned}$$

Let $S \subset \mathbb{R}^n$ such that $\sharp S = N(\mathcal{E}_0, C \cap -C)$ and $\mathcal{E}_0 \subset \mathcal{E}_0 \cap (S + C \cap -C)$. Construct a subset Λ of \mathcal{E}_0 by choosing for every $x \in S$ a point $y \in \mathcal{E}_0 \cap (x + C \cap -C)$ whenever this set is non-empty. Then observe that $x + C \cap -C \subset y + 2(C \cap -C)$. Therefore we have

$$\Lambda \subset \mathcal{E}_0, \sharp \Lambda \leq (72/c_3)^n \text{ and } \mathcal{E}_0 \subset \Lambda + 2(C \cap -C).$$

Let T be a linear transformation such that $T(\mathcal{E}_0) = B_2^n$ and set $K = T(C)$. We apply Lemma 7 to the convex set $2(K \cap -K)$ and $\sigma = \ln(72/c_3)$. We

obtain that random rank k orthogonal projections P (with probability larger than $1 - e^{-n}$) satisfy

$$c(\lambda) P(B_2^n) \subset P(K \cap -K)$$

where $c(\lambda) = c(\ln(72/c_3), \lambda)/2$ is defined by the function involved in Lemma 7.

To estimate the volume of $P(K)$ we use the covering numbers, we have

$$N(K, B_2^n) = N(C, \mathcal{E}_0) \leq N(C - C, \mathcal{E}_0) \leq 3^n \frac{|C - C|}{|\mathcal{E}_0 \cap (C - C)|} \leq (3/c_3)^n.$$

Therefore $|P(K)| \leq (3/c_3)^k |P(B_2^n)|$ so that

$$P(c(\lambda)B_2^n) \subset P(K \cap -K) \text{ and } \left(\frac{|P(K)|}{|P(c(\lambda)B_2^n)|} \right)^{1/k} \leq \frac{3}{c_3 c(\lambda)} = r(\lambda).$$

We conclude by a new scaling. □

We conclude this section with the non-symmetric statement for the QS-theorem [M4]. It can be obtained from the previous theorem and volume ratio approach of Szarek and Szarek-Tomczak-Jaegermann (see [TJ]).

Theorem 9 *Let $1 \leq k < n$ and $\lambda = k/n$. Let K be a convex compact subset of \mathbb{R}^n with non-empty interior. There exists a projection P from \mathbb{R}^n onto a subspace F of \mathbb{R}^n and a subspace E of F and an ellipsoid \mathcal{E} in E such that $\dim(E) = k$ and*

$$\mathcal{E} \subset P(K) \cap E \subset c(\lambda) \mathcal{E}$$

where $c(\lambda)$ depends only on λ .

5. M-ellipsoids; existence.

We consider in this section ellipsoids exclusively centered at 0. Let $\sigma > 0$ and let K be a convex compact subset of \mathbb{R}^n with 0 in its interior. We say that an ellipsoid \mathcal{E} of \mathbb{R}^n is an M-ellipsoid of K with constant σ , or shortly an M-ellipsoid of K , if setting $\lambda = (|K|/|\mathcal{E}|)^{1/n}$ in order that $|K| = |\lambda \mathcal{E}|$, we have

$$N(K, \lambda \mathcal{E}) \leq e^{\sigma n}.$$

It is proved in [M1] (see also [M2] and [Pi] for simplified proofs) that there exists a universal constant such that for every n , every n -dimensional symmetric convex body has an M-ellipsoid with respect to this constant. An important interest of such ellipsoids, is that they give reverse Brunn-Minkowski inequalities. Many interesting properties of centrally symmetric convex bodies and corresponding normed spaces were revealed using M-ellipsoids. We refer to a survey [M3] and [MS2].

In this section we build M-ellipsoid for arbitrary non-symmetric convex body and we show in this and the next sections that many results known in the symmetric case can be translated to the general non-symmetric case.

We will not take care below of numerical constants, we write for two positive numbers that $a \sim b$ if the ratio is bounded by two universal constants. Similarly we write $a \lesssim b$, meaning that $a \leq cb$ where $c > 0$ is a universal constant. We say for instance that an inequality such as $N(K, L)^{1/n} \lesssim c$ implies $|K \cap L|^{1/n} \gtrsim c^{-1} |K|^{1/n}$, if $N(K, L)^{1/n} \leq c$ implies $|K \cap L|^{1/n} \geq \lambda c^{-1} |K|^{1/n}$ for some universal factor $\lambda > 0$.

Lemma 10 *Let K and L be two convex compact subsets of \mathbb{R}^n with non-empty interior and with 0 as centroid. Let $c > 0$, the following properties are equivalent:*

- 1) $N(K, L)^{1/n} \lesssim c$
- 2) $|K - L|^{1/n} \lesssim c |L|^{1/n}$
- 3) $|K \cap L|^{1/n} \gtrsim c^{-1} |K|^{1/n}$
- 4) $N(L^\circ, K^\circ)^{1/n} \lesssim c$
- 5) $|L^\circ - K^\circ|^{1/n} \lesssim c |K^\circ|^{1/n}$
- 6) $|L^\circ \cap K^\circ|^{1/n} \gtrsim c^{-1} |L^\circ|^{1/n}$.

Proof: Clearly, $|K - L| \leq N(K, L)|L - L|$ and

$$N(K, L) \leq N(K, L \cap -L) \leq |2K + L \cap -L|/|L \cap -L| \leq 2^n |K - L|/|L \cap -L|.$$

Combining with the relation (1) and (4) we get

$$\frac{1}{4} \left(\frac{|K - L|}{|L|} \right)^{1/n} \leq N(K, L)^{1/n} \leq 4 \left(\frac{|K - L|}{|L|} \right)^{1/n}.$$

This shows the equivalence between the two first properties.

Now the inequalities (1) and (5) imply

$$1 \leq \frac{|K \cap L|}{|K|} \cdot \frac{|K - L|}{|L|} \leq 4^n$$

and the equivalence between 2) and 3) follows. The equivalence between 1) and 4) is Theorem 5. The other equivalences follow from Santaló inequality and its reverse and from the observation that since 0 is centroid of K then it is Santaló point of K° , so that Santaló inequality applies. \square

Remarks. 1. The previous lemma gives equivalent characterization of M-ellipsoid. Let K with 0 as centroid and let \mathcal{E} be an ellipsoid such that $|K| = |\mathcal{E}|$. Then

$$\begin{aligned} N(K, \mathcal{E})^{1/n} &\sim N(\mathcal{E}, K)^{1/n} \sim (|K - \mathcal{E}|/|\mathcal{E}|)^{1/n} \sim (|K \cap \mathcal{E}|/|K|)^{1/n} \\ &\sim N(\mathcal{E}^\circ, K^\circ)^{1/n} \sim (|\mathcal{E}^\circ - K^\circ|/|K^\circ|)^{1/n} \sim (|\mathcal{E}^\circ \cap K^\circ|/|\mathcal{E}^\circ|)^{1/n}. \end{aligned}$$

We see that if \mathcal{E} is an M-ellipsoid for K then \mathcal{E}° is an M-ellipsoid for K° .

2. The proof of the lemma gives that if K and L are two convex bodies with 0 as centroid, then

$$(|K - L|/|L|)^{1/n} \sim N(K, L)^{1/n} \sim (|K + L|/|L|)^{1/n}.$$

This shows in particular that $|K \cap L|^{1/n} \sim |K \cap -L|^{1/n}$ and $|K + L|^{1/n} \sim |K - L|^{1/n}$.

We now show that every convex body has an M-ellipsoid with some universal constant.

Theorem 11 *There exists a constant σ such that for any convex compact subset K of \mathbb{R}^n with non-empty interior and 0 as centroid, then there exist an ellipsoid \mathcal{E} of \mathbb{R}^n such that*

$$|K| = |\mathcal{E}| \text{ and } N(K, \mathcal{E}) \leq e^{\sigma n}.$$

Proof: Following a result from [M1], there exists an M-ellipsoid \mathcal{E} for the centrally symmetric body $K - K$ associated to some universal constant σ . Let $|\mathcal{E}| = |K - K|$ so that from Lemma 10 we have $N(\mathcal{E}, K - K)^{1/n} \lesssim e^\sigma$. Let $\lambda = (|K|/|\mathcal{E}|)^{1/n}$ then $\lambda \leq 1$, therefore

$$N(\lambda \mathcal{E}, K)^{1/n} \leq N(\mathcal{E}, K)^{1/n} \leq N(\mathcal{E}, K - K)^{1/n} N(K - K, K)^{1/n} \lesssim e^\sigma.$$

Using Lemma 10, we conclude that $N(K, \lambda \mathcal{E})^{1/n} \lesssim e^\sigma$. \square

Remark. Let K be a convex body with 0 as centroid. Lemma 10, the remarks following it and the proof of Theorem 11 show that K , $K \cap -K$ and $(K - K)/2$ have the same family of M-ellipsoids. As written in the introduction of this section, this means that an M-ellipsoid of one of these bodies with the parameter σ is an M-ellipsoid for the other bodies with an equivalent parameter (that is, as usual, up to a universal factor).

6. M-ellipsoids; application to global regularity

We use in this section technique which provides existence of M-ellipsoids to study global properties of convex sets. The main Theorem 13 was known in the symmetric setting. But the fact that it can be translated for general convex bodies is adding a new flavor to the theory.

Lemma 12 *Let $\sigma > 0$ and let K and K' be convex compact subsets of \mathbb{R}^n with 0 as centroid (or as Santaló point) and such that $|K| = |K'| = |B_2^n|$ and $N(K, B_2^n) \leq e^{\sigma n}$, $N(K', B_2^n) \leq e^{\sigma n}$. Then for any $T \in \mathcal{O}_n$, we have*

$$|K \cap T(K')| \geq 2^{-n} e^{-2\sigma n} |B_2^n|$$

and

$$|\text{co}(K^\circ \cup T(K')^\circ)| \leq 8^n e^{2\sigma n} |B_2^n|.$$

Proof: From the entropy estimate, we deduce that there exist subsets Λ and Λ' of \mathbb{R}^n with $\sharp \Lambda \leq e^{\sigma n}$ and $\sharp \Lambda' \leq e^{\sigma n}$ such that

$$K - T(K') \subset \Lambda - T(\Lambda') + 2B_2^n.$$

Therefore

$$|K - T(K')| \leq 2^n e^{2\sigma n} |B_2^n|.$$

Now we observe that K and $T(K')$ have the same centroid. We deduce from (1) that

$$|K \cap T(K')| \geq \frac{|K| \cdot |K'|}{|K - T(K')|} \geq 2^{-n} e^{-2\sigma n} |B_2^n|.$$

Unfortunately, we do not know Santaló inequality for $K \cap T(K')$ to finish the proof of the second inequality in an easy way. We proceed differently; from the reverse Santaló inequality (7), we have

$$|K^\circ \cap -T(K')^\circ| \geq c_0^n \frac{|B_2^n|^2}{|\text{co}(K \cup -T(K'))|} \geq c_0^n \frac{|B_2^n|^2}{|K - T(K')|} \geq 2^{-n} e^{-2\sigma n} c_0^n |B_2^n|.$$

From (5), we have

$$|\text{co}(K^\circ \cup T(K')^\circ)| \leq |K^\circ + T(K')^\circ| \leq 4^n \frac{|K^\circ| \cdot |T(K')^\circ|}{|K^\circ \cap -T(K')^\circ|}.$$

Now we note that since 0 is the centroid of K , it is the Santaló point of K° , therefore $|K^\circ| \leq |B_2^n|^2/|K| = |B_2^n|$. The same is true for K' and observe that $T(K')^\circ = T(K'^\circ)$. We deduce that

$$|\text{co}(K^\circ \cup T(K')^\circ)| \leq 4^n 2^n e^{2\sigma n} c_0^{-n} |B_2^n|.$$

The proof follows the same lines if 0 is Santaló point of K and K' . \square

Theorem 13 *Let K and K' be convex compact subsets of \mathbb{R}^n with non-empty interior and 0 as centroid and such that B_2^n is an M -ellipsoid for K and K' . Then there are positive universal constants c , r_1 and r_2 such that the relations*

$$(1/r_1) B_2^n \subset \text{co}(K \cup T(K')) \text{ and } \left(\frac{|\text{co}(K \cup T(K'))|}{|(1/r_1) B_2^n|} \right)^{1/n} \leq c$$

and

$$(1/r_1) B_2^n \subset L \cap V(L) \subset r_2 B_2^n$$

are satisfied for random rotations $T, V \in \mathcal{O}_n$, where $L = \text{co}(K \cup T(K'))$.

Proof: Let $C = K^\circ$ and $C' = K'^\circ$; since B_2^n is an M-ellipsoid for K and K' , by Lemma 10, it is also an M-ellipsoid for C and C' . To simplify, we suppose that

$$|C| = |C'| = |B_2^n| \text{ and } \max(N(C, B_2^n), N(C', B_2^n)) \leq e^{\sigma n}$$

for some universal constant σ .

We claim that there is a universal constant $r_1 > 0$ such that, for random rotation T , we have

$$C \cap T(C') \subset r_1 B_2^n.$$

Indeed for any $r_1 > 0$, define

$$C(r_1) = \{x \in C : |x| = r_1\} \text{ and } C'(r_1) = \{x \in C' : |x| = r_1\}.$$

There exist nets $(x_i)_{1 \leq i \leq N}$ and $(x'_i)_{1 \leq i \leq N'}$ of vectors of norm r_1 , such that $N, N' \leq e^{\sigma n}$ and

$$S = \cup_{1 \leq i \leq N} (x_i + B_2^n) \supset C(r_1) \text{ and } S' = \cup_{1 \leq i \leq N'} (x'_i + B_2^n) \supset C'(r_1).$$

For fixed vectors x and y on the unit sphere,

$$\mathbb{P}\text{rob}[T \in \mathcal{O}_n : |x - T(y)| \leq 2/r_1] \leq (c/r_1)^{n-1}$$

for some universal constant c (see [MS1]). Thus by homogeneity

$$\mathbb{P}\text{rob}[T \in \mathcal{O}_n : \exists 1 \leq i \leq N, 1 \leq j \leq N', |x_i - T(x'_j)| \leq 2] \leq N.N'.(c/r_1)^{n-1}.$$

We may choose r_1 such that $N.N'.(c/r_1)^{n-1} < e^{-n}$. We conclude that random rotations T satisfy $S \cap T(S') = \emptyset$ so that $C(r_1) \cap T(C'(r_1)) = \emptyset$. To finish, we observe that if T is a rotation such that $C(r_1) \cap T(C'(r_1)) = \emptyset$, then $C \cap T(C') \subset r_1 B_2^n$.

By duality we obtain

$$\text{co}(K \cup T(K')) = \text{co}(C^\circ \cup T(C')^\circ) \supset (1/r_1) B_2^n.$$

The hypothesis of Lemma 12 are satisfied, so that we have

$$|\text{co}(K \cup T(K'))| \leq 8^n e^{2\sigma n} |B_2^n| = (8r_1)^n e^{2\sigma n} |(1/r_1) B_2^n|.$$

This achieves the first part with $c = (8r_1)e^{2\sigma}$.

Now let $L = \text{co}(K \cup T(K'))$, since $|L|^{1/n} \sim |B_2^n|^{1/n}$ and $L \supset (1/r_1) B_2^n$, we get $N(L, B_2^n)^{1/n} \lesssim e^{\sigma'}$ for some numerical constant σ' . The same reasoning as before shows that for random rotation V , we have $L \cap V(L) \subset r_2 B_2^n$. \square

Let us finish this section with an application, in the spirit of [MS2].

Theorem 14 *Let K be a convex compact subsets of \mathbb{R}^n with non-empty interior and 0 as centroid. Assume that B_2^n is an M -ellipsoid for K . Assume further that for a given integer $m \geq 1$, there are m orthogonal transformations v_1, \dots, v_m and real numbers $r, c > 0$ such that*

$$r B_2^n \subset L = \frac{1}{m} \sum_{i=1}^m v_i(K) \subset cr B_2^n.$$

Then there exist c' and r' depending only on m and c such that for random rotations v , we have

$$r' B_2^n \subset L' = \frac{1}{2}(K + v(K)) \subset c'r' B_2^n.$$

In other words, L' is c' -isomorphic to an Euclidean ball.

Proof: From the remark following Theorem 11, we know that B_2^n is an M -ellipsoid for the symmetric body $(K - K)/2$. We can apply Theorem 3.1 from [MS2] to $(K - K)/2$ and

$$\tilde{L} = \frac{1}{m} \sum_{i=1}^m v_i((K - K)/2).$$

We obtain the existence of c' and r' such that for random rotation v , $\frac{K-K}{2} + v(\frac{K-K}{2})$ is c' -isomorphic to an Euclidean ball. Equivalently, $\frac{K+v(K)}{2} - \frac{K+v(K)}{2}$ is c' -isomorphic to an Euclidean ball. In particular $\frac{K+v(K)}{2}$ is contained in $c'r' B_2^n$. From an other side, the global form of Theorem 8 (see [LMP], Theorem 2') tell us that for random rotation u , $\frac{K+u(K)}{2}$ contains an Euclidean ball of “constant” radius. Indeed, to simplify, assume that $|K| = |B_2^n|$ and apply Theorem 8. Because “random” projections satisfy the conclusion of this theorem, there are two orthogonal projections P_1 and P_2 with rank $[n/2]$ and $[(n+1)/2]$ respectively, such that $P_i(B_2^n) \subset c''P_i(K \cap -K)$, for some

universal constants c'' . Denote by $I = P_1 + P_2$ the identity and $u = P_1 - P_2$. So $P_1 = (1/2)(I + u)$ and $P_2 = 1/2(I - u)$. Then u is an orthogonal transformation and for any $x \in B_2^n$ we have

$$x = P_1 x + P_2 x \in (c''/2)[(I+u)(K \cap -K) + (I-u)(K \cap -K)] \subset c''(K+u(K))/2.$$

Thus $B_2^n \subset c''(K + u(K))/2$. Choosing now $u = v$, which is allowed by the probabilistic argument, achieves the proof. \square

Remark. We give an other application, in the spirit of [MS2] (see remark after Theorem 3.2).

Let K be a convex compact subsets of \mathbb{R}^n with non-empty interior and 0 as centroid. We still assume that B_2^n is an M -ellipsoid for K . Let $m \geq 1$ be an integer. Assume further that there are orthogonal transformations u_1, \dots, u_m and v_1, \dots, v_m and real numbers $r, r', c > 0$ such that

$$r B_2^n \subset \frac{1}{m} \sum_{i=1}^m v_i(K) \subset cr B_2^n,$$

$$r' B_2^n \subset \frac{1}{m} \sum_{i=1}^m u_i(K^\circ) \subset cr' B_2^n.$$

Then there exists c' depending only on m such that K is c' -isomorphic to an Euclidean ball.

The proof is following from the previous statement by choosing u and v so that $v = u^{*-1}$. This is possible from the probabilistic argument and our definition of “random” rotation, meaning with large measure. We get thus that $\frac{K+u(K)}{2}$ and $\frac{K^\circ+v(K^\circ)}{2}$ are isomorphic to an Euclidean unit ball. Our choice $v = u^{*-1}$ implies by duality that $K \cap u(K)$ is also isomorphic to an Euclidean unit ball. Therefore K is itself isomorphic to an Euclidean unit ball.

The M -ellipsoid plays here an essential role. As observed in ([MS2], remarks after Theorem 3.1), the requirement on the “position” of K (with B_2^n as M -ellipsoid) is crucial even in the symmetric case.

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