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A general non renormalization theorem in the extended antifield formalism

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Abstract

In the context of algebraic renormalization, the extended antifield formalism is used to derive the general forms of the anomaly consistency condition and of the Callan-Symanzik equation for generic gauge theories. A local version of the latter is used to derive sufficient conditions for the vanishing of beta functions associated to terms whose integrands are invariant only up to a divergence for an arbitrary non trivial non anomalous symmetry of the Lagrangian. These conditions are independent of power counting restrictions and of the form of the gauge fixation.

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Introduction

Some of the perturbative proofs [1, 2, 3] of the vanishing of the β function for Chern-Simons theory rely on two essential steps. The first is the use of a local, non integrated version of the Callan-Symanzik equation, where the β functions are associated to strictly BRST invariant terms, the second is the fact that the Chern-Simons term is gauge, respectively BRST invariant, only up to a total divergence.

This article is motivated by the desire to understand this non renormalization mechanism independently of the gauge fixation and of power counting restrictions, so that they can be applied to effective field theories [4, 5]. Its aim is to derive general conditions under which the above mechanism can be extended to generic theories with arbitary non trivial global or local symmetries. This is done on the one hand by using the extended antifield formalism with the corresponding BRST differential to control the renormalization of the symmetries and, on the other hand, by implementing a local version of the Callan-Symanzik equation through an anticanonical transformation.

The article is organized as follows. In the first section, we review the quantum action principles, which will allow us to control the renormalization aspects of the problem.

The second section summarizes the results on the extended antifield formalism and the corresponding BRST cohomology.

A first application in section 3 concerns the derivation of the general form of the anomalous extended Zinn-Justin equation. More precisely, it is shown that by choosing suitable BRST breaking finite counterterms at each order, the anomalous insertion at the next order is characterized by the cohomology of the extended BRST differential in ghost number 1.

In the main section, power counting is implemented by a canonical transformation in the antibracket. For simplicity, the case where there are no dimensionful couplig constants and the theory is dilatation invariant is discussed next. The general form of the integrated Callan-Symanzik equation is given. Then a local form is derived toghether with sufficient conditions for the above mentioned non renormalization mechanism. As a direct application, Chern-Simons theory is discussed. In the next part, the analysis is generalized to the case where dilatation invariance is broken through the presence of dimensionful coupling constants.

In the conclusion, we comment on possible extensions for this approach.

Finally, the appendix contains three technical points needed in the main text.

1 Quantum action principle

In order to deal with the renormalization aspects of the problem, we will use the quantum action principles [6] as pioneered in [7] and elaborated for instance in [8, 9]. Let S be the classical action of the theory. If Γ denotes the renormalized generating functional for one particle irreducible vertices, one has

$$\Gamma = S + O(\hbar). \tag{1}$$

Similarily, if $\Delta \circ \Gamma$, respectively $\Delta(x) \circ \Gamma$, denotes the renormalized insertion of an (integrated) local polynomial into Γ, one has

$$\Delta \circ \Gamma = \Delta + O(\hbar). \tag{2}$$

Let g be a parameter of S, $\phi(x)$ the source for the 1PI vertex functions and $\rho(x)$ an external source coupling to a polynomial in the fields and their derivatives. We will use the quantum action principle in the following forms:

$$\begin{aligned} &\text{(non linear) field variations :} \\ &\frac{\delta\Gamma}{\delta\phi(x)}\frac{\delta\Gamma}{\delta\rho(x)} = \Delta''(x) \circ \Gamma, \\ \Delta''(x) \circ \Gamma = &\frac{\delta S}{\delta\phi(x)}\frac{\delta S}{\delta\rho(x)} + O(\hbar). \end{aligned} \tag{3}$$

$$\begin{aligned} \text{coupling constants :}\ &\frac{\partial\Gamma}{\partial g} = \Delta \circ \Gamma, \\ \Delta \circ \Gamma = &\frac{\partial S}{\partial g} + O(\hbar). \end{aligned} \tag{4}$$

This is the only input from renormalization theory that will be used in the following.

2 Extended antifield formalism

The Batalin-Vilkovisky formalism [10, 11, 12, 13] is a tool to control gauge symmetries under renormalization for generic gauge theories. A well defined quantum theory requires the introduction of a non minimal sector followed by an anticanonical transformation to a gauge fixed basis. The antifields are not set to zero, but kept as external sources, since their presence in the gauge fixed theory allows to control the renormalization of the gauge or the global symmetry. Although it is always understood in intermediate computations, we will not make the passage to the gauge fixed basis explicitly, because the considerations below rely only on the BRST cohomology of the theory. This cohomology does not depend on the non minimal sector [14] and is invariant under anticanonical transformations.

The formalism can be extended so as to include (non linear) global symmetries (see [15] and references therein), which is achieved by coupling the BRST cohomology classes in negative ghost numbers with constant ghosts. There is a further extension [16] to include the BRST cohomology classes in all the ghost numbers, which allows to take into account in a systematic way all higher order cohomological constraints due to the antibracket maps [17].

Let us briefly summarize the results of [16] needed in the following. The extended formalism is obtained by first computing a basis for the local BRST cohomology classes, containing as a subset those classes that can be obtained from the solution S of the master equation by differentiation of S with respect to an essential coupling constant[1]. The additional classes are then coupled with the help of new independent coupling constants to the solution of the master equation. This action can then be extended by terms of higher orders in the new couplings in such a way that, if we denote by ξ^A both the essential couplings and the new ones, the corresponding action $S(\xi)$ satisfies the extended master equation

$$\frac{1}{2}(S(\xi), S(\xi)) + \Delta_c S(\xi) = 0. \tag{5}$$

The BRST differential associated to the solution of the extended master

[1] A set of coupling constants g^i is essential if $\lambda^i \partial S/\partial g^i = (S, \Xi)$ for some local functional Ξ implies $\lambda^i = 0$. A parametric dependence on inessential couplings related for example to gauge fixation is always understood in the following. More details on the dependence of the theory on inessential couplings will be given elsewhere.

equation is

$$\bar{s} = (S(\xi), \cdot) + \Delta_c, \tag{6}$$

where $\Delta_c = (-)^A f^A(\xi) \frac{\partial L}{\partial \xi^A}$ is at least quadratic in the new couplings and satisfies $\Delta_c^2 = 0$. Since it does not depend on the fields and the antifields, it also satisfies $\Delta_c(A, B) = (\Delta_c A, B) + (-)^{A+1}(A, \Delta_c B)$. The local BRST cohomology classes contain the generators of all the generalized non trivial symmetries of the theory in negative ghost number, the generalized observables in ghost number zero, and the anomalies (and anomalies for anomalies) in positive ghost number. This is the reason why the extended master equation encodes the invariance of the original action under all the non trivial gauge and local symmetries, their commutator algebra as well as the antibracket algebra of all the local BRST cohomology classes.

The cohomology of $\bar{s}$ in the space F of ξ dependent local functionals in the fields, the antifields and their derivatives, over the functions in the ξ^A, is isomorphic to the cohomology of

$$s_Q = [\frac{\partial^R \cdot}{\partial \xi^A} f^A(\xi), \cdot] \tag{7}$$

in the space of graded derivations $\partial^R \cdot / \partial \xi^A \lambda^A(\xi)$, with λ^A a function of ξ alone. If $\partial^R \cdot / \partial \xi^B \mu^B(\xi)$ is a s_Q cocycle, the corresponding $\bar{s}$ cocycle is given by $\partial^R S(\xi) / \partial \xi^B \mu^B(\xi)$. Furthermore, the general solution to the equations

$$\left\{ \begin{array}{c} \frac{\partial^R S(\xi)}{\partial \xi^B} \mu^B(\xi) + \bar{s} C(\xi) = 0, \\ s_Q \frac{\partial^R \cdot}{\partial \xi^B} \mu^B(\xi) = 0, \end{array} \right. \tag{8}$$

is

$$\left\{ \begin{array}{c} C(\xi) = \bar{s} D(\xi) + \frac{\partial^R S(\xi)}{\partial \xi^B} \nu^B(\xi), \\ \frac{\partial^R \cdot}{\partial \xi^B} \mu^B(\xi) = s_Q \frac{\partial^R \cdot}{\partial \xi^B} \nu^B(\xi). \end{array} \right. \tag{9}$$

A crucial property of the extended antifield formalism is that it is stable, in the sense that every infinitesimal deformation of the solution of the extended master equation, characterized by $H^{0,n}(\bar{s}|d)$, can be extended to a complete deformation without any obstructions. This makes it an appropriate starting point for a quantum theory.

3 Anomalies to all orders

In the following, we will omit from the notation the dependence on the couplings ξ^A. The action principle applied to (5) gives

$$\frac{1}{2}(\Gamma,\Gamma)+\Delta_c\Gamma=\hbar\mathcal{A}\circ\Gamma, \tag{10}$$

where Γ is the renormalized generating functional for 1PI vertices associated to the solution S of the extended master equation and the local functional $\mathcal{A}$ is an element of F in ghost number 1. Applying $(\Gamma,\cdot)+\Delta_c$ to (10), the l.h.s vanishes identically because of the graded Jacobi identity for the antibracket and the properties of Δ_c, so that one gets the consistency condition $(\Gamma,\mathcal{A}\circ\Gamma)+\Delta_c\mathcal{A}\circ\Gamma=0$. To lowest order in $\hbar$, this gives $\bar{s}\mathcal{A}=0$, the general solution of which can be writen as

$$\mathcal{A}=\frac{\partial^R S}{\partial\xi^A}\sigma_1^A+\bar{s}\Sigma_1, \tag{11}$$

with $s_Q\partial^R\cdot/\partial\xi^A\sigma_1^A=0$. If one now defines $S^1=S-\hbar\Sigma_1$, the corresponding generating functional admits the expansion $\Gamma^1=\Gamma-\hbar\Sigma_1+O(\hbar^2)$. This implies that $\frac{1}{2}(\Gamma^1,\Gamma^1)+\Delta_c\Gamma^1=\hbar\partial^R\Gamma^1/\partial\xi^A\sigma_1^A+O(\hbar^2)$. On the other hand, the quantum action principle applied to $\frac{1}{2}(S^1,S^1)+\Delta_cS^1=O(\hbar)$ implies $\frac{1}{2}(\Gamma^1,\Gamma^1)+\Delta_c\Gamma^1=\hbar\bar{\mathcal{A}}\circ\Gamma^1$, for a local functional $\bar{\mathcal{A}}$. Comparing the two expressions, we deduce that

$$\frac{1}{2}(\Gamma^1,\Gamma^1)+\Delta_c\Gamma^1=\hbar\frac{\partial^R\Gamma^1}{\partial\xi^A}\sigma_1^A+\hbar^2\mathcal{A}'\circ\Gamma^1 \tag{12}$$

for a local functional $\mathcal{A}'$.

Applying now $(\Gamma^1,\cdot)+\Delta_c$, one gets as consistency condition

$$\frac{\partial^R}{\partial\xi^A}[\frac{\partial^R\Gamma^1}{\partial\xi^B}\sigma_1^B+\hbar\mathcal{A}'\circ\Gamma^1]\sigma_1^A+(\Gamma^1,\mathcal{A}'\circ\Gamma^1)+\Delta_c\mathcal{A}'\circ\Gamma^1=0, \tag{13}$$

giving to lowest order

$$\frac{\partial^R S}{\partial\xi^A}1/2[\sigma_1,\sigma_1]^A+\bar{s}\mathcal{A}'=0, \tag{14}$$

where $\partial^R \cdot /\partial\xi^A[\sigma_1,\sigma_1]^A = [\partial^R \cdot /\partial\xi^A\sigma_1^A, \partial^R \cdot /\partial\xi^B\sigma_1^B]$, and is a s_Q cocycle because of the graded Jacobi identity for the graded commutator. According to the previous section, the general solution to this equation is

$$\frac{\partial^R\cdot}{\partial\xi^A}[\sigma_1,\sigma_1]^A = [\frac{\partial^R\cdot}{\partial\xi^A}f^A, \frac{\partial^R\cdot}{\partial\xi^B}\sigma_2^B], \tag{15}$$

$$\mathcal{A}' = \bar{s}\Sigma_2 + \frac{\partial^R S}{\partial\xi^B}\sigma_2^B. \tag{16}$$

The redefinition $S^2 = S^1 - \hbar^2\Sigma_2$ then allows to achieve

$$\frac{1}{2}(\Gamma^2,\Gamma^2) + \Delta_c\Gamma^2 = \frac{\partial^R\Gamma^2}{\partial\xi^A}(\hbar\sigma_1^A + \hbar^2\sigma_2^A) + \hbar^3\mathcal{A}''\circ\Gamma^2, \tag{17}$$

for a local functional $\mathcal{A}''$. The reasoning can be pushed recursively to all orders with the result

$$\frac{1}{2}(\Gamma^\infty,\Gamma^\infty) + \Delta_c\Gamma^\infty = \frac{\partial^R\Gamma^\infty}{\partial\xi^A}\sigma^A, \tag{18}$$

where Γ^∞ is associated to the action $S^\infty = S - \Sigma_{k=1}\hbar^k\Sigma_k$ and $\sigma^A = \Sigma_{k=1}\hbar^k\sigma_k$ satisfies $s_Q\frac{\partial^R\cdot}{\partial\xi^A}\sigma^A = 0$.

This result agrees with the one deduced in [18, 19] without the Δ_c operator and the associated s_Q cohomology and the one in [16] in the context of dimensional regularisation. It answers the questions raised in [20] on higher order consistency conditions on the anomalies and the quantum BRST cohomology (see also [21]).

In the case where $H^1(\bar{s}) \simeq H^1(s_Q) = 0$, we see that we can achieve (10) without any anomalous insertion:

$$\frac{1}{2}(\Gamma^\infty,\Gamma^\infty) + \Delta_c\Gamma^\infty = 0. \tag{19}$$

If this equation holds, we say that the non trivial symmetries encoded in the differential $\bar{s}$ are non anomalous.

4 Integrated and local Callan-Symanzik equations

4.1 Power counting

In the antifield formalism, power counting can be implemented canonically through the operator

$$S_\eta = \int d^n x \ L_\eta = \int d^n x \ \phi^*_a (d^{(a)} + x^\mu \partial_\mu) \phi^a, \tag{20}$$

where ϕ^a is a collective notation for the original fields and the local ghosts associated to the gauge symmetries, while $d^{(a)}$ is the canonical dimension of ϕ^a in units of inverse length. The bracket around the index a means that there is no additional summation. We have $(\phi^a(x), S_\eta) = (d^{(a)} + x^\mu \partial_\mu)\phi^a(x)$ and $(\phi^*_a(x), S_\eta) = (n - d^{(a)} + x^\mu \partial_\mu)\phi^*_a(x)$, so that the canonical dimension of the antifields is choosen to be $n - d^{(a)}$. It is then straightforward to verify that for any monomial $M(x)$ in the fields, the antifields and their derivatives of homogeneous dimension d^M,

$$(M(x), S_\eta) = (d^M + x^\mu \partial_\mu) M(x). \tag{21}$$

4.2 No dimensionful coupling constants

For simplicity, we assume in a first stage that the coupling constants ξ^A as well as the inessential coupling constants all have dimension 0. In this section, we also assume that there are no anomalies, eq (19).

4.2.1 Integrated form of Callan-Symanzik equation

Because there are non dimensionful parameters, all the terms of the Lagrangian L of the solution of the extended master equation have dimension n. Hence,

$$(L, S_\eta) = (n + x^\mu \partial_\mu) L = \partial_\mu (x^\mu L). \tag{22}$$

Upon integration, we get

$$(S, S_\eta) = 0. \tag{23}$$

Furthermore, $\Delta_c S_\eta = 0$ because S_η does not depend on ξ^A. We have $(S^\infty, S_\eta) = O(\hbar)$, so that the quantum action principle gives

$$(\Gamma^\infty, S_\eta) = \hbar \mathcal{B} \circ \Gamma^\infty, \tag{24}$$

with $\mathcal{B}$ an element of F of ghost number 0.

Applying $(\Gamma^\infty, \cdot) + \Delta_c$, using the graded Jacobi identity and (19), we get the consistency condition $(\Gamma^\infty, \mathcal{B} \circ \Gamma^\infty) + \Delta_c \mathcal{B} \circ \Gamma^\infty = 0$, which implies, to lowest order in $\hbar$, $\bar{s}\mathcal{B} = 0$ and hence

$$\mathcal{B} = \frac{\partial^R S}{\partial \xi^A} \beta_1^A + \bar{s}\Xi_1. \tag{25}$$

According to the quantum action principle, we can replace $\frac{\partial^R S}{\partial \xi^A} \beta_1^A \circ \Gamma^\infty$ by $\frac{\partial^R \Gamma^\infty}{\partial \xi^A} \beta_1^A$ and the difference will be the insertion of a local functional of order $\hbar$. As shown in lemma 1 of the appendix, by adapting a reasoning of [22, 23] to the present context of algebraic renormalization, the insertion $[\bar{s}\Xi_1] \circ \Gamma^\infty$ can be replaced by $(\Gamma^\infty, \Xi_1 \circ \Gamma^\infty) + \Delta_c \Xi_1 \circ \Gamma^\infty$, and the difference will again be the insertion of a local functional of order $\hbar$.

We thus get

$$\begin{aligned}(\Gamma^\infty, [S_\eta - \hbar \Xi_1 \circ \Gamma^\infty]) + \Delta_c [S_\eta - \hbar \Xi_1 \circ \Gamma^\infty] \\ = \hbar \frac{\partial^R \Gamma^\infty}{\partial \xi^A} \beta_1^A + \hbar^2 \mathcal{B}' \circ \Gamma^\infty.\end{aligned} \tag{26}$$

Acting with $(\Gamma^\infty, \cdot) + \Delta_c$ on (26), using (19) and $s_Q \partial^R \cdot / \partial \xi^A \beta_1^A = 0$, we get the consistency condition $(\Gamma^\infty, \mathcal{B}' \circ \Gamma^\infty) + \Delta_c \mathcal{B}' \circ \Gamma^\infty = 0$ so that the reasoning can be pushed to all orders:

$$(\Gamma^\infty, [S_\eta - \Xi \circ \Gamma^\infty]) + \Delta_c [S_\eta - \Xi \circ \Gamma^\infty] = \frac{\partial^R \Gamma^\infty}{\partial \xi^A} \beta^A, \tag{27}$$

with $\Xi = \Sigma_{n=1} \hbar^n \Xi_n$ and $\beta^A = \Sigma_{n=1} \hbar^n \beta_n^A$.

Digression

Let us consider for a moment the following particular case.

(i) All the antibracket maps encoded in f^A are zero so that $\Delta_c = 0$, $\bar{s} = s = (S(\xi), \cdot)$. This happens for instance if the Kluberg-Stern and Zuber conjecture [22] holds and the BRST cohomology can be described independently of the antifields.

(ii) The only possibility (for instance for power counting reasons) for Ξ_n is $\Xi_n = -\gamma_n \int d^n x\ \phi^*_a \phi^a$, so that Ξ_n is linear in the quantum fields and $[s\Xi_n]\circ\Gamma^\infty$ can be replaced, according to the quantum action principle, at each stage in $\hbar$ by (Γ^∞, Ξ_n) up to the insertion of a local polynomial of higher order in $\hbar$. Equation (27) then takes the more familiar form of the Callan-Symanzik equations in the massless case with anomalous dimension $\gamma = \Sigma_{k=1}\hbar^k\gamma_k$ for the fields and the antifields [24]:

$$(\Gamma^\infty, S^\infty_\eta) = \frac{\partial \Gamma^\infty}{\partial \xi^A}\beta^A, \tag{28}$$

with $S^\infty_\eta = \int d^n x\ \phi^*_a(d^{(a)} + \gamma + x\cdot\partial)\phi^a$, or explicitly

$$\int d^n x \Big[\frac{\delta^R \Gamma^\infty}{\delta \phi^a(x)}(d^{(a)} + \gamma + x\cdot\partial)\phi^a(x) + \frac{\delta^R\Gamma^\infty}{\delta\phi^*_a(x)}(n - d^{(a)} - \gamma + x\cdot\partial)\phi^*_a(x)\Big] = \frac{\partial^R\Gamma^\infty}{\partial\xi^A}\beta^A. \tag{29}$$

Remarks on explicit x dependence.

Note that S_η is the generator of the dilatation symmetry of the theory. If it corresponds to a non trivial element of $H^{-1,n}(s|d)$, the question arises whether it should be coupled with a constant ghost in the extended solution $S(\xi)$ as in [25, 26]. This depends on whether or not we allow for explicit x dependence in the local functionals and the cohomology classes of s we are initially computing and then coupling to the solution of the master equation.

In the previous section, we have supposed that there is no explicit x dependence in these functionals and cohomology classes, because if we assume the absence of dimensionful couplings, we cannot control translation invariance through a corresponding cohomology class, its generator $S_\mu = \int d^n x\ \phi^*\partial_\mu\phi$ being of dimension 1.

We will assume here that one can apply the quantum action principles in the case of an explicit x dependence of the variation as in (24), at the price of allowing a priori for an explicit x dependence of the inserted local functional $\mathcal{B}$. This assumption needs to be checked by a more careful analysis of the renormalization properties of the model which is beyond the scope of this paper.

In order to prove then that $\mathcal{B}$ in eq (24) does not depend explicitly on x, we use translation invariance: classical translation invariance is expressed through $(S, S_\mu) = 0$ with quantum version $(\Gamma^\infty, S_\mu) = \hbar \mathcal{D}_\mu \circ \Gamma^\infty$, where the dimension of $\mathcal{D}_\mu$ is 1, because there are no dimensionful parameters in the theory. Applying $(\cdot, S_\mu)$ to (24), using the graded Jacobi identity for the antibracket, the commutation relation $(S_\eta, S_\mu) = -S_\mu$ and the result on the dimension of $\mathcal{D}_\mu$, i.e., the relation $(\mathcal{D}_\mu, S_\eta) = \mathcal{D}_\mu$, one finds to lowest order $(\mathcal{B}, S_\mu) = 0$. This means that $(\partial_\mu - \partial/\partial x^\mu)\mathcal{B} = 0$, and since $\partial_\mu \mathcal{B} = 0$, it which shows that $\mathcal{B}$ does not depend explicitly on x.

In the general case where we allow for dimensionful couplings considered below, we will assume that the theory is translation invariant and that the generator S_μ is coupled through the constant translation ghosts ξ^μ. One can then show that the local cohomology of the BRST operator in form degre n for the extended theory can be choosen to be independent of both x^μ and ξ^μ [27]. In the same way, Lorentz invariance can then be controled inside the formalism by coupling the appropriate generator.

4.2.2 Local form of Callan-Symanzik equation

In order to get non renormalization results for couplings associated to terms which are invariant only up to boundaries, we need a local version of the Callan-Symanzik equation. This is easily obtained from (22) by integrations by parts, giving

$$(S, L_\eta) = \partial_\mu J^\mu_\eta, \tag{30}$$

where we will refer to J^μ_η as the Callan-Symanzik current in the following. The quantum action principle gives

$$(\Gamma^\infty, L_\eta) = \partial_\mu [J^\mu_\eta \circ \Gamma^\infty] + \hbar b_\eta \circ \Gamma^\infty, \tag{31}$$

where b_η is a non integrated local polynomial. We need to know how $J^\mu_\eta \circ \Gamma^\infty$ behaves under the quantum BRST transformations. The classical descent equations obtained by applying $\bar{s}$ to (30) imply the existence of $K^{[\nu\mu]}_\eta$ such that

$$\bar{s} J^\mu_\eta = \partial_\nu K^{[\nu\mu]}_\eta. \tag{32}$$

Using lemma 1 of the appendix, we get

$$(\Gamma^\infty, J^\mu_\eta \circ \Gamma^\infty) + \Delta_c J^\mu_\eta \circ \Gamma^\infty = \partial_\nu [K^{[\nu\mu]}{}_\eta \circ \Gamma^\infty] + \hbar b^\mu_\eta \circ \Gamma^\infty, \tag{33}$$

where b^μ_η is a non integrated local polynomial. The consistency condition for (31) is then

$$(\Gamma^\infty, b_\eta \circ \Gamma^\infty) + \Delta_c b_\eta \circ \Gamma^\infty + \partial_\mu [b^\mu_\eta \circ \Gamma^\infty] = 0, \tag{34}$$

giving to lowest order $\bar{s} b_\eta + \partial_\mu b^\mu_\eta = 0$. We want to derive conditions under which, by appropriately modifying L_η, J^μ_η and $K^{[\nu\mu]}_\eta$, this equation reduces to

$$(\Gamma^\infty, b_\eta \circ \Gamma^\infty) + \Delta_c b_\eta \circ \Gamma^\infty = 0, \tag{35}$$

so that to lowest order the cocycle condition is rather $\bar{s} b_\eta = 0$.

(i) A first possibility would be the absence of non trivial BRST anomalies in the renormalization of the Callan-Symanzik current, i.e., the existence of local, finite, BRST breaking counterterms Σ^μ_k and $\Sigma^{[\nu\mu]}_k$ such that, if $J^{\infty\mu}_\eta = J^\mu_\eta - \Sigma_{k=1} \hbar^k \Sigma^\mu_k$ and $K^{\infty[\nu\mu]}_\eta = K^{[\nu\mu]}_\eta - \Sigma_{k=1} \hbar^k \Sigma^{[\nu\mu]}_k$, we have

$$(\Gamma^\infty, J^{\infty\mu}_\eta \circ \Gamma^\infty) + \Delta_c J^{\infty\mu}_\eta \circ \Gamma^\infty = \partial_\nu [K^{\infty[\nu\mu]}_\eta \circ \Gamma^\infty]. \tag{36}$$

A sufficient condition for this to occur is $H^{1+k,n-1-k}(\bar{s}|d) = 0$, for $k = 0, \ldots, n-1$. This follows from an analysis of the quantum descent equations as in [9]. Indeed, the classical descent equations are

$$\bar{s} J^{n-1-k}_\eta + d J^{n-1-k-1}_\eta = 0, \tag{37}$$

for $k = 0, \ldots n-1$ with $J^{-1}_\eta = 0$, where we can assume $J^{n-1-k}_\eta \in H^{k,n-1-k}(\bar{s}|d)$. The quantum version of the descent equations are

$$\begin{aligned}(\Gamma^\infty, J^{n-1-k}_\eta \circ \Gamma^\infty) + \Delta_c J^{n-1-k}_\eta \circ \Gamma^\infty + d[J^{n-1-k-1}_\eta \circ \Gamma^\infty] \\ = \hbar b^{n-1-k}_\eta \circ \Gamma^\infty.\end{aligned} \tag{38}$$

Applying $(\Gamma^\infty, \cdot) + \Delta_c$, we get the consistency condition $(\Gamma^\infty, b^{n-1-k}_\eta \circ \Gamma^\infty) + \Delta_c b^{n-1-k}_\eta \circ \Gamma^\infty + d[b^{n-1-k-1}_\eta \circ \Gamma^\infty] = 0$, giving to lowest order $\bar{s} b^{n-1-k}_\eta +$

$db_\eta^{n-1-k-1} = 0$, and thus, because of the assumption on the vanishing of the relevant cohomology classes, $b_\eta^{n-1-k} = \bar{s}\sigma_{\eta 1}^{n-1-k} + d\sigma_{\eta 1}^{n-1-k-1}$, so that

$$(\Gamma^\infty, [J_\eta^{n-1-k} - \hbar\sigma_{\eta 1}^{n-1-k}] \circ \Gamma^\infty) + \Delta_c([J_\eta^{n-1-k} - \hbar\sigma_{\eta 1}^{n-1-k})] \circ \Gamma^\infty) \\ + d([J_\eta^{n-1-k-1} - \hbar\sigma_{\eta 1}^{n-1-k-1}] \circ \Gamma^\infty) \\ = \hbar^2 b_\eta'^{n-1-k} \circ \Gamma^\infty, \qquad (39)$$

and the reasoning can be continued recursively to higher orders, so that we can achieve (36). If we now replace J_η^μ by $J_\eta^{\infty\mu}$ in (31) at the expense of modifying b_η appropriately, we get the desired consistency condition (35).

However, the vanishing of the cohomology groups assumed here is too restrictive to be of use. Indeed, our aim is to find conditions for the vanishing of the β functions associated to BRST cohomology goups with a non trivial descent. As shown in the appendix, the local BRST cohomology groups admit the decomposition $H^{0,n}(\bar{s}|d) \simeq lH^{1,n-1}(\bar{s}|d) \oplus rH^{0,n}(\bar{s})$, where the first group is isomorphic to the local BRST cohomology groups with a non trivial descent, $H^{0,n}_{nd}(\bar{s}|d)$. But $lH^{1,n-1}(\bar{s}|d) \subset H^{1,n-1}(\bar{s}|d)$. But by assumption, this last group vanishes, so that the assumptions unfortunately imply the absence of the terms whose non renormalization properties we wanted to show.

(ii) A second, non trivial case is when the Callan-Symanzik current is "covariantizable", i.e., when there exists $J^\mu_{\eta c} = J^\mu_\eta + \bar{s}K^\mu_\eta + \partial_\nu M_\eta^{[\nu\mu]}$ such that $\bar{s}J^\mu_{\eta c} = 0$[2]. The quantum version of this equation is $(\Gamma^\infty, J^\mu_{\eta c} \circ \Gamma^\infty) + \Delta_c J^\mu_{\eta c} \circ \Gamma^\infty = \hbar b^\mu_\eta \circ \Gamma^\infty$ and a sufficient condition for the absence of anomalies for the covariant Callan-Symanzik current, i.e., the existence of finite counterterms to $J^\mu_{\eta c} \to J^{\mu\infty}_{\eta c}$ such that

$$(\Gamma^\infty, J^{\mu\infty}_{\eta c} \circ \Gamma^\infty) + \Delta_c J^{\mu\infty}_{\eta c} \circ \Gamma^\infty = 0, \qquad (40)$$

is now the vanishing of the strict cohomology groups $H^{1,n-1}(\bar{s}) = 0$ without modulo d terms, which is of course a much less restrictive assumption.

Replacing L_η by $L_\eta - \partial_\mu K^\mu_\eta$, we get $\bar{s}(L_\eta - \partial_\mu K^\mu_\eta) = \partial_\mu J^\mu_{\eta c}$ and the quantum version

$$(\Gamma^\infty, L_\eta - (\partial_\mu K^\mu_\eta) \circ \Gamma^\infty) + \Delta_c[L_\eta - (\partial_\mu K^\mu_\eta) \circ \Gamma^\infty] \\ = \partial_\mu [J^{\mu\infty}_{\eta c}] \circ \Gamma^\infty + \hbar b_\eta \circ \Gamma^\infty, \qquad (41)$$

[2]As shown in [28], this question can again be formulated as a question of local BRST cohomology.

with the desired consistency condition (35).

From $\bar{s}b_\eta = 0$, it follows that $b_\eta = \tilde{\beta}^j c_j + \bar{s}\xi_1$ where c_j is a basis for $H^{0,n}(\bar{s})$. There is however a subtlety here, since there might be elements of $H^{0,n}(\bar{s})$, which become trivial if one allows for total divergences and could be absorbed by modifying at the same time L_η and the current. In order to deal with this aspect, we need a decomposition of strict BRST cohomology cocycles analogous to that of modulo d ones given in the appendix: $H^{g,k}(\bar{s}) = rH^{g,k}(\bar{s}) \oplus T^{g,k}$, where the space $T^{g,k}$ is defined by this equation. In other words, we split the solutions of the cocycle condition $\bar{s}k = 0$ into $k = b^{\bar{A}}k_{\bar{A}} + \bar{s}l + dm$, where $k_{\bar{A}}$ are a basis of $rH^{g,k}(\bar{s})$, and $\bar{s}m + dn = 0$. In this equation, we can assume that $m \in lH^{g,k-1}(\bar{s}|d)$, since a trivial solution to this equation can be absorbed by a redefinition of l, and by assumption, m can be lifted to give a part of the BRST cocycle k.

Thus, we take $b_\eta = \beta_{1\ \mathrm{loc}}^{\bar{j}} c_{\bar{j}} + \bar{s}\xi_1 + \partial_\mu m_1^\mu$, so that the redefinition $L_\eta - \partial_\mu K_\eta^\mu \to L_\eta^1 = L_\eta - \partial_\mu K_\eta^\mu - \hbar\xi_1$ allows to rewrite (41) as

$$(\Gamma^\infty, L_\eta^1 \circ \Gamma^\infty) + \Delta_c L_\eta^1 \circ \Gamma^\infty = \partial_\mu[J_{\eta c}^{\infty\mu} + \hbar m_1^\mu] \circ \Gamma^\infty + \hbar\beta_{1\ \mathrm{loc}}^{\bar{j}} c_{\bar{j}} \circ \Gamma^\infty + \hbar^2 b'_\eta \circ \Gamma^\infty. \tag{42}$$

In order to continue to higher orders, we need to know how the insertions $m_1^\mu \circ \Gamma^\infty, \beta_{1\ \mathrm{loc}}^{\bar{j}} c_{\bar{j}} \circ \Gamma^\infty$ behave under quantum BRST transformations. Because $m_1^\mu \in lH^{0,n-1}(\bar{s}|d) \simeq H^{-1,n-1}(\bar{s}|d)$ which represents the non trivial global currents of the theory [29], we assume that those that can appear on the right hand are covariantizable, as we did for the dilatation current. Hence, we can replace m_1^μ by m_{1c}^μ at the expense of modifying L_η^1 by terms of order at least $\hbar$. The further assumption $H^{1,n-1}(\bar{s}) = 0$ then guarantees the existence of counterterms such that $(\Gamma^\infty, m_{1c}^{\infty\mu} \circ \Gamma^\infty) + \Delta_c m_c^{\infty\mu} \circ \Gamma^\infty = 0$. From lemma 2 of the appendix, it follows that there exist finite counterterms $c_{\bar{j}} \to c_{\bar{j}}^\infty$ such that

$$(\Gamma^\infty, c_{\bar{j}}^\infty \circ \Gamma^\infty) + \Delta_c c_{\bar{j}}^\infty \circ \Gamma^\infty = \hbar a_{\bar{j}}^i k_i^\infty \circ \Gamma^\infty, \tag{43}$$

where $k_i \in H^{1,n}(\bar{s})$.

We now replace in (42) m_1^μ by $m_{1c}^{\infty\mu}$ and $c_{\bar{j}}$ by $c_{\bar{j}}^\infty$ and modify L_η^1 as well as b'_η accordingly. The consistency condition then gives to lowest order $\beta_{1\ \mathrm{loc}}^{\bar{j}} a_{1\bar{j}}^i k_i = \bar{s}b'_\eta$ so that $\beta_{1\ \mathrm{loc}}^{\bar{j}} a_{1\bar{j}}^i = 0 = \bar{s}b'_\eta$.

Going on recursively to higher orders, we find finally

$$(\Gamma^\infty, L_\eta^\infty \circ \Gamma^\infty) + \Delta_c L_\eta^\infty \circ \Gamma^\infty = \partial_\mu[J_{\eta c}^{\infty\mu} - m_c^{\infty\mu}] \circ \Gamma^\infty + \beta_{\mathrm{loc}}^{\bar{j}} c_{\bar{j}}^\infty \circ \Gamma^\infty, \tag{44}$$

where $m_c^{\infty\mu} = \Sigma_{n=1}\hbar^n m_{nc}^{\infty\mu}$ and $\beta^{\bar{j}}_{\rm loc} = \Sigma_{n=1}\hbar^n \beta^{\bar{j}}_{n\ {\rm loc}}$ and the classical approximation of $L^\infty_\eta \circ \Gamma^\infty$ is $L_\eta - \partial_\mu K^\mu_\eta$. We can now integrate to get

$$(\Gamma^\infty, S^\infty_\eta \circ \Gamma^\infty) + \Delta_c S^\infty_\eta \circ \Gamma^\infty = \hbar \beta^{\bar{j}}_{\rm loc} \int d^n x\ c^\infty_{\bar{j}} \circ \Gamma^\infty, \tag{45}$$

where the classical approximation for $S^\infty_\eta \circ \Gamma^\infty$ is the generator S_η of dilatation invariance. Comparing with (27), we thus see that the β functions of the elements of $H^{g,n}_{nd}(\bar{s}|d)$ vanish and only those associated to $rH^{g,n}(\bar{s})$ may be non vanishing.

Theorem 1 *Suppose a theory contains only dimensionless coupling constants. If (i) the extended BRST symmetry $\bar{s}$ is non anomalous, (ii) the Callan-Symanzik current and all the non trivial global conserved currents that can mix with it under renormalization are covariantizable, and (iii) the renormalization does not introduce BRST anomalies for these currents, then the β functions of cohomology classes of $\bar{s}$ with a non trivial descent vanish to all orders. A sufficient condition for (i) to hold is $H^{1,n}(\bar{s}|d) = 0$, while a sufficient condition for (iii) to hold is $H^{1,n-1}(\bar{s}) = 0$.*

4.2.3 Application: Pure semi-simple Chern-Simons theory

Consider semi-simple Chern-Simons theory without matter couplings, in a gauge without a dimensionful parameter as for instance the Landau gauge. The minimal solution to the master equation is

$$S = \int d^3x\ \epsilon^{\mu\nu\sigma} g_{ij} [A^i_\mu \partial^j_\nu A_\sigma + \frac{1}{3} f^j_{kl} A^i_\mu A^k_\nu A^l_\sigma] + A^{*\mu}_i D_\mu C^i + \frac{1}{2} C^*_i f^i_{jk} C^j C^k. \tag{46}$$

The local BRST cohomology has been worked out for instance in [30]. In ghost number 0, it contains only the Chern-Simons terms associated to the different simple factors, and all of them involve a non trivial descent. The remaining local BRST cohomology classes are at least of ghost number 3 and do not involve the antifields in a non trivial way. This means that Δ_c vanishes and there is no need to couple different local BRST cohomology classes to the solution of the master equation to ensure stability. Thus, we can take the usual BRST differential associated to the standard solution of the master equation $s = (S, \cdot)$. Condition (i) is satisfied because $H^{1,n}(s|d) = 0$. It is straightforward to verify that $J_c^\mu = 0$, the Callan-Symanzik current is trivial. This follows from

$$S_\eta = \int d^3x \;\; A_i^{*\mu}(1 + x\cdot\partial)A_\mu^i + C_i^* x\cdot\partial C^i = (S, M_\eta) + \partial_\mu N_\eta^\mu, \qquad (47)$$

with $M_\eta = 1/4 A_i^{*\mu} A_m^{*\sigma} g^{im} x^\nu \epsilon_{\nu\mu\sigma} + C_i^* x^\nu A_\mu^i$ and $N_\eta^\mu = A_i^{*\mu} x^\nu A_\mu^i$.

It is a consequence of the fact that the local BRST cohomology in negative ghost number is empty, and this both in the space of x dependent and x independent local functionals. There thus are no non nontrivial conserved currents, so that both assumptions (ii) and (iii) are satisfied. Hence, the vanishing of the β functions in pure semi-simple Chern-Simons theory can be traced back completely to the structure of the local BRST cohomology of the theory.

4.3 General broken case

4.3.1 Integrated form of Callan-Symanzik equation

Again, we assume that there are non anomalies, eq. (19). We will now allow for coupling constants ξ^A of all possible dimensions $d^{(A)}$ in the theory, which could be negative in the case of effective field theories. We have

$$(L, S_\eta) + \frac{\partial^R L}{\partial \xi^A} d^{(A)} \xi^A = \partial_\mu (x^\mu L). \qquad (48)$$

Integrating, one gets

$$\mathcal{C} S = 0, \qquad (49)$$

with $\mathcal{C} = (\cdot, S_\eta) + \frac{\partial^R \cdot}{\partial \xi^A} d^{(A)} \xi^A$. Using (5), we get $s_Q \partial^R \cdot / \partial \xi^A d^{(A)} \xi^A = [\bar{s}, \mathcal{C}] = 0$. The quantum version of (49) is

$$\mathcal{C}\Gamma^\infty = \hbar \mathcal{B} \circ \Gamma^\infty. \qquad (50)$$

and, applying $(\Gamma^\infty, \cdot) + \Delta_c$, the consistency condition to lowest order implies $\bar{s}\mathcal{B} = 0$. We can then get as in the previous section the general form of the integrated Callan-Symanzik equation:

$$(\Gamma^\infty, [S_\eta - \Xi \circ \Gamma^\infty]) + \Delta_c [S_\eta - \Xi \circ \Gamma^\infty] = \frac{\partial^R \Gamma^\infty}{\partial \xi^A} (\beta^A - d^{(A)} \xi^A), \qquad (51)$$

with $\Xi = \Sigma_{n=1} \hbar^n \Xi_n$ and $\beta^A = \Sigma_{n=1} \hbar^n \beta^A_n$.

In the derivation above, we have neglected the fact that there could be other dimensionful parameters α^i in the theory, besides the essential ones discussed so far. They could for instance come from the gauge fixation of the theory. Differentiation of the extended master equation and of $s_Q^2 = 0$ with respect to α^i and using (8), (9) implies

$$\frac{\partial^R S}{\partial \alpha^i} = \bar{s}\Xi_i + \frac{\partial^R S}{\partial \xi^A} \kappa_i^A, \tag{52}$$

with

$$\frac{\partial^R \cdot}{\partial \xi^A} \frac{\partial^R f^A}{\partial \alpha_i} = s_Q \frac{\partial^R \cdot}{\partial \xi^A} \kappa_i^A. \tag{53}$$

If d^i is the dimension of α^i, we have to add the term $\partial^R S / \partial \alpha^i d^{(i)} \alpha^i$ in (49). Using (52) and $\Delta_c S_\eta = 0$, (49) becomes

$$\bar{s}(S_\eta + \Xi_i d^{(i)} \alpha^i) + \frac{\partial^R S}{\partial \xi^A} (d^{(A)} \xi^A + \kappa_i^A d^{(i)} \alpha^i) = 0. \tag{54}$$

Together with (19) this equation can again be used to prove that

$$s_Q \partial^R \cdot / \partial \xi^A (d^{(A)} \xi^A + \kappa_i^A d^{(i)} \alpha^i) = 0. \tag{55}$$

The quantum analysis then proceeds exactly as before, and we get as quantum equation (51) with S_η replaced by $S_\eta + \Xi_i d^{(i)} \alpha^i$ and $d^{(A)} \xi^A$ by $d^{(A)} \xi^A + \kappa_i^A d^{(i)} \alpha^i$.

4.3.2 Local form of Callan-Symanzik equation

To get a local Callan-Symanzik equation, we integrate the first term in (48) by parts:

$$(S, L_\eta) + \frac{\partial^R L}{\partial \xi^A} d^{(A)} \xi^A = \partial_\mu J_\eta^\mu. \tag{56}$$

[Again, dimensionful inessential parameters can be incorporated by the substitutions $(S, L_\eta) \to \bar{s}(L_\eta + \xi_i d^{(i)} \alpha^i)$, $d^{(A)} \xi^A \to d^{(A)} \xi^A + \kappa_i^A d^{(i)} \alpha^i$ and finally $J_\eta^\mu \to J_\eta^\mu - j_i^\mu d^{(i)} \alpha^i$, where

$$\frac{\partial^R L}{\partial \alpha^i} = \bar{s}\xi_i + \frac{\partial^R L}{\partial \xi^A} \kappa_i^A + \partial_\mu j_i^\mu.]$$

Let us now assume that there exists K^μ_η, $M^{[\nu\mu]}_\eta$ such that if

$$J^\mu_{\eta c} = J^\mu_\eta + \bar{s} K^\mu_\eta + \partial_\nu M^{[\nu\mu]}_\eta, \tag{57}$$

then

$$\bar{s} J^\mu_{\eta c} = 0. \tag{58}$$

The assumption on the Callan-Symanzik current J^μ_η implies, by acting with $\bar{s}$,

$$\bar{s} \frac{\partial^R L}{\partial \xi^A} d^{(A)} \xi^A = 0. \tag{59}$$

This is a strong restriction on the terms of the (extended) Lagrangian coupled with dimensionful couplings: differentiation with respect to ξ^B and putting the ξ's to zero requires all these terms to be strictly BRST invariant. This means that the non renormalization theorem to be derived holds only if the terms that are invariant up to boundaries are all of dimension n.

Equation (56) becomes

$$\bar{s}[L_\eta - \partial_\mu K^\mu_\eta] + \frac{\partial^R L}{\partial \xi^A} d^{(A)} \xi^A = \partial_\mu J^\mu_{\eta c}. \tag{60}$$

The quantum version of this equation is

$$\begin{aligned} [(\Gamma^\infty, \cdot) + \Delta_c][L_\eta - \partial_\mu K^\mu_\eta] \circ \Gamma^\infty + \frac{\partial^R L}{\partial \xi^A} d^{(A)} \xi^A \circ \Gamma^\infty \\ = \partial_\mu [J^\mu_{\eta c} \circ \Gamma^\infty] + \hbar b_\eta \circ \Gamma^\infty. \end{aligned} \tag{61}$$

If the Callan-Symanzik current and $\frac{\partial^R L}{\partial \xi^A} d^{(A)} \xi^A$ renormalize without anomalies, i.e., if there exist counterterms such that

$$(\Gamma^\infty, J^{\infty\mu}_{\eta c} \circ \Gamma^\infty) + \Delta_c J^{\infty\mu}_{\eta c} \circ \Gamma^\infty = 0 \tag{62}$$

respectively

$$(\Gamma^\infty, [\frac{\partial^R L}{\partial \xi^A} d^{(A)} \xi^A]^\infty \circ \Gamma^\infty) + \Delta_c [\frac{\partial^R L}{\partial \xi^A} d^{(A)} \xi^A]^\infty \circ \Gamma^\infty = 0, \tag{63}$$

which is guaranteed if both $H^{1,n-1}(\bar{s})$ and $H^{1,n}(\bar{s})$ vanish, we can replace $J^{\mu}_{\eta c}$ by $J^{\infty\mu}_{\eta c}$ and $[\partial^R L/\partial\xi^A d^{(A)}\xi^A]$ by $[\partial^R L/\partial\xi^A d^{(A)}\xi^A]^\infty$ in (61) at the expense of modifying b_η appropriately. The consistency condition obtained by acting with $(\Gamma^\infty,\cdot)+\Delta_c$ is then the same as in the previous section, $(\Gamma^\infty, b_\eta\circ\Gamma^\infty)+\Delta_c b_\eta\circ\Gamma^\infty=0$ and we are back to the situation studied before.

The final result is

$$(\Gamma^\infty, L^\infty_\eta\circ\Gamma^\infty)+\Delta_c L^\infty_\eta\circ\Gamma^\infty+[\frac{\partial^R L}{\partial\xi^A}d^{(A)}\xi^A]^\infty\circ\Gamma^\infty$$
$$=\partial_\mu[J^{\infty\mu}_{\eta c}-m^{\infty\mu}_c]\circ\Gamma^\infty+\beta^{\bar{j}}_{\rm loc}c^\infty_{\bar{j}}\circ\Gamma^\infty, \qquad (64)$$

where again $m^{\infty\mu}_c=\Sigma_{n=1}\hbar^n m^{\infty\mu}_{nc}$ and $\beta^{\bar{j}}_{\rm loc}=\Sigma_{n=1}\hbar^n\beta^{\bar{j}}_{n\ {\rm loc}}$ and the classical approximation of $L^\infty_\eta\circ\Gamma^\infty$ is $L_\eta-\partial_\mu K^\mu_\eta$. Integration gives to

$$(\Gamma^\infty, S^\infty_\eta\circ\Gamma^\infty)+\Delta_c S^\infty_\eta\circ\Gamma^\infty+\int d^n x[\frac{\partial^R L}{\partial\xi^A}d^{(A)}\xi^A]^\infty\circ\Gamma^\infty$$
$$=\hbar\beta^{\bar{j}}_{\rm loc}\int d^n x\ c^\infty_{\bar{j}}\circ\Gamma^\infty. \qquad (65)$$

As before, the classical approximation for $S^\infty_\eta\circ\Gamma^\infty$ is the generator S_η of dilatation invariance, while the one for $\int d^n x[\frac{\partial^R L}{\partial\xi^A}d^{(A)}\xi^A]^\infty\circ\Gamma^\infty$ is $\frac{\partial^R S}{\partial\xi^A}d^{(A)}\xi^A$. Comparing with (27), we see again that the β functions of the elements of $H^{g,n}_{nd}(\bar{s}|d)$ vanish and only those associated to $rH^{g,n}(\bar{s})$ may be non vanishing.

Theorem 2 *If (i) the extended BRST symmetry $\bar{s}$ is non anomalous and the insertion $\partial^R L/\partial\xi^A d^{(A)}\xi^A\circ\Gamma^\infty$ renormalizes without anomaly, (ii) the Callan-Symanzik current and all the non trivial global conserved currents that can mix with it under renormalization are covariantizable, and (iii) the renormalization does not introduce BRST anomalies for these currents, then the β functions of cohomology classes of $\bar{s}$ with a non trivial descent vanish to all orders. A sufficient condition for (i) to hold is $H^{1,n}(\bar{s}|d)=0$, while a sufficient condition for (iii) to hold is $H^{1,n-1}(\bar{s})=0$.*

4.3.3 Application: Semi-simple Chern-Simons theory coupled to matter

We take as a starting point Chern-Simons theory based on a semi-simple Lie algebra to which are coupled matter fields. The minimal solution to the master equation is given by

$$S_{CSM} = S + \int d^3x \; L_M(y^m, D^T_\mu y^m, \ldots, D^T_{\mu_1} \ldots D^T_{\mu_k} y^m) + C^i (T_i y)^m y^*_m, \quad (66)$$

with S as in (46). The matter field Lagrangian is supposed to be gauge invariant, but the whole theory does not admit additional local symmetries. In particular, it is not restricted to be power counting renormalizable. The matter field Lagrangian admits the decomposition $L_M = L^{kin} + k^A K_A$, where the K_A are strictly gauge invariant polynomials and the k^A some coupling constants.

The local BRST cohomology [30] in ghost number 0 is exhausted by the Chern-Simons terms associated to the various simple factors and the different invariant matter field polynomials. The latter are strictly invariant, while the former involve a non trivial descent. Neither depend on the antifields in a non trivial way. The theory itself is thus stable. The only antifield dependent cohomology classes are in ghost number -1 and are related to the global symmetries of the theory. Since we are not interested here in controling them, we will not couple the corresponding generators. Hence, we can assume $\Delta_c = 0$ and $\bar{s} = s$. We have furthermore:

(i) the terms with dimensionful couplings are all strictly invariant,

(ii) the theory is anomaly free because $H^{1,n}(s|d) = 0$,

(iii) there is no anomaly in the renormalization of the invariant terms since $H^{1,n}(s) = 0$,

(iv) all the non trivial conserved currents are covariantizable [28] and $H^{1,n-1}(s) = 0$.

Thus, the only thing left to check is that the Callan-Symanzik current is covariantizable. Because it is not the current of a symmetry of the theory, we need to modify the reasoning of [28] in order to do so. Let us decompose $S_{CSM} = S_0 + \int d^3x \; k^\alpha K_\alpha$, where S_0 is the solution of the master equation where all the dimensionful k^α of the k^A have been set to zero. We have $(S_0, L_\eta) = \partial_\mu J^\mu_{0\eta}$, with

$$L_\eta = A^{*\mu}_i (1 + x \cdot \partial) A^i_\mu + C^*_i x \cdot \partial C^i + y^*_m (d^{(m)} + x \cdot \partial) y^m, \quad (67)$$

so that dilatation is a true global symmetry of the theory. It follows from [28] that both the generator and its current can be covariantized. Explicitly, if $M = C_i^* x^\nu A_\nu^i$ and $N^\mu = A_i^{*\mu} x^\nu A_\nu^i$, we have

$$\begin{aligned} L_\eta^c &= L_\eta - (S_0, M) - \partial_\mu N^\mu \\ &= A_i^{*\mu} x^\nu F_{\nu\mu}^i + y_m^* d^{(m)} y^m + y_m^* x^\nu D_\nu^T y^m, \end{aligned} \tag{68}$$

with $(S_0, L^{\eta c}) = \partial_\mu J^\mu_{0\eta c}$ and $(S_0, J^\mu_{0\eta c}) = 0$.

On the other hand, we have $(S_{CSM}, L_\eta) + K_\alpha d^{(\alpha)} k^\alpha = \partial_\mu J^\mu_\eta$. Furthermore, $L_\eta - (S_{CSM}, M) - \partial_\mu N^\mu = L_\eta^c$ because $(\int d^3x\ k^\alpha K_\alpha, M) = 0$. This means that $(S_{CSM}, L_\eta^c) + K_\alpha d^{(\alpha)} k^\alpha = \partial_\mu [J^\mu_\eta - (S_{CSM}, N^\mu)]$. As explained in [28], it follows that one can replace $J^\mu_\eta - (S_{CSM}, N^\mu)$ by $J^\mu_{\eta c}$ with $(S_{CSM}, J^\mu_{\eta c}) = 0$.

This proves the vanishing of the β functions associated to the Chern-Simons terms for this power counting non renormalizable model, independently of the choice of the gauge fixing fermion. Note that invariant terms containing covariant derivatives of the Yang-Mills field strength, which are all cohomologically trivially can be incorporated by coupling them with inessential parameters as discussed in detail in [31]. The covariant version of the generator of dilatation is also crucial in [3], where it has been obtained by using the trace of the improved energy-momentum tensor.

Conclusion

To summarize, the main idea of the paper is that the terms associated to the β functions can be considered as anomalies for (broken) dilatation invariance and as such, they have to satisfy consistency conditions. It is possible to formulate a local version of the Callan-Symanzik equation, where, under suitable assumptions, these conditions require the dilatation anomalies to be strictly invariant, and not only invariant up to a total divergence, as follows from the usual integrated Callan-Symanzik equation.

The general conditions for the validity of the non renormalization theorem given here can of course be checked in a straightforwrd way for other (topological) models than Chern-Simons theory, as soon as the local BRST cohomology of the theory is known.

The main assumption for the non renormalization mechanism given here is the non anomalous renormalization of the Callan-Symanzik current. If this current is not covariantizable, i.e., if it does not admit a representative which has no non trivial descent, the anomalies in the renormalization of this current are characterized by $H^{1,n-1}(\bar{s}|d)$ and this group will not be vanishing if there are terms of ghost number 0 and form degree n involving a non trivial descent.

Trying to understand the non renormalization theorems for the axial or the non abelian anomaly or the ones in supersymmetric theories by similar cohomological techniques will be the object of future investigations.

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Appendix

Lemma 1 *In the anomaly free case defined by (19), the equation*

$$[\bar{s}\Xi]\circ\Gamma^\infty = (\Gamma^\infty, \Xi\circ\Gamma^\infty) + \Delta_c[\Xi\circ\Gamma^\infty] + \hbar\mathcal{D}\circ\Gamma^\infty \tag{69}$$

holds, for local functionals $\mathcal{D}$ and Ξ.

Proof. Consider $S_\rho = S^\infty + \Xi\rho$. Then $\frac{1}{2}(S_\rho, S_\rho) + \Delta_c S_\rho = \bar{s}\Xi\rho + O(\hbar) + O(\rho^2)$. The quantum action principle applied to this equation gives $\frac{1}{2}(\Gamma_\rho, \Gamma_\rho) + \Delta_c\Gamma_\rho = \mathcal{D}(\rho)\circ\Gamma_\rho$. Putting ρ to zero and using (19), we get $\mathcal{D}(0) = 0$, so that $\mathcal{D}(\rho) = \mathcal{D}'(\rho)\rho$. If we now differentiate with respect to ρ and set ρ to zero, we get $(\Gamma^\infty, \Xi\circ\Gamma^\infty) + \Delta_c[\Xi\circ\Gamma^\infty] = \Delta'(0)\circ\Gamma^\infty$. To lowest order in $\hbar$, we find $\Delta'(0) = \bar{s}\Xi$. □

A different proof of this lemma can be obtained by using the so called extended BRST technique [32, 31].

Decomposition of BRST modulo d cohomology classes: [33]

A BRST cocycle modulo d is given by a k-form b of ghost number g satisfying $\bar{s}b + dc = 0$ for some c. The equivalence classes $[b]$ under the equivalence relation $b \sim b + \bar{s}f + dg$ are the elements of $H^{g,k}(\bar{s}|d)$. The descent equations, following from the triviality of the cohomology of d in form degree less than n, imply that $\bar{s}c + de = 0$ for some e, so that $[c] \in H^{g+1,n-1}(s|d)$. Consider the map $\mathcal{D} : H^{g,n}(\bar{s}|d) \longrightarrow H^{g+1,n-1}(\bar{s}|d)$ defined by $\mathcal{D}[b] = [c]$. It is straightforward to verify that (i) this map is well defined, i.e., it does not depend on the choice of representatives, (ii) $Ker\ \mathcal{D}$ is determined by $[b]$ with $\bar{s}b = 0$; this space will be denoted by $rH^{g,k}(\bar{s})$ and corresponds to those elements of $H^{g,k}(\bar{s})$ which remain non trivial under the more general $\bar{s}$ modulo d coboundary condition, and (iii) $Im\ \mathcal{D}$ is given by those elements $[c] \in H^{g+1,k-1}(\bar{s}|d)$, which can be lifted; we denote the corresponding subspace by $lH^{g+1,k-1}(\bar{s}|d)$.

We thus have the direct sum decomposition $H^{g,k}(\bar{s}|d) = rH^{g,k}(\bar{s}) \oplus H^{g,k}_{nd}(\bar{s}|d)$, where $H^{g,k}_{nd}(\bar{s}|d) \simeq lH^{g+1,n-1}(\bar{s}|d)$. The subspace $H^{g,k}_{nd}(\bar{s}|d)$ is the subspace of cohomology classes with a non trivial descent.

Lemma 2 *In the anomaly free case defined by (19), any integrated BRST cohomology class $K_\alpha = \int d^n x\ k_\alpha$ in ghost number g_α, $k_\alpha \in H^{g_\alpha,n}(\bar{s}|d)$, there exist counterterms $K_\alpha \rightarrow K^\infty_\alpha$ such that*

$$(\Gamma^\infty, K^\infty_\alpha \circ \Gamma^\infty) + \Delta_c K^\infty_\alpha \circ \Gamma^\infty = \hbar a^\beta_\alpha K^\infty_\beta \circ \Gamma^\infty, \tag{70}$$

where $k_\beta \in H^{g_\alpha+1,n}(\bar{s}|d)$ and $a^\beta_\alpha a^\gamma_\beta = 0$ as a power series in $\hbar$.

The same result holds for non integrated strict BRST cohomology classes $k_\alpha \in H^{g_\alpha,k}(\bar{s})$.

Proof. We have $\bar{s}K_\alpha = 0$ so that according to lemma 1, $(\Gamma^\infty, K_\alpha \circ \Gamma^\infty) + \Delta_c K_\alpha \circ \Gamma^\infty = \hbar \mathcal{A}_\alpha \circ \Gamma^\infty$. The consistency implies to lowest order that $\mathcal{A}_\alpha = a^\beta_{\alpha 1} K_\beta + \bar{s}\Sigma_{1\ \alpha}$, with $k_\beta \in H^{g_\alpha+1,n}(\bar{s}|d)$. Hence, through the redefinition $K^1_\alpha = K_\alpha - \hbar\Sigma^1_\alpha$, we can achieve

$$(\Gamma^\infty, K^1_\alpha \circ \Gamma^\infty) + \Delta_c K^1_\alpha \circ \Gamma^\infty = \hbar a^\beta_{\alpha 1} K^1_\beta \circ \Gamma^\infty + \hbar^2 \mathcal{A}'_\alpha \circ \Gamma^\infty. \tag{71}$$

To lowest order, the consistency condition for this equation gives $a^\beta_{\alpha 1} a^\gamma_{\beta 1} K_\gamma = \bar{s}\mathcal{A}'_\alpha$, implying $a^\beta_{\alpha 1} a^\gamma_{\beta 1} = 0 = \bar{s}\mathcal{A}'_\alpha$. The reasoning can be pushed recursively to higher orders in $\hbar$, with the announced result. The proof for the strict cohomology classes proceeds in the same way. □

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