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THE NEGATIVE DISCRETE SPECTRUM OF A CLASS OF TWO-DIMENTIONAL SCHRÖDINGER OPERATORS WITH MAGNETIC FIELDS

BY

A. Laptev and O. Safronov

Abstract. We obtain an asymptotic formula for the number of negative eigenvalues of a class of two-dimensional Schrödinger operators with small magnetic fields. This number increases as a coupling constant of the magnetic field tends to zero.

1.Introduction and Notations. Let us consider a selfadjoint Schrödinger operator in $L^2(\mathbb{R}^2)$ formally written as

$$\mathcal{H}_\beta = (i\nabla + \beta A)^2 - V, \qquad \beta > 0. \tag{1.1}$$

Here the electric potential $V = V(x)$, $x \in \mathbb{R}^2$, is a nonnegative function and $A = A(x) = (A_1(x), A_2(x))$ is a vector potential. The precise definition of the operator $\mathcal{H}_\beta$ can be given via the corresponding quadratic form

$$\mathfrak{h}[u] = \int_{\mathbb{R}^2} \Big(|i\nabla u + \beta\, A\, u|^2 - V|u|^2 \Big)\, dx, \quad u \in H^1(\mathbb{R}^2). \tag{1.2}$$

Under certain conditions on V and A the form (1.2) is closed on the Sobolev space $H^1(\mathbb{R}^2)$ and generates the selfadjoint operator $\mathcal{H}_\beta$ in $L^2(\mathbb{R}^2)$. We shall study a class of operators whose discrete negative spectrum is infinite when $\beta = 0$ and finite when $\beta > 0$. The number of the negative eigenvalues of the operator $\mathcal{H}_\beta$, $\beta > 0$ is denoted by $N(\beta) = N(\mathcal{H}_\beta) = N(\beta, \mathfrak{h})$ depending on what is more convenient. In this paper we obtain an asymptotic formula for the value $N(\beta)$, as $\beta \to 0$ and therefore give a quantative description of how a magnetic field "lifts up" the spectrum of the operator $\mathcal{H}_\beta$. This setting becomes possible because the spectral point zero is the resonance state for a two-dimensional Schrödinger operator and is thus sensitive to perturbations either by electric or vector potentials.

In order to formulate the main result we consider a *gauge transformation* which annihilates the radial component of the vector potential A (see [T, Section 8.4.2]) and which sometimes is called by the transversal (or Poincaré) gauge. The latter corresponds to a unitary transformation of the original operator and does not change its spectrum. Most of our discussions will be described in polar coordinates

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$(r,\theta) \in [0,\infty) \times \mathbb{T}$ at $x \in \mathbb{R}^2$, where $\mathbb{T} = \{x \in \mathbb{R}^2 : |x| = 1\}$. We denote by $e_r = x/r$ the radial unit vector and by e_θ the unit vector which completes e_r up to the oriented orthonormal basis at $x \neq 0$. Proposition 3.1 shows that if A is smooth enough, then one can gauge away the radial component of the vector potential A or find real functions ψ and a, such that

$$A + \nabla\psi = a(x)e_\theta. \tag{1.3}$$

Then (1.2) transfers into a more convenient form for which we use the same notation

$$\mathfrak{h}[u] = \int_0^\infty \int_0^{2\pi} \Big(|u_r'|^2 + |ir^{-1}u_\theta' + \beta\, au|^2 - V|u|^2\Big)\, r\, d\theta dr. \tag{1.4}$$

In the next section we formulate our main result (Theorem 2.1) concerning the asymptotics of the value $N(\beta)$ as $\beta \to 0$ and give its proof in Section 4. Theorem 2.1 deals with the special behaviour of the vector and electric potentials near infinity and covers the important applications including Aharonov-Casher magnetic fields. If $\beta = 0$ then the infinite number of the negative eigenvalues is prescribed by the behaviour of V at ∞. In particular, our result shows that a compactly supported magnetic field with a small coupling constant β (the case $\nu = 0$ in Theorem 2.1) makes the negative spectrum finite (an immediate influence at ∞) and gives the rate of increase of the number of negative eigenvalues as the coupling constant β turns to zero.

We show in Section 5 that similar effects hold true if a has some prescribed asymptotic properties at $x_0 \in \mathbb{R}^2$ and finally we apply our results to vector potentials corresponding to the Aharonov-Bohm effect in Section 6.

The result of Theorem 2.1 can be easily generalized to the case when Condition (2) (see Section 2) is satisfied only asymptotically as $r \to \infty$. In fact a similar result holds true for a much wider class of magnetic Schrödinger operators without a prescribed asymptotic behavior at infinity, but involving some cumbersome quantative assumptions on the mean values of V and a over $\mathbb{T}$. Finally we decided to consider here a simple case covering, however, the most important applications.

Denote by $< \cdot, \cdot >$ the real scalar product of vectors in $\mathbb{R}^2$. Let

$$\mathcal{B}(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, dt$$

be the Beta function. An open ball with its center at $x \in \mathbb{R}^2$ and radius $r > 0$ is denoted by $B(x,r)$; $\mathbb{R}_+ := (0,\infty)$, $\mathbb{N}$ and $\mathbb{Z}$ are sets of positive and all integers respectively. For the scalar product in $L^2(\mathbb{T})$ we use $(\cdot,\cdot)_{\mathbb{T}}$. The class of functions $u \in L^p(\mathbb{R}^2)$, $1 \le p \le \infty$, whose distributional derivatives belong to $L^p(\mathbb{R}^2)$, is denoted by $W^{1,p}(\mathbb{R}^2)$. The corresponding local class of functions is $W^{1,p}_{\rm loc}(\mathbb{R}^2)$. Let us also agree that the number of negative eigenvalues corresponding to a quadratic form will sometimes be denoted by its reference number, say $N(\beta,\mathfrak{h}) = N(\beta,(1.2))$. Let G be a function of β and $G(\beta) \to \infty$, as $\beta \to 0$. In a similar way we shall denote the functionals

$$\Delta_G(\mathfrak{h}) = \Delta_G((1.2)) = \limsup_{\beta\to 0} N(\beta,\mathfrak{h})/G(\beta) \tag{1.5}$$
$$\text{and} \quad \delta_G(\mathfrak{h}) = \delta_G((1.2)) = \liminf_{\beta\to 0} N(\beta,\mathfrak{h})/G(\beta).$$

2. Assumptions and Results. It is now convenient to formulate our main assumptions in terms of the function a introduced in (1.3) (see also Proposition 3.1).

Introduce the mean values $\bar{a}$ and $\bar{V}$ of the functions a and V over $\mathbb{T}$

$$\bar{a} = \frac{1}{2\pi}\int_0^{2\pi} a(r,\theta)\, d\theta \qquad \text{and} \qquad \bar{V} = \frac{1}{2\pi}\int_0^{2\pi} V(r,\theta)\, d\theta. \tag{2.1}$$

We assume now that the following four conditions are satisfied:

There exist constants $C_1, C_2 > 0$, a function $\Psi \in L^\infty(\mathbb{T})$ and $r_0 > 1$ such that

(1) $\frac{C_1}{r^2(\log r)^\mu} \le V(r) \le \frac{C_2}{r^2(\log r)^\mu}, \quad r \ge r_0, \quad 0 < \mu < 2,$

(2) $a(r,\theta) = \frac{\Psi(\theta)}{r(\log r)^{\nu/2}} \quad r \ge r_0, \quad 0 \le \nu < \mu,$

Suppose also that there is a constant $C > 0$, such that for any $r_1, r_2 > r_0$

”(31)”

$$|\bar{a}^{-2}(r_1)\bar{V}(r_1) - \bar{a}^{-2}(r_2)\bar{V}(r_2)| \le C|\log(r_1 r_2^{-1})|(\min(\log r_1, \log r_2) + 1)^{-1-\mu+\nu}.$$

In order to characterize the behaviour of V, a for $r < r_0$ we will suppose that

(1) $a, V \in L^\infty(\mathbb{R}^2)$.

Conditions (1) and (3) follow from the corresponding conditions in Theorem XIII.82 [RS] applied to the effective potential Φ given in (4.9). If Conditions (1)-(4) are satisfied, then the quadratic form (1.4) is semi-bounded, closed on $H^1(\mathbb{R}^2)$ and therefore defines the selfadjoint operator $\mathcal{H}_\beta$.

Note that for the mean value of Ψ we have

$$\bar{\Psi} = r(\log r)^{\nu/2}\, \bar{a}, \quad r > r_0.$$

Therem 2.1. *Let the functions V and a satisfy the conditions (1)-(4), $\bar{\Psi} \ne 0$ and $0 \le \nu < \mu < 2$. Then the number of the negative eigenvalues $N(\mathcal{H}_\beta)$ of the operator (1.1) satisfies the asymptotic formula*

$$\lim_{\beta \to 0} N(\mathcal{H}_\beta)/G(\beta) = 1, \tag{2.2}$$

where

$$G(\beta) = \frac{1}{2\pi^2}\int_{\mathbb{R}^2} \left(\bar{V}(|x|) - \beta^2\, \bar{a}^2\right)_+^{1/2} \frac{dx}{|x|}.$$

Remark 2.1. If a bounded potential $V(x) \sim c|x|^{-2}\log^{-\mu}|x|$ as $|x| \to \infty$ and $\mu > 0$, $c > 0$, then the Schrödinger operator $-\Delta - V$ in $L^2(\mathbb{R}^d)$, $d \ge 3$ has finite number of negative eigenvalues (see [BS] and [L]). However, if $d = 2$ and $0 < \mu < 2$, then this fact is no longer true and the negative spectrum of the operator $-\Delta - V$ is infinite due to the lack of the Hardy inequality (see [BL], [Sol]). Theorem 1.1 shows that a nontrivial magnetic field “lifts” the negative spectrum up and gives quantative characteristics of this lifting in terms of asymptotics with respect to a small coupling constant at the vector potential.

Remark 2.2. The asymptotic formula (2.2) depends only on the mean values of the potentials V and a over $\mathbb{T}$ and, for example, holds true even when the supports of these functions do not intersect.

Remark 2.3. The value $\bar{a}$ appearing in Theorem 2.1 is invariant with respect to gauge transformations. Indeed, for any $\nu \geq 0$ and $r > r_0$ by the Stocks formula we find

$$\bar{a} = \frac{1}{2\pi} r^{-1} \int_0^{2\pi} r\, a(r,\theta)\, d\theta = \frac{1}{2\pi} r^{-1} \int_{B(0,r)} (\partial_{x_1} A_2 - \partial_{x_2} A_1)\, dx. \tag{2.3}$$

Remark 2.4. If $\nu = 0$, then the magnetic field $B = \operatorname{curl} A = 0$ for $r > r_0$. Thus Theorem 2.1 establishes the following phenomena: A small perturbation by a compactly supported magnetic field can make the infinite negative spectrum of a Schrödinger operator finite. The number of its negative eigenvalues satisfies the asymptotic formula (2.2), as the coupling constant $\beta \to 0$.

In some special cases the formula (2.2) can be simplified.

Corollary 2.2. *Let Conditions (2) and (4) be satisfied. Assume also that*

$$V(r,\theta) = \frac{\Upsilon(\theta)}{r^2 (\log r)^\mu}, \quad r \geq r_0, \quad 0 < \mu < 2.$$

Then

$$\lim_{\beta \to 0} \beta^{\varkappa} N(\mathcal{H}_\beta) = c_{\mu,\nu} \bar{\Upsilon}^{(\varkappa+1)/2} \bar{\Psi}^{-\varkappa},$$

$$\text{where} \quad c_{\mu,\nu} = \pi^{-1} (\mu - \nu)^{-1} \mathcal{B}\Big(\frac{\varkappa}{2}, \frac{3}{2}\Big), \quad \varkappa = \frac{2-\mu}{\mu - \nu}.$$

3. Preliminary Results. We shall show here that under some assumptions of smoothness of the vector potential A, its radial component can be gauged away.

Proposition 3.1. *Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector potential in $\mathbb{R}^2$ and assume that its radial component $A_{\mathrm{rad}} = \langle A, e_r \rangle$ satisfy the following conditions*

$$r^{-1} A_{\mathrm{rad}} \in L^1_{\mathrm{loc}}(\mathbb{R}^2) \qquad \text{and} \qquad \frac{1}{r} \frac{\partial A_{\mathrm{rad}}}{\partial \theta} \in L^1_{\mathrm{loc}}(\mathbb{R}^2).$$

Then there exists a real function $\psi \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$, such that

$$A(x) + \nabla \psi(x) = a(x) e_\theta, \qquad \text{a.e.}$$

where

$$a(x) = a(r,\theta) = \langle A, e_\theta \rangle - r^{-1} \frac{\partial}{\partial \theta} \int_0^r A_{\mathrm{rad}}(\rho,\theta)\, d\rho. \tag{3.1}$$

Proof. Let $\psi(r,\theta) = -\int_0^r A_{\mathrm{rad}}(\rho,\theta)\, d\rho$ (for almost all θ this function is defined for for all $r > 0$). Then $\psi \in W^{1,1}_{\mathrm{loc}}$ and

$$\nabla \psi = \psi'_r e_r + r^{-1} \psi'_\theta e_\theta = -\langle A, e_r \rangle e_r - r^{-1} \Big(\frac{\partial}{\partial \theta} \int_0^r A_{\mathrm{rad}}(\rho,\theta)\, d\rho \Big) e_\theta,$$

and thus

$$< A + \nabla\psi, e_r >= 0, \qquad < A + \nabla\psi, e_\theta >= a \qquad \text{a.e.}$$

The proof is completed. □

Remark 3.1. Proposition 3.1 can be easily generalized into a corresponding statement

for vector potentials in $\mathbb{R}^d$ for any $d \geq 2$.

We shall see later that the asymptotics does not change if we change the potentials a and V on a compact set. Therefore we assume that the function a is smooth for $|x| < r_0$.

When studying the quadratic form (1.4) we use spectral properties of the self-adjoint differential operator $K_{\alpha,r}$ defined by

$$(3.2) \qquad (K_{\beta,r}\,\varphi)(\theta) = i\frac{\partial}{\partial\theta}\varphi(\theta) + \beta\, r\, a(r,\theta)\varphi(\theta), \quad 0 \leq \theta < 2\pi, \quad \varphi \in H^1(\mathbb{T})$$

and depending on r and β as parameters. The spectrum of this operator is discrete and its eigenvalues $\{\lambda_k\}_{k\in\mathbb{Z}}$ and the complete orthonormal system of eigenfunctions $\{\varphi_k\}_{k\in\mathbb{Z}}$ are given by the expressions:

$$(3.3) \qquad \lambda_k = \lambda_k(\beta, r) = k + \beta\, r\, \bar{a}(r), \qquad k \in \mathbb{Z},$$

and

$$(3.4) \qquad \varphi_k = \varphi_k(\beta, r, \theta) = \frac{1}{\sqrt{2\pi}} e^{-i(\theta\,\lambda_k(\beta,r) - \beta\, r\, \int_0^\theta a(r,\tau)\,d\tau)}, \qquad k \in \mathbb{Z}.$$

In particular, λ_k and φ_k are differentiable with respect to r and we also always have

$$(3.5) \qquad (\varphi_k, (\varphi_k)'_r)_{\mathbb{T}} = 0, \quad r > r_0, \quad k \in \mathbb{Z}.$$

Condition (2) also gives us that for $r > r_0$

$$(3.6) \quad \omega(r) = \frac{1}{\beta^2}\int_0^{2\pi} |(\varphi_0)'_r|^2\, d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \left| \int_0^\theta (r(a-\bar{a}))'_r\, d\tau \right|^2 d\theta$$
$$= \frac{1}{2\pi}\left| \left(\log^{-\nu/2} r\right)'_r \right|^2 \int_0^{2\pi} \left| \int_0^\theta (\Psi(\tau) - \bar{\Psi})\, d\tau \right|^2 d\theta$$
$$= o(r^{-2}(\log r)^{-\nu}), \qquad \text{as} \quad r \to \infty.$$

We introduce now the following decomposition

$$(3.7) \qquad u = z + g, \qquad u \in L^2(\mathbb{R}^2),$$

where

$$(3.8) \quad z = z(\beta, r, \theta) = (u, \varphi_0)_{\mathbb{T}}\, \varphi_0(\beta, r, \theta)$$
$$= \varphi_0(\beta, r, \theta)\int_0^{2\pi} u(r,\tau)\overline{\varphi_0(\beta, r, \tau)}\, d\tau, \quad r\text{-a.e.},$$

and

$$g = u - z. \tag{3.9}$$

Obviously

$$(g, \varphi_0)_{\mathbb{T}} = \int_0^{2\pi} g(\beta, r, \theta)\overline{\varphi_0(\beta, r, \theta)}\, d\theta = 0$$

for almost all r. Therefore the equations (3.7)-(3.9) induce the corresponding orthogonal

decomposition of the Hilbert space $L^2(\mathbb{R}^2)$. They also give a direct (not necessary orthogonal) decomposition of the space $H^1(\mathbb{R}^2)$. Using the property $\varphi_0 \in W^{1,\infty}(\mathbb{R}^2)$ we find that $v(r)\varphi_0(r,\theta) \in H^1(\mathbb{R}^2)$ if and only if $v \in H^1(\mathbb{R}_+)$. This fact will be important for us when we study the restriction of the form $\mathfrak{h}$ on the class of functions (3.8).

The decomposition (3.7) is not invariant with respect to the action of the operator $\mathcal{H}_\beta$. We show, however, that there is an "asymptotic invariance" as $\beta \to 0$, which allows us to compute the asymptotic formulae (2.2) and (2.3).

4. Proof of Theorem 2.1. We shall now discuss two side estimates for the quadratic form (1.4) employing the above mentioned asymptotic invariance of this form restricted on the functions (3.8) and (3.9).

Upper spectral estimate. In order to obtain an upper estimate for the negative spectrum of the operator (1.1) we begin by estimating the form (1.4) from below

$$\begin{aligned} &\mathfrak{h}[u] = \mathfrak{h}[z+g] \qquad (4.1)\\ &= \int_0^\infty \int_0^{2\pi} \Big(|(z+g)'_r)|^2) + |i\, r^{-1}(z+g)'_\theta + \beta\, a(z+g)|^2 - V(r,\theta)|z+g|^2 \Big)\, r\, d\theta dr. \end{aligned}$$

Let $0 < \varepsilon < 1$. Applying the Cauchy inequality we obtain

$$\mathfrak{h}[u] \geq \mathfrak{h}_1[z] + \mathfrak{h}_2[g], \tag{4.2}$$

where

$$\mathfrak{h}_1[z] = \int_0^\infty \int_0^{2\pi} \Big((1-\varepsilon)|z'_r|^2 + |i\, r^{-1} z'_\theta - \beta\, a\, z|^2 - (1+\varepsilon)V|z|^2 \Big)\, r\, d\theta dr \tag{4.3}$$

and

$$\mathfrak{h}_2[g] = \int_0^\infty \int_0^{2\pi} \Big((1-\varepsilon^{-1})|g'_r|^2 + |i\, r^{-1} g'_\theta - \beta\, a\, g|^2 - (1+\varepsilon^{-1})V)|g|^2 \Big)\, r\, d\theta dr. \tag{4.4}$$

Since $(g, \varphi_0)_{\mathbb{T}} = 0$ we find

$$\int_0^{2\pi} |i\, r^{-1} g'_\theta - \beta\, a\, g|^2\, d\theta \geq r^{-2} \lambda_1^2(\beta, r) \int_0^{2\pi} |g|^2\, d\theta,$$

where λ_1 is defined in (3.3) ($k = 1$). It is now easy to see that there exists $\beta_0 > 0$, such that the operator generated by the form (4.4) has a finite number of negative eigenvalues uniformly with respect to $\beta < \beta_0$. Indeed, from (3.3) and Condition (3) we observe that if $\beta_0 \bar{\Psi} < 1/2$ and $r > r_0$, then for any $\beta < \beta_0$ we have $\lambda_1(\beta, r) > 1/2$. Since the potential V is a bounded function satisfying $V(r,\theta) = o(r^{-2})$, $r \to \infty$, we conclude that $[(2r)^{-2} - (1+\varepsilon^{-1})V]_-$ is a bounded function whose support is a compact set and thus the negative spectrum of the form $\mathfrak{h}_2$ is finite.

Let us study the form $\mathfrak{h}_1[z]$. By substituting $z(r,\theta) = v(r)\varphi_0(r,\theta)$ and using (3.5) we obtain

$$\mathfrak{h}_1[z] = \int_0^\infty \int_0^{2\pi} \Big((1-\varepsilon)|(v\varphi_0)'_r|^2 + r^{-2}\lambda_0^2(\beta,r)|v|^2|\varphi_0|^2 - (1+\varepsilon)V|v|^2|\varphi_0|^2 \Big)\, r\, d\theta dr \tag{4.5}$$

$$= \int_0^\infty \Big((1-\varepsilon)|v'|^2 + \big((1-\varepsilon)\,\beta^2\,\omega + \beta^2 \bar{a}^2 - (1+\varepsilon)\bar{V}\big)\big)|v|^2 \Big)\, r\, dr, \tag{4.6}$$

where ω is defined in (3.6). Note that in order to obtain a lower estimate for the form (4.6), we can omit the term with ω, since this function is positive.

Let us fix an arbitrary $r_1 > 0$. Using rather standard variational arguments we can reduce the study of the upper estimate for the value $N(\alpha, (4.6))$ to that of the upper estimate for $N(\alpha, (4.7))$, where

$$\int_{r_1}^\infty \Big((1-\varepsilon)|v'|^2 + \big(\beta^2\bar{a}^2 - (1+\varepsilon)\bar{V}\big)\big)|v|^2 \Big)\, r\, dr, \quad v \in C_0^\infty(r_1,\infty). \tag{4.7}$$

so the term with the function ω is omitted and we take the integration over the interval (r_1,∞). Without loss of generality we can assume that

$$a(r,\theta) = \Psi(\theta)\, r^{-1}(\log r)^{-\nu/2}$$

for any $r > r_1$ i.e. a is substituted by its behavior valid for $r > r_0$. In order to prove this fact it is sufficient to note that the number of the negative spectrum of the form

$$\int_0^{r_1} ((1-\varepsilon)|v'|^2 - (1+\varepsilon)\bar{V}\,|v|^2)\, r\, dr$$

is finite and independent of β. Thus by the splitting principle, the asymptotics for $N(\alpha, (4.6))$ does not change if we change the function a on a compact set and integrate in (4.6) from r_1 to ∞.

Returning to the study of the quadratic form (4.7) we introduce the substitution

$$y(t) = (\log r)^{-\nu/4} v(r), \qquad r = \exp(((2-\nu)t/2)^{\frac{2}{2-\nu}}). \tag{4.8}$$

We also put

$$\Phi(t) = \bar{V}(e^{((2-\nu)t/2)^{\frac{2}{2-\nu}}})/\bar{a}^2(e^{((2-\nu)t/2)^{\frac{2}{2-\nu}}}), \tag{4.9}$$

and choose

$$r_1 = \exp(((2-\nu)/2)^{\frac{2}{2-\nu}}).$$

Then we find that $N(\alpha,(4.7)) = N(\alpha,(4.10))$, where

$$(4.10)\quad \int_1^\infty \Big((1-\varepsilon)|y'|^2 + (1-\varepsilon)ct^{-2} + \big(\beta^2 - (1+\varepsilon)\Phi(t)\big)\bar\Psi^2\,|y|^2\Big)\,dt, \qquad y \in C_0^\infty(1,\infty),$$

with some constant $c > 0$ which can be precisely calculated. Applying Theorem XIII.82 [RS] to the operator generated by the form (4.10), we arrive at

$$(4.11)\quad \Delta_G(\mathfrak{h}_1) \le \Delta_G((4.10))$$
$$= \limsup_{\beta\to 0}\,(G(\beta))^{-1}\,\frac{1}{\pi\sqrt{1-\varepsilon}}\int_1^\infty \big(\bar\Psi^2\,((1+\varepsilon)\Phi - \beta^2\,) + (1-\varepsilon)ct^{-2}\big)_+^{1/2}\,dt$$
$$= \limsup_{\beta\to 0}\,(G(\beta))^{-1}\,\frac{1}{\pi\sqrt{1-\varepsilon}}\int_0^\infty \big((1+\varepsilon)\bar V - \beta^2\bar a^2\big)_+^{1/2}\,dr.$$

Let us show that ε appearing in the right hand side of (4.11) can be substituted by 0. Indeed, using the inequality $\sqrt{(a+b)_+} - \sqrt{a_+} \le \sqrt{b_+}$ and Condition (1) we have

$$\limsup_{\beta\to 0}\Big((G(\beta))^{-1}\int\big((1+\varepsilon)\bar V - \beta^2\bar a^2\big)_+^{1/2}dr - 1\Big)$$
$$\le \sqrt{\varepsilon}\,\limsup_{\beta\to 0}\,(G(\beta))^{-1}\int_{(1+\varepsilon)\bar V \ge \beta^2\bar a^2} \bar V^{1/2}\,dr \le C\sqrt{\varepsilon},$$

where $C = C(\bar V, \bar a)$. Since ε is arbitrary this implies $\Delta_G(\mathfrak{h}_1) \le 1$ which is equivalent to the inequality

$$\limsup_{\beta\to 0}(G(\beta))^{-1}N(\mathcal{H}_\beta) \le 1.$$

Lower spectral estimate. The proof of the estimate of the value $N(\mathcal{H}_\beta)$ from below coincides with the corresponding proof of the upper estimate. We only need to write the opposite inequalities, change ε by $-\varepsilon$ and use the functional δ_G instead of Δ_G. MOreover, we should not ignore the term containing $\tilde\varphi_0$ but use the estimate (3.6). This gives us

$$\limsup_{\beta\to 0}(G(\beta))^{-1}N(\mathcal{H}_\beta) \ge 1.$$

The theorem is proved. $\square$

5. Local singularities. Let us now consider the problem generated by the form (1.4), where a and V have compact supports and satisfy the following local conditions at $x = 0$:

There exist constants $C_1, C_2 > 0$, a function $\Psi \in L^\infty(\mathbb{T})$ and a number $r_0 < 1$ such that

(1*)
$$\frac{C_1}{r^2(\log r)^\mu} \le V(r) \le \frac{C_2}{r^2(\log r)^\mu}, \quad r \le r_0, \quad 0 < \mu < 2,$$

(2*)
$$a(r,\theta) = \frac{\Psi(\theta)}{r(\log r)^{\nu/2}} \quad r \le r_0, \quad 0 \le \nu < \mu,$$

Suppose also that there is a constant $C > 0$, such that for any $0 < r_1, r_2 < r_0$

(3*)

$$|\bar{a}^{-2}(r_1)\bar{V}(r_1) - \bar{a}^{-2}(r_2)\bar{V}(r_2)| \\ \le C|\log(r_1 r_2^{-1})|(\min(\log r_1, \log r_2) + 1)^{-1-\mu+\nu}.$$

(4*) the functions a and V are compactly supported and

$$a, V \in L^\infty(\mathbb{R}^2 \setminus B(0, r_0)).$$

If Conditions (1*)-(4*) are satisfied, then the quadratic form (1.4) is semi-bounded on $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, therefore its closure generates the selfadjoint operator (1.1) for which we use the same notation $\mathcal{H}_\beta$.

Nothing new is needed in order to obtain the following result:

Therem 5.1. *Let the functions V and a satisfy Conditions (1*)-(4*), $\bar{\Psi} \neq 0$ and $0 \le \nu < \mu < 2$ then the asymptotic formula (2.2) is fulfilled.*

Obviously we can also consider a combined problem where the functions a and V are singular both at infinity and at zero (or at any other (several) point(s)). In order to prove the corresponding results one needs to use the splitting arguments together with Theorems 2.1 and 5.1.

6. Aharonov-Bohm-type magnetic fields. Finally we would like to discuss a special case, where

$$a(r,\theta) = \frac{\Psi(\theta)}{r}, \qquad \Psi \in L^\infty(\mathbb{T}). \tag{6.1}$$

The corresponding magnetic field is zero everywhere except $x = 0$. In this case the exponential change of variables (4.8) applied to the whole $\mathbb{R}_+$ leads to a study of the number of the negative eigenvalues of a Schrödinger operator on $\mathbb{R}$ and we obtain:

Theorem 6.1. *Let a be defined by (6.1). Assume that there is $0 < \mu < 2$ such that*

$$r^2(1 + |\log r|^\mu)V(r,\theta) \in L^\infty(\mathbb{R}^2)$$

and either

$$\liminf_{r\to\infty} r^2(\log r)^\mu \bar{V}(r,\theta) > 0, \quad \text{or} \quad \limsup_{r\to 0} r^2|\log r|^\mu \bar{V}(r,\theta) > 0.$$

Suppose also that there is a constant $C > 0$, such that for any $r_1, r_2 \in \mathbb{R}_+$

$$|r_1^2\bar{V}(r_1) - r_2^2\bar{V}(r_2)| \le C|\log(r_1 r_2^{-1})|(\min(\log r_1, \log r_2) + 1)^{-1-\mu}.$$

If now $\bar{\Psi} \neq 0$, then

$$\lim_{\alpha\to 0} N(\mathcal{H}_\beta)/G(\beta) = 1,$$

where

$$G(\beta) = \frac{1}{2\pi^2} \int_{\mathbb{R}^2} \Big(\bar{V}(|x|) - \beta^2 \, \frac{\bar{\Psi}^2}{|x|^2} \Big)_+^{1/2} \frac{dx}{|x|}.$$

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References

[BL] M.Sh.Birman and A.Laptev, *The negative discrete spectrum of a two-dimensional Schrödinger operator*, Comm. Pure Appl. Math. **XLIX** (1996), 967–997.

[BS] M.Sh.Birman and M.Z.Solomyak, *Estimates for the number of negative eigenvalues of the Schrödinger operator and its generalizations.*, Advances in Soviet Math. **7** (1991), 1–55.

[L] A. Laptev, *Asymptotics of the negative discrete spectrum of a class of Schrödinger operators with large coupling constant.*, Proc. of AMS **119** (1993), 481–488.

[RS] M.Reed and B.Simon, *Methods of mordern mathematical physics. 4*, vol. 4, Academic Press, New York, San Francisco, London, 1978.

[Sol] M.Solomyak, *Piecewise-polynomial approximation of functions from* $H^l((0,1)^d)$, $2l = d$, *and applications to the spectral theory of Schrödinger operator*, Israel J. Math. **86** (1994), 253–276.

[T] B. Thaller, *The Dirac equation.*, Texis and Monographs in Physics., Springer-Verlag, 1992.

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