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A left-handed simplicial action for euclidean general relativity

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Abstract

An action for simplicial euclidean general relativity involving only left-handed fields is presented. The simplicial theory is shown to converge to continuum general relativity in the Plebanski formulation as the simplicial complex is refined.

An entirely analogous hypercubic lattice theory, which approximates Plebanski's form of general relativity is also presented.

1 Introduction

It has been known for some time that in general relativity (GR) the gravitational field can be represented entirely by left-handed fields, i.e. connections and tensors that transform only under the left-handed, or self-dual subgroup of the frame rotation group.[1]

[1] In euclidean GR the frame rotation group is $SO(4)$ which can be written as the product $SU(2)_R \otimes SU(2)_L/Z_2$. Left handed tensors are transform only under the $SU(2)_L$ factor.

The present paper presents a simplicial model of GR with an internal $SU(2)$ gauge symmetry which, at least in the continuum limit, corresponds to the left handed frame rotation group. The gravitational field is represented by spin 2 and spin 1 $SU(2)$ tensors, and $SU(2)$ parallel propagators, associated with the 4-simplices, and with 2-cells and edges constructed from the 4-simplices, respectively.

This is meant to provide a step in the construction of a covariant path integral, or sum over histories, formulation of loop quantized GR. In Ashtekar's reformulation of classical canonical GR [Ash86], [Ash87] the canonical variables are the left handed part of the spin connection on space and, conjugate to it, the densitized dreibein. The connection can thus be taken as the configuration variables, opening the door to a loop quantization of GR [GT86], [RS88], [RS90], [ALMMT95]. In loop quantization one supposes that the state can be written as a power series in the spatial Wilson loops of the connection (which coordinatize the connections up to gauge), so the fundamental excitations are loops created by the Wilson loop operators. The kinematics of loop quantized canonical GR requires that geometrical observables measuring lengths [Thi96b], areas [RS95] [AL96a], and volumes [RS95], [AL96b], [Lew96], [Thi96a] have discrete spectra and finite, Plank scale lowest non-zero eigenvalues, suggesting that GR thus quantized has a natural UV cutoff. One would therefore expect that a path integral formulation of this theory would have, in addition to manifest covariance, also reasonable UV behaviour.

A step toward such a path integral formulation is the construction of the analogous formulation in a simplicial approximation to GR. Loop quantization can be applied to any spacetime lattice[2] theory in which the boundary data is a connection on the boundary.[3] Moreover, for such theories it has been shown [Rei94] that the evolution operator can be written as a sum over the worldsheets of the loop excitations,[4] that is, as a path integral. All that

Examples are left handed spinors and self-dual antisymmetric tensors, i.e. tensors a that satisfy $a^{[IJ]} = \epsilon^{IJ}{}_{KL} a^{KL}$.

[2]The lattice need not be hypercubic or regular in any way.

[3]When the boundary is compact (a finite lattice), and all the irreducible representations of the group are finite polynomials of the fundamental representation (as is the case in $SU(2)$) then any regular, gauge invariant function of the connection (i.e. of the parallel propagators on the links) on the boundary can be written as a power series in the Wilson loops.

[4]The proof is given for theories whose actions are functions of the connection only. How-

is needed for a path integral formulation of loop quantized GR is a lattice action for GR in terms of the left handed part of the spin connection, and other fields, such that the connection is the boundary data.

Plebanski [Ple77] found precisely such an action in the continuum. [5] Here a simplicial lattice analogue of this action is presented. The corresponding path integral formulation of loop quantized simplicial GR will appear in a forthcoming paper.

The present work might also be useful in numerical relativity, since a simple modification of the new simplicial lattice action yields a hypercubic lattice action for GR.

In section 2 the simplicial model is presented. Field equations and boundary terms are discussed in section 3. The continuum limit is taken in section 4. Finally section 5 contains some comments on the results, and also states the hypercubic action. Appendix A gives a metrical interpretation of the simplicial field e_s, and in appendix B some lemmas used to establish the continuum limit are proved.

2 The model

Plebanski gave the following action for general relativity (GR) in terms of left-handed fields [Ple77][CDJM91]:[6]

$$I_P = \int \Sigma_i \wedge F^i - \frac{1}{2}\phi^{ij}\Sigma_i \wedge \Sigma_j \tag{1}$$

(The euclidean theory is obtained when all fields are real). This action has internal gauge group $SU(2)$, with Σ a 2-form and an $SU(2)$ vector (spin 1),

ever, if the connection is the only boundary datum then all other fields can be integrated out to obtain a local action in terms of only the connection.

[5]The left handed part of the spin connection is also the boundary data of the GR action of Samuel [Sam87], and Jacobson and Smolin [JS87], [JS88]. However we shall not try to build a lattice analogue of that action here.

[6]The definition of exterior multiplication used here is $[a \wedge b]_{\alpha_1\ldots\alpha_m\beta_1\ldots\beta_n} = a_{[\alpha_1\ldots\alpha_m}b_{\beta_1\ldots\beta_n]}$, where spacetime indices are labeled by lower case greek letters $\{\alpha,\beta,\gamma,...\}$. Forms are integrated according to $\int_A a = \int_A \epsilon^{u_1\ldots u_m} a_{u_1\ldots u_m}\ d^m\sigma$ where A is an m dimensional manifold, σ^u are coordinates on A, the indices u_i run from 1 to m, and $\epsilon^{u_1\ldots u_m}$ is the m dimensional Levi-Civita symbol ($\epsilon^{12\ldots m} = 1$ and ϵ is totally antisymmetric).

F the curvature of an $SU(2)$ connection A, and ϕ a spacetime scalar and spin 2 $SU(2)$ tensor. The action is written in terms of the components of these fields in the adjoint, or $SO(3)$, representation of $SU(2)$. (Indices in this representation run over $\{1, 2, 3\}$ and will be indicated by lowercase roman letters $\{i, j, k, l, ...\}$). ϕ is thus represented by a traceless symmetric matrix ϕ^{ij}.

On non-degenerate[7] solutions these fields can be expressed in terms of more conventional variables. Σ is the self-dual part of the vierbein wedged with itself:[8]

$$\Sigma_i = 2[e \wedge e]^{+\,0i} \equiv e^0 \wedge e^i + \frac{1}{2}\epsilon_{ijk} e^j \wedge e^k, \tag{2}$$

which transforms as a spin 1 vector under $SU(2)_L$, the left-handed subgroup of the frame rotation group $SO(4) = SU(2)_R \otimes SU(2)_L/Z_2$, and as a scalar under $SU(2)_R$. A is the self-dual ($SU(2)_L$) part of the spin connection, and ϕ turns out to be the left-handed Weyl curvature spinor. The non-degenerate solutions correspond in this way exactly to the set of solutions to Einstein's equations with non-degenerate spacetime metric.

Ashtekar's canonical variables are just the purely spatial parts of A and Σ (the dual of the spatial part of Σ is the densitized triad), and, in the non-degenerate sector, the canonical theory derived from (1) is identical to Ashtekar's [CDJM91][Rei95]. Since this is precisely the sector of non-degenerate spatial metric it is of course also equivalent to the ADM theory [ADM62]. However, when the metric is degenerate the canonical theory differs from Ashtekar's [Rei95]. Since Plebanski's theory defines an extension of GR to degenerate geometries, and this extension is not the only one possible, I will refer to this theory as Plebanski's theory.

The approximate model of GR presented here makes use of a somewhat intricate cellular spacetime structure. (See Fig. 1). Spacetime is represented by an orientable simplicial complex Δ.[9] For any simplex μ one may define a center C_μ as the average of its vertices (using any of the family of linear coordinates defined by the affine structure of the simplex). For a given 4-

[7] $\Sigma_k \wedge \Sigma^k \neq 0$

[8] Note that upstairs and downstairs $SO(3)$ indices are the same.

[9] We shall always assume that the simplicial complex is a combinatorial manifold, so it has every nice property that one would expect a simplicial representation of spacetime to have. See [SW93] for details.

simplex ν and 2-simplex $\sigma < \nu$[10] there are two 3-simplices $\tau_1,\ \tau_2 < \nu$ which share σ. In the definition of the model a central role will be played by the plane quadrilateral $s(\sigma\nu)$ formed by the centers of ν, τ_1, σ, and τ_2.[11]

$s(\sigma, \nu)$ will be called a "wedge" and will often be denoted by just s or $\sigma\nu$. 4-simplices will be denoted by ν, with some additional subscripts or markings to distinguish different 4-simplices. 3-simplices will similarly be denoted by τ, and 2-simplices by σ. 1-simplices i.e. vertices, will be denoted by latin capitals $P, Q, R, ...$

The 4-simplices will be given a uniform orientation throughout Δ, and the orientation of each wedge $s(\sigma\nu)$ will be determined by the orientations of σ and ν through the requirement that a positively oriented basis on σ concatenated with a positively oriented basis on s forms a positively oriented basis on ν.

In our model of GR we associate to each wedge $s(\sigma\nu)$ an $SU(2)$ spin 1 vector $e_{s\,i}$, which will more or less play the role of Plebanski's Σ_i field.[12] The role of A is played by $SU(2)$ parallel propagators along the edges of the $s(\sigma\nu)$. Specifically, there is an $h_l \in SU(2)$ for each edge $l(\nu\tau)$ from the center of 4-simplex ν to that of 3-simplex $\tau < \nu$, and there is a $k_r \in SU(2)$ for each edge $r(\tau\sigma)$ from the center of 3-simplex τ to that of 2-simplex $\sigma \in \tau$.

Finally, a spin 2 $SU(2)$ tensor φ_ν (represnted by a symmetric, traceless matrix, φ_ν^{ij}) is associated with each 4-simplex. φ_ν^{ij} plays the role of ϕ^{ij}.

The action for the model is

$$I_\Delta = \sum_{\nu<\Delta} [\sum_{s<\nu} e_{s\,i}\theta_s^i - \frac{1}{60}\varphi_\nu^{ij} \sum_{s,\bar{s}<\nu} e_{s\,i} e_{\bar{s}\,j} sgn(s, \bar{s})]. \tag{3}$$

θ_s^i is a measure of the curvature on s. It is a function of the $SU(2)$ parallel propagators via

$$\theta_s^i = tr[J^i g_{\partial s}], \tag{4}$$

[10] $\mu < \rho$ denotes that μ is a subcell of ρ. This means that μ can be a subsimplex of ρ, which can be a simplex or simplicial complex, or the restriction to ρ of a cell of the dual simplicial complex, which will be defined shortly.

[11] That s is plane can be seen as follows. Think of the simplices in terms of their vertices: $\sigma = PQR$, $\tau_1 = PQRS$, $\nu = PQRST$, and $\tau_2 = PQRT$, and take $C_\sigma = 1/3(P + Q + R)$ to be the origin. Then $C_{\tau_1} = 1/4\,S$, $C_{\tau_2} = 1/4\,T$, so $C_\nu = 1/5(T + S)$ is in the span of C_{τ_1} and C_{τ_2}, proving that s is a plane.

[12] We define $e_{\sigma\nu}$ to reverse sign when the orientation of σ is reversed.

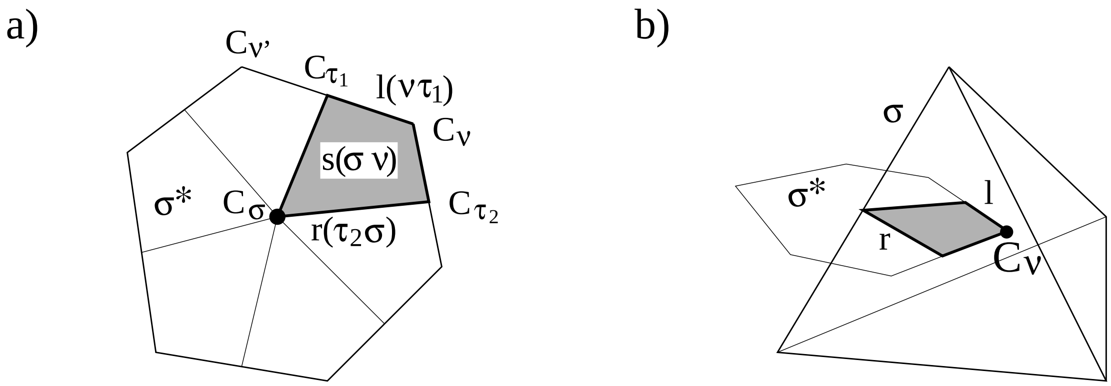

Figure 1: Panel a) illustrates the definitions of $s(\sigma\nu)$ and the edges $l(\nu\tau)$ and $r(\tau\sigma)$. In the middle lies the center of a 2-simplex σ. The corners are the centers of all the 4-simplices $\nu, \nu', \ldots$ that share σ. The curve $C_\nu C_{\tau_1} C_{\nu'}$ connecting C_ν and $C_{\nu'}$ is a realization of the edge τ_1^* in the cellular complex Δ^* dual to Δ, which is dual to τ_1. Similarly $\cup_\nu s(\sigma\nu)$ is a realization of the 2-cell $\sigma^* < \Delta^*$ dual to σ. One may think of $s(\sigma\nu)$ as the wedge of σ^* in ν: $s(\sigma\nu) = \sigma^* \cap \nu$. Likewise, $l(\nu\tau) = \tau^* \cap \nu$ and the radial edge $r(\tau\sigma) = \sigma^* \cap \tau$.

Panel b) shows the analogous structure in a 3-dimensional complex with a 3-simplex playing the role of ν, 2-simplices as τ_1 and τ_2, and a 1-simplex as σ.

where $g_{\partial s}$ the holonomy around ∂s, and the J_i are 1/2 the Pauli sigma matrices[13] $g_{\partial s}$ and θ_s may be written in terms of a rotation vector ρ^i as

$$g_{\partial s} = e^{i\rho\cdot J} = \cos\frac{|\rho|}{2}\mathbf{1} + 2i\sin\frac{|\rho|}{2}\hat{\rho}\cdot J \tag{6}$$

$$\theta_s = 2\sin\frac{|\rho|}{2}\hat{\rho} \tag{7}$$

($\hat{\rho} = \rho/|\rho|$). The rotation vector is essentially the curvature on the wedge s, and, when the holonomy is close to one, i.e. the curvature is small, θ_s approximates the rotation vector.[14]

$sgn(s,\bar{s}) \equiv sgn(\sigma,\bar{\sigma})$ is essentially the sign of the oriented 4-volume spanned by the 2-simplices σ and $\bar{\sigma}$ associated with s and $\bar{s}$: If σ and $\bar{\sigma}$ share only one vertex (the minimum number when both belong to the same 4-simplex), then the orientations of σ and $\bar{\sigma}$ define an orientation for ν, namely the orientation of the basis produced by concatenating positively oriented bases of σ and $\bar{\sigma}$. If this orientation matches that already chosen for ν then $sgn(s,\bar{s}) = 1$. If it is the opposite $sgn(s,\bar{s}) = -1$. If σ and $\bar{\sigma}$ share 2 or 3 vertices they lie in the same 3-plane and span no 4-volume. In this case $sgn(s,\bar{s}) = 0$.

A nice, very explicit, formula can be given for the sum in the second term (3). If the vertices of the 4-simplex are numbered 1, 2, 3, 4, 5, so that 12, 13, 14, 15 form a positively oriented basis then

$$\sum_{s,\bar{s}<\nu} e_{s\,i} e_{\bar{s}\,j} sgn(s,\bar{s}) = \frac{1}{4}\sum_{P,Q,R,S,T\in\{1,2,3,4,5\}} e_{PQR\,i} e_{PST\,j}\epsilon^{PQRST}, \tag{8}$$

where $e_{PQR} = e_{s(PQR,\nu)}$, and PQR indicates the 2-simplex with positively ordered vertices P, Q, R.

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[13]

$$J_1 = \tfrac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad J_2 = \tfrac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad J_3 = \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{5}$$

[14]Note that θ_s reverses sign when the orientation of s reverses because the direction of the boundary ∂s reverses, which, in turn, means that $g_{\partial s} \rightarrow g_{\partial s}^{-1}$, $\rho_s \rightarrow -\rho_s$, and, finally, $\theta_s \rightarrow -\theta_s$.

3 Field equations and boundary terms

Extremization of (3) with respect to $h_{l(\nu,\tau)}$ is most easily carried out by parametrizing variations of h_l via $h_l + \delta h_l = h_l exp[i\alpha_l \cdot J]$. Then

$$itr[h_l J_i \frac{\partial I_\Delta}{\partial h_l}] = \frac{\partial I_\Delta}{\partial \alpha_l^i}|_{\alpha_l=0}. \tag{9}$$

On the other hand $l(\nu,\tau) \subset \partial s(\sigma,\nu)$ when $\sigma < \tau$, so such a variation of h_l induces

$$g_{\partial\sigma\nu} \rightarrow g_{\partial\sigma\nu} e^{i\alpha_l \cdot J}. \tag{10}$$

(Here we have oriented each $\sigma < \tau$ to match $\partial\tau$ and τ to match $\partial\nu$, with the effect that l is positively oriented in $\partial s(\sigma,\nu)$). Thus

$$\delta\theta^i = \alpha_l^j 2tr[J_j J^i g_{\partial s}] \tag{11}$$

and

$$\delta I_\Delta = a g_l \cdot \sum_{\sigma<\tau} w_{\sigma\nu}, \tag{12}$$

with

$$w_{s\,j} = 2tr[J_j J^i g_{\partial s}] e_{s\,i} = \cos\frac{|\rho|}{2} e_{s\,j} + 2\sin\frac{|\rho|}{2}[\hat{\rho} \times e_s]_j \tag{13}$$

Extremization with respect to $h_{l(\nu\tau)}$ thus requires[15]

$$0 = itr[h_l J_i \frac{\partial I_\Delta}{\partial h_l}] = \sum_{\sigma<\tau} w_{\sigma\nu\,i} \tag{14}$$

Similarly, extremization with respect to $k_{r(\tau\sigma)}$ requires

$$u_{\sigma\nu_1} = u_{\sigma\nu_2} \tag{15}$$

where ν_1 and ν_2 are the two 4-simplices sharing τ, and

$$u_{\sigma\nu_1\,i} = U^{(1)}[k_{r(\tau\sigma)} h_{l(\nu_1\tau)}]_i^{\ j} w_{\sigma\nu_1\,j}. \tag{16}$$

($U^{(1)}(g)$ is the spin 1 representation of $g \in SU(2)$). In general $u_{\sigma\nu}$ is $w_{\sigma\nu}$ parallel transported from C_ν along the boundary $\partial s(\sigma\nu)$ in a positive sense[16] to C_σ. (See Fig. 2). u_s is thus, like w_s, e multiplied by a factor which goes

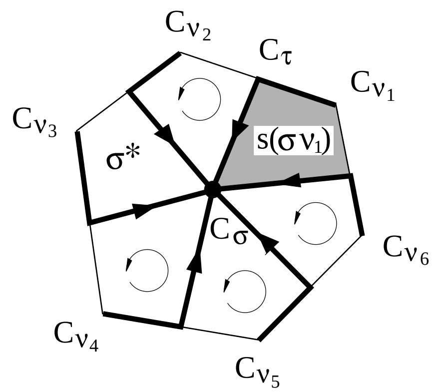

Figure 2: The dual 2-cell $\sigma^* < \Delta^*$ is shown with its uniform orientation indicated by an arrow circling in a positive sense. The routes by which the $w_{\sigma\nu_n}$ are parallel transported from C_{ν_n} to C_σ to form the $u_{\sigma\nu_n}$ are indicated by bold lines.

to one as the group elements h_l and k_r approach **1**. Note that (15) implies that all $u_{\sigma\nu}$ for a given 2-simplex σ have a common value u_σ.

Extremization with respect to φ_ν^{ij} yields

$$\frac{\partial I_\Delta}{\partial \varphi_\nu^{ij}} = \frac{1}{60} \sum_{s,\bar{s}<\nu} e_{s\,i} e_{\bar{s}\,j} sgn(s,\bar{s}) \propto \delta_{ij} \qquad (17)$$

(since φ_ν^{ij} is traceless).

Finally, extremization with respect to $\bar{e}_s$ implies

$$\frac{\partial I_\Delta}{\partial e_{s\,i}} = \theta_s^i - \frac{1}{30} \varphi_\nu^{ij} \sum_{\bar{s}<\nu} e_{\bar{s}\,j} sgn(s,\bar{s}) = 0. \qquad (18)$$

What about boundary terms? Suppose the simplicial complex has a boundary $\partial\Delta$ which doesn't cut through any 4-simplex, so it is itself a 3 dimensional simplicial complex. No boundary term needs to be added to the action (3), If the connection is held fixed at that boundary. That is to say, if the group elements k_r on the edges $r(\tau,\sigma)$, on the boundary are held fixed.

[15]If $\nu = PQRST$, $\tau = PQRS$ then $\sum_{\sigma<\tau} w_{s(\sigma\nu)} = -w_{PQR} + w_{QRS} - w_{RSP} + w_{SPQ}$.

[16]ν_1, ν_2 are numbered with index increasing in a positive sense around $\partial\sigma^*$. Note that the definition of the orientation of $s(\sigma,\nu)$ in terms of that of σ and ν defines a uniform orientation on σ^*, since the orientations of the 4-simplices is uniform in the complex Δ.

($r(\tau\sigma)$ is is the intersection of τ and the 1-cell in $[\partial\Delta]^*$, the complex dual to $\partial\Delta$, dual to σ).

Another natural field to hold fixed is u_σ for $\sigma < \partial\Delta$ (since it lives on the boundary, unlike $e_{\sigma\nu}$ and $w_{\sigma\nu}$ which live in the internal space at C_ν, and thus always off the boundary). This corresponds more closely to what is usually done in Regge calculus [Reg61], that is, holding the lengths of the edges fixed on the boundary, because $e_{\sigma\nu}$, and therefore u_σ, is essentially the metrical field variable in our simplicial model. (See Appendix A for more on the relation of $e_{\sigma\nu}$ and u_σ to metric geometry). In this case the action must be modified: In the modified action the θ_s for s abutting the boundary are evaluated with k_r replaced by $\mathbf{1}$ $\forall r < \partial\Delta$ (but the definition of u_σ, (16), is unchanged).

4 The continuum limit

The form of the action (3) is clearly analogous to that of the Plebanski action (1). Plebanski's field equations

$$\frac{\delta I_P}{\delta\phi^{ij}} = \frac{1}{2}\Sigma_i \wedge \Sigma_j \propto \delta_{ij} \qquad (19)$$

$$\frac{\delta I_P}{\delta A^i} = D \wedge \Sigma_i = 0 \qquad (20)$$

$$\frac{\delta I_P}{\delta \Sigma_i} = F^i - \phi^{ij}\Sigma_j = 0 \qquad (21)$$

resemble simplicial field equations (17), (14), and (18) respectively. We shall see that in the continuum limit these resemblances become exact. The simplicial field equation (15) has no continuum analogue, it is an identity in the continuum limit we will define.

In order to take the continuum limit of the simplicial theory we define below a map Ω_Δ of continuum fields on a spacetime manifold M into simplicial fields on a simplicial decomposition Δ of M, which allows us to represent continuum field histories by simplicial ones, and a class of sequences $\{\Delta_n\}_{n=0}^\infty$ of simplicial decompositions of spacetime which become infinitely fine everywhere in a nice way as $n \to \infty$. Any continuum field history (A, Σ, ϕ) then defines a sequence of increasingly faithful images $\Omega_n(A, \Sigma, \phi) = (h, k, e, \varphi)_n$

on the complexes Δ_n, and corresponding evaluations $I_n(A, \Sigma, \phi)$ of the simplicial action.

The continuum limit of the simplicial theory is defined by letting its solutions be just those continuum field configurations for which the variations δI_n of the simplicial action under variations δA, $\delta\Sigma$, $\delta\phi$ of the continuum fields vanish to first order as $n \to \infty$. In this section it will be demonstrated that the continuum limit, in this sense, of the theory defined by (3) is exactly Plebanski's theory (1).

Roughly speaking, the claim is that in the continuum limit the simplicial field equations agree with the Plebanski field equations. More precisely it is that the Plebanski field equations, integrated against the continuum field variations $(\delta A, \delta\Sigma, \delta\phi)$ are the limits of the simplicial field equations integrated against the variations $(\delta h, \delta k, \delta e, \delta\varphi)_n$ which are the images of $(\delta A, \delta\Sigma, \delta\phi)$, and are thus, as $n \to \infty$, smooth in a certain sense.

It will also be shown that the action of the simplicial theory converges to that of the Plebanski theory in the continuum limit. That is, as $n \to \infty$ $I_n(A, \Sigma, \phi) \to I_P(A, \Sigma, \phi)$. This result almost implies the convergence of the field equations claimed above:

$$\delta I_P = \delta I_n + \delta \Delta I_n \tag{22}$$

where $\Delta I_n = I_P - I_n$. If $\lim_{n\to\infty} \delta\Delta I_n = 0$ then $\lim_{n\to\infty} \delta I_n = \delta I_P$ at fixed (A, Σ, ϕ), so in the limit, the zeros - the solutions, agree. Now $\lim_{n\to\infty} \Delta I_n = 0$ so, unless ΔI_n has a more and more undulatory dependence on A, Σ, ϕ as $n \to \infty$ $\lim_{n\to\infty} \delta\Delta I_n = 0$. Nevertheless, the convergence of the field equations will be proved independently in the following, because it is interesting to see how the continuum field equations emerge from the simplicial ones.

One might hope to show that the Plebanski theory is the continuum limit of the simplicial theory in a stronger sense than has been claimed, namely that the solutions of Plebanski's theory are exactly the limit points as $n \to \infty$ of continuum fields A, Σ, ϕ corresponding to solutions of the simplicial theory in some reasonable topology on the space of continuum field configurations. A proof of this will not be attempted here.

However, if the simplicial and Plebanski theories can be approximated by the same theories with a spacetime cutoff (larger than the scale of the simplices), that is, by versions in which the actions are only depend on modes of the fields A, Σ, ϕ which have local wavelength everywhere larger than a local cutoff length,[17] then our results show that the simplicial and Plebanski theories approximate each other as $n \to \infty$. In the cutoff theories only the field equations required by extremization with respect to field modes allowed by the cutoff appear, and these we know to be the same in the simplicial and Plebanski theories when $n \to \infty$.

Furthermore, if such cut off versions of the theories can be used to define a path integral quantization, then, by the equality of the actions as $n \to \infty$ (and assuming corresponding integration measures are chosen), the quantum theories agree as $n \to \infty$.

Now to the details.

Definition 1: The map $\Omega_\Delta : (A, \Sigma, \phi) \mapsto (h, k, e, \varphi)$ of continuum fields on M to simplicial fields on the simplicial decomposition Δ of M is defined by

$$h_l = \mathcal{P} e^{i \int_l A \cdot J} \tag{23}$$

$$k_r = \mathcal{P} e^{i \int_r A \cdot J} \tag{24}$$

$$u_{\sigma \nu\, i} = \int_\sigma v(C_\sigma, x)_i{}^j \Sigma_j(x) \tag{25}$$

$$\varphi_\nu^{ij} = \phi^{ij}(C_\nu) \tag{26}$$

with $e_{\sigma\nu}$ defined from u_σ via (16) and (13), [18]

$\mathcal{P}$ denotes path ordering, and $v(C_\sigma, x) = U^{(1)}(\mathcal{P} e^{i \int_{C_\sigma}^x A \cdot J})$ is the parallel propagator of spin 1 $SU(2)$ vectors along a straight line from x to C_σ (according to the affine structure of σ).

This definition of Ω_Δ is not the only one possible. Other maps also lead to equivalence of the continuum limit of the simplicial theory and the Plebanski theory. For instance maps such that h, k, e, φ converge to those of Definition 1 as the simplicial complex is refined are viable alternatives. However the Ω_Δ chosen here seems to lead to the cleanest proofs.

[17] This is always what is assumed when, as is usually done, a field theory is defined as the limit of UV regulated theories.

[18] The map (13) which defines w_s in terms of e_s is invertible except when the trace of the holonomy around ∂s vanishes. However, when the connection A is continous we may, by choosing a sufficiently fine simplicial complex make all holonomies around wedges close to one.

Before I define the simplicial complexes to be used note that we will be concerned with compact spacetimes or compact pieces of spacetimes, which always admit finite simplicial decompositions[19]

As the sequence of progressively finer simplicial decompositions we will use *uniformly refining* sequences, which are defined as follows.

Definition 2: $\{\Delta_n\}_{n=0}^{\infty}$ is a uniformly refining sequence of simplicial decompositions of a compact manifold M if

1) Δ_0 is a finite simplicial decomposition of M,
2) Δ_{n+1} is a finite refinement of Δ_n,

and, in a fixed positive definite metric g_0 which is constant on each $\nu < \Delta_0$ (in linear coordinates on ν),

3) r_n, the maximum of the radii r_ν of the 4-simplices $\nu < \Delta_n$ approaches zero as $n \to \infty$, and
4) r_ν^4 divided by the 4-volume of ν is uniformly bounded for all 4-simplices $\nu < \Delta_n$ as $n \to \infty$.

In Appendix B it is proven that conditions 3) and 4) are independent of the particular metric chosen.

The following lemma, also proven in Appendix B, will be useful

Lemma 1: If h is a continous d-form on a compact manifold M, $\{\Delta_n\}_{n=0}^{\infty}$ is a uniformly refining sequence of simplicial decompositions of M, and g_0 is a positive definite metric constant on each $\nu < \Delta_0$, then $\forall \epsilon > 0$ N can be chosen sufficiently large so that for any d-subcell c of a 4-simplex $\nu < \Delta_N$

$$|\int_c h - \int_c h_{C_\nu}| < \epsilon r_\nu^d, \tag{27}$$

where h_{C_ν} is the constant d-form (in linear coordinates on ν) which agrees with h at C_ν, and r_ν is the g_0 radius of ν.

[19]This follows from the arguments on p. 488 of [SW93].

Lemma 1 can be conveniently restated in terms of the *characteristic tensor* of c, which can be defined in linear coordinates x^α on ν by

$$t_c^{\alpha_1...\alpha_d} = \int_c dx^{\alpha_1} \wedge ... \wedge dx^{\alpha_d}. \tag{28}$$

t_c might be called the coordinate volume tensor of c, it is the d-dimensional generalization of the coordinate length vector of an edge and the coordinate area bivector of a 2-cell. When c is a d-simplex with vertices $P_1, ..., P_{d+1}$ $t_c^{\alpha_1...\alpha_d} = (P_1P_2)^{[\alpha_1}...(P_1P_{d+1})^{\alpha_d]}$. Using t_c equation (27) in Lemma 1 can be written as

$$\int_c h = \int h(C_\nu)_{\alpha_1...\alpha_2} t_c^{\alpha_1...\alpha_2} + O(\epsilon r_\nu^d), \tag{29}$$

where $O(\epsilon r_\nu^d)$ denotes a quantity Q_c that vanishes faster than r_ν^d as $n \to \infty$, that is, $max\{Q_c/r_\nu^d | c < \nu < \Delta_n\} \to 0$ as $n \to \infty$.

Now we are ready to prove that the simplicial action (3) converges to the Plebanski action in the continuum limit.

Theorem 1: If $\{\Delta_n\}_{n=0}^\infty$ is a uniformly refining sequence of simplicial decomposition of an orientable, compact 4-manifold M, A, Σ, θ are Plebanski fields on M with Σ and ϕ continous and A contiously differentiable, and $I_n(A, \Sigma, \phi)$ is the evaluation of the simplicial action (3) on the simplicial fields $(h, k, e, \varphi)_n = \Omega_{\Delta_n}(A, \Sigma, \phi)$ defined on Δ_n by (23) - (26), then

$$lim_{n\to\infty} I_n(A, \Sigma, \phi) = I_P(A, \Sigma, \phi). \tag{30}$$

Proof: Choose a positive definite metric g_0 awhich is constant on each $\nu < \Delta_0$. By several applications of Lemma 1 one shows that $\forall \epsilon > 0$ N may be chosen large enough so that

$$|\theta^i_{\sigma\nu} \quad - \quad F(C_\nu)^i_{\alpha\beta} t^{\alpha\beta}_{s(\sigma\nu)}| < \epsilon r_\nu^2 \tag{31}$$

$$|e_{\sigma\nu\, i} \quad - \quad \Sigma(C_\nu)_{\alpha\beta\, i} t^{\alpha\beta}_\sigma| < \epsilon r_\nu^2 \tag{32}$$

$\forall \sigma < \nu < \Delta_n$. Thus

$$\begin{aligned} I_N \quad = \quad & \sum_{\nu<\Delta_N} [\Sigma_{i\,\alpha\beta} F^i_{\gamma\delta}|_{C_\nu} \sum_{\sigma<\nu} t^{\alpha\beta}_\sigma t^{\gamma\delta}_{s(\sigma,\nu)} \\ & - \frac{1}{60} \phi^{ij} \Sigma_{i\,\alpha\beta} \Sigma_{j\,\gamma\delta}|_{C_\nu} \sum_{\sigma,\bar\sigma<\nu} t^{\alpha\beta}_\sigma t^{\gamma\delta}_{\bar\sigma} sgn(\sigma,\bar\sigma) + \Delta I_{N\nu}], \end{aligned} \tag{33}$$

where the error term $\Delta I_{N\nu}$ is bounded by

$$|\Delta I_{N\nu}| < \epsilon r_\nu^4[10||F|| + (10 + \frac{1}{2}||\phi||)(||\Sigma|| + \epsilon)]|_{C_\nu} \quad (34)$$

(Here the norm $||T||$ of a tensor $T^{i_1 \ldots i_m}_{\alpha_1 \ldots a g_n}$ is $||T|| = (T^{i_1 \ldots i_m}_{\alpha_1 \ldots a g_n} T^{j_1 \ldots j_m}_{\beta_1 \ldots b g_n} \delta_{i_1 j_1} \ldots g_0^{\alpha_1 \beta_1} \ldots)^{\frac{1}{2}}$). Since F, Σ, and ϕ are continous on the compact manifold M they are bounded, so $|\Delta I_{N\nu}| < \epsilon r_\nu^4 \kappa$ with κ a finite constant.

The ratio of r_ν^4 to the 4-volume V_ν of ν is uniformly bounded. Let this bound be R then $r_\nu^4 < RV_\nu$ implying that $|\Delta I_{N\nu}| < \epsilon\kappa RV_\nu$. Therefore the sum of the errors is bounded by

$$|\sum_{\nu < \Delta_N} \Delta I_{N\nu}| < \epsilon\kappa RV_M, \quad (35)$$

where V_M is the g_0 volume of M, a finite number.

Straightforward calculations show

$$\sum_{\sigma, \bar{\sigma} < \nu} t_\sigma^{\alpha\beta} t_{\bar{\sigma}}^{\gamma\delta} sgn(\sigma, \bar{\sigma}) = 30 t_\nu^{\alpha\beta\gamma\delta} \quad (36)$$

$$\sum_{\sigma < \nu} t_\sigma^{\alpha\beta} t_{s(\sigma\nu)}^{\gamma\delta} = t_\nu^{\alpha\beta\gamma\delta}. \quad (37)$$

Thus, putting everything together, we get

$$I_N = [\sum_\nu (\Sigma_{i\,\alpha\beta} F^i_{\gamma\delta} - \frac{1}{2}\phi^{ij}\Sigma_{i\,\alpha\beta}\Sigma j\,\gamma\delta)|_{C_\nu} t_\nu^{\alpha\beta\gamma\delta}] + \Delta I_N \quad (38)$$

$$\overrightarrow{N \to \infty} \int_M \Sigma_i \wedge F^i - \frac{1}{2}\phi^{ij}\Sigma_i \wedge \Sigma_j = I_P(A, \Sigma, \phi). \quad (39)$$

∎

To prove the equivalence of the continuum limit of the simplicial model with Plebanski's theory at the level of field equations we first show that the Euler-Lagrange (E-L) functions, i.e. the variational derivatives of the action with respect to the fields, are just the integrals of the Euler-Lagrange d-forms of the Plebanski theory over certain d-cells, modulo corrections which become negligible as the simplicial complex is refined.

The precise statement is

Lemma 2: For any simplicial field history corresponding to a continuum field history via (23) - (26) (and under the other hypothesies of Theorem 1).

1) The simplicial field equation (15) $tr[J_i k_r \frac{\partial I_n}{\partial k_r}] = 0$ holds identically for all n.

2) $\forall \nu < \Delta_n, \quad s, l < \nu$

$$
\begin{aligned}
\frac{\partial I_n}{\partial \phi_\nu^{ij}} &= \int_\nu \frac{\delta I_P}{\delta \phi^{ij}} + O(\epsilon r_\nu^4) && (40)\\
\frac{\partial I_n}{\partial e_{s\,i}} &= \int_s \frac{\delta I_P}{\delta \Sigma_i} + O(\epsilon r_\nu^2) && (41)\\
itr[h_l J_i \frac{\partial I_n}{\partial h_l}] &= \int_\nu \frac{\delta I_P}{\delta \phi^{ij}} + O(\epsilon r_\nu^3) && (42)
\end{aligned}
$$

where τ is the 3-simplex in ν dual to l, and in the integrals the integrands are parallel transported to C_ν along straight lines.

3) The Plebanski field equations are fully represented by the simplicial field equations in the sense that if the Plebanski E-L forms are not almost everywhere zero, then on a sufficiently fine simplicial complex the simplicial E-L functions will not all vanish.

Proof: 1) follows immediately from the definition (25) of $u_{\sigma\nu}(A, \Sigma)$. (40) and (42) from the corresponding Plebanski field equations (19) and (20) and

$$
\begin{aligned}
\frac{\partial I_n}{\partial \phi_\nu^{ij}} &= \frac{1}{60} \sum_{\sigma, \bar\sigma < \nu} e_{\sigma\nu\,i} e_{\bar\sigma\nu\,j} sgn(\sigma, \bar\sigma) && (43)\\
&= (\Sigma_{i\,\alpha\beta} \Sigma_{j\,\gamma\delta})|_{C_\nu} \frac{1}{60} \sum_{\sigma, \bar\sigma < \nu} t_\sigma^{\alpha\beta} t_{\bar\sigma}^{\gamma\delta} sgn(\sigma, \bar\sigma) + O(\epsilon r_\nu^4) && (44)\\
&= \frac{1}{2} (\Sigma_i \wedge \Sigma_j)|_{C_\nu\ \alpha\beta\gamma\delta} t_\nu^{\alpha\beta\gamma\delta} + O(\epsilon r_\nu^4) && (45)\\
itr[h_l J_i \frac{\partial I_n}{\partial h_l}] &= \sum_{\sigma < \tau} w_{\sigma\nu\,i} = h_{l\,i}^{-1j} \sum_{\sigma < \tau} k_{r(\tau\sigma\,j}^{-1}{}^k u_{\sigma\,k} && (46)\\
&= (D \wedge \Sigma_i)|_{C_\nu\ \alpha\beta\gamma} t_\tau^{\alpha\beta\gamma} + O(\epsilon r_\nu^3). && (47)
\end{aligned}
$$

Similarly (41) follows from (21) and the identity $\sum_{\bar\sigma < \nu} t_{\bar\sigma}^{\alpha,\beta} sgn(\sigma, \bar\sigma) = 30 t_{s(\sigma,\nu)}^{\alpha\beta}$ via

$$
\frac{\partial I_n}{\partial e_{s\,i}} = \ = \theta_{\sigma\nu}^i - \frac{1}{30} \varphi_\nu^{ij} \sum_{\bar\sigma < \nu} e_{\bar\sigma\nu\,j} sgn(\sigma, \bar\sigma) \qquad (48)
$$

$$= \quad (F^i - \phi^{ij}\Sigma_j)|_{C_\nu\ \alpha\beta} t^{\alpha\beta}_{s(\sigma\nu)} + O(\epsilon r_\nu^2). \tag{49}$$

3) follows as a corrollary of 2). Condition 4) in Definition 2, which defines Δ_n, prevents the 4-simplices from having zero volume, and therefore ensures that in each such 4-simplex $\nu < \Delta_n$ the t_s, t_τ, t_ν span the spaces of 2, 3, and 4 index antisymmetric tensors at C_ν. ∎

Using (9) and the fact that under variations of A, Σ, ϕ the variations of the corresponding simplicial fields h, e, φ are given by

$$\delta\varphi^{ij}_\nu = \delta\phi^{ij}(C_\nu) \tag{50}$$

$$\delta e_{\sigma\nu\, i} = \int_\sigma \delta\Sigma_i + O(\epsilon r_\nu^2) \tag{51}$$

$$\delta\alpha^i_l = \int_l \delta A^i + O(\epsilon r_\nu) \tag{52}$$

(where the integrands are parallel transported to C_ν along straight lines). one finds that

Theorem 2: Under the hypothesies of Theorem 1, under variations A, Σ, and ϕ

$$\lim_{n\to\infty} \delta I_n(A,\Sigma,\phi) = \delta I_P(A,\Sigma,\phi). \tag{53}$$

5 Comments

• A hypercubic lattice action for GR can be defined in complete analogy with the simplicial one. One defines the centers of n-cubes as the averages of their vertices, and the dual complex from these centers as in the simplicial context. In particular, the wedges s, and the edges l and r are defined by replacing n-simplices with n-cubes in their simplicial definitions. (See Fig. 3).

The action of the hypercubic model is

$$I_\Box = \sum_{\nu<\Box}[\sum_{s<\nu} e_{s\,i}\theta^i_s - \frac{1}{8}\varphi^{ij}_\nu \sum_{s,\bar{s}<\nu} e_{s\,i}e_{\bar{s}\,j}\ sgn(s,\bar{s})], \tag{54}$$

where $\Box$ is the hypercubic lattice and ν labels 4-cubes. $sgn(s,\bar{s}) = sgn(\sigma,\bar{\sigma})$ is the sign of the 4-volume spanned by the 2-cubes (squares) σ and $\bar{\sigma}$ dual to s

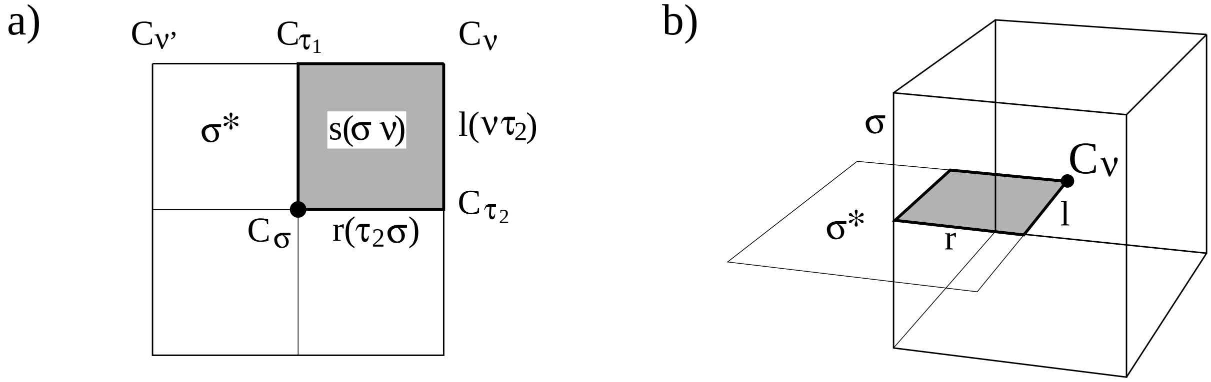

Figure 3: This is the analogue of Fig. 1 for a hypercubic lattice. Panel a) illustrates the definitions of $s(\sigma\nu)$ and the edges $l(\nu\tau)$ and $r(\tau\sigma)$. In the middle lies the center of a 2-cube σ. The corners are the centers of all the 4-cubes $\nu, \nu', ...$ that share σ. The curve $C_\nu C_{\tau_1} C_{\nu'}$ connecting C_ν and $C_{\nu'}$ is a realization of the edge τ_1^* in the lattice $\Box^*$ dual to $\Box$, which is dual to τ_1. Similarly $\cup_\nu s(\sigma\nu)$ is a realization of the 2-cell $\sigma^* < \Box^*$ dual to σ.

Panel b) shows the analogous structure in a 3-dimensional cubic lattice with a 3-cube playing the role of ν, 2-cubes as τ_1 and τ_2, and a 1-cube as σ.

and $\bar{s}$ respectively, provided σ and $\bar{\sigma}$ share one vertex. Otherwise $sgn(s, \bar{s}) = 0$. In particular this means that it is zero when the 2-cubes share no vertices.

Plebanski theory is the continuum limit of the hypercubic theory (54) in the same sense as it is the continuum limit of the simplicial theory. (The proofs of section 4 go through with minor adjustments).

• Boström, Miller and Smolin [BMS94] found a hypercubic lattice action corresponding to the CDJ action [CDJ91] (which is closely related to that of Plebanski) by following the method of Regge [Reg61] and evaluating the continuum action on field configurations in which the curvature has support only on the 2-dimensional faces of a spacetime lattice. Unfortunately, the continuum action is not unambigously defined on such field configurations. Nevertheless, Boström *et. al.* present an action corresponding to a particular disambiguation of this expression. This action, written in terms of the discrete curvature variable of Boström *et. al.* is formally similar to that one obtains from (54) by eliminating e_s using the field equations. However, their curvature is defined very differently from our curvature (θ_s) in terms of the fundamental fields, which in their case is a discrete connection and in our case parallel propagators.

• The simplicial theory presented here converges in the continuum limit to Plebanski's theory, which is not equivalent is not equivalent to Ashtekar's canonical theory when the spatial metric is degenerate. Thus one would expect a quantization of our simplicial model to approximate a quantization of the constraints of [Rei95] rather than Ashtekar's constraints.

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A The simplicial field e_s and metric geometry

On non-degenerate solutions Plebanski's fields A, Σ, ϕ have metrical interpretations. Thus simplicial fields defined via (23) - (26) should also have a metrical interpretation. In fact, when Σ_i satisfies the field equation (19) $\Sigma_i \wedge \Sigma_j \propto \delta_{ij}$, and the non-degeneracy requirement $\Sigma_k \wedge \Sigma^k \neq 0$, it defines a non-degenerate cotetrad $e^I{}_\alpha$ (unique up to $S0(3)_L$ transformations) and thus a non-degenerate metric[20]

$$g_{\alpha\beta} = e^I{}_\alpha \delta_{IJ} e^J{}_\beta. \tag{55}$$

Furthermore, when A obeys the field equation $D \wedge \Sigma_i = 0$, the Plebanski action becomes the Einstein-Hilbert action of the metric (55),[21] so this metric is the physical metric.

On solutions of (19) $\Sigma_i = 2[e \wedge e]^{+\,0i}$, where

$$[e \wedge e]^{+\,IJ} = \frac{1}{2} e^I \wedge e^J + \frac{1}{4} \epsilon^{IJ}{}_{KL} e^K \wedge e^L \tag{56}$$

is the self-dual part of $e \wedge e$. Therefore, by (25), $e_{\sigma\nu}$ is the self-dual part of the metric area bivector of σ (modulo corrections that vanish faster than the area as the simplicial complex is refined):

$$e_{\sigma\nu\, i} = \int_\sigma \Sigma_i + O(\epsilon r_\nu^2) = 2 a_\sigma^{+\,0i} + O(\epsilon r_\nu^2), \tag{57}$$

with $a_\sigma^{IJ} = e^I{}_\alpha e^J{}_\beta |_{C_\nu} t_\sigma^{\alpha\beta}$ the metric area bivector, equal to the coordinate area bivector in metric normal coordinates.

If we choose the normal coordinates x^I so that σ lies in a spatial hypersurface ($x^0 = \mathit{constant}$) then $e_{\sigma\nu\, i} = \frac{1}{2}\epsilon^i{}_{jk} a_\sigma^{jk}$ - exactly the normal area vector of σ.

It would be nice to give a metric interpretation of $e_{\sigma\nu}$ also away from the continuum limit, to make possible a direct comparison of our simplicial theory with Regge calculus [Reg61]. This is difficult however since, unlike in Regge calculus, the simplices of our model are not flat. The wedges, s, carry curvature, θ_s, which does not generally vanish, even on solutions.

[20] A spinorial proof of this result is given in [CDJM91]. A proof in the language of $SO(3)$ tensors is provided in Appendix B of [Rei95].

[21] For a proof see [Rei95], Section 3.

However, when the holonomies $g_{\partial s}$ are all $\mathbf{1}$[22] the metric interpretation of the continuum limit extends to arbitrary simplicial complexes. Since $g_{\partial s} = \mathbf{1}$, $w_s = e_s$, so field equation (14) requires

$$\sum_{\sigma<\tau} e_{\sigma\nu\, i} = 0. \tag{58}$$

(58) imposes 12 independent linear constraints on the 30 $e_{\sigma\nu\, i}$, which imply just that there is a 2-form $\Sigma_{\nu\, i\, \alpha\beta}$ (18 components) such that

$$e_{\sigma\nu\, i} = \Sigma_{\nu\, i\, \alpha\beta} t_\sigma^{\alpha\beta}. \tag{59}$$

Field equation (17) and the non-degeneracy condition $\sum_{\sigma,\bar{\sigma}<\nu} e_{\sigma\nu} \cdot e_{\bar{\sigma}\nu} sgn(\sigma,\bar{\sigma}) \neq 0$ are equivalent to $\Sigma_{\nu\, i} \wedge \Sigma_{\nu\, j} \propto \delta_{ij}\ \Sigma_{\nu\, k} \wedge \Sigma_\nu^k \neq 0$. As for the continuum Σ fields this implies Σ defines a non-degenerate metric, and that $e_{\sigma\nu}$ is the self-dual part of the metric area bivector.

The roles played by the field equations are worth noting. A co-tetrad that is constant on a simplex defines an image of the simplex in an affine 4-space $\mathcal{E}$ with metric δ_{IJ} and a fixed orthonomal basis $\{v_0, v_1, v_2, v_3\}$. (58) ensures that any 3-simplex can be mapped into $\mathcal{E}$ such that the of $e_{\sigma\nu\, i}$ of its faces are the self-dual parts of the area bivectors. This requirement fixes the image (the "geometrical image") of the 3-simplex up to $SU(2)_R$ transformations. [23] (17) and the non-degeneracy condition ensure that the 3-simplices can all be mapped geometrically into $\mathcal{E}$ by the same co-tetrad, so they ensure that the image 3-simplices all can fit together to form a 4-simplex. Finally, in the gauge in which $e_{\sigma\nu} = u_\sigma$ field equation (15) ensures that the geometrical images of the 3-simplex faces of neighboring 4-simplices match up modulo an $SU(2)_R$ transformation. The only remaining field equation, (18), simply requires $\varphi_\nu = 0$.

The field equations thus restrict the simplicial fields to ones corresponding to a Regge type simplicial geometry determined by edge lengths. Moreover,

---

[22] The only solutions of the simplicial theory satisfying this requirement exactly are flat spacetimes, even though in continuum euclidean GR there are curved solutions with vanishing self-dual curvature.

[23] *Proof*: clearly the simplex can be uniquely mapped into the spatial (123) hyperplane of $\mathcal{E}$ such that the $e_{\sigma\nu\, i}$ are the spatial normal area vectors. These are equal to the self-dual parts of the spacetime area bivectors, which are invariant under $SU(2)_R$ transformations. Thus any $SU(2)_R$ transformed image of the 3-simplex fulfills the requirement. With some more effort one can show that these are the only allowed transformations.

since the transformation needed to match up the geometrical images of the 3-simplex shared by two 4-simplices is right-handed, the holonomy around a 2-simplex of the metric compatible connection, which by definition transports the one image of the 3-simplex into the other, is purely right-handed. Since the metric compatible holonomy leaves the image of the central 2-simplex invariant, it can only be right-handed if it is **1**.

In a curved field configuration (14) and (15) contain curvature terms which spoil the exact metrical interpretation of e_s (though it is of course recovered as the continuum limit is approached).

B Lemmas for the continuum limit

Lemma B.1: If $\{\Delta_n\}_{n=0}^{\infty}$ is a uniformly refining sequence of simplicial decompositions with repsect to one positive definite metric g_0 which is constant on each 4-simplex of Δ_0, then it is a uniformly refining sequence with respect to any other such metric g'_0.

Proof: Since all Δ_n $n > 0$ are refinements of Δ_0 and Δ_0 has a finite number of simplices it is sufficient to prove that conditions 3) and 4) of Definition 2 hold with respect to g'_0 *in each 4-simplex* $\nu_0 < \Delta_0$. Inside ν_0 g_0 and g'_0 are constant metrics, and since they are both positive definite they are related by a non-singular linear transformation. Hence there exist non-zero, finite constants a and b so that

$$r'_\nu < ar_\nu \qquad (60)$$

$$V'_\nu = bV_\nu. \qquad (61)$$

($'$ quantities are calculated with g'_0).

Thus 3), $r_n \to 0$ as $n \to \infty$, implies $r'_n < ar_n$ also approaches zero in this limit, that is 3), also holds with respect to g'_0. Furthermore, if $\exists c > 0$ such that $V_\nu > cr_\nu^4$ then $V'_\nu = bV_\nu > cbr_\nu^4 > cb/a^4 r'^4_\nu$, so 4) with respect to g_0 implies 4) with respect g'_0. ∎

Lemma A.2: If f is a continous function on a compact manifold M and $\{\Delta_n\}_{n=0}^{\infty}$ is a uniformly refining sequence of simplicial decompositions of M then $\forall \epsilon > 0$ N can be chosen sufficiently large that $|f(x) - f(y)| < \epsilon$ $\forall x, y \in \nu < \Delta_N$.

Proof: Since the simplicial decomposition Δ_0 is finite, it is sufficient to prove the lemma in each $\nu_0 < \Delta_0$ separately. The continuity of f implies that, with respect to any constant, positive definite metric g_0 on ν_0, there exists for each point $x \in \nu_0$ an open ball $B(x, r_x)$ of radius $r_x > 0$ about x such that $|f(x) - f(y)| < \epsilon/2 \ \forall y \in B(x, r_x)$.

Since ν_0 is compact it can be covered by a finite set of balls $\{B(x_m, 1/3r_{x_m})\}_{m=1}^{M}$. Now let $r = 1/3max\{r_{x_m}\}$, and note that $\forall x \in \nu_0 \ B(x,r) \cap \nu_0 \subset B(x_m, r_m) \cap \nu_0$ for some $m \in \{1, ..., M\}$. This implies that $|f(x) - f(y)| < \epsilon \ \forall x, y \in B(x,r) \cap \nu_0$. Since N can be chosen large enough that $r_\nu < r \ \forall \nu < \Delta_N \cap \nu_0$ the requirements of the lemma are satisfied in ν_0. ▮

Corrollary (Lemma 1): If h is a continous d-form on M and g_0 is a positive definite metric adapted to Δ_0, then $\forall \epsilon > 0 \ N$ can be chosen sufficiently large so that for any d-cell $c < \nu < \Delta_N$

$$|\int_c h - \int_c h_{C_\nu}| < \epsilon r_\nu^d, \tag{62}$$

where h_{C_ν} is the constant d-form (according to the affine structure of ν) which agrees with h at C_ν, and r_ν is the g_0 radius of ν.

Proof: Fix on each ν_0 normal coordinates x^α of g_0.

$$|\int_c h - \int_c h_{C_\nu}| < \sum_{\alpha_1 ... \alpha_d} \int_c |h(x)_{\alpha_1 ... \alpha_d} - h(C_\nu)_{\alpha_1 ... \alpha_d}||dx^{\alpha_1} ... dx^{\alpha_d}|. \tag{63}$$

Furthermore, since the components $h_{\alpha_1 ... \alpha_d}$ are continous functions, one may choose N large enough that $|h(x)_{\alpha_1 ... \alpha_d} - h(C_\nu)_{\alpha_1 ... \alpha_d}| < \epsilon(dimension \ M)^{-d} 2^{-d}$. Thus

$$|\int_c h - \int_c h_{C_\nu}| \ < \ \epsilon max\{\int_c |dx^{\alpha_1} ... dx^{\alpha_d}|\} \tag{64}$$

$$< \ \epsilon r_\nu^d. \tag{65}$$

▮

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