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Rank gradient for Artin groups and their relatives

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Abstract

We prove that the rank gradient vanishes for mapping class groups of genus bigger than 1, $\mathrm{Aut}(F_n)$ for all n, $\mathrm{Out}(F_n)$, $n \geq 3$ and any Artin group whose underlying graph is connected. We compute the rank gradient and verify that it is equal to the first L^2-Betti number for some classes of Coxeter groups.
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1 Introduction

Let G be a finitely generated group. We denote by $d(G)$ the minimal size of a generating set of G (setting $d(\{1\}) = 0$) and for a subgroup $H < G$ of finite index define

$$r(G, H) = (d(H) - 1)/[G : H].$$

Let $H_1 > H_2 > \cdots$ be a chain of normal subgroups of finite index in G. We define the rank gradient of G with respect to the chain (H_i) to be

$$RG(G, (H_i)) = \lim_{i \to \infty} r(G, H_i).$$

The notion of rank gradient was first defined by Lackenby in [11] for the study of Kleinian groups and further investigated in [1] and [2]. It is not known whether the limit $RG(G, (H_i))$ depends on the choice of the chain (H_i) under the condition that $\cap_i H_i = \{1\}$. However, as shown in [2], when G contains a normal infinite amenable subgroup or is a non-uniform lattice in a higher rank Lie group then $RG(G, (H_i)) = 0$ for any normal chain (H_i) in G with trivial intersection.

We remark that rank gradient with respect to an infinite chain in a group G is a natural upper bound for its first L^2-Betti number $\beta_1^{(2)}(G)$. Indeed it is well known that $d(H) \geq \beta_1^{(2)}(H)$ giving that $r(G, H_i) \geq \beta_1^{(2)}(G) - [G : H_i]^{-1}$ and letting $[G : H_i] \to \infty$ we get

$$RG(G, (H_i)) \geq \beta_1^{(2)}(G). \tag{1}$$

In particular if the rank gradient of G is zero with respect to some infinite chain then the first L^2-betti number of G is also zero. It is not known whether the inequality (1) can be strict for a normal chain (H_i) with trivial intersection.

In this note we compute the rank gradient (with respect to all chains) for several well-known families of groups: Artin groups, mapping class groups, $\mathrm{Aut}(F_n)$, $\mathrm{Out}(F_n)$ and some Coxeter groups. A result of Gaboriau [7] states that the first L^2-Betti number vanishes for any group that contains an infinite normal finitely generated subgroup of infinite index and in this situation one can sometimes find a chain with respect to which rank gradient is zero, see [10]. This applies to Artin groups of finite type, mapping class groups and (outer-) automorphism groups of free groups. Additionally, the L^2-Betti numbers of Artin groups and many Coxeter groups have been computed. This paper is an attempt to compare these results with the rank gradient. In all of the groups considered here, the rank gradient is found to be independent of the normal chain and moreover is equal to $\beta_1^{(2)}(G)$. In fact our arguments yield the stronger result that these groups have *fixed price*. For definition and properties of cost and fixed price we refer the reader to the book [9]. In particular Proposition 32.1 from [9] provides the cost analogue of section 2.

Singer's conjecture predicts that the L^2-Betti numbers of a closed aspherical n-manifold are zero in all dimensions except for $n/2$. The conjecture has been verified for different classes of groups, including some right-angled Coxeter groups [4]. We suspect that more generally if G is a virtual Poincare duality group of dimension at least three then the rank gradient of G is zero with respect to any normal chain with trivial intersection.

2 Basic results on rank gradient

In this section, we collect all the basic results about rank gradient that we will use in the paper. All of them may be proved with elementary techniques. The first result is a corollary of the more general Theorem 4 in [2] but in this special case the proof is nearly trivial.

Proposition 2.1 *Let G be a group containing an infinite central cyclic subgroup C. Then $RG(G,(H_i)) = 0$ for any chain of subgroups H_i with $[C : C \cap H_i] \to \infty$.*

Proof. Let $h_i = [C : C \cap H_i]$ and $a_i = [G : H_iC]$. Then $d(H_iC) \leq a_i d(G)$ and hence H can be generated by $d(C \cap H_i) + d(H_i/(C \cap H_i)) = 1 + d(H_iC/C) \leq 1 + a_i d(G)$ elements. On the other hand $[G : H_i] = a_i h_i$ and so $r(G, H_i) \leq (1 + a_i d(G))/(a_i h_i) \to 0$ since $h_i \to \infty$. □

Proposition 2.2 *Let $G = A *_C B$ be an amalgam of two finitely generated groups over a finite subgroup C. Let (H_i) be a normal chain in G such that $C \cap (\cap_i H_i) = \{1\}$. Then*

$$RG(G,(H_i)) = RG(A,(A \cap H_i)) + RG(B,(B \cap H_i)) + 1/|C|$$

.

Proof. This is an easy computation with the Bass-Serre tree T of G. The key observation is that for every normal subgroup H_j such that $H_j \cap C = \{1\}$ we have that H_j is a free product of several copies of $A \cap H_j$, $B \cap H_j$ and a free group equal to the fundamental group of the graph $H_j \backslash T$. Applying the Grushko-Neumann theorem to H_j shows that

$$d(H_j)-1 = [G : H_j A](d(A\cap H_j)-1)+[G : H_j B](d(B\cap H_j)-1)+\frac{[G : H_j]}{|C|}$$

giving the result. □

Proposition 2.3 *[2, Prop. 9] Let G be a group generated by two subgroups A and B such that $C := A \cap B$ is infinite. Then*

$$RG(G,(H_i)) \leq RG(A,(A \cap H_i)) + RG(B,(B \cap H_i))$$

for any normal chain (H_i) in G such that $[C : (C \cap H_j)] \to \infty$.

2.1 Artin groups

The first family we consider are the Artin groups. We recall their definition below. Given a graph Γ with edges labelled by integers ≥ 2, the Artin group A_Γ is the group with presentation given by a set of generators a_v where v ranges over the set of vertices of Γ and relations

$$\underbrace{a_v a_w a_v \cdots}_{n} = \underbrace{a_w a_v a_w \cdots}_{n}$$

for every edge labelled $n \in \mathbb{N}$ joining pair of vertices v, w of Γ.

A basic fact about Artin groups says that if one takes any subset W of the vertices V of the defining graph, then the subgroup generated by W in A_Γ is precisely the Artin group A_Ω, where Ω is the subgraph generated by W in Γ (see [12]).

We note that it is unknown whether all Artin groups are residually finite. Let G_0 denote the intersection of all finite index subgroups of a group G. For an Artin group $A = A_\Gamma$ we consider normal chains (H_i) in A with $\cap H_i = A_0$.

Our result is:

Theorem 2.4 *Let $A = A_\Gamma$ be an Artin group and let (H_i) be a normal chain with intersection A_0. Then $RG(A,(H_i)) = b - 1$ where b is the number of connected components of Γ.*

Proof. Note that a free product of residually finite groups is residually finite, therefore in a free product of two groups $G = C * D$ one has $C \cap G_0 = C_0$ and $D \cap G_0 = D_0$. In view of Proposition 2.2 we only need to prove that $RG(A_\Gamma,(H_i)) = 0$ whenever Γ is connected.

We begin with an observation: Let $\pi : A_\Gamma \to \mathbb{Z}$ be the homomorphism sending all generators a_v of A to the canonical generator of $\mathbb{Z}$. Clearly $\ker \pi \geq A_0$ and therefore if T is any cyclic subgroup of A with $T \cap \ker \pi = \{1\}$ then $T \cap A_0 = T \cap (\cap_i H_i) = 1$. In particular this applies to all vertex subgroups $\langle a_v \rangle$ of A. Let $e = (v, w)$ be an edge joining vertices v and w labelled by some integer $n > 1$ of Γ. The subgroup $A_e = \langle a_v, a_w \rangle$ has infinite cyclic center $Z_e = Z(A_e)$ generated by $(a_v a_w)^{n/2}$ if n is even and by $(a_v a_w)^n$ if n is odd. In both cases we have that $Z_e \cap \ker \pi = \{1\}$ and therefore $Z_e \cap (\cap H_i) = \{1\}$. The following Lemma is the crux of the proof of the Theorem.

Lemma 2.5 *Let A_Γ be an Artin group, where Γ is a connected graph. Let (H_i) be a normal chain in A_Γ such that $A_v \cap (\cap H_i) = \{1\}$ for every vertex $v \in V\Gamma$ and $Z_e \cap (\cap H_i) = \{1\}$ for every edge $e \in E\Gamma$. Then, $RG(A_\Gamma, (H_i)) = 0$.*

Proof of Lemma 2.5. The Lemma is proved by induction on the number of vertices $n = |V\Gamma|$. If $n = 1$, $A_\Gamma \cong \mathbb{Z}$ and if $n = 2$, then the centre of A_Γ is infinite cyclic. In both cases the result follows directly from Proposition 2.1. We may assume that the number n of vertices is at least 3 and that the Lemma holds for all graphs with fewer vertices.

Let $v \in V\Gamma$. If removing v disconnects the graph Γ, set $\Gamma_j, j = 1, \ldots, k$ to be the connected components of $\Gamma \backslash \{v\}$. Then, the original Artin group A_Γ is an amalgam of the subgroups $A_{\Gamma_j \cup \{v\}}$, $j = 1, \ldots, k$, along the subgroup A_v. As $k \geq 2$, the induction hypothesis applies to each of the components $A_{\Gamma_j \cup \{v\}}$ and the Lemma follows in this case from Proposition 2.2.

Suppose then that removing v does not disconnect the graph. Let $e = (v, w)$ be an edge. Clearly, A_e and $A_{\Gamma \backslash \{v\}}$ generate the group A_Γ. Moreover A_e and $A_{\Gamma \backslash \{v\}}$ intersect in the subgroup A_w. The proof of the Lemma is now completed by applying the induction hypothesis to the subgroup $A_{\Gamma \backslash \{v\}}$ and invoking Proposition 2.3. □

The Theorem provides an elementary proof for the following well-known result.

Corollary 2.6 *The first L^2-Betti number for an Artin group A_Γ is 0 if the underlying graph is connected. More generally, if Γ has n connected components, then the first L^2-Betti number of A_Γ is precisely $n - 1$.*

3 $\mathrm{Aut}(F_n)$, $\mathrm{Out}(F_n)$ and $\mathrm{MCG}(S_g)$

Theorem 3.1 *Let G be one of the groups $\mathrm{Aut}(F_n)$, $n \geq 2$, $\mathrm{Out}(F_n)$, $n \geq 3$ and $\mathrm{MCG}(S_g)$, $g \geq 2$. The rank gradient for G is zero for any normal chain with trivial intersection.*

Here, $\mathrm{MCG}(S_g)$ denotes the mapping class group of the closed surface S_g of genus g. Note that S_1 is the torus and its mapping class group is $SL(2, \mathbb{Z})$. It is well known that $SL(2, \mathbb{Z})$ is isomorphic to the amalgam $\mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$ and therefore its rank gradient is $1/12$.

Proof of Theorem 3.1. To prove the result for the mapping class groups, we recall that $\mathrm{MCG}(S_g)$ is generated by the $2g + 1$ Humphries generators $m_1, m_2, a_1, \ldots, a_g$ and $c_1, \ldots, c_{g-1}$ (see Figure 4.1 from [6]). Consider the subgroups arising from specific pairs of generators as listed below:

$$\langle m_1, a_1 \rangle,\ \langle a_1, c_1 \rangle,\ \langle c_1, a_2 \rangle,\ \langle a_2, m_2 \rangle,\ \langle a_2, c_2 \rangle,\ \langle c_2, a_3 \rangle, \ldots, \langle c_{g-1}, a_g \rangle.$$

Each pair listed here corresponds to a pair of simple closed non-seperating curves with intersection number 1. This means that each of these 2-generated subgroups is precisely the braid group B_3 on 3 strands. Since braid groups are Artin groups by Theorem 2.4 the rank gradient for B_3 is zero for any normal chain with trivial intersection. Every two consecutive subgroups in the list above intersect in an infinite cyclic subgroup. Therefore, by Proposition 2.3, the Theorem follows for all mapping class groups for $g \geq 2$.

To deal with $\mathrm{Aut}(F_n)$ first we note the well-known fact that B_n embeds in $\mathrm{Aut}(F_n)$ such that the centre Z_n of B_n is $B_n \cap \mathrm{Inn}(F_n)$. This shows that B_n/Z_n is residually finite because $\mathrm{Out}(F_n)$ is residually finite [8]. For $n \geq 4$ let $A = \langle \sigma_1, \ldots, \sigma_{n-2} \rangle$ and $B = \langle \sigma_2, \sigma_3, \ldots, \sigma_{n-1} \rangle$ denote the two canonical copies of B_{n-1} in B_n. From the description of Z_n as the subgroup generated by the Garside element Δ_n we see that $A \cap Z_n = B \cap Z_n = \{1\}$ and therefore the images of A and B in B_n/Z_n are two groups with rank gradient zero and infinite intersection. With an application of Proposition 2.3 we have therefore proved the following.

Proposition 3.2 *Let $n \geq 4$. The rank gradient of B_n/Z_n is zero for any normal chain with trivial intersection.*

Now $\mathrm{Aut}(F_2)$ contains B_4/Z_4 as a subgroup of index two [5] and therefore its rank gradient is also zero. Let $n > 2$ and let $x_1, \ldots, x_n$ be the free generators of F_n. For $i = 1, \ldots, n-1$, let A_i be the copy of $\mathrm{Aut}(F_2)$ in $\mathrm{Aut}(F_n)$ which preserves $\langle x_i, x_{i+1} \rangle$ and fixes all the generators except x_i and x_{i+1}. Consider the automorphisms $f_i \in A_i$ given by $f_i(x_i) = x_i x_{i+1} x_i^{-1}, f(x_{i+1}) = x_i$. The group D generated by $f_1, \ldots, f_{n-1}$ is isomorphic to B_n. Note that $\mathrm{Aut}(F_n)$ is generated by the A_i, and moreover $A_i \cap D \geq \langle f_i \rangle$ is infinite. Using that the rank gradient of each of A_i and D is zero, Theorem 3.1 follows again from Proposition 2.3.

We now turn to the outer automorphism groups $\mathrm{Out}(F_n)$. Again let $x_1, \ldots, x_n$ denote a set of free generators for F_n. If $n \geq 4$, then $\mathrm{Out}(F_n)$ is generated by the two copies of $\mathrm{Aut}(F_{n-1})$ in $\mathrm{Out}(F_n)$, one which fixes x_1 and the other fixes x_n. Clearly $\mathrm{Aut}(F_2)$ is contained in the intersection of these two subgroups. We can therefore invoke Proposition 2.3 and the vanishing of the rank gradient for $\mathrm{Aut}(F_n)$ to see that the Theorem holds for $\mathrm{Out}(F_n)$, $n \geq 4$.

We now deal with $\mathrm{Out}(F_3)$. First we shall define three copies of $\mathrm{Aut}(F_2)$ in $\mathrm{Aut}(F_3)$.

Let X be the copy of $\mathrm{Aut}(F_2)$ acting on $\langle x_1, x_2 \rangle$ and fixing x_3, Y be the copy of $\mathrm{Aut}(F_2)$ acting on $\langle x_2, x_3 \rangle$ and fixing x_1 and let Z be the copy of $\mathrm{Aut}(F_2)$ acting on $\langle x_1, x_2 \rangle$ and fixing $x_1 x_3$. Define $\alpha, \beta, \gamma \in \mathrm{Aut}(F_3)$ as follows:

$$\alpha(x_1) = x_1,\ \alpha(x_2) = x_1 x_2,\ \alpha(x_3) = x_3,$$
$$\beta(x_1) = x_1 x_2,\ \beta(x_2) = x_2,\ \beta(x_1 x_3) = x_1 x_3,$$
$$\gamma(x_1) = x_1 x_2^{-1},\ \gamma(x_2) = x_2,\ \gamma(x_3) = x_3.$$

Now $\alpha \in X \cap Z, \beta \in Z, \gamma \in X$ and $\beta(x_3) = x_2^{-1} x_3$. The composition $\gamma \circ \beta$ fixes x_1 and x_2 and sends x_3 to $x_2^{-1} x_3$.

Let us write $\bar{a}$, $\bar{A}$ for the image of the element a, and respectively subgroup A of $\mathrm{Aut}(F_3)$ in $\mathrm{Out}(F_3)$. The groups $\bar{X}, \bar{Y}, \bar{Z}$ are all isomorphic to $\mathrm{Aut}(F_2)$ which has rank gradient zero. The intersection $\bar{X} \cap \bar{Z} \geq \langle \bar{\alpha} \rangle$ is infinite and therefore the group $\langle \bar{X}, \bar{Z} \rangle$ has rank gradient zero. Now $\langle \bar{X}, \bar{Z} \rangle \cap \bar{Y}$ contains $\bar{\gamma} \circ \bar{\beta}$ and is therefore infinite giving that the rank gradient of $\langle \bar{X}, \bar{Y}, \bar{Z} \rangle$ is zero. Finally we note that $\bar{X}, \bar{Y}$ generate $\mathrm{Out}(F_3)$. This completes the proof of Theorem 3.1. ■

4 Coxeter groups

As before, let Γ be a finite graph labelled with integers bigger than 1. The Coxeter group C_Γ is defined to be the image of the Artin group A_Γ with the extra relations that the generators a_v have order 2.

$$C_\Gamma = \langle A_\Gamma \mid a_v^2 = 1, \forall v \in V \rangle.$$

In many ways Coxeter groups are better undestood than Artin groups, for example they are all linear groups (and so residually finite), with a concrete finite dimensional $K(\pi, 1)$ complex. However their L^2-Betti numbers have been computed only in some special situations even in the case of right angled Coxeter groups, see for example [4].

Our result on rank gradient is even more special. Let $\mathcal{C}_0$ be the class of Coxeter groups which are virtually abelian, virtually free or virtually surface groups. Let $\mathcal{C}$ be the smallest class of Coxeter groups which contains $\mathcal{C}_0$ and is closed under amalgamation along subgroups K with $\beta_1^{(2)}(K) = 0$ (for example K can be any virtually abelian group).

Theorem 4.1 *Let G be an infinite Coxeter group in $\mathcal{C}$ and let (H_i) be an infinite normal chain in G with trivial intersection. Then $RG(G, (H_i)) = \beta_1^{(2)}(G)$.*

Proof. If $G \in \mathcal{C}_0$ the claim is straightforward: If G is virtually a surface or virtually a free group then some member of the chain N_j is either a surface group or a free group, in which case the equality between rank gradient and first L^2-Betti number is well known. In general the definition of $\mathcal{C}$ says that G can be obtained by a sequence of amalgamations starting from some groups in $\mathcal{C}_0$. By induction on the number of amalgamation steps it is enough to prove the following: if $G = G_1 *_K G_2$ such that $\beta_1^{(2)}(K) = 0$ and G_1, G_2 satisfy the claim in the Theorem, then $RG(G, (N_i)) = \beta_1^{(2)}(G)$. To establish the claim, we use the following equality for L^2-Betti numbers (with the convention that $1/|G| = 0$ if G is infinite):

$$\beta_1^{(2)}(G) = \beta_1^{(2)}(G_1) - \frac{1}{|G_1|} + \beta_1^{(2)}(G_2) - \frac{1}{|G_2|} + \frac{1}{|K|} \tag{2}$$

which holds whenever $\beta_1^{(2)}(K) = 0$, see the appendix of [3]. When K is finite then the claim follows from Proposition 2.2 and the assumption on G_1, G_2. When K is infinite then Proposition 2.3 gives

$$RG(G, (N_i)) \leq RG(G_1, (G_1 \cap N_i)) + RG(G_1, (G_1 \cap N_i)) =$$

$$= \beta_1^{(2)}(G_1) + \beta_1^{(2)}(G_2) = \beta_1^{(2)}(G) \leq RG(G, (N_i))$$

and hence again equality must hold. □

We give an application.

Theorem 4.2 *Suppose that* Γ *is a planar graph without circuits of length less than 6. Then for any normal chain* (N_i) *in* C_Γ *with trivial intersection we have*

$$RG(C_\Gamma, (N_i)) = \beta_1^{(2)}(C_\Gamma) = \frac{|V|}{2} - 1 - \sum_{e \in EV} \frac{1}{2l_e}$$

where l_e *is the label of the edge* e *of* Γ.

Proof. We show first that Γ must have a vertex, say v of valency at most two. Indeed, if every vertex has valency ≥ 3, then the number $|E|$ of edges of Γ is at least $3|V|/2$. On the other hand the number of regions of the plane cut out by Γ is at most $|E|/3$ (because every region has at least 6 edges on the boundary). Now Euler's formula $1 = |V| - |E| + |F| \leq |V| - 2|E|/3 \leq 0$ derives a contradiction.

Now suppose that v has valency 2 and let w_1, w_2 be the two neighbours of v. Set $A = \langle a_v, a_{w_1}, a_{w_2} \rangle$, a Coxeter group whose graph is the two edges $e_1 = (v, w_1)$ and $e_2 = (v, w_2)$. One can check, for instance, by using (2) that the rank gradient of the virtually free group A is equal to $\beta_1^{(2)}(A) = \frac{1}{2} - \frac{1}{2l_{e_1}} - \frac{1}{2l_{e_2}}$.

Let B be the subgroup generated by all a_u for $u \in V \backslash \{v\}$. Then B is a Coxeter group with graph $\Gamma' = \Gamma - \{v\}$ and C_Γ is the amalgam of A and B along the intersection $\langle a_{w_1}, a_{w_2} \rangle \simeq D_\infty$. By induction on $|V|$ we may assume that the Theorem holds for B, in particular

$$\beta_1^{(2)}(B) = \frac{|V| - 1}{2} - 1 - \sum_{e \in E\Gamma'} \frac{1}{2l_e}.$$

Therefore from the proof of Theorem 4.1 the rank gradient and first L^2-Betti number of C_Γ is $\beta_1^{(2)}(A) + \beta_1^{(2)}(B)$ which is what had to be proved. The case when v has valency zero or 1 is similar. □

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