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Symmetries of Gaussian Measures and Operator Colligations

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Symmetries of Gaussian measures and operator colligations

Yury Neretin[1]

Consider an infinite-dimensional linear space equipped with a Gaussian measure and the group $\mathrm{GLO}(\infty)$ of linear transformations that send the measure to equivalent one. Limit points of $\mathrm{GLO}(\infty)$ can be regarded as 'spreading' maps (polymorphisms). We show that the closure of $\mathrm{GLO}(\infty)$ in the semigroup of polymorphisms contains a certain semigroup of operator colligations and write explicit formulas for action of operator colligations by polymorphisms of the space with Gaussian measure.

1 Introduction. Polymorphisms, Gaussian measures, and colligations

1.1. The group $\mathrm{Gms}(M)$**.** Let $M = (M, \mu)$ be a Lebesgue space M with a probability measure μ ([29], see, also [14]), let $L^p(M, \mu)$ be the space of measurable functions on M with norm

$$\|f\|_p = \Bigl(\int_M |f(m)|^p \, d\mu(m)\Bigr)^{\frac{1}{p}}, \qquad \text{where } 1 \leqslant p \leqslant \infty.$$

Denote by $\mathrm{Gms}(M)$ the group of all bijective a.s. maps $M \to M$ that send the measure μ to an equivalent measure. For $g \in \mathrm{Gms}(M)$ we denote by $g'(m)$ the Radon–Nikodym derivative of g.

Fix $\lambda \in \mathbb{C}$ lying in the strip $0 \leqslant \operatorname{Re} \lambda \leqslant 1$,

$$\lambda = \tfrac{1}{p} + is, \qquad \text{where } 1 \leqslant p \leqslant \infty,\ s \in \mathbb{R}. \tag{1.1}$$

For any $g \in \mathrm{Gms}(M)$ we define the linear operator $T_\lambda(g)$ by

$$T_\lambda(g) f(m) = f(mg) g'(m)^\lambda. \tag{1.2}$$

Evidently, the operators $T_\lambda(g)$ form a representation of the group $\mathrm{Gms}(M)$ by isometric operators in the Banach space $L^p(M, \mu)$. For $p = 2$ we get a unitary representation in $L^2(M, \mu)$.

Polymorphims, which are introduced below, are "limit points" of the group $\mathrm{Gms}(M)$.

1.2. Gaussian measures. Consider $\mathbb{R}$ equipped with the Gaussian measure $\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$. Let $n = 1, 2, \ldots, \infty$. Denote by $\mathbb{R}^\omega$ the product of n copies of $\mathbb{R}$ equipped with the product measure $\mu_\omega = \mu \times \mu \times \ldots$. We denote elements of $\mathbb{R}^\omega$ by $x = \bigl(x_1, x_2, \ldots\bigr)$.

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Proposition 1.1 *If* $\sum b_j^2 < \infty$, *then the series* $\sum b_j x_j$ *converges a.s. on* $\mathbb{R}^\infty$ *with respect to the measure* μ_∞.

This is a special case of the Kolmogorov–Hinchin theorem about series of independent random variables, see, e.g., [32].

1.3. Groups of symmetries of Gaussian measures. Denote by $\mathrm{O}(\infty)$ the infinite-dimensional orthogonal group, i.e., the group of all infinite real matrices A satisfying the conditions

$$AA^t = A^t A = 1,$$

where ^{t} denotes the transposition.

For an invertible real infinite matrix A we consider the polar decomposition $A = SU$, where $U \in \mathrm{O}(\infty)$, and S is a positive self-adjoint operator. We define the group $\mathrm{GLO}(\infty)$ consisting of matrices $A = SU$ such that $S - 1$ is a Hilbert–Schmidt[2] operator. Equivalently, we can represent A as $A = \exp(T)U$, where $U \in \mathrm{O}(\infty)$ and T is a Hilbert–Schmidt self-adjoint operator.

Thus the set $\mathrm{GLO}(\infty)$ is the product of $\mathrm{O}(\infty)$ and the space of self-adjoint Hilbert–Schmidt matrices. We take the weak operator topology[3] on $\mathrm{O}(\infty)$ and the natural topology on the space of Hilbert–Schmidt matrices[4] . We equip $\mathrm{GLO}(\infty)$ with the topology of product. Then $\mathrm{GLO}(\infty)$ is a topological group with respect to this topology (*the Shale topology,* [30]).

Consider an infinite matrix $A = \{a_{ij}\}$. Apply it to a vector $x \in \mathbb{R}^\infty$,

$$xA = \begin{pmatrix} x_1 & x_2 & \dots \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \sum x_i a_{i1} & \sum x_i a_{i2} & \dots \end{pmatrix} \tag{1.3}$$

Let A be an operator bounded in the space ℓ_2. By Proposition 1.1 the vector xA is defined for almost all $x \in (\mathbb{R}^\infty, \mu_\infty)$.

Theorem 1.2 a) *For* $A \in \mathrm{O}(\infty)$ *the map* $x \mapsto xA$ *preserves measure* μ_∞.

b) *For* $A \in \mathrm{GLO}(\infty)$, *the map* $x \mapsto xA$ *is defined a.s. on* $(\mathbb{R}^\infty, \mu_\infty)$ *and sends the measure* μ_∞ *to an equivalent measure* $\mu(xA)$.

c) *Let* $A = (1 + T)U$, *where* $A \in \mathrm{O}(\infty)$ *and* T *is in the trace class*[5]. *Then the Radon–Nikodym derivative is given by the formula*

$$\frac{d\mu(xA)}{d\mu(x)} = |\det A| \cdot \exp(-\tfrac{1}{2}\langle xA, xA\rangle + \tfrac{1}{2}\langle x, x\rangle) := \\ := |\det(1+T)| \cdot \exp\big(-\langle xT, x\rangle - \tfrac{1}{2}\langle xT, xT\rangle\big) \tag{1.4}$$

[2]An operator T is Hilbert–Schmidt, if $\sum_{ij} |t_{ij}|^2 < \infty$, see, e.g., [28]
[3]See e.g., [28].
[4]See e.g. [28].
[5]See, [28].

d) *Let* $A = 1 + T$, *where* T *is a diagonal matrix with entries* $t_j > -1$ *satisfying* $\sum_j t_j^2 < \infty$. *Then the Radon–Nikodym derivative is given by*

$$\prod_{j=1}^{\infty}(1+t_j)e^{-(2t_j+t_j^2)x_j^2/2},$$

the product converges a.s. on $(\mathbb{R}^\infty, \mu_\infty)$.

e) *For* A, $B \in \mathrm{GLO}(\infty)$ *the identity*

$$(xA)B = x(AB)$$

holds a.s. on $(\mathbb{R}^\infty, \mu)$.

The theorem is a reformulation of the Feldman–Hajeck Theorem on equivalence of Gaussian measures (see, e.g., [11], [4]), the most comprehensive exposition is in [31].

REMARK. For $A \in \mathrm{GLO}_1(\infty)$, the absolute value of determinant $|\det(A)| := |\det(1+T)|$ is well-defined (see, e.g, [17]), it satisfies

$$|\det(A_1 A_2)| = |\det(A_1)| \cdot |\det(A_2|.$$

The $\det(A)$ makes no sence. $\square$

REMARK. In our definition the action is defined a.s, and the identity $x(AB) = (xA)B$ also is valid a.s. The removing of "a.s." is impossible, the group $\mathrm{O}(\infty)$ can not act pointwise by measure preserving transformations, see [8]. $\square$

1.4. Polymorphisms (spreading maps), for details, see [22]. [17], [20]). Denote by $\mathbb{R}^\times$ the multiplicative group of positive real numbers, denote by t the coordinate on $\mathbb{R}^\times$, by $\alpha * \beta$ we denote the convolution of measures on $\mathbb{R}^\times$. Let $M = (M, \mu)$, $N = (N, \nu)$ be Lebesgue spaces with probability measures. A *polymorphism*[6] $\mathfrak{P} : (M, \mu) \rightsquigarrow (N, \nu)$ is a measure $\mathfrak{P} = \mathfrak{P}(m, n, t)$ on $M \times N \times \mathbb{R}^\times$ satisfying two conditions:

a) the projection of $\mathfrak{P}(m, n, t)$ to M is μ;

b) the projection of $t \cdot \mathfrak{P}(m, n, t)$ to N is ν.

We denote by $\mathrm{Pol}(M, N)$ the set of all polymorphisms $(M, \mu) \rightsquigarrow (N, \nu)$.

There is a well-defined associative multiplication

$$\mathrm{Pol}(M, N) \times \mathrm{Pol}(N, K) \to \mathrm{Pol}(M, K)$$

1.5. Convergence of polymorphisms. For $\mathfrak{P} \in \mathrm{Pol}(M, N)$ and measurable subsets $A \subset M$, $B \subset N$ we consider the projection $A \times B \times \mathbb{R}^\times \to \mathbb{R}^\times$ and denote by $\mathfrak{p}[A \times B]$ the pushforward of $\mathfrak{P}$ under this projection.

[6] These objects were introduced in [16], see also [17]. The term was proposed be Vershik [33], who used it for measures on $M \times N$, see also "bistochastic kernels" from [10]. On some appearances of polymorphisms in variation problems and mathematical hydrodynamics, see [2].

We say that a sequence $\mathfrak{P}_j \in \mathrm{Pol}(M,N)$ *converges* to $\mathfrak{P}$ if for any $A \subset M$, $B \subset N$ we have weak convergences

$$\mathfrak{p}[A \times B] \to \mathfrak{p}[A, \times B], \qquad t \cdot \mathfrak{p}_j[A \times B] \to t \cdot \mathfrak{p}[A \times B].$$

Proposition 1.3 *The product of polymorphisms is separately continuous, i.e. if $\mathfrak{P}_j$ converges to $\mathfrak{P}$ in $\mathrm{Pol}(M,N)$ and $\mathfrak{Q}_j$ converges to $\mathfrak{Q}$ in $\mathrm{Pol}(N,K)$, then $\mathfrak{Q} \diamond \mathfrak{P}_j$ converges to $\mathfrak{Q} \diamond \mathfrak{P}$ and $\mathfrak{Q}_j \diamond \mathfrak{P}$ converges to $\mathfrak{Q} \diamond \mathfrak{P}$.*

Note that there is no joint continuity, generally $\mathfrak{Q}_j \mathfrak{P}_j$ does not converge to $\mathfrak{Q} \diamond \mathfrak{P}$.

1.6. Embedding $\mathfrak{I} : \mathrm{Gms}(M) \to \mathrm{Pol}(M,M)$. Now let a measure μ on M be continuous. We consider the embedding

$$\mathfrak{I} : \mathrm{Gms}(M) \to \mathrm{Pol}(M,M) \tag{1.5}$$

given by the following way. Take the map $M \mapsto M \times M \times \mathbb{R}^\times$ given by $m \mapsto \big(m, g(m), g'(m)\big)$. Then the pushforward of the measure μ is a polymorphism $\mathfrak{I}(g) : M \to M$.

Proposition 1.4 ([16], [22]) *The group* $\mathrm{Gms}(M)$ *is dense in* $\mathrm{Pol}(M,M)$.

1.7. Formulation of problem. We wish to describe the closure of $\mathrm{GLO}(\infty)$ in the semigroup of polymorphisms[7] of $\mathbb{R}^\infty$. Our solution is not final, we show a large semigroup (see the next subsection) in this closure.

1.8. Operator colligations. Fix $\omega = 0, 1, \dots, \infty$. Denote by $\mathrm{GLO}(\omega+\infty)$ the group consisting of $(\omega+\infty) \times (\omega+\infty)$ matrices g that are elements of the group GLO (i.e, $\mathrm{GLO}(\omega+\infty)$ is another notation for $\mathrm{GLO}(\infty)$). Consider the subgroup $\mathrm{O}(\infty) \subset \mathrm{GLO}(\omega+\infty)$ consisting of block $(\omega+\infty)\times(\omega+\infty)$ matrices $\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$, where u is an orthogonal matrix.

We say that an *operator colligation* is an element g of $\mathrm{GLO}(\omega+\infty)$ defined up to the equivalence

$$g \sim h_1 g h_2, \qquad \text{where } h_1,\, h_2 \in \mathrm{O}(\infty),$$

or, in more details,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$$

where u, v are orthogonal matrices. Denote by $\mathrm{Coll}(\omega)$ the set of all operator colligations. In other words, $\mathrm{Coll}(\omega)$ is the double coset space

$$\mathrm{Coll}(\omega) = \mathrm{O}(\infty) \setminus \mathrm{GLO}(\omega+\infty)/\mathrm{O}(\infty).$$

[7] The closure of $\mathrm{O}(\infty)$ gives action of the semigroup of all contractive linear operators by polymorphisms of $\mathbb{R}^\infty$, see Nelson [15], .

The *product of operator colligations* is defined by the formula

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\circ\begin{pmatrix}\varphi & \psi\\ \theta & \varkappa\end{pmatrix} := \begin{pmatrix}\alpha & \beta & 0\\ \gamma & \delta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\varphi & 0 & \psi\\ 0 & 1 & 0\\ \theta & 0 & \varkappa\end{pmatrix} = \begin{pmatrix}\alpha\varphi & \beta & \alpha\psi\\ \gamma\varphi & \delta & \gamma\psi\\ \theta & 0 & \varkappa\end{pmatrix}$$

The resulting matrix has size

$$\bigl(\omega+(\infty+\infty)\bigr)\times\bigl(\omega+(\infty+\infty)\bigr) \quad = \quad (\omega+\infty)\times(\omega+\infty),$$

i.e., we again get an element of $\mathrm{Coll}(\omega)$.

Proposition 1.5 *The product $\circ$ is a well-defined associative operation on the set* $\mathrm{Coll}(\omega)$.

This can be verified by a straightforward calculation. For a clarification of this operation, see [17], Section IX.5. Classical operator colligations are matrices determined up to the equivalence

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\sim\begin{pmatrix}1 & 0\\ 0 & u\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & u^{-1}\end{pmatrix}.$$

Colligations, their multiplication, and characteristic functions appeared in the spectral theory of non-self-adjoint operators (M. S. Livshits, V. P. Potapov, 1946–1955, [12], [13], [27], see survey in [3], see also algebraic version in [7]).

1.9. Results of the paper. First (Theorem 3.2), we prove the following statements:

— The closure of $\mathrm{GLO}(\infty)$ in polymorphisms of $(\mathbb{R}^\infty, \mu_\infty)$ contains the semigroup $\mathrm{Coll}(\infty)$.

— For $n<\infty$ the semigroup $\mathrm{Coll}(n)$ admits a canonical embedding to semigroup of polymorphisms of the space $(\mathbb{R}^n, \mu_n)$.

Our main purpose is to write explicit formulas (Theorems 5.2, 6.1) for this embedding.

1.10. A general problem. Many interesting actions of infinite dimensional groups on spaces with measures are known, see survey [18] and recent 'new' constructions [9], [26], [21], [1]. In all cases there arises the problem of description of closure of the group in polymorphisms, in all the cases this gives semigroups that essentially differ from the initial groups[8] . In this work and in [20] the problem was solved in two the most simple cases (Gaussian and Poisson measures). In both cases we get unusual interesting formulas.

[8]This is counterpart of Olshanski problem about weak closure of image of unitary representation, see [24]; for a finite-dimensional counterpart, see [6].

2 Polymorphisms. Preliminaries

First, we need some preliminaries on polymorphisms.

2.1. Measures on $\mathbb{R}^\times$. Denote by $\mathbb{R}^\times$ the multiplicative group of positive real numbers, denote by t the coordinate on $\mathbb{R}^\times$, by $\varphi * \psi$ we denote *convolution* of finite measures φ and ψ on $\mathbb{R}^\times$, it defined by

$$\int_{\mathbb{R}^\times} f(t)\, d(\varphi * \psi)(t) = \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} f(pq)\, d\psi(p)\, d\varphi(q).$$

Recall that a sequence of finite measures ψ_j on $\mathbb{R}^\times$ *weakly converges* to a measure ψ if for any continuous function f on $\mathbb{R}^\times$ we have the convergence

$$\int_{\mathbb{R}^"} f(t)\, d\psi_j(t) \longrightarrow \int_{\mathbb{R}^\times} f(t)\, d\psi(t).$$

2.2. Product of polymorphisms. Here we give a formal definition of the product of polymorphisms, but actially we use Theorem 2.4 instead of the definiton. For details, see [22].

Let p be a function on $M \times N$ taking values in finite measures on $\mathbb{R}^\times$. Such a function determines a measure $\mathfrak{P}$ on a product $M \times N \times \mathbb{R}^\times$,

$$\iiint\limits_{M\times N\times\mathbb{R}^\times} f(m,n,t)\, d\mathfrak{P}(m,n,t) := \iint\limits_{A\times B} \int\limits_{\mathbb{R}^\times} f(m,n,t)\, dp(m,n)(t)\, d\nu(n)\, d\mu(m).$$

If p satisfies two identities

$$\int_A \int_N \int_{\mathbb{R}^\times} dp(m,n)(t)\, dp(m,n)(t)\, d\nu(n)\, d\mu(m) = \mu(A),$$
$$\int_M \int_B \int_{\mathbb{R}^\times} t\, dp(m,n)(t)\;\, dp(m,n)(t)\, d\nu(n)\, d\mu(m) = \nu(B)$$

for any measurable subsets $A \subset M$, $B \subset N$, then $\mathfrak{P}$ is a polymorphism. If $\mathfrak{P}$ has such aform, we say that $\mathfrak{P}$ is absolutely continuous.

Now let $\mathfrak{P} \in \mathrm{Pol}(M,N)$, $\mathfrak{Q} \in \mathrm{Pol}(N,K)$ be absolutely continuous polymorphisms, p, q be the correspondin functions. Then the function r on $M \times K$ is determined by

$$r(a,c) = \int_N p(m,n) * q(n,k)\, d\nu(n).$$

The integral is convergent a.s.

Theorem 2.1 *This product admits a unique separately continuous extension to an operation* $\mathrm{Pol}(M,N) \times \mathrm{Pol}(N,K) \to \mathrm{Pol}(M,K)$.

2.3. Involution in the category of polymorphisms. Let $\mathfrak{P} : M \rightsquigarrow N$ be a polymorphism. We define the polymorphism $\mathfrak{P}^\star : N \rightsquigarrow M$ by

$$\mathfrak{P}^\star(n,m,t) = t \cdot \mathfrak{P}(m,n,t^{-1})$$

For any polymorphisms $\mathfrak{P} : M \rightsquigarrow N$, $\mathfrak{Q} : N \rightsquigarrow K$, the following property holds

$$(\mathfrak{Q} \diamond \mathfrak{P})^\star = \mathfrak{P}^\star \diamond \mathfrak{Q}^\star.$$

If $g \in \mathrm{Gms}(M)$, then

$$\mathfrak{I}(g)^\star = \mathfrak{I}(g^{-1}).$$

Our next purpose is to extend the operators (1.2) to arbitrary polymorphisms.

2.4. Mellin transform of polymorphisms. Here we present without proof some simple statements from [22]. Notice that below we use Theorem 2.4 and do not refer to the definition of product of polymorphisms.

Fix $\lambda = \frac{1}{p} + is \in \mathbb{C}$ as above (1.1). Let q is defined from $\frac{1}{p} + \frac{1}{q} = 1$. For a polymorphism $\mathfrak{P} : M \rightsquigarrow N$ we consider the bilinear form on $L^p(M, \mu) \times L^q(N, \nu) \to \mathbb{C}$ given by

$$S_\lambda(f, g) = \iiint_{M \times N \times \mathbb{R}^\times} f(m) g(n) t^\lambda \, d\mathfrak{P}(m, n, t).$$

Proposition 2.2 ([22]) a)

$$|S_\lambda(f, g)| \leqslant \|f\|_{L_p} \cdot \|g\|_{L_q}.$$

b) $\mathfrak{P}$ *is uniquely determined by the family of forms* $S_\lambda(\cdot, \cdot)$.

Corollary 2.3 a *There exists a unique linear operator*

$$T_\lambda(\mathfrak{P}) : L^p(N, \nu) \to L^p(M, \mu)$$

such that

$$S(f, g) = \int_M f(m) \cdot T_\lambda(\mathfrak{P}) \cdot g(m) \, d\mu(m).$$

b) $\|T_\lambda(\mathfrak{P})\| \leqslant 1$, *where a norm is the norm of an operator* $L^p(N, \nu) \to L^p(M, \mu)$.

c) *A polymorphism* $\mathfrak{P}$ *is uniquely determined by the operator-valued function* $\lambda \mapsto T_\lambda(\mathfrak{P})$, *and, moreover, by its values on each line* $\frac{1}{p} + is$ *for fixed* p.

For $h \in \mathrm{Gms}(M)$, we have

$$T_\lambda(\iota(h)) = T_\lambda(h),$$

where $T_\lambda(h)$ is defined by (1.2).

Theorem 2.4 T_λ *is a representation of a category, i.e.*

$$T_\lambda(\mathfrak{Q} \diamond \mathfrak{P}) = T_\lambda(\mathfrak{Q}) T_\lambda(\mathfrak{P}). \tag{2.1}$$

2.5. Convergence.

Theorem 2.5 a) *$T_\lambda(\mathfrak{P})$ is weakly continuous, i.e., if $\mathfrak{P}_j$ converges to $\mathfrak{P}$, then*

$$\int_M f(m) \cdot T_\lambda(\mathfrak{P}_j) g(m))\, d\mu(m) \quad \text{converges to} \quad \int_M f(m) T_\lambda(\mathfrak{P}) g(m)\, d\mu(m) \tag{2.2}$$

for any $f \in L^q(M)$, $g \in L^p(N)$.

b) *Conversely, if* (2.2) *holds for each λ in the strip $0 \leqslant \operatorname{Re}\lambda \leqslant 1$, then $\mathfrak{P}_j$ converges to $\mathfrak{P}$. Moreover, it is sufficient to require the convergences on the lines $\operatorname{Re}\lambda = 0$ and $\operatorname{Re}\lambda = 1$.*

3 Abstract statement

3.1. Polymorphisms $\mathfrak{l}_n$. Let (M, μ) be a space with measure. Denote by $\Delta(m, m')$ the measure on $M \times M$ supported by the diagonal of $M \times M$ such that the projection of Δ to the first factor M is μ.

Let $\omega = 0, 1, \ldots, \infty$. Consider the space $\mathbb{R}^\omega \times \mathbb{R}^\infty$ equipped with the measure $\mu_{\omega+\infty} = \mu_\omega \times \mu_\infty$. Let x, x' range in $\mathbb{R}^\omega$, y in $\mathbb{R}^\infty$, t in $\mathbb{R}^\times$. Consider the polymorphism

$$\mathfrak{l}_\omega : (\mathbb{R}^\omega, \mu_\omega) \rightsquigarrow (\mathbb{R}^\omega \times \mathbb{R}^\infty, \mu_\omega \times \mu_\infty)$$

given by

$$\mathfrak{l}_\omega(x'; x, y; t) = \Delta(x, x') \times \mu_\infty(y) \times \delta(t-1),$$

where δ is the delta-function.

The following statement is straightforward.

Lemma 3.1 a) *For a function f on $\mathbb{R}^\omega$ we have*

$$T_\lambda(\mathfrak{l}_\omega) f(x, y) = f(x)$$

b) *For a function $g(x, y)$ on $\mathbb{R}^{\omega+\infty}$, we have*

$$T_\lambda(\mathfrak{l}_\omega^\star) g(x) = \int_{\mathbb{R}^\infty} g(x, y)\, d\mu_\infty(y)$$

c) *$\mathfrak{l}_\omega^\star \diamond \mathfrak{l}_\omega : \mathbb{R}^\omega \rightsquigarrow \mathbb{R}^\omega$ is $\Delta(x, x') \times \delta(t-1)$.*

d) *The polymorphism*

$$\mathfrak{t}_\omega := \mathfrak{l}_\omega \diamond \mathfrak{l}_\omega^\star : \mathbb{R}^{\omega+\infty} \rightsquigarrow \mathbb{R}^{\omega+\infty}$$

equals

$$\Delta(x, x') \times \mu_\infty(y) \times \mu_\infty(y') \times \delta(t-1),$$

where (x, y) is in the first copy of $\mathbb{R}^{\omega+\infty}$ and (x', y') is in the second copy.

e) *The operator corresponding to $\mathfrak{t}_\omega$ is*

$$T_\lambda(\mathfrak{t}_\omega) f(x, y) = \int_{\mathbb{R}^\infty} f(x, z)\, d\mu_\infty(z).$$

In particular, in L^2 this operator is the orthogonal projection to the space of functions independent on y.

f) *Consider a sequence* $h_j = \begin{pmatrix} 1 & 0 \\ 0 & u_j \end{pmatrix} \in \mathrm{O}(\infty)$ *where u_j weakly converges to* 0. *Then $\mathfrak{I}(h_j)$ converges to* $\mathfrak{t}_\omega = \mathfrak{l}_\omega \diamond \mathfrak{l}_\omega^\star$.

An example of a sequence u_j is

$$u_j = \begin{array}{cl} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{l} \}j \\ \}j \\ \}\infty \end{array} \end{array}$$

3.2. Action of colligations. Let $\omega = 0, 1, \dots, \infty$. Let $\mathfrak{a} \in \mathrm{Coll}(\omega)$, let A be its representative in $\mathrm{GLO}(\omega + \infty)$. Consider the polymorphism

$$\tau^{(\omega)}(\mathfrak{a}) : (\mathbb{R}^\omega, \mu_\omega) \rightsquigarrow (\mathbb{R}^\omega, \mu_\omega)$$

given by

$$\tau^{(\omega)}(\mathfrak{a}) = \mathfrak{l}_\omega \mathfrak{I}(A) \mathfrak{l}_\omega^\star.$$

Theorem 3.2 *The map $\tau^{(\omega)} : \mathrm{Coll}(\omega) \to \mathrm{Pol}(\mathbb{R}^\omega, \mathbb{R}^\omega)$ is a homorphism of semigroups.*

Theorem 3.3 *For $\omega = \infty$ the image $\tau^{(\infty)}(\mathrm{Coll}(\infty)) \subset \mathrm{Pol}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ is contained in the closure of $\mathfrak{I}\big(\mathrm{GLO}(\infty)\big)$.*

3.3. Proof of Theorem 3.2. We must verify the identity

$$T_\lambda(\mathfrak{a}_1) T_\lambda(\mathfrak{a}_2) = T_\lambda(\mathfrak{a}_1 \circ \mathfrak{a}_2). \tag{3.1}$$

or, equivalently,

$$T_\lambda(\mathfrak{t}_\omega A_1 \mathfrak{t}_\omega) T_\lambda(\mathfrak{t}_\omega A_2 \mathfrak{t}_\omega) = T_\lambda^{(\omega)}(\mathfrak{t}_\omega A_1 A_2 \mathfrak{t}_\omega).$$

Let ρ be a unitary representation of $\mathrm{GLO}(\omega + \infty) \simeq \mathrm{GLO}(\infty)$ continuous with respect to the Shale topology. Denote by $H(\omega)$ the space of $\mathrm{O}(\infty)$-invariant vectors. Denote by $P(\omega)$ the orthogonal projection on $H(\omega)$. For $A \in \mathrm{GLO}(\omega + \infty)$, we define the operator

$$\rho^{(\omega)}(\mathfrak{a}) := P(\omega) \rho(A) : \; H(\omega) \to H(\omega). \tag{3.2}$$

It can be easily checked that $\rho^{(\mathfrak{a})}(g)$ depends on a operator colligation $\mathfrak{a}$ and not on A itself.

Theorem 3.4 *We get a representation of the semigroup $\mathrm{Coll}(\omega)$ in the space $H(\omega)$.*

$$\rho^{(\omega)}(\mathfrak{a}_1) \rho^{(\omega)}(\mathfrak{a}_2) = \rho^{(\omega)}(\mathfrak{a}_1 \circ \mathfrak{a}_2). \tag{3.3}$$

See [24], [17], see a simple proof in [23].

We need this theorem for representations $T_{1/2+is}$ of the group $\mathrm{GLO}(\omega+\infty)$ in $L^2(\mathbb{R}^{\omega+\infty}), \mu_{\omega+\infty}$, in this case $P(\omega)$ is $T_{1/2+is}(\mathfrak{t})$,

$$T_{1/2+is}(\mathfrak{a}) = T_{1/2+is}(\mathfrak{t})T_{1/2+is}(A)T_{1/2+is}(\mathfrak{t}),$$

the identity 3.3 can be written as

$$T^{(\omega)}_{1/2+is}(\mathfrak{a}_1)T^{(\omega)}_{1/2+is}(\mathfrak{a}_2) = T^{(\omega)}_{1/2+is}(\mathfrak{a}_1 \circ \mathfrak{a}_2) \tag{3.4}$$

Since T_λ depends holomorphically in λ, we get (3.1).

Remark. Identity 3.4 can be verified by a long straightforward calculation (and in fact this was done in [24]).

3.4. Proof of Theorem 3.3. Let $\mathfrak{a} \in \mathrm{Coll}(\infty)$, let $A \in \mathrm{GLO}(\infty+\infty)$ be its representative. We define the polymorphism

$$\sigma(\mathfrak{a}) : (\mathbb{R}^{\infty+\infty}, \mu_{\infty+\infty}) \rightsquigarrow (\mathbb{R}^{\infty+\infty}, \mu_{\infty+\infty})$$

by

$$\sigma(\mathfrak{a}) = \mathfrak{t}_\infty \diamond \tau(A) \diamond \mathfrak{t}^\star_\infty.$$

By Lemma 3.1.f, the element $\mathfrak{t}_\infty$ is contained in the closure of $\mathrm{O}(\infty)$. By separate continuity of the product, $\mathfrak{t}_\infty \diamond \tau(A) \diamond \mathfrak{t}^\star_\infty$ is contained in the closure of $\mathrm{GLO}(\infty+\infty)$

Next, represent the set of natural numbers $\mathbb{N}$ as a union of two disjoint sets I, J. Consider the monotonic bijections $I \to \mathbb{N}$, $J \to \mathbb{N}$. In this way we identify $\mathbb{R}^\infty$ and $\mathbb{R}^{\infty+\infty}$. Denote by $\sigma(\mathfrak{a}; I) : \mathbb{R}^\infty \rightsquigarrow \mathbb{R}^\infty$ the image of the polymorphism $\sigma(\mathfrak{a})$ under this identification. By construction $\sigma(\mathfrak{a}, I)$ is contained in the closure of $\mathrm{GLO}(\infty)$.

Now take

$$I_k = \{1,\, 2,\, 3,\, \dots,\, k,\, k+2,\, k+4,\, k+6, \dots\},$$

Then $\sigma(\mathfrak{a}, I_k)$ converges to $\tau(\mathfrak{a})$. □

3.5. Injectivity. We formulate without proof the following statement.

Theorem 3.5 *The maps* $\mathrm{Coll}(\omega) \to \mathrm{Pol}(\mathbb{R}^\omega, \mathbb{R}^\omega)$ *are injective.*

This is equivalent to the statement: the family of representations $\mathfrak{a} \mapsto P(\omega)T_\lambda(\mathfrak{a})P(\omega)$ separates points of $\mathrm{Coll}(\omega)$.

4 Canonical forms

4.1. Canonical forms. Let $n < \infty$, $\mathfrak{g} \in \mathrm{Coll}(n)$. Let $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ be a representative of $\mathfrak{g}$.

Lemma 4.1 *Assume that rank of g_{12} is maximal. Then $\mathfrak{g}$ has a representative of the form*

$$G = \underbrace{\begin{pmatrix} a & b \\ c & d \\ 0 & H \end{pmatrix}}_{n \;\; n+\infty} \begin{matrix} \}n \\ \}n \\ \}\infty \end{matrix} \quad = \quad \underbrace{\begin{pmatrix} a & b_1 & b_2 \\ c & d_1 & d_2 \\ 0 & 0 & h \end{pmatrix}}_{n \;\; n \;\; \infty} \begin{matrix} \}n \\ \}n \\ \}\infty \end{matrix} \tag{4.1}$$

where h is a diagonal matrxix with positive entries h_j, $\sum (h_j - 1)^2 < \infty$.

Lemma 4.2 *Any $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{O}(n+\infty)$ admits a representation in the form*

$$g = (1 + S) \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix},$$

where S is a Hilbert–Schmidt matrix and $u \in \mathrm{O}(\infty)$.

Proof of Lemma 4.2. The matrix $\delta^t \delta - 1$ is Hilbert–Schmidt and δ is Fredholm of index 0, therefore δ can be represented as

$$\delta = vHu,$$

where u, $v \in \mathrm{O}(\infty)$, and H is a diagonal matrix, the matrix $H - 1$ is Hilbert–Schmidt. Therefore g has the form

$$g = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} \alpha & \beta' \\ \gamma' & H \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$$

The middle factor is (1+ Hilbert–Schmidt matrix). Finally, we get a desired representation

$$g = \left[\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} \alpha & \beta' \\ \gamma' & H \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}^{-1} \right] \cdot \left[\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \right]$$

Proof of Lemma 4.1. By Lemma 4.2, we can assume that $G - 1$ is a Hilbert–Schmidt matrix. Since $\operatorname{rk} g_{12} = n$, a left multiplication by an orthogonal matrix w can reduce g_{12} to the form $\begin{pmatrix} c \\ 0 \end{pmatrix}$.

Thus we get a matrix $R' = \begin{pmatrix} a & b \\ c & d \\ 0 & H \end{pmatrix}$ such that $R' - 1$ is Hilbert–Schmidt. We transform R' by

$$\begin{pmatrix} a & b \\ c & d \\ 0 & H \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ 0 & H \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & v_{11} & v_{12} \\ 0 & v_{21} & v_{22} \end{pmatrix},$$

where u and $\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$ are orthogonal matrices. Consider $(n+\infty)\times\infty$ matrix $J = \begin{pmatrix} 0 & 1 \end{pmatrix}$. Then $H - J$ is a Hilbert–Shmidt operator, therefore the Fredholm index of H equals n. Since G is invertible, $\ker H = 0$, Hence $\operatorname{codim} \operatorname{Im} H = n$. Such H can be reduced to the form $\begin{pmatrix} 0 & h \end{pmatrix}$, where h is diagonal. The standard proof of the theorem about singular values (see [28]) can be adapted to this case. □

4.2. Coordinates. Take a colligation reduced to a canonical form (4.1). We pass to *Potapov coordinates* (see [27]) on the space of matrices,

$$\begin{pmatrix} P & Q \\ R & T \end{pmatrix} := \begin{pmatrix} b - ac^{-1}d & -ac^{-1} \\ c^{-1}d & c^{-1} \end{pmatrix}$$

or

$$\begin{pmatrix} P_1 & P_2 & Q \\ R_1 & R_2 & T \end{pmatrix} := \begin{pmatrix} b_1 - ac^{-1}d_1 & b_2 - ac^{-1}d_2 & -ac^{-1} \\ c^{-1}d_1 & c^{-1}d_2 & c^{-1} \end{pmatrix},$$

the size of the block matrices is $(n + \infty + n) \times (n + n)$. Formulas below are written in the terms of P, Q, R, T, and h.

5 Calculations. Finite matrices

5.1. Measures $\Phi[b, M; t]$. Let $M \geqslant 0$, $b \in \mathbb{R}$. We define the measure $\Phi[b, M; t]$ on $\mathbb{R}^\times$ by

— for $b > 0$

$$\Phi[b, M; t] = \begin{cases} \frac{1}{\sqrt{2\pi}} t^{1/b}(-b\ln t)^{-1/2} \cosh\sqrt{-\frac{4M}{b}\ln t}\,\frac{dt}{t} & \text{if } 0 < t < 1; \\ 0 & \text{if } t > 1. \end{cases}$$

— for $b = 0$

$$\Phi[0, M; t] = e^M \delta(t - 1)$$

— for $b < 0$,

$$\Phi[b, M; t] = \begin{cases} 0 & \text{if } 0 < t < 1 \\ \frac{1}{\sqrt{2\pi}} t^{-1/b}(4Mb\ln t)^{-1/2} \cosh\sqrt{\frac{4M}{b}\ln t}\,\frac{dt}{t} & \text{if } t > 1 \end{cases}$$

Lemma 5.1

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^\times} t^\lambda \Phi[b, M; t] = \frac{1}{\sqrt{1 + b\lambda}} \exp\Bigl\{\frac{M}{1 + b\lambda}\Bigr\}.$$

PROOF. To be definite, set $b > 0$. We must evaluate

$$\frac{1}{\sqrt{2\pi}} \int_0^1 t^{\lambda+1/b}(-b\ln t)^{-1/2} \cosh\sqrt{-\frac{4M}{b}\ln t}\,\frac{dt}{t}.$$

We substitute $y = \ln t$ and get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(\lambda+1/b)y} (-by)^{-1/2} \cosh \sqrt{-\frac{4M}{b} y}\, dy.$$

Next, we set $z = -\frac{4M}{b} y$, and come to

$$\frac{1}{\sqrt{2\pi} \cdot \sqrt{4M}} \int_0^{\infty} e^{-\frac{1}{4M}(b\lambda+1)z} z^{-1/2} \cosh \sqrt{z}\, dz = \\ = \frac{1}{\sqrt{2\pi} \cdot \sqrt{M}} \int_0^{\infty} e^{-\frac{1}{4M}(b\lambda+1)u^2} \cosh u\, du.$$

Writing $\cosh u = \frac{1}{2}(e^u + e^{-u})$, we get

$$\frac{1}{\sqrt{2\pi} \cdot 2\sqrt{M}} \int_{-\infty}^{\infty} e^{-\frac{1}{4M}(b\lambda+1)u^2} e^u\, du = \frac{1}{\sqrt{1+b\lambda}} \exp\Bigl\{ \frac{M}{1+b\lambda} \Bigr\}.$$

5.2. Formula. We consider coordinates on $\mathrm{Coll}(n)$ defined above. For x, $u \in \mathbb{R}^n$ we define the following δ-measure $dN_{x,u}(t)$ on $\mathbb{R}^\times$

$$dN_{x,u}(t) = A(x,u)\, \delta\bigl(t - B(x,u)\bigr),$$

where

$$A(x,u) = |\det T| \exp\Bigl\{ -\frac{1}{2} \|xQ + uT\|^2 - \frac{1}{2} \|(xP + uR)H^t (1 - HH^t)^{-1}\|^2 \Bigr\},$$

$$B(x,u) = |\det G| \exp\Bigl\{ \frac{1}{2} \bigl(\|xQ + uT\|^2 - \|x\|^2 + \|u\|^2 - \\ - (xP + uR)(1 - H^t H)^{-1} (xP + uR)^t \bigr) \Bigr\}, \quad (5.1)$$

where $\|\cdot\|$ is the standard norm in $\mathbb{R}^n$.

Denote by h_j the diagonal entries of the matrix h. Denote by $(\psi_1, \psi_2, \dots)$ the coordinates of the vector $xP_2 + uR_2$.

Theorem 5.2 *Let $\mathfrak{g} \in \mathrm{Coll}(n)$ have a representative*

$$G = \begin{array}{cl} \begin{pmatrix} a & b_1 & b_2 & 0 \\ c & d_1 & d_2 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{l} \}n \\ \}n \\ \}m-n \\ \}\infty \end{array} \\ \underbrace{\;\;}_{n}\underbrace{\;\;}_{n}\underbrace{\;\;}_{m-n}\underbrace{\;\;}_{\infty} & \end{array} \quad (5.2)$$

and $h_j \neq 1$. Then the polymorphism $\tau(\mathfrak{a})$ is given by

$$\Bigl(N_{x,u}(t) * \mathop{\Huge *}_{j=1}^{m-n} \Phi\Bigl[h_j^2 - 1, \frac{h_j^2 |\psi_j|^2}{2(1-h_j^2)}; t \Bigr] \Bigr) dx\, du, \quad (5.3)$$

where $*$ denotes the convolution in $\mathbb{R}^\times$ and $\mathop{\text{\Large$ $}}$ is the symbol of multiple convolution with respect to j.*

5.3. Transformation of the determinant. Note that

$$\det G = \det \begin{pmatrix} a & b_1 & b_2 \\ c & d_1 & d_2 \\ 0 & 0 & h \end{pmatrix} =$$
$$= \det \begin{pmatrix} a & b_1 \\ c & d_1 \end{pmatrix} \cdot \det(h) = \pm \det(c) \det(b_1 - ac^{-1} d_1) \det(h).$$

Thus

$$|\det G| = \left| \frac{\det(P_1)\det(H)}{\det(T)} \right|.$$

5.4. Calculation. We wish to write explicitly operators (3.2) for the representations $T_\lambda(G)$.

$$T_\lambda^{(n)}(G) = T_\lambda(\mathfrak{l}) T_\lambda(G) T_\lambda(\mathfrak{l}^\star).$$

Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^{m-n}$, $\xi \in \mathbb{R}^\infty$. The operator $T_\lambda(\mathfrak{l}^\star)$ sends a function $f(x)$ on $\mathbb{R}^n$ to the same function $f(x)$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{m-n} \times \mathbb{R}^\infty$. We apply $T_\lambda(G)$ and come to

$$|\det G|^\lambda f(xa + yc) \exp \left\{ -\frac{\lambda}{2} \begin{pmatrix} x & y & z \end{pmatrix} (GG^t - 1) \begin{pmatrix} x^t \\ y^t \\ z^t \end{pmatrix} \right\}. \tag{5.4}$$

Next, the operator $T_\lambda(\mathfrak{l})$ is the average with respect to variables $(y, z, \xi) \in \mathbb{R}^n \times \mathbb{R}^{m-n} \times \mathbb{R}^\infty$. Since the function (5.4) is independent on ξ, we take average with respect to (y, z). We come to

$$T_\lambda^{(n)}(G) f(x) = |\det G|^\lambda \iint\limits_{\mathbb{R}^n \times \mathbb{R}^{m-n}} f(xa + yc) \times$$
$$\times \exp \left\{ -\frac{\lambda}{2} \begin{pmatrix} x & y & z \end{pmatrix} (GG^t - 1) \begin{pmatrix} x^t \\ y^t \\ z^t \end{pmatrix} \right\} d\mu_n(y)\, d\mu_{m-n}(z) =$$
$$= \frac{|\det(G)|^\lambda}{(2\pi)^{m/2}} \cdot e^{\frac{1}{2}x^2} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^{m-n}} f(xa + yc) \times$$
$$\times \exp \left\{ -\frac{\lambda}{2} \begin{pmatrix} x & y & z \end{pmatrix} GG^t \begin{pmatrix} x^t \\ y^t \\ z^t \end{pmatrix} + \frac{\lambda - 1}{2} \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} x^t \\ y^t \\ z^t \end{pmatrix} \right\} dy\, dz \tag{5.5}$$

We change variable y by u according

$$u = xa + yc, \qquad y = uc^{-1} - xac^{-1}.$$

Then

$$\begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} x & u & z \end{pmatrix} S,$$

where

$$S = \begin{pmatrix} 1 & -ac^{-1} & 0 \\ 0 & c^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Quadratic form in (5.5) transforms to

$$\left\{ -\frac{\lambda}{2} \begin{pmatrix} x & u & z \end{pmatrix} SGG^tS^t \begin{pmatrix} x^t \\ u^t \\ z^t \end{pmatrix} + \frac{\lambda - 1}{2} \begin{pmatrix} x & u & z \end{pmatrix} SS^t \begin{pmatrix} x^t \\ u^t \\ z^t \end{pmatrix} \right\}$$

Passing to Potapov coordinates, we get

$$SS^t = \begin{pmatrix} 1 + QQ^t & QT^t & 0 \\ TQ^t & TT^t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$SG = \begin{pmatrix} 0 & P \\ 1 & R \\ 0 & H \end{pmatrix} \qquad SGG^tS^t = \begin{pmatrix} PP^t & PR^t & PH^t \\ RP^t & 1 + RR^t & RH^t \\ HP^t & HR^t & HH^t \end{pmatrix}$$

We come to the expression of the form

$$T_\lambda^{(n)}(G)\, f(x) = \int_{\mathbb{R}^n} \mathcal{K}(x,u) f(u)\, du,$$

where the kernel $\mathcal{K}$ is given by

$$\mathcal{K}(x,u) = (2\pi)^{-n/2} |\det(G)|^\lambda |\det c|^{-1} \exp\{V(x,u)\} \int_{\mathbb{R}^{m-n}} \exp\{U(x,u,z)\}\, dz,$$

where

$$\begin{aligned} \exp\{V(x,u)\} = \exp\Big\{ \frac{1}{2} xx^t + \frac{\lambda-1}{2} \begin{pmatrix} x & u \end{pmatrix} \begin{pmatrix} QQ^t + 1 & QT^t \\ TQ^t & TT^t \end{pmatrix} \begin{pmatrix} x^t \\ u^t \end{pmatrix} - \\ - \frac{\lambda}{2} \begin{pmatrix} x & u \end{pmatrix} \begin{pmatrix} PP^t & PR^t \\ RP^t & RR^t + 1 \end{pmatrix} \begin{pmatrix} x^t \\ u^t \end{pmatrix} \Big\} = \\ = \exp\Big\{ -\frac{\lambda}{2} \|xP + uR\|^2 + \frac{\lambda - 1}{2} \|xQ + uT\|^2 + \frac{\lambda}{2} (\|x\|^2 - \|u\|^2) \Big\} \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} \int_{\mathbb{R}^{m-n}} \exp\Big\{U(x,u,z)\Big\}\, dz = \\ = (2\pi)^{-(m-n)/2} \int_{\mathbb{R}^{m-n}} \exp\Big\{\frac{1}{2} z(-\lambda HH^t + \lambda - 1) z^t\Big\} \exp\Big\{-\lambda z H(P^t x^t + R^t u^t)\Big\}\, dz = \\ = \det(\lambda HH^t - \lambda + 1)^{-1/2} \times \\ \times \exp\Big\{ \frac{\lambda^2}{2} (xP + uR) H^t (\lambda HH^t - \lambda + 1)^{-1} H (xP + yR)^t \Big\} \end{aligned} \tag{5.7}$$

We wish to examine the exponential factor in (5.7). Recall that H is an $(m \times n)$ matrix of the form

$$H = \begin{pmatrix} 0 & \dots & 0 & h_1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & h_2 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & h_{m-n} \end{pmatrix}$$

Therefore HH^t is the diagonal matrix with entries h_j^2 and $H^t(\lambda HH^t - \lambda + 1)^{-1}H$ is the diagonal matrix with entries 0 (n times) and $\frac{h_j^2}{\lambda h_j^2 - \lambda + 1}$. Therefore, (5.7) equals

$$(2\pi)^{n-m} \prod_{j=1}^{m-n} \big(1 + \lambda(h_j^2 - 1)\big)^{-1/2} \exp\Big\{\frac{\lambda^2 h_j^2 |\psi_j|^2}{2(\lambda h_j^2 - \lambda + 1)}\Big\} \tag{5.8}$$

Next, we write

$$\frac{\lambda^2 h_j^2}{\lambda h_j^2 - \lambda + 1} = \frac{\lambda h_j^2}{h_j^2 - 1} - \frac{h_j^2}{(h_j^2-1)^2} + \frac{h_j^2}{(h_j^2-1)^2} \cdot \frac{1}{\lambda h_j^2 - \lambda + 1} \tag{5.9}$$

and represent the product (5.8) as

$$\begin{aligned} &\exp\Big\{-\frac{1}{2}(xP + uR)H^t(1 - HH^t)^{-2}H(xP+uR)^t\Big\}\times \\ &\qquad \times \exp\Big\{-\frac{\lambda}{2}(xP+uR)H^t(1-HH^t)^{-1}H(xP+uR)^t\Big\}\times \\ &\qquad \times \prod_{j=1}^{m-n} (\lambda(h_j^2-1)+1)^{-1/2} \exp\Big\{\frac{h_j^2\|\psi_j\|^2}{2(h_j^2-1)^2} \cdot \frac{1}{\lambda(h_j^2-1)+1}\Big\} \end{aligned} \tag{5.10}$$

Uniting (5.6) and (5.10), we come to a final expression for the kernel of integral operator

$$\mathcal{K}_\lambda(x,u) =$$

$$=|\det c|^{-1} \exp\Big\{-\frac{1}{2}\|xQ + uT\|^2 - \frac{1}{2}\|(xP+uR)H^t(1-HH^t)^{-1}\|^2\Big\}\times \tag{5.11}$$

$$\times |\det(G)|^\lambda \cdot \exp\Big\{\frac{\lambda}{2}\big(\|xQ+uT\|^2 + \|x\|^2 - \|u\|^2 - \tag{5.12}$$

$$- (xP+uR)(1-H^tH)^{-1}(xP+yR)^t\big)\Big\}\times \tag{5.13}$$

$$\times \prod_{j=1}^{m-n} (\lambda(h_j^2-1)+1)^{-1/2} \exp\Big\{\frac{h_j^2\|\psi_j\|^2}{2(h_j^2-1)^2} \cdot \frac{1}{\lambda(h_j^2-1)+1}\Big\}. \tag{5.14}$$

Now we must represent the kernel as a Mellin transform of a measure

$$\mathcal{K}_\lambda(x,u) = \int_0^\infty t^\lambda dM_{x,u}(t).$$

The expression for $\mathcal{K}_\lambda(x,u)$ is a product, therefore its Mellin transform is a convolution. We must evaluate inverse Mellin transform for all factors. The first factor (5.11) is constant. The second factor (5.12)–(5.13) has the form $e^{\lambda a(x,u)}$, we have

$$e^{\lambda a(x,u)} = \int_0^\infty t^\lambda \delta\bigl(t - e^{a(x,u)}\bigr).$$

For factors in (5.14) the inverse Mellin transform was evaluated in Lemma 5.1.

This proves Theorem 5.2.

6 Convergent formula

6.1. Formula. Now consider arbitrary $\mathfrak{g} \in \mathrm{Coll}(n)$ being in the canonical form (4.1),

$$\begin{pmatrix} a & b_1 & b_2 \\ c & d_1 & d_2 \\ 0 & 0 & h \end{pmatrix}$$

To write a formula that is valid in general case, we rearrange factors in (5.3). First, we define δ-measures on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$dN^\circ_{x,u}(t) = A^\circ(x,u)\delta\bigl(t - B^\circ(x,u)\bigr),$$

where

$$A^\circ(x,u) = \det(T)\, \exp\Bigl\{-\frac{1}{2}\|xQ + uT\|^2\Bigr\}$$

$$B^\circ(x,u) = \frac{|\det P_1|}{|\det T|} \exp\Bigl\{\frac{1}{2}\bigl(\|xQ + uT\|^2 - \|xP_1 + uR_1\|^2 - \|x\|^2 + \|u\|^2\bigr)\Bigr\}.$$

In fact, $dN^\circ_{x,u}(t)$ is the measure $dN_{x,u}(t)$ defined for the matrix $\begin{pmatrix} a & b_1 \\ c & d_1 \end{pmatrix}$.

Next, we define the following probability measures $\Xi_j = \Xi[h_j, \psi_j]$ on $\mathbb{R}^\times$:

$$\Xi[h_j,\psi_j] = \\ = \exp\Bigl\{-\frac{|\psi_j|^2 h_j^2}{2(1-h_j^2)^2}\Bigr\} \cdot \delta\Bigl(t - h_j \exp\Bigl\{\frac{|\psi_j|^2}{2(1-h_j^2)}\Bigr\}\Bigr) * \Phi\Bigl[h_j^2 - 1, \frac{h_j^2|\psi_j|^2}{2(1-h_j^2)^2}; t\Bigr] \tag{6.1}$$

if $h_j \neq 1$. For $h_j = 1$ we set

$$\Xi[1,\psi_j] = \frac{1}{|\psi_j|} e^{-\frac{1}{8}|\psi_j|^2} \exp\Bigl\{-\frac{\ln^2 t}{2|\psi_j|^2}\Bigr\}\frac{dt}{t^{3/2}}, \qquad \Xi[1,0] = \delta(t-1).$$

Theorem 6.1 *Let $\mathfrak{a} \in \mathrm{Coll}(n)$ be arbitrary. Then the polymorphism $\tau(\mathfrak{a})$ is given by*

$$\Bigl(dN^\circ_{x,u}(t) * \mathop{\Huge *}_{j=1}^{\infty} \Xi[h_j,\psi_j]\Bigr) dx\, du. \tag{6.2}$$

Lemma 6.2 a) *Measures* $\Xi[h_j,\psi_j]$ *are probabilistic.*

b) *The products*

$$\mathop{\ast}_{j=1}^{\infty} \Xi[h_j,\psi_j], \qquad\qquad \mathop{\ast}_{j=1}^{\infty} \bigl(t\cdot \Xi[h_j,\psi_j]\bigr) \tag{6.3}$$

weakly converge in the semigroup of measures on $\mathbb{R}^\times$.

Theorem 6.3 a) *For a matrix* g *denote by denote by* $g^{(m)}$ *the matrix* $\begin{pmatrix} z&0\\0&1\end{pmatrix}$, *where* z *is the upper left* $(n+m)\times(n+m)$ *corner of the matrix* g. *Then the the polynorphism* $\tau(\mathfrak{g}^{(m)})$ *coincides with*

$$\Bigl(dN^\circ_{x,u}(t) * \mathop{\ast}_{j=1}^{m-n} \Xi[h_j,\psi_j]\Bigr) dx\,du. \tag{6.4}$$

b) *The sequence of polymorphisns* (6.4) *converges in semigroup of polymorphisms of* $(\mathbb{R}^n,\mu_n)$. *to* $\tau(\mathfrak{a})$.

6.2. Rearrangement of factors (Lemma 6.3.a. First, rearrange factors in (5.11)–(5.14):

$$\mathcal{K}_\lambda(x,u) = |\det T|\exp\Bigl\{-\frac{1}{2}\|xQ+uT\|^2\Bigr\}\Bigl(\frac{|\det(P_1)|}{\det(T)|}\Bigr)^\lambda\times \tag{6.5}$$

$$\times\exp\Bigl\{\frac{\lambda}{2}\bigl(\|xQ+uT\|^2+\|x\|^2-\|u\|^2-\|xP_1+uR_1\|^2\bigr)\Bigr\} \tag{6.6}$$

$$\times\prod_{j=1}^{m-n}\biggl(\exp\Bigl\{\frac{h_j^2|\psi_j|^2}{2(1-h_j^2)^2}\Bigr\}\cdot h_j^\lambda\exp\Bigl\{\frac{\lambda|\psi_j|^2}{2(1-h_j^2)}\Bigr\}\times \tag{6.7}$$

$$\times\bigl(\lambda(h_j^2-1)+1\bigr)^{-1/2}\exp\Bigl\{\frac{h_j^2\|\psi_j\|^2}{2(h_j^2-1)^2}\cdot\frac{1}{\lambda(h_j^2-1)+1}\Bigr\}\biggr) \tag{6.8}$$

Factors in the product (6.5)–(6.6) looks as singular near $h_j=1$. But this singularity is artificial, it appears due division in the line (5.9). Returning to the previous line (5.8) of the calculation, we get for $h_j=1$ the following factor

$$\exp\Bigl\{-\frac{1}{2}\lambda|\psi_j|^2+\frac{1}{2}\lambda^2|\psi_j|^2\Bigr\} = \frac{1}{|\psi_j|}e^{-\frac{1}{8}|\psi_j|^2}\int_0^\infty t^\lambda\exp\Bigl\{-\frac{\ln^2 t}{2|\psi_j|^2}\Bigr\}\frac{dt}{t^{3/2}}$$

6.3. Proof of Lemma 6.3.b).

Lemma 6.4 *The embedding* $\iota:\mathrm{GLO}(\infty)\to\mathrm{Pol}(\mathbb{R}^\infty,\mathbb{R}^\infty)$ *is continuous.*

Proof. According Proposition 2.5.b it is sufficient to prove that the representations $T_\lambda(g)$ of $\mathrm{GLO}(\infty)$ are weakly continuous for all λ. It is sufficient to take $f = e^{iax}$ and $g = e^{ibx}$ in (2.2) and to verify continuity of the corresponding matrix elements with respect to the Shale topology. $\square$

Let g be of the form (4.1). For finite matrices formulas (5.3) and (6.2) coincide. Denote by $g^{(m)}$ the matrix $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$, where z is the upper left $(n+m)\times(n+m)$ corner of the matrix g. For $g^{(m)}$ the formula (6.4) gives a correct result. Next, $g^{(m)}$ converges to g in the Shale topology. Therefore $\tau(g^{(m)})$ converges to $\tau(g)$ as $g \to \infty$. This proves the last statement of the theorem.

6.4. Proof of Theorem 6.1. We must prove convergence of the infinite convolution in (6.3). The characteristic function of $\Xi[h_j, \psi_j]$ is given by

$$\int_0^\infty t^\lambda \Xi_j[h_j, \psi_j] = h_j^\lambda \bigl(1 + \lambda(h_j^2 - 1)\bigr)^{-1/2} \exp\Bigl\{\frac{\lambda^2 h_j^2 |\psi_j|^2}{2(\lambda h_j^2 - \lambda + 1)} - \frac{\lambda}{2}|\psi_j|^2\Bigr\}$$

We have $\sum (h_j - 1)^2 < \infty$, $\sum |\psi_j|^2 < \infty$. Under these conditions we have a convergence of the product in the strip $0 \leqslant \operatorname{Re}\lambda \leqslant 1$. This implies the weak convergence of measures on $\mathbb{R}^\times$.

The convergence is uniform on compacts sets with respect to x, u, and this implies coincidence of (6.2) and limit of (6.4).

References

[1] Borodin, A.; Olshanski, G. *Harmonic analysis on the infinite- dimensional unitary group and determinantal point processes.* Ann. of Math. (2) 161 (2005), 1319–1422.

[2] Brenier, Y., *Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations.* Comm. Pure Appl. Math. 52 (1999), no. 4, 411–452.

[3] Brodskij, M.S. *Unitary operator colligations and their characteristic functions.* Russ. Math. Surv. 33, No.4, 159–191 (1978)

[4] Bogachev, V. I. *Gaussian measures.* American Mathematical Society, Providence, RI, 1998.

[5] Danilenko, A.I.; Ryzhikov, V.V. *Mixing constructions with infinite invariant measure and spectral multiplicities.* Ergodic Theory Dynam. Systems 31 (2011), no. 3, 853???873

[6] De Concini, C.; Procesi, C. *Complete symmetric varieties.* Invariant theory (Montecatini, 1982), 1–44, Lecture Notes in Math., 996 (1983).

[7] Dym, H. *Linear algebra in action.* American Mathematical Society, Providence, RI, 2007

[8] Glasner, E.; Tsirelson, B.; Weiss, B. *The automorphism group of the Gaussian measure cannot act pointwise.* Israel J. Math. 148 (2005), 305–329.

[9] Kerov, S., Olshanski, G., Vershik, A. *Harmonic analysis on the infinite symmetric group.* Invent. Math. 158, No. 3, 551-642 (2004)

[10] Krengel, U., *Ergodic theorems.* Walter de Gruyter, Berlin, 1985. Springer, 1983.

[11] Kuo, Hui Hsiung, *Gaussian measures in Banach spaces.* Lecture Notes in Mathematics, Vol. 463. Springer-Verlag, Berlin-New York, 1975.

[12] Livshits, M.S. *On a certain class of linear operators in Hilbert space.* Mat. Sb., N. Ser. 19(61), 239-262 (1946); English transl. in Amer. Math. Soc. Transl. (Ser. 2), Vol. 13, 61-83 1960

[13] Livshits, M. S. *On spectral decomposition of linear nonself-adjoint operators.* Mat. Sbornik N.S. 34(76), (1954). 145–199. English transl. in Amer. Math. Soc. Transl. (Ser. 2), Vol 5, 1957, 67–114

[14] Martin, N. F. G.; England, J. W.*Mathematical theory of entropy.* Addison-Wesley Publishing Co., Reading, Mass., 1981

[15] Nelson, E. *The free Markoff field.* J. Funct. Anal. 12 (1973), 211–227

[16] Neretin, Yu.A. *Categories of bistochastic measures and representations of some infinite- dimensional groups.* Mat. Sb. 183, No.2, 52-76 (1992); English transl.: Sbornik Math. 75, No.1, 197-219 (1993);

[17] Neretin, Yu. A. *Categories of symmetries and infinite-dimensional groups.* Oxford University Press, New York, 1996; Russian transl.: URSS, 1998.

[18] Neretin, Yu.A. *Fractional diffusions and quasi-invariant actions of infinite-dimensional groups.* Tr. Mat. Inst. Steklova 217, 135-181 (1997); English transl.: Proc. Steklov Inst. Math. 217, 126-173 (1997

[19] Neretin, Yu. A. *Notes on affine isometric actions of discrete groups.* in Heyer, H.; Marion J. (eds.) *Analysis on infinite-dimensional Lie groups and algebras (Marseille, 1997)*, 274–320, World Sci. Publ., River Edge, NJ, 1998

[20] Neretin, Yu. A. *Spreading maps (polymorphisms), symmetries of Poisson processes, and matching summation.* J. Math. Sci. (N. Y.) 126 (2005), no. 2, 1077–1094

[21] Neretin, Yu. A. *Hua-type integrals over unitary groups and over projective limits of unitary groups.* Duke Math. J. 114 (2002), no. 2, 239–266.

[22] Neretin, Yu.A. *On the boundary of the group of transformations leaving a measure quasi-invariant,* Preprint ESI, 2339

[23] Neretin, Yu. A. *Sphericity and multiplication of double cosets for infinite-dimensional classical groups.* Funct.Anal. Appl., 2011, N3.

[24] Olshanski, G.I. *Unitary representations of infinite dimensional pairs (G,K) and the formalism of R. Howe.* in *Representation of Lie groups and related topics,* Adv. Stud. Contemp. Math. 7, 269-463 (1990).

[25] Olshanski, G. I. *On semigroups related to infinite-dimensional groups.* In: *Topics in representation theory* (A. A. Kirillov, ed.). Advances in Soviet Math., vol. 2. Amer. Math. Soc., Providence, R.I., 1991, 67-101.

[26] Pickrell, D. *Measures on infinite-dimensional Grassmann manifolds,* J. Funct. Anal. 70, No. 2 (1987), 323–356

[27] Potapov, V. P. *The multiplicative structure of J-contractive matrix functions.* Trudy Moskov. Mat. Obshchestva. 4 (1955), 125–236; English transl. Amer. Math. Soc. Transl. (2) 15 (1960) 131–243

[28] Reed, M., Simon, B. *Methods of modern mathematical physics. I. Functional analysis.* Academic Press, New York-London, 1972.

[29] Rohlin, V. A. *On the fundamental ideas of measure theory.* Mat. Sbornik N.S. 25(67), (1949). 107–150. Amer. Math. Soc. Translation 1952, (1952). no. 71.

[30] Shale, D. *Linear symmetries of free boson fields.* Trans. Amer. Math. Soc. 103 1962 149–167.

[31] Shilov, G. E.; Fan Dyk Tin *Integral, measure and derivative on linear spaces.* Nauka, Moscow 1966.

[32] Shiryaev, A. N. *Probability.* Second edition. Graduate Texts in Mathematics, 95. Springer-Verlag, New York, 1996.

[33] Vershik, A.M. *Multivalued mappings with invariant measure* (*polymorphisms*) *and Markov operators.* Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova 72, 26-61 (1977). English transl.: J. Sov. Math. 23, 2243-2266 (1983).

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