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DICHOTOMY THEOREMS FOR COUNTABLY INFINITE DIMENSIONAL ANALYTIC HYPERGRAPHS

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Abstract. We give classical proofs, strengthenings, and generalizations of Lecomte's characterizations of the class of analytic ω-dimensional hypergraphs on Hausdorff spaces which have countable Borel chromatic number.

1. Introduction

An ω-*dimensional* (*directed*) *hypergraph* on a set X is a family $G \subseteq {}^{\omega}X$ of non-constant sequences. A (Y-)*coloring* of G is a function $c \colon X \to Y$ which sends sequences in G to non-constant sequences in ${}^{\omega}Y$. More generally, a *homomorphism* from an ω-dimensional hypergraph G on X to an ω-dimensional hypergraph H on Y is a function $\varphi \colon X \to Y$ which sends sequences in G to sequences in H.

In [3], Kechris-Solecki-Todorcevic isolated an acyclic $D_2(\mathbf{\Sigma}^0_1)$ graph on ${}^{\omega}2$ that is minimal among all analytic graphs which do not have Borel ω-colorings. In [4], Lecomte proved that an analogous ω-dimensional hypergraph is minimal among all analytic ω-dimensional hypergraphs which do not have Borel ω-colorings.

Here we provide a classical proof of a slight strengthening of Lecomte's result, which allows us to provide new insight into the curious fact that the notion of minimality appearing in the ω-dimensional case is weaker than that appearing in the Kechris-Solecki-Todorcevic theorem. We also give generalizations of Lecomte's result to κ-Souslin graphs. We work in `ZF` except where stated otherwise.

2. Preliminaries

A topological space is *analytic* if it is the continuous image of a closed subset of ${}^{\omega}\omega$. Given a set $R \subseteq \prod_{i \in I} X_i$, we say that a sequence $(A_i)_{i \in I}$ is R-*discrete* if $A_i \subseteq X_i$ for all $i \in I$ and $\prod_{i \in I} A_i$ is disjoint from R.

Proposition 1. *Suppose that $(X_i)_{i \in I}$ is a countable sequence of Hausdorff spaces, $R \subseteq \prod_{i \in I} X_i$ is analytic, and $(A_i)_{i \in I}$ is an R-discrete sequence of analytic sets. Then there exist a Borel set $S \subseteq \prod_{i \in I} X_i$*

and an S-discrete sequence $(B_i)_{i \in I}$ of Borel sets such that $R \subseteq S$ and $A_i \subseteq B_i$ for all $i \in I$.

Proof. This is a straightforward generalization of the Novikov separation theorem (see, for example, Theorem 28.5 of [2]). ⊠

The *restriction* of G to a set $A \subseteq X$ is given by $G \restriction A = G \cap {}^{\omega}A$. We say that A is *G-discrete* if $G \restriction A = \emptyset$.

Proposition 2. *Suppose that X is a Hausdorff space, G is an analytic ω-dimensional hypergraph on X, and $A \subseteq X$ is a G-discrete analytic set. Then there is a G-discrete Borel set $B \subseteq X$ such that $A \subseteq B$.*

Proof. By Proposition 1, there is a G-discrete sequence $(B_n)_{n \in \omega}$ of Borel subsets of X such that $A \subseteq B_n$ for all $n \in \omega$, and it easily follows that the set $B = \bigcap_{n \in \omega} B_n$ is as desired. ⊠

For each set $I \subseteq {}^{<\omega}\omega$, let G_I denote the ω-dimensional hypergraph on ${}^{\omega}\omega$ given by $G_I = \{(s \frown i \frown x)_{i \in \omega} \mid s \in I \text{ and } x \in {}^{\omega}\omega\}$. We say that I is *dense* if $\forall s \in {}^{<\omega}\omega \exists t \in I \ (s \sqsubseteq t)$.

Proposition 3. *Suppose that $I \subseteq {}^{<\omega}\omega$ is dense and $A \subseteq {}^{\omega}\omega$ is a non-meager set with the Baire property. Then A is not G_I-discrete.*

Proof. Fix $s \in {}^{<\omega}\omega$ such that A is comeager in $\mathcal{N}_s$, fix $t \in I$ such that $s \sqsubseteq t$, and fix $x \in {}^{\omega}\omega$ such that $t \frown i \frown x \in A$ for all $i \in \omega$. As $(t \frown i \frown x)_{i \in \omega} \in G_I$, it follows that A is not G_I-discrete. ⊠

Fix sequences $s_n \in {}^{n}\omega$ such that the set $I = \{s_n \mid n \in \omega\}$ is dense, and put $G_0(\omega) = G_I$.

3. Dichotomy theorems

The primary dichotomy in [4] concerns the existence of continuous homomorphisms from $G_0(\omega) \restriction X_0$ to G, where X_0 denotes the dense G_δ set of sequences $x \in {}^{\omega}\omega$ such that $s_n \frown 0 \sqsubseteq x$ for infinitely many $n \in \omega$. We will establish the analogous result concerning the existence of continuous homomorphisms from $G_0(\omega) \restriction X_z$ to G, where $z \in {}^{\omega}\omega$ is strictly increasing and X_z denotes the dense G_δ set of sequences $x \in {}^{\omega}\omega$ such that $x \restriction n \in {}^{n}z(n)$ for infinitely many $n \in \omega$.

Note that if $z(n) > \max_{k \in m \in n} s_m(k)$ for all $n \in \omega$, then $X_0 \subseteq X_z$, so the inclusion map is a continuous homomorphism from $G_0(\omega) \restriction X_0$ to $G_0(\omega) \restriction X_z$. The following fact therefore yields the original result:

Theorem 4. *Suppose that X is a Hausdorff space and G is an analytic ω-dimensional hypergraph on X. Then for all strictly increasing sequences $z \in {}^{\omega}\omega$, exactly one of the following holds:*

(1) *There is a Borel ω-coloring of G.*
(2) *There is a continuous homomorphism from $G_0(\omega) \upharpoonright X_z$ to G.*

Proof. To see that (1) and (2) are mutually exclusive suppose, towards a contradiction, that $c \colon X \to \omega$ is an ω-universally Baire measurable coloring of G and $\varphi \colon X_z \to X$ is a Baire measurable homomorphism from $G_0(\omega) \upharpoonright X_z$ to G. Then the function $c_0 = c \circ \varphi$ is a Baire measurable coloring of $G_0(\omega) \upharpoonright X_z$. As X_z is comeager, there exists $n \in \omega$ such that $c_0^{-1}(\{n\})$ is non-meager and $G_0(\omega)$-discrete, which contradicts Proposition 3.

It remains to show that at least one of (1) and (2) holds. We can clearly assume that G is non-empty, in which case there are continuous surjections $\varphi_G \colon {}^\omega\omega \to G$ and $\varphi_X \colon {}^\omega\omega \to \mathrm{dom}(G)$, where

$$\mathrm{dom}(G) = \{x \in X \mid \exists y \in G \exists n \in \omega\ (x = y(n))\}.$$

Suppose that $n \in \omega$. A *global* $(n\text{-})$*approximation* is a pair of the form $p = ((u^p_m)_{m \in n+1}, (v^p_m)_{m \in n+1})$, where $u^p_m \colon {}^m z(m) \to {}^m\omega$ and $v^p_m \colon {}^{<m} z(m) \to {}^m\omega$ for all $m \in n+1$, with the property that for all $l \in m \in n+1$, the following conditions are satisfied:

(a) $\forall l \in m \in n+1 \forall s \in {}^l z(l) \forall t \in {}^m z(m)\ (s \sqsubseteq t \Longrightarrow u^p_l(s) \sqsubseteq u^p_m(t))$.
(b) $\forall l \in m \in n+1 \forall s \in {}^{<l} z(l) \forall t \in {}^{<m} z(m)$
$((s \sqsubseteq t \text{ and } m - l = |t| - |s|) \Longrightarrow v^p_l(s) \sqsubseteq v^p_m(t))$.

Fix an enumeration $(p_k)_{k \in \omega}$ of the set of all global approximations.

An *extension* of a global m-approximation p is a global n-approximation q such that $u^p_l = u^q_l$ and $v^p_l = v^q_l$ for all $l \in m+1$. In the special case that $n = m+1$, we say that q is a *one-step extension* of p.

A *local* $(n\text{-})$*approximation* is a pair of the form $l = (f^l, g^l)$, where $f^l \colon {}^n\omega \to {}^\omega\omega$ and $g^l \colon {}^{<n}\omega \to {}^\omega\omega$, with the property that

$$\forall k \in n \forall t \in {}^{n-(k+1)}\omega\ (\varphi_G \circ g^l(t) = (\varphi_X \circ f^l(s_k \frown i \frown t))_{i \in \omega}).$$

We say that l is *compatible* with a global n-approximation p if the following conditions are satisfied:

(i) $\forall m \in n+1 \forall s \in {}^m z(m) \forall t \in {}^n\omega\ (s \sqsubseteq t \Longrightarrow u^p_m(s) \sqsubseteq f^l(t))$.
(ii) $\forall m \in n+1 \forall s \in {}^{<m} z(m) \forall t \in {}^{<n}\omega$
$((s \sqsubseteq t \text{ and } n - m = |t| - |s|) \Longrightarrow v^p_m(s) \sqsubseteq g^l(t))$.

We say that l is *compatible* with a set $Y \subseteq X$ if $\varphi_X \circ f^l[{}^n\omega] \subseteq Y$.

Suppose now that $Y \subseteq X$ is a Borel set, α is a countable ordinal, and $c \colon Y^c \to \omega \cdot \alpha$ is a Borel coloring of $G \upharpoonright Y^c$. Associated with each global n-approximation p is the set $L_n(p, Y)$ of local n-approximations which are compatible with both p and Y, as well as the set

$$A_n(p, Y) = \{\varphi_X \circ f^l(s_n) \mid l \in L_n(p, Y)\}.$$

We say that p is Y*-terminal* if $L_{n+1}(q, Y) = \emptyset$ for all one-step extensions q of p. Let $T_n(Y)$ denote the set of Y-terminal global n-approximations, and put $T(Y) = \bigcup_{n \in \omega} T_n(Y)$.

Lemma 5. *Suppose that $n \in \omega$, p is a global n-approximation, and $A_n(p, Y)$ is not G-discrete. Then p is not Y-terminal.*

Proof of lemma. Fix local approximations $l_i \in L_n(p, Y)$ for $i \in \omega$ with the property that $(\varphi_X \circ f^{l_i}(s_n))_{i \in \omega} \in G$. Then there exists $x \in {}^\omega\omega$ such that $\varphi_G(x) = (\varphi_X \circ f^{l_i}(s_n))_{i \in \omega}$. Let l denote the local $(n+1)$-approximation given by $f^l(s^\frown i) = f^{l_i}(s)$ for $i \in \omega$ and $s \in {}^n\omega$, $g^l(\emptyset) = x$, and $g^l(t^\frown i) = g^{l_i}(t)$ for $i \in \omega$ and $t \in {}^{<n}\omega$. As l is compatible with a one-step extension of p, it follows that p is not Y-terminal. ⊠

Proposition 2 and Lemma 5 ensure that for each Y-terminal global n-approximation p, there is a G-discrete Borel set $B_n(p, Y) \subseteq X$ such that $A_n(p, Y) \subseteq B_n(p, Y)$. Set

$$Y' = Y \setminus \bigcup \{B_n(p, Y) \mid n \in \omega \text{ and } p \in T_n(Y)\},$$

and for each $y \in Y \setminus Y'$, put

$$k(y) = \min\{k \in \omega \mid \exists n \in \omega \ (p_k \in T_n(Y) \text{ and } y \in B_n(p_k, Y))\}.$$

Define $c' \colon (Y')^c \to \omega \cdot (\alpha + 1)$ by

$$c'(y) = \begin{cases} c(y) & \text{if } y \in Y^c \text{ and} \\ \omega \cdot \alpha + k(y) & \text{otherwise.} \end{cases}$$

Lemma 6. *The function c' is a coloring of the hypergraph $G \restriction (Y')^c$.*

Proof of lemma. Suppose, towards a contradiction, that there exist $\beta \in \omega \cdot (\alpha + 1)$ and $(y_i)_{i \in \omega} \in G \restriction (Y')^c$ such that $c'(y_i) = \beta$ for all $i \in \omega$. Then there exists $k \in \omega$ with $\beta = \omega \cdot \alpha + k$, thus p_k is Y-terminal and $(y_i)_{i \in \omega} \in G \restriction B(p_k, Y)$, the desired contradiction. ⊠

Lemma 7. *Suppose that p is a global approximation whose one-step extensions are all Y-terminal. Then p is Y'-terminal.*

Proof of lemma. Fix $n \in \omega$ such that p is a global n-approximation. Suppose, towards a contradiction, that there is a one-step extension q of p for which there exists $l \in L_{n+1}(q, Y')$. Then $\varphi_X \circ f^l(s_{n+1}) \in B_{n+1}(q, Y)$ and $B_{n+1}(q, Y) \cap Y' = \emptyset$, the desired contradiction. ⊠

Recursively define Borel sets $Y_\alpha \subseteq X$ and Borel colorings $c_\alpha \colon Y_\alpha^c \to \omega \cdot \alpha$ of $G \restriction Y_\alpha^c$ for $\alpha \in \omega_1$ by

$$(Y_\alpha, c_\alpha) = \begin{cases} (X, \emptyset) & \text{if } \alpha = 0, \\ (Y'_\beta, c'_\beta) & \text{if } \alpha = \beta + 1, \text{ and} \\ (\bigcap_{\beta \in \alpha} Y_\beta, \lim_{\beta \to \alpha} c_\beta) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

As there are only countably many approximations, there exists $\alpha \in \omega_1$ such that $T(Y_\alpha) = T(Y_{\alpha+1})$.

If the unique global 0-approximation p^0 is Y_α-terminal, then the fact that $A(p^0, Y_\alpha) = \mathrm{dom}(G) \cap Y_\alpha$ ensures that c_α extends to a Borel $(\omega \cdot \alpha + 1)$-coloring of G, thus there is a Borel ω-coloring of G.

Otherwise, by repeatedly applying Lemma 7 we obtain one-step extensions p^{n+1} of p^n for all $n \in \omega$, none of which are Y_α-terminal. For each $k \in \omega$, let $X_{z,k}$ denote the dense G_δ set of sequences $x \in {}^\omega\omega$ with $x \restriction n \in {}^n z(k+n+1)$ for infinitely many $n \in \omega$. Define continuous functions $\psi_X \colon X_z \to {}^\omega\omega$ and $\psi_k \colon X_{z,k} \to {}^\omega\omega$ for $k \in \omega$ by

$$\psi_X(x) = \lim_{n \to \omega} u^{p^n}(x \restriction n) \text{ and } \psi_k(x) = \lim_{n \to \omega} v^{p^{k+n+1}}(x \restriction n),$$

where the limits are taken over all $n \in \omega$ for which the maps are defined.

To see that $\varphi_X \circ \psi_X$ is a homomorphism from $G_0(\omega) \restriction X_z$ to G, it is enough to show that $\varphi_G \circ \psi_k(x) = (\varphi_X \circ \psi_X(s_k \frown i \frown x))_{i \in \omega}$ for all $n \in \omega$ and $x \in X_{z,k}$. By the continuity of φ_G and φ_X, it is enough to show that for every open neighborhood U of $\psi_k(x)$ and every open neighborhood V of $(\psi_X(s_k \frown i \frown x))_{i \in \omega}$, there exists $(y, (y_i)_{i \in \omega}) \in U \times V$ with $\varphi_G(y) = (\varphi_X(y_i))_{i \in \omega}$. Towards this end, fix $m \in \omega$ and an open set $W \subseteq {}^m({}^\omega\omega)$ such that $(\psi_X(s_k \frown i \frown x))_{i \in m} \in W$ and $W \times {}^\omega({}^\omega\omega) \subseteq V$. Then there exists $n \in \omega$ such that $s_k \frown i \frown (x \restriction n) \in {}^{k+n+1}z(k+n+1)$ for all $i \in m$, $\mathcal{N}_{\psi_k(x)} \subseteq U$, and $\prod_{i \in m} \mathcal{N}_{\psi_X(s_k \frown i \frown x) \restriction (k+n+1)} \subseteq W$. Fix a local approximation $l \in L(p^{k+n+1}, Y_\alpha)$. Then the points $y = g^l(x \restriction n)$ and $y_i = f^l(s_n \frown i \frown (x \restriction n))$ for $i \in \omega$ are as desired. ⊠

The following fact implies Lecomte's result that $G_0(\omega) \restriction X_z$ cannot be replaced with $G_0(\omega)$ in the statement of Theorem 4:

Proposition 8. *Suppose that $z \in {}^\omega\omega$ is strictly increasing. Then there is no continuous homomorphism from $G_0(\omega)$ to $G_0(\omega) \restriction X_z$.*

Proof. We will use the following straightforward corollary of the proof of Theorem 3 of [4]:

Lemma 9 (Lecomte)**.** *Suppose that $\varphi \colon {}^\omega\omega \to {}^\omega\omega$ is a continuous homomorphism from $G_0(\omega)$ to $G_0(\omega)$. Then there exist a co-infinite set $I \subseteq \omega$ and $y_0 \in {}^\omega\omega$ such that $\forall y \in {}^\omega\omega$ $(y \restriction I = y_0 \restriction I \implies y \in \varphi[{}^\omega\omega])$.*

Suppose now, towards a contradiction, that $\varphi \colon {}^\omega\omega \to X_z$ is a continuous homomorphism from $G_0(\omega)$ to $G_0(\omega) \upharpoonright X_z$. Fix $I \subseteq \omega$ and $y_0 \in {}^\omega\omega$ as in Lemma 9, let $(i_k)_{k \in \omega}$ denote the strictly increasing enumeration of I^c, and define $y \in {}^\omega\omega$ by

$$y(n) = \begin{cases} y_0(n) & \text{if } n \in I \text{ and} \\ z(i_{k+1}) & \text{if } n = i_k. \end{cases}$$

Then $y \upharpoonright n \notin {}^n z(n)$ for all $n > i_0$, so $y \notin X_z$, a contradiction. ⊠

As originally noted by Lecomte, there is nevertheless a weak version of Theorem 4 in which we can replace $G_0(\omega) \upharpoonright X_z$ with $G_0(\omega)$:

Theorem 10 (Lecomte)**.** *Work in* `ZFC`*. Suppose that X is a Hausdorff space and G is an analytic ω-dimensional hypergraph on X. Then exactly one of the following holds:*

(1) *There is a Borel ω-coloring of G.*
(2) *There is a Baire measurable homomorphism from $G_0(\omega)$ to G.*

Proof. The proof of Theorem 4 shows that (1) and (2) are mutually exclusive. To see that at least one of these holds, fix a strictly increasing sequence $z \in {}^\omega\omega$. By Theorem 4, it is enough to show that there is a Baire measurable homomorphism from $G_0(\omega)$ to $G_0(\omega) \upharpoonright X_z$. As X_z is comeager, every function from ${}^\omega\omega$ to ${}^\omega\omega$ whose support is disjoint from X_z is Baire measurable, so it is enough to show that for all $x \in {}^\omega\omega$, there is a homomorphism from $G_0(\omega) \upharpoonright [x]_{E_0(\omega)}$ to $G_0(\omega) \upharpoonright X_z$. As the sets of the form X_w are $E_0(\omega)$-invariant and together cover ${}^\omega\omega$, this follows from Theorem 4. ⊠

Theorem 4, Proposition 8, and Theorem 10 lead to the following:

Question 11 (Lecomte)**.** Can the homomorphism in part (2) of Theorem 10 be taken to be Borel? Equivalently, is there a Borel homomorphism from $G_0(\omega)$ to $G_0(\omega) \upharpoonright X_z$ for every (or some) strictly increasing sequence $z \in {}^\omega\omega$?

In light of Theorem 10, perhaps the most natural attempt at producing a negative answer to Question 11 is to find a finer Polish topology τ on ${}^\omega\omega$, compatible with the underlying Borel structure of ${}^\omega\omega$, with the property that for no τ-comeager set $X \subseteq {}^\omega\omega$ is there a τ-Baire measurable homomorphism from $G_0(\omega)$ to $G_0(\omega) \upharpoonright X$. Similarly, one could look for a σ-finite measure μ on ${}^\omega\omega$ with the property that for no μ-conull set $X \subseteq {}^\omega\omega$ is there a μ-measurable homomorphism from $G_0(\omega)$ to $G_0(\omega) \upharpoonright X$.

Theorem 4 immediately implies that neither strategy can succeed: simply choose $z \in {}^\omega\omega$ such that X_z is τ-comeager or μ-conull, and proceed as in the proof of Theorem 10. In fact, by combining this with a straightforward recursive construction, we obtain the following:

Theorem 12. *Work in* $\mathtt{ZFC} + \mathtt{add}(\mathtt{null}) = \mathfrak{c}$. *Suppose that* X *is a Hausdorff space and* G *is an analytic* ω*-dimensional hypergraph on* X. *Then exactly one of the following holds:*

1. *There is a Borel* ω*-coloring of* G.
2. *There is a homomorphism from* $G_0(\omega)$ *to* G *which is universally measurable and* ω*-universally Baire measurable.*

We close by noting generalizations of Lecomte's results to broader classes of definable sets. Suppose that κ is an infinite aleph. A topological space is κ-*Souslin* if it is the continuous image of a closed subset of ${}^{\omega}\kappa$. By removing our use of Proposition 1 from the proof of Theorem 4 and replacing ω with κ as appropriate, we obtain:

Theorem 13. *Suppose that* κ *is an infinite aleph,* X *is a Hausdorff space, and* G *is a* κ*-Souslin* ω*-dimensional hypergraph on* X. *Then for all strictly increasing* $z \in {}^{\omega}\omega$, *at least one of the following holds:*

1. *There is a* κ*-coloring of* G.
2. *There is a continuous homomorphism from* $G_0(\omega) \restriction X_z$ *to* G.

By employing techniques of Kanovei [1], we can do even better:

Theorem 14. *Suppose that* κ *is an infinite aleph,* X *is a Hausdorff space, and* G *is a* κ*-Souslin* ω*-dimensional hypergraph on* X. *Then for all strictly increasing* $z \in {}^{\omega}\omega$, *at least one of the following holds:*

1. *There is a* κ^+*-Borel* κ*-coloring of* G.
2. *There is a continuous homomorphism from* $G_0(\omega) \restriction X_z$ *to* G.

Question 15. Is there a classical proof of Theorem 14?

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