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A PALEY–WIENER THEOREM FOR PERIODIC SCATTERING WITH APPLICATIONS TO THE KORTEWEG–DE VRIES EQUATION

IRYNA EGOROVA AND GERALD TESCHL

Abstract. Consider a one-dimensional Schrödinger operator which is a short-range perturbation of a quasi-periodic, finite-gap operator. We give necessary and sufficient conditions on the left, right reflection coefficient such that the difference of the potentials has finite support to the left, right, respectively. Moreover, we apply these results to show a unique continuation type result for solutions of the Korteweg–de Vries equation in this context. By virtue of the Miura transform an analogous result for the modified Korteweg–de Vries equation is also obtained.

1. Introduction

Since the seminal work of Gardner et al. [12] in 1967 the inverse scattering transform is one of the main tools for solving the Korteweg–de Vries (KdV) equation

$$q_t(x,t) = -q_{xxx}(x,t) + 6q(x,t)q_x(x,t). \tag{1.1}$$

Since it very much resemblances the use of the classical Fourier transform method to solve linear partial differential equations, the inverse scattering transform is also known as the nonlinear Fourier transform. Moreover, the linear and nonlinear Fourier transform share many other properties one of which, namely the Paley–Wiener theorem, will be the main subject of this paper.

Let $L_q = -\frac{d^2}{dx^2} + q(x)$ be the one-dimensional Schrödinger operator. Assume that $q(x)$ decays sufficiently fast such that one can associate left/right reflection coefficients $R_\pm(\lambda)$ with it. In their seminal paper Deift and Trubowitz [3] observed that if L_q has no eigenvalues, then $q(x)$ has support in $(-\infty, a)$ if $R_+(\lambda)$ has an analytic extension satisfying the growth condition $\sqrt{\lambda}\, R_+(\lambda) = O(\mathrm{e}^{-2a\mathrm{i}\sqrt{\lambda}})$. Combining this result with some Hardy space theory enabled Zhang [24] to prove unique continuation results for the KdV equation. To be able to use the result from Deift and Trubowitz, commutation methods (see [3], [14], [15]) were used to remove all eigenvalues. If one wants to avoid this extra step, this raises the question what is needed in addition to the growth condition on $R_+(\lambda)$ in the case when eigenvalues are present. It seems that Aktosun [1] was the first to realize that there is an extra condition on the residue of $R_+(\lambda)$ at an eigenvalue. However, it seems he did not notice that this condition, together with the growth estimate, is also sufficient. This Paley–Wiener type theorem will be our fist main result, Theorem 4.1.

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In fact, we will establish the result for more general case of potentials which are asymptotically close to a real-valued, quasi-periodic, finite-gap potential. We then apply this to solutions of the KdV equation and prove a unique continuation result (Theorem 5.3) for the KdV equation in this setting. Again we extend the results from [24] to solutions which are not decaying but rather are asymptotically close to some quasi-periodic, finite-gap solution $p(x,t)$. While these results are only special cases of some more general results which can be proven using modern harmonic analysis (see for example [7] and the references therein), we still present them here since the proof is much simpler and does not require advanced harmonic analysis (note that in the discrete case an even simpler argument is possible [17]).

For further results on the Cauchy problem of the KdV equation with initial conditions supported on a half-line see Rybkin [21] (cf. also Tarama [22]) and the references therein.

2. Some general facts on quasi-periodic, finite-gap potentials

In this section we briefly recall some basic facts on finite gap potentials needed later one. For further information we refer to, for example, [13], [16], [18], or [20].

Let L_p be a one-dimensional Schrödinger operator with a finite gap potential $p(x)$ associated with the hyperelliptic Riemann surface of the square root $Y(\lambda)^{1/2}$, where

$$Y(\lambda) = -\prod_{j=0}^{2r}(\lambda - E_j), \quad E_0 < E_1 < \cdots < E_{2r}.$$

The spectrum of L_p consists of $r+1$ bands:

$$\sigma = \sigma(L_p) = [E_0, E_1] \cup \cdots \cup [E_{2j-2}, E_{2j-1}] \cup \cdots \cup [E_{2r}, \infty)$$

and the potential $p(x)$ is uniquely determined by its associated Dirichlet divisor

$$\{(\mu_1, \sigma_1), \dots, (\mu_r, \sigma_r)\},$$

where $\mu_j \in [E_{2j-1}, E_{2j}]$ and $\sigma_j \in \{+1, -1\}$.

We denote by $\psi_\pm(\lambda, x)$ the corresponding Weyl solutions of $L_p \psi_\pm = \lambda \psi_\pm$, normalized according to $\psi_\pm(\lambda, 0) = 1$ and satisfying $\psi_\pm(\lambda, .) \in L^2((0, \pm\infty))$ for $\lambda \in \mathbb{C} \setminus \sigma$. These functions are meromorphic for $\lambda \in \mathbb{C}\backslash\sigma$ with continuous limits (away from its singularities described below) on σ from the upper and lower half plane. Unless otherwise stated we will always chose the limit from the upper half plane (the one from the lower half plane producing just the corresponding complex conjugate number).

When there is the need to distinguish between these limits we will cut the complex plane along the spectrum σ and denote the upper and lower sides of the cuts by σ^{u} and σ^{l}. The corresponding points on these cuts will be denoted by λ^{u} and λ^{l}, respectively. Moreover, we will write

$$f(\lambda^{\mathrm{u}}) := \lim_{\varepsilon \downarrow 0} f(\lambda + \mathrm{i}\varepsilon), \qquad f(\lambda^{\mathrm{l}}) := \lim_{\varepsilon \downarrow 0} f(\lambda - \mathrm{i}\varepsilon), \qquad \lambda \in \sigma.$$

Let $m_\pm(\lambda) = \frac{\partial}{\partial x}\psi_\pm(\lambda, 0)$ be the Weyl functions of operator L_p. Due to our normalization, for every Dirichlet eigenvalue μ_j the Weyl functions might have poles. If μ_j is in the interior of its gap, precisely one Weyl function $m_+(\lambda)$ or $m_-(\lambda)$ will have a simple pole. Otherwise, if μ_j sits at an edge, both will have a square root singularity. Hence we divide the set of poles accordingly:

$$M_+ = \{\mu_j \mid \mu_j \in (E_{2j-1}, E_{2j}) \text{ and } m_+(\lambda) \text{ has a simple pole}\},$$
$$M_- = \{\mu_j | 1 \le j \le r\} \backslash M_+.$$

In addition, we set

$$\delta_\pm(z) := \prod_{\mu_j \in M_\pm} (z - \mu_j), \quad \tilde{\psi}_\pm(\lambda, x) := \delta_\pm(\lambda)\psi_\pm(\lambda, x) \tag{2.1}$$

such that $\tilde{\psi}_\pm$ are analytic for $\lambda \in \mathbb{C}\backslash\sigma$. Note that we have chosen M_- such that

$$\delta_-(\lambda)\delta_+(\lambda) = \prod_{j=1}^{r} (\lambda - \mu_j). \tag{2.2}$$

Finally, introduce the function

$$g(\lambda) = -\frac{\prod_{j=1}^{r}(\lambda - \mu_j)}{2Y^{1/2}(\lambda)} = \frac{1}{W(\psi_+(\lambda), \psi_-(\lambda))}, \tag{2.3}$$

where the branch of the square root is chosen such that

$$\frac{1}{\mathrm{i}} g(\lambda^{\mathrm{u}}) = \mathrm{Im}(g(\lambda^{\mathrm{u}})) > 0 \quad \text{for} \quad \lambda \in \sigma,$$

where $W(f, g) = f(x)g'(x) - f'(x)g(x)$ is the usual Wronski determinant.

Recall also the well-known asymptotics

$$g(\lambda) = \frac{\mathrm{i}}{2\sqrt{\lambda}} + O(\lambda^{-1}) \tag{2.4}$$

and

$$\psi_\pm(\lambda, x, t) = \mathrm{e}^{\pm \mathrm{i}\sqrt{\lambda}x} \left(1 + O\Big(\frac{1}{\sqrt{\lambda}}\Big)\right), \tag{2.5}$$

as $\lambda \to \infty$.

3. Scattering theory in a nutshell

In this section we give a brief review of scattering theory with respect to quasi-periodic, finite-gap backgrounds. We refer to [2] for further details and proofs (see also [8], [9], [10], [19]).

Let L_p be a Schrödinger operators with a real-valued, quasi-periodic, finite-gap potentials $p(x)$ as in the previous section. Let $q(x)$ be a real-valued function satisfying

$$\int_{\mathbb{R}} (1 + |x|^2)|q(x) - p(x)| dx < \infty \tag{3.1}$$

and let

$$L_q := -\frac{d^2}{dx^2} + q(x), \quad x \in \mathbb{R},$$

be the "perturbed" operator. The spectrum of L_q consists of a purely absolutely continuous part σ plus a finite number of eigenvalues situated in the gaps,

$$\sigma^d := \{\lambda_1, \dots, \lambda_s\} \subset \mathbb{R} \setminus \sigma.$$

The Jost solutions of the equation

$$\left(-\frac{d^2}{dx^2}+q(x)\right)\phi(x)=\lambda\phi(x),\quad \lambda\in\mathbb{C},$$

that are asymptotically close to the Weyl solutions of the background operators as $x\to\pm\infty$ can be represented with the help of the transformation operators as

$$\phi_\pm(\lambda,x)=\psi_\pm(\lambda,x)\pm\int_x^{\pm\infty}K_\pm(x,y)\psi_\pm(\lambda,y)dy, \tag{3.2}$$

where $K_\pm(x,y)$ are real-valued functions satisfying

$$K_\pm(x,x)=\pm\frac{1}{2}\int_x^{\pm\infty}(q(y)-p(y))dy. \tag{3.3}$$

$$|K_\pm(x,y)|\le C(x_0)\int_{\frac{x+y}{2}}^{\pm\infty}|q(z)-p(z)|dz,\quad \pm y>\pm x>\pm x_0. \tag{3.4}$$

Representation (3.2) shows, that the Jost solutions inherit all singularities of the background Weyl solutions as well as the asymptotics

$$\phi_\pm(\lambda,x,t)=\mathrm{e}^{\pm\mathrm{i}\sqrt{\lambda}x}\left(1+O\Big(\frac{1}{\sqrt{\lambda}}\Big)\right),\quad \lambda\to\infty. \tag{3.5}$$

Hence we set (recall (2.1))

$$\tilde\phi_\pm(\lambda,x)=\delta_\pm(\lambda)\phi_\pm(\lambda,x)$$

such that the functions $\tilde\phi_\pm(\lambda,x)$ have no poles in the interior of the gaps of σ. For every eigenvalue we can then introduce the corresponding norming constants

$$\left(\gamma_k^\pm\right)^{-1}=\int_{\mathbb{R}}\tilde\phi_\pm^2(\lambda_k,x)dx.$$

Since at every eigenvalue the two Jost solutions must be linearly dependent, we have

$$\tilde\phi_+(\lambda_k,x)=c_k\tilde\phi_-(\lambda_k,x). \tag{3.6}$$

Furthermore, introduce the scattering relations

$$T(\lambda)\phi_\pm(\lambda,x)=\overline{\phi_\mp(\lambda,x)}+R_\mp(\lambda)\phi_\mp(\lambda,x),\quad \lambda\in\sigma^{\mathrm{u,l}}, \tag{3.7}$$

where the transmission and reflection coefficients are defined as usual,

$$T(\lambda):=\frac{W(\overline{\phi_\pm(\lambda)},\phi_\pm(\lambda))}{W(\phi_\mp(\lambda),\phi_\pm(\lambda))},\quad R_\pm(\lambda):=-\frac{W(\phi_\mp(\lambda),\overline{\phi_\pm(\lambda)})}{W(\phi_\mp(\lambda),\phi_\pm(\lambda))},\quad \lambda\in\sigma^{\mathrm{u,l}}. \tag{3.8}$$

Since

$$T(\lambda)=\frac{W(\psi_+(\lambda),\psi_-(\lambda))}{W(\phi_+(\lambda),\phi_-(\lambda))}=\frac{1}{g(\lambda)W(\phi_+(\lambda),\phi_-(\lambda))}$$

the transmission coefficient has a meromorphic extension to the set $\mathbb{C}\backslash\sigma$ with simple poles at the eigenvalues λ_k and residues given by (cf. [2])

$$\operatorname*{Res}_{\lambda=\lambda_k}T(\lambda)=2Y^{1/2}(\lambda_k)c_k^{\pm1}\gamma_k^\pm. \tag{3.9}$$

It is important to emphasize that the reflection coefficients in general do not have a meromorphic extension.

The sets

$$\mathcal{S}_\pm(q) := \Big\{ R_\pm(\lambda),\ \lambda\in\sigma;\ \lambda_1,\dots,\lambda_s\in\mathbb{R}\setminus\sigma,\ \gamma_1^\pm,\dots,\gamma_s^\pm\in\mathbb{R}_+ \Big\}$$

are called the right, left scattering data, respectively. Given $p(x)$, the potential $q(x)$ can be uniquely recovered from each one of them as follows:

The kernels $K_\pm(x,y)$ of the transformation operators satisfy the Gelfand-Levitan-Marchenko (GLM) equations

$$K_\pm(x,y) + F_\pm(x,y) \pm \int_x^{\pm\infty} K_\pm(x,z)F_\pm(z,y)dz = 0, \quad \pm y > \pm x, \tag{3.10}$$

where [1]

$$\begin{aligned} F_\pm(x,y) = {}& \frac{1}{2\pi\mathrm{i}}\oint_\sigma R_\pm(\lambda)\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)g(\lambda)d\lambda \\ &+ \sum_{k=1}^s \gamma_k^\pm \tilde{\psi}_\pm(\lambda_k,x)\tilde{\psi}_\pm(\lambda_k,y). \end{aligned} \tag{3.11}$$

Conversely, given $\mathcal{S}_\pm(q)$ we can compute $F_\pm(x,y)$ and solve (3.10) for $K_\pm(x,y)$. The potential $q(x)$ can then be recovered from (3.3).

4. Perturbations with finite support on one side and the nonlinear Paley–Wiener theorem

In this section we want to look at the special case where $q(x)$ will be equal to $p(x)$ for $x\le a$ or $x\ge b$. Our main result in this section is the following theorem:

Theorem 4.1 (Nonlinear Paley–Wiener)**.** *Suppose $q(x)$ satisfies* (3.1)*. Then we have $q(x)=p(x)$ for $x\le a$ if and only if $\delta_+(\lambda)R_-(\lambda)$ has an analytic extension to $\mathbb{C}\backslash(\sigma\cup\sigma^d)$ such that*

$$\operatorname*{Res}_{\lambda=\lambda_k} \frac{g(\lambda)}{\delta_-(\lambda)^2}R_-(\lambda) = \gamma_k^-, \tag{4.1}$$

$$\sqrt{\lambda}R_-(\lambda) = O(\mathrm{e}^{2a\mathrm{i}\sqrt{\lambda}}) \quad \text{as } \lambda\to\infty. \tag{4.2}$$

Similarly, we have $q(x)=p(x)$ for $x\ge b$ if and only if $\delta_-(\lambda)R_+(\lambda)$ has an analytic extension to $\mathbb{C}\backslash(\sigma\cup\sigma^d)$ such that

$$\operatorname*{Res}_{\lambda=\lambda_k} \frac{g(\lambda)}{\delta_+(\lambda)^2}R_+(\lambda) = \gamma_k^+, \tag{4.3}$$

$$\sqrt{\lambda}R_+(\lambda) = O(\mathrm{e}^{-2\mathrm{i}b\sqrt{\lambda}}) \quad \text{as } \lambda\to\infty. \tag{4.4}$$

Proof. Suppose first that $q(x)=p(x)$ for $x\le a$. Then we have

$$\phi_+(\lambda,x) = \alpha(\lambda)\psi_+(\lambda,x) + \beta(\lambda)\psi_-(\lambda,x), \qquad x\le a,$$

and thus

$$\begin{aligned} \alpha(\lambda) &= \frac{W(\psi_-(\lambda),\phi_+(\lambda))}{W(\psi_-(\lambda),\psi_+(\lambda))} = -g(\lambda)W(\psi_-(\lambda),\phi_+(\lambda)), \\ \beta(\lambda) &= -\frac{W(\psi_+(\lambda),\phi_+(\lambda))}{W(\psi_-(\lambda),\psi_+(\lambda))} = g(\lambda)W(\psi_+(\lambda),\phi_+(\lambda)), \end{aligned}$$

[1] Here we have used the notation $\oint_\sigma f(\lambda)d\lambda := \int_{\sigma^{\mathrm{u}}} f(\lambda)d\lambda - \int_{\sigma^{\mathrm{l}}} f(\lambda)d\lambda$.

where the Wronskians can be evaluated at any $x \le a$. In particular, $\alpha(\lambda)$ is analytic in $\mathbb{C}\backslash\sigma$ and $\beta(\lambda)$ is meromorphic in $\mathbb{C}\backslash\sigma$ with the only simple poles at $\lambda \in M_+$. Note also that $\beta(\lambda)$ has simple zeros at $\lambda \in M_-$and thus

$$\tilde{\beta}(\lambda) = \frac{\delta_+(\lambda)}{\delta_-(\lambda)}\beta(\lambda) \tag{4.5}$$

is analytic in $\mathbb{C}\backslash\sigma$. Hence, since $\alpha(\lambda)$ vanishes at each eigenvalue λ_k, evaluating

$$\tilde{\phi}_+(\lambda, x) = \alpha(\lambda)\tilde{\psi}_+(\lambda, x) + \tilde{\beta}(\lambda)\tilde{\psi}_-(\lambda, x), \qquad x \le a,$$

at λ_k shows $\tilde{\beta}(\lambda_k) = c_k$ and formula (4.1) follows from (2.2), (2.3), (3.9), (4.5) and

$$R_-(\lambda) = \frac{\beta(\lambda)}{\alpha(\lambda)},$$

respectively,

$$g(\lambda)R_-(\lambda)\delta_-^{-2}(\lambda) = \tilde{\beta}(\lambda)T(\lambda)(2Y^{1/2}(\lambda))^{-1}$$

The asymptotic behavior (4.2) follows using the well-known asymptotical formula $\alpha(\lambda) = T(\lambda)^{-1} = 1 + o(1)$, (2.4), (2.5), and (3.5). This finishes the first part.

To see the converse, note that the growth estimate implies that we can evaluate the integral in (3.11) by the residue theorem by using a large circular arc of radius r whose contribution will vanish as $r \to \infty$ by the Jordan Lemma. Hence the integral in (3.11) is just the sum over the residues which are precisely at the eigenvalues λ_k and by our conditions (4.1) on the poles of integrand it will cancel with the other sum in (3.11). Thus $F(x, y) = 0$ for $y < x < a$ which by the GLM equation implies $K_-(x, y) = 0$ for $y < x < a$ which finally implies $p(x) - q(x) = 0$ for $y < x < a$. □

As a consequence note that the scattering data $\mathcal{S}_\pm(q)$ are determined by $R_\pm(\lambda)$ alone in such a situation since the eigenvalues and norming constants can be read off from the poles of $R_\pm(\lambda)$. In particular, combining this result with the results from [2] we can give the following characterization of scattering data which give rise to potential supported on a half line.

Let

$$\omega_{zz^*} = \left(\frac{Y^{1/2}(z)}{\lambda - z} + P_{zz^*}(\lambda)\right)\frac{d\lambda}{Y^{1/2}(\lambda)} \tag{4.6}$$

be the normalized Abelian differential of the third kind with poles at z and z^* on the Riemann surface associated with the function $Y^{1/2}(\lambda)$. Here $P_{zz^*}(z)$ is a polynomial of degree $g - 1$ which has to be chosen such that ω_{zz^*} has vanishing a-periods (the a-cycles are chosen to surround the gaps of the spectra, changing sheets twice). Furthermore, let

$$B(\lambda, z) = \exp\left(\int_{E_0}^{\lambda} \omega_{zz^*}\right) \tag{4.7}$$

be the Blaschke factor on this surface (see e.g. [19] or [23] for more details). Then, as a corollary of Theorem 4.1 and Theorem 4.3 of [2] we obtain

Theorem 4.2. *(Characterization) Suppose* $q(x)$ *satisfies* (3.1) *and* $q(x) = p(x)$ *for* $x \le a$.

Then a function $R_-(\lambda)$ *is the reflection coefficient for an operator* L_q *with such a potential if and only if the following conditions are fulfilled:*

- *The function* $R_-(\lambda)$ *is continuous on the set* $\sigma^{\mathrm{u}} \cup \sigma^{\mathrm{l}}$ *and possess the symmetry property* $R_-(\lambda^{\mathrm{u}}) = \overline{R_-(\lambda^{\mathrm{l}})}$. *Moreover,* $|R_-(\lambda)| < 1$ *for* $\lambda \notin \partial\sigma$ *and*
  $|R_-(\lambda)| \le 1 - C|\lambda - E|$ *in a small vicinity of each point* $E \in \partial\sigma$.
  If $|R_-(E)| = 1$, *then*
  $$R_-(E) = \begin{cases} -1 & \text{for } E \notin M_-, \\ 1 & \text{for } E \in M_-. \end{cases}$$
- *The function* $R_-(\lambda)\delta_+(\lambda)$ *admits an analytic continuation to* $\mathbb{C} \setminus \{\sigma \cup \sigma_d\}$, *where* $\sigma_d = \{\lambda_1, ..., \lambda_s\} \subset \mathbb{R} \setminus \sigma$ *is a finite number of real points. Moreover, the function* $g(\lambda)\delta_-(\lambda)^{-2}R_-(\lambda)$ *has simple poles at the points* λ_k *with*
  $$\operatorname*{Res}_{\lambda=\lambda_k} \frac{g(\lambda)}{\delta_-(\lambda)^2} R_-(\lambda) > 0.$$
- *For all large* $\lambda \in \mathbb{C}$
  $$\sqrt{\lambda} R_-(\lambda) = O(\mathrm{e}^{2a\mathrm{i}\sqrt{\lambda}}).$$
- *The function* $Y^{1/2}(\lambda)T^{-1}(\lambda)$, *where*
  $$T(\lambda) = \prod_{k=1}^{s} B^{-1}(\lambda, \lambda_k) \exp\left(\frac{1}{2\pi\mathrm{i}} \oint_\sigma \log(1 - |R_-|^2) w_{\lambda\lambda^*} \right),$$
  is continuous up to the boundary $\sigma^{\mathrm{u}} \cup \sigma^{\mathrm{l}}$.
- *The function*
  $$F_{+,c}(x, y) = \oint_\sigma \overline{R_-(\lambda)T^{-1}(\lambda)}\, T(\lambda)\psi_+(\lambda, x)\psi_+(\lambda, y) g(\lambda) d\lambda$$
  satisfies the estimates
  $$|F_{+,c}(x, y)| + \left| \frac{\partial}{\partial x} F_{+,c}(x, y) \right| \le Q\,(x + y)\,,$$
  $$\int_0^\infty \left| \frac{d}{dx} F_{+,c}(x, x) \right| (1 + x^2)\, dx < \infty,$$
  where $Q(x)$ *is a continuous, positive, decaying as* $x \to +\infty$, *function with* $xQ(x) \in L^1(0, \infty)$.

Note that given a reflection coefficient $R_-(\lambda)$ as in the previous theorem, we could form a set of scattering data by choosing arbitrary eigenvalues plus corresponding norming constants. Then, as long as we take the known algebraic constraints (see [23]) into account, we still get a potential $q(x)$ satisfying (3.1) by inverse scattering. However, unless (4.2) holds, this potential will not satisfy $q(x) = p(x)$.

5. Applications to KdV

Finally, we want to show how our main result can be used to prove unique continuation results for the KdV and MKdV equations.

Let $p(x,t)$ be a real-valued, quasi-periodic, finite-gap solution of the KdV equation (1.1) and suppose $q(x,t)$ is a (classical) solution of (1.1), satisfying

$$\int_{\mathbb{R}} \big(|q(x,t) - p(x,t)| + |q_t(x,t) - p_t(x,t)|\big)(1 + |x|^2) dx < \infty \tag{5.1}$$

for all $t \in \mathbb{R}$. For the existence of such solutions we refer to [6], [4] (see also [11]). Then all considerations from the previous section apply to the operator $L_q(t)$ if we consider t as an additional parameter. Moreover, the time evolution of the scattering data can be computed explicitly and is given in the next lemma:

Lemma 5.1 ([6]). *Let $q(x,t)$ be a solution of the KdV equation satisfying* (5.1). *Then $\lambda_k(t) = \lambda_k(0) \equiv \lambda_k$;*

$$R_\pm(\lambda,t) = R_\pm(\lambda,0)\mathrm{e}^{\alpha_\pm(\lambda,t)-\overline{\alpha_\pm(\lambda,t)}}, \quad \lambda \in \sigma, \tag{5.2}$$

$$T(\lambda,t) = T(\lambda,0), \quad \lambda \in \mathbb{C}, \tag{5.3}$$

$$\gamma_k^\pm(t) = \gamma_k^\pm(0)\,\frac{\delta_\pm^2(\lambda_k,0)}{\delta_\pm^2(\lambda_k,t)}\,\mathrm{e}^{2\alpha_\pm(\lambda_k,t)}, \tag{5.4}$$

where $\delta_\pm(\lambda,t)$ is defined as in (2.1) *with $\mu_j^\pm = \mu_j^\pm(t)$,*

$$\alpha_\pm(\lambda,t) := \int_0^t \left(2(p(0,s)+2\lambda)m_\pm(\lambda,s) - \frac{\partial p(0,s)}{\partial x}\right) ds \tag{5.5}$$

and $m_\pm(\lambda,t)$ are the Weyl functions of operator $L_p(t)$.

Our first result reads

Theorem 5.2. *Let $p(x,t)$ be a quasi-periodic, finite-gap solution of the KdV equation and $q(x,t)$ a solution of the KdV equation satisfying* (5.1). *Suppose that $q(x,t) = p(x,t)$ for $x < a$ at two times $t_0 \neq t_1$. Then $q(x,t) = p(x,t)$ for all $(x,t) \in \mathbb{R}^2$.*

Proof. Without loss we can choose $t_0 = 0$. Then $\sqrt{\lambda}R_-(\lambda,0) = O(\mathrm{e}^{2a\mathrm{i}\sqrt{\lambda}})$. If $q(.,0) \neq p(.,0)$ we can choose a maximal and this estimate cannot be improved! Thus $\alpha_-(\lambda,t) = -8\mathrm{i}t\lambda^{3/2}(1+o(1))$ shows that the same estimate cannot hold for another $t \neq 0$ and we are done. □

This is a special case of a much stronger result from [7] which states that if q_1 and q_2 are strong solutions of the KdV equation such that

$$q_1(\cdot,t_0) - q_2(\cdot,t_0),\ q_1(\cdot,t_1) - q_2(\cdot,t_1) \in H^1(\mathbb{R}, \mathrm{e}^{a\max(0,x^{3/2})}dx) \tag{5.6}$$

for any $a > 0$, then $q_1 \equiv q_2$.

With the help of Theorem 5.2 we also obtain the following unique continuation result for our situation:

Theorem 5.3. *Let $p(x,t)$ be a quasi-periodic, finite-gap solution of the KdV equation and $q(x,t)$ a solution of* (1.1) *satisfying* (5.1). *Suppose that $q(x,t) = p(x,t)$ for (x,t) in some open set $U \subset \mathbb{R}^2$. Then $q(x,t) = p(x,t)$ for all $(x,t) \in \mathbb{R}^2$.*

Proof. Let $[a,b] \times [t_0,t_1] \subset U$ and define

$$\tilde{q}(x,t) = \begin{cases} p(x,t), & x \leq a, \\ q(x,t), & x \geq a, \end{cases}$$

for $t \in [t_0,t_1]$. Then Theorem 5.2 implies $\tilde{q}(x,t) = p(x,t)$ for $(x,t) \in \mathbb{R}\times[t_0,t_1]$ and consequently $q(x,t) = p(x,t)$ for $(x,t) \in [a,\infty)\times[t_0,t_1]$. Hence another application of Theorem 5.2 finishes the proof. □

Let $u(x,t)$ be a quasi-periodic, finite-gap solution of the mKdV equation and suppose $v(x,t)$ is a (classical) solution of the mKdV equation

$$v_t(x,t) = -v_{xxx}(x,t) + 6v(x,t)^2 v_x(x,t). \tag{5.7}$$

Then by virtue of the Miura transform (see, e.g., [13], [14]),

$$p(x,t) = u(x,t)^2 + u_x(x,t), \tag{5.8}$$

is a quasi-periodic, finite-gap solution of the KdV equation and

$$q(x,t) = v(x,t)^2 + v_x(x,t) \tag{5.9}$$

is a solution of the KdV equation. We will suppose again that $q(x,t)$ satisfies (5.1) for every t. For the existence of such solutions we refer to [5].

Corollary 5.4. *Let $u(x,t)$ be a quasi-periodic, finite-gap solution of the mKdV equation and $v(x,t)$ a solution of the mKdV equation such that $q(x,t)$ defined by* (5.9) *is a solution of KdV satisfying* (5.1) *with $p(x,t)$ defined by* (5.8)*. Suppose that $v(x,t) = u(x,t)$ for (x,t) in some open set $U \subset \mathbb{R}^2$. Then $v(x,t) = u(x,t)$ for all $(x,t) \in \mathbb{R}^2$.*

Proof. Since $v(x,t) = u(x,t)$ for $(x,t) \in U$ implies $q(x,t) = p(x,t)$ for $(x,t) \in U$, Theorem 5.3 shows $q(x,t) = p(x,t)$ for $(x,t) \in \mathbb{R}^2$. Hence $w_x(x,t) + w(x,t)^2 + 2u(x,t)w(x,t) = 0$, where $w(x,t) = v(x,t) - u(x,t)$ and the standard uniqueness result for ordinary differential equations yields $w(x,t) = 0$ for $(x,t) \in \mathbb{R}\times\{t|(x_0,t) \in U \text{ for some } x_0\}$. Thus uniqueness of solutions of the mKdV equation [5, Thm. 4.1] finally implies $w(x,t) = 0$ for all $(x,t) \in \mathbb{R}^2$. □

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