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Spectral Decomposition of Compactly Supported Poincaré Series and Existence of Cusp Forms

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SPECTRAL DECOMPOSITION OF COMPACTLY SUPPORTED POINCARÉ SERIES AND EXISTENCE OF CUSP FORMS

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Abstract. Let G be a semisimple algebraic group defined over a number field k. We discuss the existence of various types of cusp forms in $\mathcal{A}_{cusp}(G(k)\backslash G(\mathbb{A}))$ using Hecke operators and p–adic representation theory.

Introduction

Existence and construction of cusp forms is a fundamental problem in the modern theory of automorphic forms ([1], [23], [21], [22], [13]). In this paper we address the issue of existence of cusp forms using an extension and refinement of a classical method of (adelic) compactly supported Poincaré series ([15], [25], [24]). Our approach is based on the spectral decomposition of compactly supported Poincaré series. This method was successfully applied in the case of a cocompact discrete subgroup of a semisimple Lie group [19] to give some quantitative information on the decomposition of the corresponding L^2–space. The main theorem of this paper (see Theorem 6-3 (iv)) develops this idea further using the adelic language.

This is not the only application of our main theorem (see Theorem 6-3). The other application that we have in mind is the one with which we start this introduction. To explain it, let us introduce some notation first.

Let G be a semisimple algebraic group defined over a number field k. We write V_f (resp., V_∞) for the set of finite (resp., Archimedean) places. For $v \in V_\infty \cup V_f$, we write k_v for the completion of k at v; if $v \in V_f$, then we let $\mathcal{O}_v$ be the ring of integers of k_v. Let $G_\infty = \prod_{v \in V_\infty} G(k_v)$. This is a semisimple Lie group with finite center; let K_∞ and $\mathfrak{g}_\infty$ be a maximal compact subgroup and the (real) Lie algebra of G_∞, respectively. Let $G(\mathbb{A}_f)$ be the restricted product of all $G(k_v)$, $v \in V_f$. Let $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))$ be the space of K_∞–finite cusp forms for $G(\mathbb{A})$ (see [9], or Section 1). This is a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$–module. In particular, it is a smooth $G(k_v)$–module for $v \in V_f$. This fact enables us to apply the Bernstein's theory and decompose according to the Bernstein classes $\mathfrak{M}_v$ (see Section 4)

$$\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A})) = \oplus_{\mathfrak{M}_v} \mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))(\mathfrak{M}_v).$$

If $\mathfrak{M}_v$ is a Bernstein's class of (M_v, ρ_v), where M_v is a Levi subgroup of $G(k_v)$ and ρ_v is an (irreducible) supercuspidal representation of M_v, then, by definition, $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))(\mathfrak{M}_v)$ is the largest $G(k_v)$–submodule of $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))$ such that its every irreducible subquotient is a subquotient of $\mathrm{Ind}_{P_v}^{G(k_v)}(\chi_v \rho_v)$, for some unramified character χ_v of M_v. Here P_v is an arbitrary parabolic subgroup of $G(k_v)$ containing M_v as a Levi subgroup. Obviously, this is also a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$–module decomposition. Further, we can iterate this for v ranging over a finite set of places, and as a result, we arrive at the question of non–triviality of a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$–module $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))(\mathfrak{M}_v;\ v \in T)$, where $T \subset V_f$ is a finite and non–empty set of places. Our

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main Theorem 6-3 easily implies the affirmative answer to this question. But this is not a quite right question. The right question is the following: Let T be a non–empty finite set of places of k such that G is unramified over k_v for $v \notin V_f - T$. For each $v \in T$, let $\mathfrak{M}_v$ be a Bernstein's class of $G(k_v)$. Is the space of invariants under the open–compact group $\prod_{v\in V_f-T} G(\mathcal{O}_v)$ of $\prod_{v\in V_f-T} G(k_v)$ in $\mathcal{A}_{cusp}(G(k)\setminus G(\mathbb{A}))(\mathfrak{M}_v;\ v\in T)$ non–trivial? Stated differently, it is the question of existence of irreducible cuspidal automorphic representations $\pi = \pi_\infty \otimes_{v\in V_f} \pi_v$ such that π_v belongs to the class $\mathfrak{M}_v$ for $v \in T$, and π_v is unramified for $v \in V_f - T$. The main result of this paper (see Theorem 6-3) shows that this is true in a significant number of (new) cases.

Theorem 0-1. *Assume that one of the following holds:*

- (i) *For at least one element $v \in T$, the Bernstein's class is supercuspidal i.e., $\mathfrak{M}_v$ is a class of $(G(k_v), \rho_v)$, where ρ_v is a supercuspidal representation.*
- (ii) *$G(k)\setminus G(\mathbb{A})$ is compact.*
- (iii) *G has at least two classes of associated standard maximal k–parabolic subgroups, and for each class $\mathfrak{V}$ of associated maximal k–parabolic subgroups G, we may select a place $v_{\mathfrak{V}} \in T$ such that to a different classes correspond different places. Assume that, for each class $\mathfrak{V}$, $\mathfrak{M}_{v_{\mathfrak{V}}}$ is the class of $(M(k_{v_{\mathfrak{V}}}), \rho_{p_{\mathfrak{V}}})$, where the Levi subgroup $M(k_{v_{\mathfrak{V}}})$ belongs to a parabolic subgroup in the class $\mathfrak{V}$.*

Then for a sufficiently small open–compact subgroup $L \subset G(\mathbb{A}_f)$ of the form $L = \prod_{v\in T} L_v \times \prod_{v\in V_f-T} G(\mathcal{O}_v)$, there exist infinitely many K_∞–types δ which depends on L such that a $(\mathfrak{g}_\infty, K_\infty)\times G(\mathbb{A}_f)$–module $\mathcal{A}_{cusp}(G(k)\setminus G(\mathbb{A}))(\mathfrak{M}_v;\ v\in T)$ contains infinitely many irreducible representations of the form $\pi^j_\infty \otimes_{v\in V_f} \pi^j_v$, where π^j_v is unramified for $v \in V_f - T$, π^j_v belongs to the class $\mathfrak{M}_v$ and it contains a non–trivial vector invariant under L_v for $v \in T$, and where the irreducible unitarizable $(\mathfrak{g}_\infty, K_\infty)$–module π^j_∞ contains δ: the set of equivalence $\{\pi^j_\infty\}$ is infinite.

We consider a particular case of Theorem 0-1 as an example. Let $G = SL_{2n+1}$, $n \geq 2$. Then the representatives for the classes $\mathfrak{V}$ are given by the maximal parabolic subgroups associated to the following partitions of $2n+1$:

$$(i, 2n+1-i), \quad 1 \leq i \leq n.$$

We write $P_i = M_i U_i$ for the standard Levi decomposition of a maximal parabolic subgroup associated to i.

Corollary 0-2. *Assume that $v_1, \ldots, v_n \in V_f$ are different places. For each $1 \leq i \leq n$, we choose a supercuspidal representation ρ_i of $M_i(k_{v_i})$. Then there exists infinitely many cuspidal automorphic representations $\pi = \pi_\infty \otimes_{v\in V_f} \pi_v$ of $SL_{2n+1}(\mathbb{A})$ ($n \geq 2$) such that π_{v_i} is a subquotient of $\mathrm{Ind}_{P_i(k_{v_k})}^{G(k_{v_k})}(\chi_{v_k}\rho_{v_k})$, for some unitary unramified character χ_{p_k}, i.e., π_{v_i} is tempered but not in the discrete series, and π_v is unramified for $v \notin \{v_1, \ldots, v_i\}$, $v \in V_f$.*

Proof. Everything follows from Theorem 0-1 except the fact that the unramified character χ_{v_k} must be unitary. But this follows from the fact that π_{v_i} must be unitary. More precisely, since P_i is not associated to itself, the induced representation $\mathrm{Ind}_{P_i(k_{v_k})}^{G(k_{v_k})}(\chi_{v_k}\rho_{v_k})$ is irreducible and never unitary for non–unitary χ_{v_k}. (This is a general well–known fact which follows from ([11], Section 7).) $\square$

Theorem 0-1 is a consequence of the spectral decomposition of adelic compactly supported Poincaré series. We start by introducing this notion. Let $f \in C_c^\infty(G(\mathbb{A}))$. Then the adelic compactly supported Poincaré series $P(f)$ is defined as follows:

$$P(f)(g) = \sum_{\gamma \in G(k)} f(\gamma \cdot g).$$

It is well–known [12] that the right–regular representation of $G(\mathbb{A})$ on $L^2_{cusp}(G(k) \setminus G(\mathbb{A}))$ is decomposed into a countable direct sum of irreducible $G(\mathbb{A})$–invariant subspaces:

$$L^2_{cusp}(G(k) \setminus G(\mathbb{A})) = \oplus_j \mathfrak{H}_j.$$

Then we define *the cuspidal spectral decomposition of* $P(f)$ as follows:

$$\text{the orthogonal projection of } P(f) \text{ to } L^2_{cusp}(G(k) \setminus G(\mathbb{A})) = \sum_j \psi_j, \quad \psi_j \in \mathfrak{H}^j.$$

In order to make this concept useful we employ the following approach. We fix an arbitrary function $\otimes_{v \in V_f} f_v \in C_c^\infty(G(\mathbb{A}_f))$ which does not vanish at 1, and we select an open compact group $L \subset G(\mathbb{A}_f)$ such that this function is right–invariant under L. Then, in section Section 3 we study possible K_∞–types δ which appear in $L^2(K_\infty \cap \Gamma_L \setminus K_\infty)$ and $f_\infty \in C_c^\infty(G_\infty)$ such that the following holds:

(a) The Poincaré series $P(f)$ and its restriction to G_∞ are non–trivial, where $f \overset{\text{def}}{=} f_\infty \otimes_{v \in V_f} f_v \in C_c^\infty(G(\mathbb{A}))$.
(b) $P(f)$ is right–invariant under L and transforms according to δ on the right.
(c) The support of $P(f)|_{G_\infty}$ is contained in the set of the form $\Gamma_L \cdot C$, where C is a compact set which is right–invariant under K_∞, and $\Gamma_L \cdot C$ is not whole G_∞. [1]

The precise description of the K_∞–types is given by Theorem 3-2. The requirement that δ belongs to $L^2(K_\infty \cap \Gamma_L \setminus K_\infty)$ is explained in ([19], Theorem 2-1). This is necessary in order to apply a non–vanishing criterion from ([19], Section 3). We remark that $P(f)$ has a quite large support because of (b). Hence, its non–vanishing is difficult. The condition (c) is fundamental in establishing that the number of cusp forms in Theorem 0-1 is infinite. This is done in the main result of Section 6 (see Theorem 6-3 (iv)). It is based on a principle explained in ([19], Section 3).

To make the results of Section 3 useful, in Section 4, for each finite places $v \in V_f$ we apply Bernstein's theory to the right–regular smooth representation of $C_c(G(k_v))$. The main results of that section are the principle of local cuspidality along a parabolic subgroup (see Lemma 4-1) and non–triviality of Bernstein components for the the right–regular smooth representation of $C_c(G(k_v))$ (see Lemma 4-2). The global consequence on cuspidality of Poincaré series is discussed in Proposition 4-3. In Section 5, we show that the analogue theory does not exists in the archimedean case (see Proposition 5-1). Finally, in Section 6 (see Theorem 6-3), we explain the spectral decomposition of cuspidal Poincaré series constructed in Theorem 3-2 of Section 3. Theorem 0-1 is a reformulation of Theorem 6-3 when $P(f)$ is cuspidal.

We believe that when combined with p–adic theory of types (see [17], [3]), the main results of this paper will be even more useful in the construction of cuspidal automorphic representations. We leave some of this for another paper [20].

We remark that completely different adelic Poincaré series were studied in [18]. There we established their cuspidality and non–vanishing.

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[1] We remaind the reader that to an open compact subgroup $L \subset G(\mathbb{A}_f)$, we can attach a congruence subgroup $\Gamma_L \subset G_\infty$ (see Section 1).

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1. Preliminary results

In this section we fix the notation used in this paper. We let G be a semisimple algebraic group defined over a number field k. We write V_f (resp., V_∞) for the set of finite (resp., Archimedean) places. For $v \in V := V_\infty \cup V_f$, we write k_v for the completion of k at v. If $v \in V_f$, we let $\mathcal{O}_v$ denote the ring of integers of k_v. Let $\mathbb{A}$ be the ring of adeles of k. For almost all places of k, G is defined over $\mathcal{O}_v$. The group of adelic points $G(\mathbb{A}) = \prod'_v G(k_v)$ is an restricted product over all places of k of the groups $G(k_v)$: $g = (g_v)_{v \in V} \in G(\mathbb{A})$ if and only if $g_v \in G(\mathcal{O}_v)$ for almost all v. $G(\mathbb{A})$ is a locally compact group and $G(k)$ is embedded diagonally as a discrete subgroup of $G(\mathbb{A})$.

For a finite subset $S \subset V$, we let

$$G_S = \prod_{v \in S} G(k_v).$$

In addition, if S contains all Archimedean places V_∞, we let $G^S = \prod'_{v \notin S} G(k_v)$. Then

$$G(\mathbb{A}) = G_S \times G^S. \tag{1-1}$$

We let $G_\infty = G_{V_\infty}$ and $G(\mathbb{A}_f) = G^{V_\infty}$.

The group G_∞ is a semisimple Lie group. It might not be connected but it has a finite center. The group $G(\mathbb{A}_f)$ is a totally disconnected group. Let $K_\infty \subset G_\infty$ be a maximal compact subgroup. Let $\mathfrak{g}_\infty = Lie(G_\infty)$ be the (real) Lie algebra of G_∞. Let $\mathcal{U}(\mathfrak{g}_\infty)$ be the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}_{\infty,\mathbb{C}} = \mathfrak{g}_\infty \otimes_{\mathbb{R}} \mathbb{C}$. Let $\mathcal{Z}(\mathfrak{g}_\infty)$ be the center of $\mathcal{U}(\mathfrak{g}_\infty)$. The maximal compact subgroup K_∞ comes as a fixed point set of a Cartan involution Θ of G_∞. The differential θ of Θ has the following decomposition of $\mathfrak{g}_\infty$:

$$\mathfrak{g}_\infty = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are $+1$ and -1 eigenspaces of θ, respectively. We have $\mathfrak{k} = Lie(K_\infty)$. Let $\mathfrak{a}_\infty$ be a maximal Abelian subalgebra of $\mathfrak{p}$. We choose some ordering of the roots $\Sigma(\mathfrak{a}_\infty, \mathfrak{g}_\infty)$ so that we determine the positive roots $\Sigma^+(\mathfrak{a}_\infty, \mathfrak{g}_\infty)$. Let N_∞ be the corresponding unipotent radical. This determines minimal parabolic subgroup $P_\infty = M_\infty A_\infty N_\infty$ of G_∞, where $A_\infty = \exp(\mathfrak{a}_\infty)$ and $M_\infty = Z_{K_\infty}(A_\infty)$. We have the following diffeomorphism:

$$N_\infty \times A_\infty \times K_\infty \xrightarrow{(n,a,k) \mapsto n \cdot a \cdot k} G_\infty = N_\infty A_\infty K_\infty.$$

The Iwasawa decomposition implies that there exist unique C^∞–functions $a : G_\infty \to A_\infty$, $n : G_\infty \to N_\infty$, and $k : G_\infty \to K_\infty$ such that

$$g = n(g) \cdot a(g) \cdot k(g), \quad g \in G_\infty. \tag{1-2}$$

Let $\hat{K}_\infty$ be the set of equivalence of irreducible representations of K_∞. Let $\delta \in \hat{K}_\infty$, then we write $d(\delta)$ and ξ_δ the degree and character of δ, respectively. We fix the normalized Haar measure dk on K_∞. Let π be a Banach representation of G_∞ on a Banach space $\mathcal{B}$. Then, for $b \in \mathcal{B}$ and $\delta \in \hat{K}_\infty$, we let

$$E_\delta(b) = \int_{K_\infty} d(\delta)\overline{\xi_\delta(k)}\pi(k) b \, dk.$$

It belongs to the δ–isotypic component $\mathcal{B}(\delta)$ of $\mathcal{B}$.

We say that a continuous function $f : G(\mathbb{A}) \to \mathbb{C}$ is smooth if $f(\cdot, g_f) \in C^\infty(G_\infty)$ for all $g_f \in G(\mathbb{A}_f)$, and there exists an open compact subgroup $L \subset G(\mathbb{A}_f)$ such that $f(g_\infty, g_f \cdot l) = f(g_\infty, g_f)$ for all $(g_\infty, g_f) \in G_\infty \times G(\mathbb{A}_f)$ and $l \in L$. Here we consider f as a function of two variables $f(g) = f(g_\infty, g_f)$, where $g = (g_\infty, g_f)$. We write $C^\infty(G(\mathbb{A}))$ for the vector space of all smooth functions on $G(\mathbb{A})$. We let $C_c^\infty(G(\mathbb{A}))$ be the space of all smooth compactly supported functions on $G(\mathbb{A})$. It is easy to show that $C_c^\infty(G(\mathbb{A}))$ is a span of the functions $f_\infty \otimes_{v \in V_f} f_v$ where $f_\infty \in C_c^\infty(G_\infty)$, $f_v \in C_c^\infty(G(k_v))$ $(v \in V_f)$, and $f_v = char_{G(\mathcal{O}_v)}$ for almost all v.

By definition, we let $C^\infty(G(k) \setminus G(\mathbb{A})) \subset C^\infty(G(\mathbb{A}))$ be the subspace consisting of all functions $f \in C^\infty(G(\mathbb{A}))$ such that $f(\gamma \cdot g) = f(g)$ for all $\gamma \in G(k)$ and $g \in G(\mathbb{A})$.

Let $X \in \mathfrak{g}_\infty$. Let $f \in C^\infty(G(\mathbb{A}))$. Then we let $X.f(g_\infty, g_f) = d/dt|_{t=0} f(g_\infty \exp(tX), g_f)$. This gives the structure of a $\mathcal{U}(\mathfrak{g}_\infty)$–module on $C^\infty(G(\mathbb{A}))$. The subspace $C^\infty(G(k) \setminus G(\mathbb{A}))$ is a $\mathcal{U}(\mathfrak{g}_\infty)$–submodule. In fact, both are invariant under the action of $G(\mathbb{A})$ by the right translation.

The function $f \in C^\infty(G(\mathbb{A}))$ is K_∞–finite (on the right) if $span_{\mathbb{C}}\{(g_\infty, g_f) \to f(g_\infty k_\infty, g_f);\ k_\infty \in K_\infty\}$ is finite dimensional. Similarly, $f \in C^\infty(G(\mathbb{A}))$ is $\mathcal{Z}(\mathfrak{g}_\infty)$–finite if the space spanned by $z.f$, $z \in \mathcal{Z}(\mathfrak{g}_\infty)$ is finite dimensional. In other words, the annhilator of f in $\mathcal{Z}(\mathfrak{g}_\infty)$ has finite codimension. By a well–known result, if $f \in C^\infty(G(\mathbb{A}))$ is K_∞–finite and $\mathcal{Z}(\mathfrak{g}_\infty)$–finite, then it is real–analytic in g_∞. We write $C^\infty(G(\mathbb{A}))_{K_\infty, Z(\mathfrak{g}_\infty)-\text{finite}}$ for the space of all $f \in C^\infty(G(\mathbb{A}))$ which are K_∞–finite and $\mathcal{Z}(\mathfrak{g}_\infty)$–finite on the right. Similarly, we define $C^\infty(G(k) \setminus G(\mathbb{A}))_{K_\infty, Z(\mathfrak{g}_\infty)-\text{finite}}$. The space $C^\infty(G(\mathbb{A}))_{K_\infty, Z(\mathfrak{g}_\infty)-\text{finite}}$ is no longer $G(\mathbb{A})$–invariant but it is $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$–module. The space $C^\infty(G(k) \setminus G(\mathbb{A}))_{K_\infty, Z(\mathfrak{g}_\infty)-\text{finite}}$ is its submodule.

An automorphic form is a function $f \in C^\infty(G(k) \setminus G(\mathbb{A}))_{K_\infty, Z(\mathfrak{g}_\infty)-\text{finite}}$ which satisfies certain growth condition (see [9], 4.2). The space of all automorphic forms we denote by $\mathcal{A}(G(k) \setminus G(\mathbb{A}))$. It is a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$–submodule of $C^\infty(G(k) \setminus G(\mathbb{A}))_{K_\infty, Z(\mathfrak{g}_\infty)-\text{finite}}$. The subspace of cuspidal automorphic forms we denote by $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))$. By definition $f \in \mathcal{A}(G(k) \setminus G(\mathbb{A}))$ is a cuspidal automorphic form if

$$\int_{U_P(k) \setminus U_P(\mathbb{A})} f(ng)dn = 0 \ \ \text{(for all } g \in G(\mathbb{A})), \tag{1-3}$$

for all proper k–parabolic subgroups P of G. In this paper we write U_P for the unipotent radical of k–parabolic subgroup P of G. In general, we say that a locally integrable function $f : G(k) \setminus G(\mathbb{A}) \to \mathbb{C}$ is a cuspidal function if it satisfies (1-3) for almost all $g \in G(\mathbb{A})$.

The space of cuspidal automorphic forms $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))$ is a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$–submodule of $\mathcal{A}(G(k) \setminus G(\mathbb{A}))$.

The topological space $G(k) \setminus G(\mathbb{A})$ has a finite volume $G(\mathbb{A})$–invariant measure:

$$\int_{G(k) \setminus G(\mathbb{A})} P(f)(g)dg = \int_{G(\mathbb{A})} f(g)dg, \ \ (f \in C_c^\infty(G(\mathbb{A}))), \tag{1-4}$$

where *the adelic compactly supported Poincaré series* $P(f)$ is defined as follows:

$$P(f)(g) = \sum_{\gamma \in G(k)} f(\gamma \cdot g) \in C_c^\infty(G(k) \setminus G(\mathbb{A})). \tag{1-5}$$

We say that $P(f)$ is a *an adelic compactly supported cuspidal Poincaré series* if the function $P(f)$ is a cuspidal function.

The measure introduced in (1-4) enables us to introduce the Hilbert space $L^2(G(k) \setminus G(\mathbb{A}))$, and its closed subspaces $L^2_{cusp}(G(k) \setminus G(\mathbb{A}))$ consisting of all cuspidal functions in $L^2(G(k) \setminus G(\mathbb{A}))$.

Both of them are unitary representations of $G(\mathbb{A})$. Moreover, we have the following result from the representation theory (see [12]):

Theorem 1-6. *The space $L^2_{cusp}(G(k)\setminus G(\mathbb{A}))$ can be decomposed into a direct sum of irreducible unitary representations of $G(\mathbb{A})$ each occuring with a finite multiplicity.*

Let $L \subset G(\mathbb{A}_f)$ be an open–compact subgroup. Then the intersection

$$\Gamma = \Gamma_L = G(k) \cap L \subset G(\mathbb{A}_f), \tag{1-7}$$

which is taken in $G(\mathbb{A}_f)$, is a discrete subgroup of G_∞. It is called a congruence subgroup [9]. It is well–known that we can fix a finite volume G_∞–invariant measure on $\Gamma \setminus G_\infty$:

$$\int_{\Gamma\setminus G_\infty} P(f_\infty)(g)dg = \int_{G_\infty} f_\infty(g)dg \tag{1-8}$$

for $f_\infty \in C_c^\infty(G_\infty)$, where *the compactly supported Poincaré series (for Γ)* is defined as follows:

$$P(f_\infty)(g) \overset{\text{def}}{=} \sum_{\gamma\in\Gamma} f_\infty(\gamma\cdot g). \tag{1-9}$$

The function $P(f_\infty)$ belongs to the space $C_c^\infty(\Gamma\setminus G_\infty)$ (the subspace of $C^\infty(G_\infty)$ consisting of all left Γ–invariant functions compactly supported modulo Γ). We use the measure on $\Gamma\setminus G_\infty$ to define a Hilbert space $L^2(\Gamma\setminus G_\infty)$ which is a unitary representation of G_∞. Similarly as before, we define the notion of cuspidality by letting $U_{P,\infty}$ to be the product

$$U_{P,\infty} = \prod_{v\in V_\infty} U_P(k_v), \tag{1-10}$$

and integrating over $U_{P,\infty}\cap\Gamma\setminus U_{P,\infty}$, for any proper k–parabolic subgroup P of G. The analogue of Theorem 1-6 is valid (see [12]).

2. Restriction of an Adelic Compactly Supported Poincaré Series to G_∞

In this section, we study the restriction of an adelic Poincaré series (1-5) to G_∞. As before, we write $g = (g_\infty, g_f) \in G(\mathbb{A}) = G_\infty \times G(\mathbb{A}_f)$. We have the following:

$$P(f)(g_\infty, 1) = \sum_{\gamma\in G(k)} f_\infty(\gamma\cdot g_\infty, \gamma). \tag{2-1}$$

Now, we show the following simple but important proposition:

Proposition 2-2. *Let $f \in C_c^\infty(G(\mathbb{A}))$. Assume that L is an open compact subgroup of $G(\mathbb{A}_f)$ such that f is right—invariant under L. We define a congruence subgroup of G_∞ using (1-7). Then the function in (2-1) is a compactly supported Poincaré series attached on G_∞ for Γ_L. Moreover, if f is cuspidal, then the function in (2-1) is cuspidal for Γ_L.*

Proof. Since f is compactly supported, we can find $c_1, \ldots, c_l \in G(\mathbb{A}_f)$ and $f_{\infty,1}, \ldots, f_{\infty,l} \in C_c^\infty(G_\infty)$ such that $f = \sum_{i=1}^l f_{\infty,i} \otimes char_{c_i\cdot L}$. Then (2-1) implies

$$P(f)(g_\infty, 1) = \sum_{\gamma\in G(k)} f_\infty(\gamma\cdot g_\infty, \gamma) = \sum_{i=1}^{l} \sum_{\gamma\in G(k)\cap c_i\cdot L} f_{\infty,i}(\gamma\cdot g_\infty).$$

It might happen that $G(k) \cap c_i \cdot L = \emptyset$ since $G(k)$ is not necessarily dense in $G(\mathbb{A}_f)$. Nevertheless, if $G(k) \cap c_i \cdot L \neq \emptyset$, we may assume that $c_i \in G(k)$. Hence $G(k) \cap c_i \cdot L = c_i \cdot \Gamma_L$. Thus

$$P(f)(g_\infty, 1) = \sum_{\substack{1 \leq i \leq l \\ G(k)\cap c_i \cdot L \neq \emptyset}} \sum_{\gamma \in \Gamma_L} f_{\infty,i}(c_i \cdot \gamma \cdot g_\infty).$$

This function belongs to $C_c^\infty(\Gamma_L \setminus G_\infty)$ and it is a compactly supported Poincaré series for Γ_L.

Let P be a k–parabolic subgroup of G. Then, for a fixed $g_\infty \in G_\infty$, the function $u \mapsto P(f)(u \cdot (g_\infty, 1))$ is right–invariant under the open–compact subgroup $L_v \overset{\text{def}}{=} L \cap U_v(\mathbb{A}_f)$. Now, Lemma 2-3 shows that the cuspidality of $P(f)$ implies the Γ_L–cuspidality of the function given by (2-1). □

It remains to prove Lemma 2-3. Let P be a k–parabolic subgroup of G. We remaind the reader that $U_{P,\infty}$ is defined by (1-10). We fix Haar measures du_∞, du_f, and du on $U_{P,\infty}$, $U_P(\mathbb{A}_f)$, and $U_P(\mathbb{A})$, respectively, such that the following holds:

$$\int_{U_P(\mathbb{A})} \varphi(u)du = \int_{U_{P,\infty}} \int_{U_P(\mathbb{A}_f)} \varphi(u_\infty, u_f)du_\infty du_f, \quad \varphi \in C_c(U_P(\mathbb{A})).$$

Lemma 2-3. *Let $\psi : U_P(k) \setminus U_P(\mathbb{A}) \to \mathbb{C}$ be a continuous function which is right–invariant under an open compact subgroup $L_P \subset U_P(\mathbb{A}_f)$. Then, if we let $vol_{U_P(\mathbb{A}_f)}(L_P) = \int_{U_P(\mathbb{A}_f)} char_{L_P} du_f$, then we have the following formula:*

$$\int_{U_P(k)\setminus U_P(\mathbb{A})} \psi(u)du = vol_{U_P(\mathbb{A}_f)}(L_P) \cdot \int_{\Gamma_{L_P}\setminus U_{P,\infty}} \psi(u_\infty)du_\infty,$$

where Γ_{L_P} is a discrete subgroup of $U_{P,\infty}$ defined as follows: $\Gamma_{L_P} = U_P(k) \cap L_P$.

Proof. By the usual integration theory, we can find a compactly supported continuous function $\varphi : U_P(\mathbb{A}) \to \mathbb{C}$ such that $\psi = P(\varphi)$, where $P(\varphi)(u) \overset{\text{def}}{=} \sum_{\gamma \in U_P(k)} \varphi(\gamma \cdot u)$ $(u \in U_P(\mathbb{A}))$. Since ψ is right–invariant under the open compact subgroup L_P, we can assume that φ satisfies the same. Now, we can find $u_1, \ldots, u_l \in U_P(\mathbb{A}_f)$, and continuous compactly supported functions $\varphi_1, \ldots, \varphi_l$ on $U_{P,\infty}$, such that $\varphi = \sum_{i=1}^l \varphi_i \otimes char_{u_i L_P}$, where we consider φ as a function of two variables $u = (u_\infty, u_f) \in U_P(\mathbb{A}) = U_{P,\infty} \times U_P(\mathbb{A}_f)$. Next, the strong approximation implies that $U_P(\mathbb{A}) = U_P(k)U_{P,\infty}L_P$. Hence $U_P(\mathbb{A}_f) = U_P(k)L_P$. This implies that we can assume $u_1, \ldots, u_l \in U_P(k)$. This is used to determine the restriction of ψ to $U_{P,\infty}$. As in the proof of Proposition 2-2, we obtain the following:

$$\psi(u_\infty) = \sum_{i=1}^{l} \sum_{\gamma \in \Gamma_{L_P}} (l(u_i^{-1})\varphi_i)(\gamma \cdot u_\infty),$$

where l denotes the left–translation. Hence

$$\int_{\Gamma_{L_P}\setminus U_{P,\infty}} \psi(u_\infty)du_\infty = \sum_{i=1}^{l} \int_{\Gamma_{L_P}\setminus U_{P,\infty}} \left(\sum_{\gamma \in \Gamma_{L_P}} (l(u_i^{-1})\varphi_i)(\gamma \cdot u_\infty) \right) du_\infty =$$

$$= \sum_{i=1}^{l} \int_{U_{P,\infty}} (l(u_i^{-1})\varphi_i)(u_\infty)du_\infty = \sum_{i=1}^{l} \int_{U_{P,\infty}} \varphi_i(u_\infty)du_\infty.$$

Again we compute by definition

$$\int_{U_P(k)\backslash U_P(\mathbb{A})} \psi(u)du = \int_{U_P(k)\backslash U_P(\mathbb{A})} \left(\sum_{\gamma\in U_P(k)} \varphi(\gamma\cdot u)\right) du = \int_{U_P(\mathbb{A})} \varphi(u)du =$$

$$= \int_{U_{P,\infty}} \left(\int_{U_P(\mathbb{A}_f)} \varphi(u_\infty, u_f)du_f\right) du_\infty = vol_{U_P(\mathbb{A}_f)}(L_P)\cdot \int_{U_{P,\infty}} \left(\sum_{i=1}^{l} \varphi_i(u_\infty)\right) du_\infty.$$

Combining the last two displayed formula, we obtain the lemma. □

3. Non–Vanishing of Adelic Compactly Supported Poincaré Series

In this section we develop a non–vanishing criterion for (1-5) which controls not only a non–vanishing of (1-5) but also a non–vanishing of the restriction to G_∞ (see Section 2). The criterion is based on a non–vanisihing criterion given by ([19], Lemma 3.2).

We introduce some notation. Let S be a finite set of places, containing V_∞, large enough that G is defined over $\mathcal{O}_v$ for $v\notin S$. We use the decomposition of $G(\mathbb{A})$ given by (1-1). We let

$$\Gamma_S = \left(\prod_{v\notin S} G(\mathcal{O}_v)\right)\cap G(k) \text{ the intersection is taken in } G^S.$$

This can be considered as a subgroup of G_S using the diagonal embedding of $G(k)$ into the product (1-1) and then the projection to the first component. Since $G(k)$ is a discrete subgroup of $G(\mathbb{A})$, it follows that Γ_S is a discrete subgroup of G_S.

For $v\in S-V_\infty$, we choose an open–compact subgroup L_v. We put

$$\Gamma = \left(\prod_{v\in S-V_\infty} L_v \times \prod_{v\notin S} G(\mathcal{O}_v)\right)\cap G(k) = \Gamma_S \cap \left(\prod_{v\in S-V_\infty} L_v\right). \tag{3-1}$$

This is a discrete subgroup of G_∞. Now, we have the following non–vanishing criterion:

Theorem 3-2. *Let S be a finite set of places, containing V_∞, large enough such that G is defined over $\mathcal{O}_v$ for $v\notin S$. Assume that for each $v\in V_f$ we have $f_v\in C_c^\infty(G(k_v))$ such that $f_v(1)\neq 0$, and $f_v = char_{G(\mathcal{O}_v)}$ for all $v\notin S$. For $v\in S-V_\infty$, we choose an open–compact subgroup L_v such that f_v is right–invariant under L_v. Then the intersection $\Gamma_S\cap\left[K_\infty\times\prod_{v\in S-V_\infty}\text{supp }(f_v)\right]$ is a finite set and it can be written as follows:*

$$\cup_{j=1}^{l}\gamma_j\cdot(K_\infty\cap\Gamma). \tag{3-3}$$

Next, we let

$$c_j = \prod_{v\in S-V_\infty} f_v(\gamma_j).$$

Then the K_∞–invariant map $C^\infty(K_\infty)\to C^\infty(K_\infty\cap\Gamma\setminus K_\infty)$ given by

$$\alpha\mapsto\left(k\mapsto\hat{\alpha}(k)\overset{def}{=}\sum_{j=1}^{l}\sum_{\gamma\in K_\infty\cap\Gamma} c_j\cdot\alpha(\gamma_j\gamma\cdot k)\right) \tag{3-4}$$

is non–trivial, and, for every $\delta \in \hat{K}_\infty$, contributing in the decomposition of the closure of the image of (3-4) in $L^2(K_\infty \cap \Gamma \setminus K_\infty)$, we can find a non–trivial $f_\infty \in C_c^\infty(G_\infty)$ such that the following hold:

(i) $E_\delta(f_\infty) = f_\infty$.

(ii) *The Poincaré series $P(f)$ and its restriction to G_∞ (which is a Poincaré series for Γ_L) are non–trivial, where $f \overset{def}{=} f_\infty \otimes_{v \in V_f} f_v \in C_c^\infty(G(\mathbb{A}))$.*

(iii) $E_\delta(P(f)) = P(f)$ *and $P(f)$ is right–invariant under L.*

(iv) *The support of $P(f)|_{G_\infty}$ is contained in the set of the form $\Gamma_L \cdot C$, where C is a compact set which is right–invariant under K_∞, and $\Gamma_L \cdot C$ is not whole G_∞.*

Proof. Arguing as in the proof of ([19], Lemma 3-2), we can find a neighborhood of $1 \in G_\infty$ of the form UVK_∞, where $U \subset N_\infty$ and $V \subset A_\infty$ are neighborhoods of identities, such that

$$\Gamma_S \cap \left[(UVK_\infty) \times \prod_{v \in S - V_\infty} \text{supp } (f_v)\right] = \Gamma_S \cap \left[K_\infty \times \prod_{v \in S - V_\infty} \text{supp } (f_v)\right]. \tag{3-5}$$

Obviously, the intersection in (3-5) is finite. It can be described as the set of all $\gamma \in \Gamma_S$ satisfying

$$\gamma \in K_\infty \text{ and } \prod_{v \in S - V_\infty} f_v(\gamma) \neq 0. \tag{3-6}$$

The set of all $\gamma \in \Gamma_S$ satisfying $\prod_{v \in S - V_\infty} f_v(\gamma) \neq 0$ is clearly right–invariant under Γ. Hence, the characterization of the intersection in (3-5), given by (3-6), shows that the intersection in (3-5) is right–invariant under $K_\infty \cap \Gamma$ and it can be written as a disjoint union (3-3).

We show that the map (3-4) is non–trivial. First of all, our assumption $f_v(1) \neq 0$ ($v \in V_f$) and the characterization of the intersection (3-5) given by (3-6) enables us to assume that $\gamma_1 = 1$. Then $c_1 = \prod_{v \in S - V_\infty} f_v(1) \neq 0$.

Next, let W a neighborhood of $\gamma_1 = 1 \in K_\infty$ such that W intersects the finite set (3-3) exactly in $\{\gamma_1\}$. Let $\alpha \in C^\infty(K_\infty)$, which vanishes outside W, such that $\alpha(\gamma_1) \neq 0$. Then, for $k = 1$, the right–hand side of (3-4) becomes

$$\sum_{j=1}^{l} \sum_{\gamma \in K_\infty \cap \Gamma} c_j \cdot \alpha(\gamma_j \gamma) = c_1 \alpha(\gamma_1) \neq 0.$$

This shows the non–triviality of the map (3-4).

Let $\alpha \in C^\infty(K_\infty)$ be any function such that the right–hand side of (3-4) is non–trivial. We can write its spectral expansion in $L^2(K_\infty \cap \Gamma \setminus K_\infty)$ as follows:

$$\hat{\alpha} = \sum_{\delta \in \hat{K}_\infty} E_\delta(\hat{\alpha}), \tag{3-7}$$

where

$$E_\delta(\hat{\alpha})(k) = \int_{K_\infty} d(\delta) \overline{\xi_\delta(k')} \hat{\alpha}(kk') \; dk'.$$

As we explain at the beginning of Section 3 in [19], only δ's containing a non–trivial vector invariant under $K_\infty \cap \Gamma$ can contribute to the spectral expansion given by (3-7). For $\delta \in \hat{K}_\infty$ such that $E_\delta(\hat{\alpha}) \neq 0$, $E_\delta(\hat{\alpha})$ is a linear combination of matrix coefficients of the form (3-1) in [19]. In particular, since $\widehat{E_\delta(\alpha)} = E_\delta(\hat{\alpha})$, this shows the existence of α such that $E_\delta(\alpha) = \alpha$ and $\hat{\alpha} \neq 0$, for every δ appearing in the decomposition of the closure of the image of the map $\alpha \mapsto \hat{\alpha}$ under K_∞.

Now, we fix δ appearing in the decomposition under K_∞ of the closure in $L^2(K_\infty \cap \Gamma \setminus K_\infty)$ of the image of the map $\alpha \mapsto \hat{\alpha}$, and select an arbitrary $\xi \in C^\infty(K_\infty)$ such that $E_\delta(\xi) = \xi$ and $\hat{\xi} \neq 0$. We also take $\zeta \in C_c^\infty(U)$ and $\eta \in C_c^\infty(V)$ such that $\zeta(1) \neq 0$ and $\eta(1) \neq 0$. We define $f_\infty \in C_c^\infty(G_\infty)$ by

$$f_\infty(uvk) = \zeta(u)\eta(v)\xi(k).$$

Then, by a short calculation, we obtain $E_\delta(f_\infty) = f_\infty$. This proves (i). Also, it immediately implies $E_\delta(P(f)) = P(f)$ which is the first claim in (iii). The right invariance under L in (iii) is obvious.

By construction, we see

$$\text{supp }(f_\infty) \subset UVK_\infty. \tag{3-8}$$

This is used to prove the following observation:

Lemma 3-9. *Let $\gamma \in \Gamma_S$, such that $\prod_{v \in S - V_\infty} f_v(\gamma) \neq 0$, and $k \in K_\infty$. Then, $f_\infty(\gamma \cdot k) \neq 0$ implies $\gamma \in \Gamma_S \cap \left[K_\infty \times \prod_{v \in S - V_\infty} \text{supp }(f_v)\right]$.*

Proof. Indeed, (3-8) implies that

$$\gamma \cdot k \in \Gamma_S \cap \left[(UVK_\infty) \times \prod_{v \in S - V_\infty} \text{supp }(f_v)\right].$$

Hence

$$\gamma \in \Gamma_S \cap \left[(UVK_\infty \cdot k^{-1}) \times \prod_{v \in S - V_\infty} \text{supp }(f_v)\right] = \Gamma_S \cap \left[(UVK_\infty) \times \prod_{v \in S - V_\infty} \text{supp }(f_v)\right].$$

Now, we apply (3-5). □

Finally, for $k \in K_\infty$, using Lemma 3-9, we compute

$$P(f)(k,1) = \sum_{\gamma \in \Gamma_S} \left(\prod_{v \in S - V_\infty} f_v(\gamma)\right) \cdot f_\infty(\gamma \cdot k) = \sum_{j=1}^{l} \sum_{\gamma \in K_\infty \cap \Gamma} c_j \cdot f_\infty(\gamma_j \gamma \cdot k) = \zeta(1)\eta(1)\hat{\xi}(k). \tag{3-10}$$

In particular, $P(f)$ is not identically zero on K_∞. This proves (ii). Finally, we prove (iv). Since f is factorizable, using the notation form the proof of Propositiom 2-2, we see that $f_{\infty,1} = \cdots = f_{\infty,l} = f_\infty$ in the expression for f given at the begining of the proof of Propositiom 2-2. Now, the same proof gives the following expression for the restriction to G_∞:

$$P(f)(g_\infty, 1) = \sum_{\substack{1 \leq i \leq l \\ G(k) \cap c_i \cdot L \neq \emptyset}} \sum_{\gamma \in \Gamma_L} f_\infty(c_i \cdot \gamma \cdot g_\infty).$$

(We remaind the reader that those c_i's are not the one form the present theorem but the ones from the proof of Proposition 2-2. In particular, if $G(k) \cap c_i \cdot L \neq \emptyset$, then we take $c_i \in G(k)$.) Since (3-8) holds, we see that the restriction has the support contained in

$$\cup_{\substack{1 \leq i \leq l \\ G(k) \cap c_i \cdot L \neq \emptyset}} \Gamma_L \cdot c_i^{-1} \cdot UVK_\infty.$$

One can easily show that this is different than G_∞ if we shrink U and V. (One can adjust the argument given in the proof of Lemma 3-4 in [19].) This completes the proof of (iv). □

We finish this section with the followng remark:

Lemma 3-11. *Maintaing the assumptions of Theorem 3-2 , there are infinitely many $\delta \in \hat{K}_\infty$ contributing in the decomposition of the closure of the image of (3-4).*

Proof. Indeed, it is enough to prove that given different elements $k_1, \dots, k_l \in K_\infty$ and non–zero $c_1, \dots, c_l \in \mathbb{C} - \{0\}$, the map $C^\infty(K_\infty) \to C^\infty(K_\infty)$ given by

$$\alpha \mapsto \left(k \mapsto \hat{\alpha}(k) \overset{\mathrm{def}}{=} \sum_{i=1}^{l} c_i \cdot \alpha(k_i \cdot k) \right)$$

has no finite image. To accomplish this, we select a neighborhood U of $1 \in K_\infty$ such that $k_i k_j^{-1} U \cap U = \emptyset$ for all i, j, $i \neq j$. Then if α is supported in U, we easily see that $\hat{\alpha} \neq 0$. □

4. Construction of Cuspidal Compactly Supported Adelic Poincaré Series

In this section we use Bernstein's decomposition of the category of smooth complex representations of reductive p–adic groups [2] to construct adelic cuspidal compactly supported Poincaré series on $G(\mathbb{A})$.

Let us fix a place $v \in V_f$. We introduce some notation following standard references [4] and [5]. A parabolic subgroup of $G(k_v)$ is a group of k_v–points of a k_v–parabolic subgroup of G. We consider the category of smooth (or algebraic) representations of $G(k_v)$. Let P_v be a parabolic subgroup of $G(k_v)$ given by a Levi decomposition $P_v = M_v U_v$, where M_v is a Levi factor and U_v is the unipotent radical of P_v. If σ_v is a smooth representation of M_v we extended trivially across U_v to a representation of P_v, then we denote the normalized induction by $\mathrm{Ind}_{P_v}^{G(k_v)}(\sigma_v)$. If π_v is a smooth representation of $G(k_v)$, then we denote by $Jacq^{P_v}_{G(k_v)}(\pi_v)$ a normalized Jacquet module of π_v with respect to P_v. When restricted to U_v, $Jacq^{P_v}_{G(k_v)}(\pi_v)$ is a direct sum of (possibly infinitely many) copies of a trivial representation. Therefore, when M_v is fixed, we write $Jacq^{M_v}_{G(k_v)}(\pi_v) = Jacq^{P_v}_{G(k_v)}(\pi_v)$. Let M_v^0 be the subgroup of M_v given as the intersection of the kernels of all characters $m_v \mapsto |\chi_v(m_v)|_v$, where χ_v ranges over the group of all k_v–rational algebraic characters $M_v \to k_v^\times$. We say that a character $\chi_v : M_v \to \mathbb{C}^\times$ is unramified if it is trivial on M_v^0. We say that an irreducible representation ρ_v of M_v is supercuspidal if $Jacq^{Q_v}_{M_v}(\rho_v) = 0$ for all proper parabolic subgroups Q_v of M_v.

Now, following Bernstein [2], on the set of pairs (M_v, ρ_v), where M_v is a Levi subgroup of $G(k_v)$ and ρ_v is a smooth irreducible supercuspidal representation of M_v, we introduce the relation of equivalence as follows: (M_v, ρ_v) and (M_v', ρ_v') are equivalent if we can find $g_v \in G(k_v)$ and an unramified character χ_v of M_v' such that $M_v' = g_v M_v g_v^{-1}$ and $\rho_v' \simeq \chi_v \rho_v^{g_v}$ i.e.,

$$\rho_v^{g_v}(m_v') = \chi_v(m_v') \rho_v(g_v^{-1} m_v' g_v), \quad m_v' \in M_v'.$$

In the discussion below, we write $\mathfrak{M}_v$ for the Bernstein's equivalence class of a pair (M_v, ρ_v).

Let V be a smooth complex representations of $G(k_v)$. Let

$$V(\mathfrak{M}_v)$$

be the largest smooth submodule of V such that every irreducible subquotient of V is a subquotient of $\mathrm{Ind}_{P_v}^{G(k_v)}(\chi_v \rho_v)$, for some unramified character χ_v of M_v. Here P_v is an arbitrary parabolic subgroup of $G(k_v)$ containing M_v as a Levi subgroup. The fundamental result of Bernstein is the following decomposition:

$$V = \oplus_{\mathfrak{M}_v} V(\mathfrak{M}_v).$$

Now, we prove the following lemma:

Lemma 4-1. *We fix a Bernstein's equivalence class* $\mathfrak{M}_v$ *(of a pair* (M_v, ρ_v)*). Let us consider* $C_c^\infty(G(k_v))$ *as a smooth representation of* $G(k_v)$ *acting by right–translations. Let* $f_v \in C_c^\infty(G(k_v))(\mathfrak{M}_v)$*. Let* $P'_v = M'_v U'_{P_v}$ *be a parabolic subgroup of* $G(k_v)$ *such that* M'_v *does not contain a conjugate of* M_v*. Then*

$$\int_{U'_{P_v}} f_v(g_v u_v g'_v) du_v = 0, \quad \text{for all } g_v, g'_v \in G(k_v).$$

Proof. Assume that we can find $g_v, g'_v \in G(k_v)$ such that

$$0 \neq \int_{U'_{P_v}} f_v(g_v u_v g'_v) du_v = \int_{(g'_v)^{-1} U'_{P_v} g'_v} f_v(g_v g'_v u_v) du_v = \int_{(g'_v)^{-1} U'_{P_v} g'_v} F_v(u_v) du_v,$$

where F_v is defined by $F_v(x) \overset{\text{def}}{=} f_v(g_v g'_v \cdot x)$ $(x \in G(k_v))$. Since the action of $G(k_v)$ by left–translations commutes with the action by right–translations, we obtain $F_v \in C_c^\infty(G(k_v))(\mathfrak{M}_v)$. This enables us to assume $g_v = g'_v = 1$. Let $X(f_v)$ be a subrepresentation of $C_c^\infty(G(k_v))(\mathfrak{M}_v)$ generated by f_v. Since $\int_{U'_{P_v}} f_v(u_v) du_v \neq 0$, we see

$$Jacq_{G(k_v)}^{P'_v}(X(f_v)) \neq 0.$$

The set of parabolic subgroups of $G(k_v)$ contained in P'_v is partially ordered by the inclusion. Let P''_v be the minimal parabolic subgroup contained in P'_v such that $Jacq_{G(k_v)}^{P''_v}(X(f_v)) \neq 0$. We write $P''_v = M''_v U''_v$ for some Levi decomposition of P''_v. By the standard theory ([4], 2.6) there exists an irreducible smooth representation ρ''_v of M''_v which is a subquotient of $Jacq_{G(k_v)}^{P''_v}(X(f_v))$. We claim that ρ''_v is supercuspidal. If ρ''_v is not supecuspidal, we can find a parabolic subgroup Q''_v of M''_v such that $Jacq_{M''_v}^{Q''_v}(\rho''_v) \neq 0$. Then $R''_v = Q''_v U''_v$ is a proper parabolic subgroup of P''_v. The transitivity of Jacquet modules ([5], Proposition 2.3) implies

$$Jacq_{G(k_v)}^{R''_v}(X(f_v)) = Jacq_{M''_v}^{Q''_v}\left(Jacq_{G(k_v)}^{P''_v}(X(f_v))\right).$$

Now, the exactness of Jacquet functor implies that $Jacq_{G(k_v)}^{R''_v}(X(f_v)) \neq 0$. This is a contradiction. This proves that ρ''_v is supercuspidal.

Now, since ρ''_v is supercuspidal, ([5], Theorem 2.4 (c)) implies that

$$\mathrm{Hom}_{M''_v}\left(Jacq_{G(k_v)}^{P''_v}(X(f_v)),\ \rho''_v\right) \neq 0.$$

Thus, Frobenius reciprocity implies

$$\mathrm{Hom}_{G(k_v)}\left(X(f_v),\ \mathrm{Ind}_{P''_v}^{G(k_v)}(\rho''_v)\right) \simeq \mathrm{Hom}_{M''_v}\left(Jacq_{G(k_v)}^{P''_v}(X(f_v)),\ \rho''_v\right) \neq 0.$$

Since $X(f_v)$ is a subrepresentation of $C_c^\infty(G(k_v))(\mathfrak{M}_v)$, this implies

$$(M''_v, \rho''_v) \in \mathfrak{M}_v.$$

In particular, M''_v is conjugate to M_v. But, since $P''_v \subset P'_v$, by fixing some appropriate minimal parabolic subgroup of $G(k_v)$ contained in P'_v and the corresponding maximal split torus, we see that M''_v is conjugate to a Levi in M'_v. This is a contradiction. □

The next lemma gives further information on the decomposition of $C_c^\infty(G(k_v))$.

Lemma 4-2. *Let* $\mathfrak{M}_v$ *be a Bernstein's equivalence class. Let* (M_v, ρ_v) *represent the class* $\mathfrak{M}_v$*. Then we have the following:*

(i) $C_c^\infty(G(k_v))(\mathfrak{M}_v) \neq 0$

(ii) *Let* π_v *be a smooth irreducible representation of* $G(k_v)$. *Assume* $f_v \in C_c^\infty(G(k_v))(\mathfrak{M}_v)$. *If* $\pi_v(f_v) \neq 0$, *then the contragredient representation* $\widetilde{\pi}_v$ *belongs to the class* $\mathfrak{M}_v$ *i.e., there exists a parabolic subgroup* P_v *of* $G(k_v)$ *which has* M_v *as a Levi factor and an unramified character* χ_v *of* M_v *such that* $\widetilde{\pi}_v$ *is an irreducible subquotient of* $\mathrm{Ind}_{P_v}^{G(k_v)}(\chi_v\rho_v)$. *In other words,* π_v *belongs to the class of* $(M_v, \widetilde{\rho}_v)$.

Proof. As before, in this proof the group $G(k_v)$ acts on $C_c^\infty(G(k_v))$ by right–translations. We begin the proof by the following observation. Let (π_v, V_v) be a smooth (not necessarily irreducible) representation of $G(k_v)$. We write $(\widetilde{\pi}_v, \widetilde{V}_v)$ for the contragredient representation of π_v. We denote by $\langle\ ,\ \rangle : V_v \times \widetilde{V}_v \to \mathbb{C}$ for a canonical $G(k_v)$–invariant pairing. The functions $f_v \in C_c^\infty(G(k_v))$ act as follows:

$$\pi_v(f_v)v_v = \int_{G(k_v)} f_v(g_v)\pi_v(g_v)v_v\ dg_v, \quad v_v \in V_v.$$

For a fixed $\widetilde{v}_v \in \widetilde{V}_v$, this implies the following $G(k_v)$–invariant pairing:

$$(f_v, v_v) \mapsto \langle \pi_v(f_v)v_v, \widetilde{v}_v\rangle = \int_{G(k_v)} f_v(g_v)\langle \pi_v(g_v)v_v, \widetilde{v}_v\rangle\ dg_v.$$

If $\pi_v(f_v)$ is not trivial, then we can select $\widetilde{v}_v$ such that the pairing is non–trivial when restricted to $X(f_v) \times V_v$, where $X(f_v)$ is a $G(k_v)$–subrepresentation of $C_c^\infty(G(k_v))$ generated by f_v. Hence it implies

$$\mathrm{Hom}_{G(k_v)}\left(X(f_v), \widetilde{\pi}_v\right) \neq 0.$$

This proves (ii) by the definition of $C_c^\infty(G(k_v))(\mathfrak{M}_v)$.

Let $\pi_v \overset{\mathrm{def}}{=} \mathrm{Ind}_{P_v}^{G(k_v)}(\widetilde{\rho}_v)$. Then we can select some $f_v \in C_c^\infty(G(k_v))$ such that $\pi_v(f_v) \neq 0$. (For example, a characteristic function of a sufficiently small open compact subgroup would do.) Then, the first part of the proof implies

$$\mathrm{Hom}_{G(k_v)}\left(X(f_v),\ \mathrm{Ind}_{P_v}^{G(k_v)}(\rho_v)\right) \neq 0.$$

If we decompose according to the Bernstein classes:

$$X(f_v) = \oplus_{\mathfrak{N}_v} X(f_v)(\mathfrak{N}_v),$$

and apply (ii), then we see that

$$\mathrm{Hom}_{G(k_v)}\left(X(f_v)(\mathfrak{M}_v),\ \mathrm{Ind}_{P_v}^{G(k_v)}(\rho_v)\right) \neq 0.$$

In particular, $X(f_v)(\mathfrak{M}_v) \neq 0$. This implies (i). □

Now, we go back to a global theory. We prove the following proposition:

Proposition 4-3. *Let* $f = f_\infty \otimes_{v\in V_f} f_v \in C_c^\infty(G(\mathbb{A}))$. *Let* P *be a* k*–parabolic subgroup of* G. *Assume that there is a finite place* w *and a equivalence class* $\mathfrak{M}_w$ *(represented by* (M_w, ρ_w)*) such that a Levi subgroup of* $P(k_w)$ *does not contain a conjugate of* M_w *and* $f_w \in C_c^\infty(G(k_w))(\mathfrak{M}_w)$. *Then the constant term of* $P(f)$ *along* P *vanishes.*

Proof. By definition, the constant term of $P(f)$ with respect to a k–parabolic subgroup P of G is the following:

$$\int_{U_P(k)\backslash U_P(\mathbb{A})} P(f)(ug)du = \int_{U_P(k)\backslash U_P(\mathbb{A})} \left(\sum_{\gamma \in G(k)} \varphi(\gamma \cdot ug) \right) du$$

$$(4\text{-}4) \qquad = \int_{U_P(\mathbb{A})} \left(\sum_{\gamma \in G(k)/U_P(k)} f(\gamma \cdot ug) \right) du$$

$$= \sum_{\gamma \in G(k)/U_P(k)} \int_{U_P(\mathbb{A})} f(\gamma \cdot ug) du.$$

Since f is factorizable i.e., $f = f_\infty \otimes_{v \in V_f} f_v \in C_c^\infty(G(\mathbb{A}))$, every term on the right–hand side of the formula given by (4-4) is zero because of Lemma 4-1:

$$\int_{U_P(\mathbb{A})} f(\gamma \cdot ug)du = \left(\int_{U_{P,\infty}} f_\infty(\gamma \cdot u_\infty \cdot g_\infty) du_\infty \right) \cdot \prod_{v \in V_f} \int_{U_P(k_v)} f_v(\gamma \cdot u_v \cdot g_v) du_v = 0.$$

□

5. A comment on a archimedean case

In this section we show that the analogue of the results of Section 4 in the archimedean case does not give anything interesting.

Proposition 5-1. *Let P be a proper parabolic subgroup of a Lie group G_∞. We write $U_{P,\infty}$ for its unipotent radical. Let $\varphi \in C_c^\infty(G_\infty)$. Then, if $\int_{U_{P,\infty}} \varphi(g_1 \cdot u \cdot g_2)du = 0$, for all $g_1, g_2 \in G_\infty$, then $\varphi = 0$.*

Proof. We remind the reader that N_∞ is the unipotent radical of the minimal parabolic subgroup of G_∞ fixed in Section 1. We show that the assumption of the lemma implies

$$(5\text{-}2) \qquad \int_{N_\infty} \varphi(g_1 \cdot n \cdot g_2)dn = 0, \quad \text{for all } g_1, g_2 \in G_\infty.$$

Indeed, after conjugation by an element of G_∞, we may assume that $U_{P,\infty} \supset N_\infty$. Now,

$$\int_{N_\infty} \varphi(g_1 n g_2)dn = \int_{U_{P,\infty}\backslash N_\infty} \left(\int_{U_{P,\infty}} \varphi(g_1 u u' g_2)du \right) du' = 0.$$

This proves (5-2).

Having established (5-2), let P now denotes an arbitrary standard parabolic subgroup of G_∞ (i.e., it contains P_∞). We write the Langlands decomposition of P as follows: $P = A_P M_P^1 U_{P,\infty}$. The Haar measure is given by the formula ($f \in C_c^\infty(G_\infty)$):

$$(5\text{-}3) \qquad \int_{G_\infty} f(g)dg = \int_{U_{P,\infty}} \int_{A_P} \int_{M_P^1} \int_{K_\infty} f(uamk)\delta_P^{-1}(a)\ du\ da\ dm\ dk,$$

where we require that the Haar measure dk is normalized: $\int_{K_\infty} dk = 1$.

We assume that M_P^1 has representations in the discrete series. Let $\nu \in \mathfrak{a}_P^* \otimes_\mathbb{R} \mathbb{C}$ and $\sigma \in \hat{M}_P^1$ be a representation in the discrete series acting on a Hilbert space $\mathfrak{H}_\sigma$ with a M_P^1–invariant scalar product $(\ ,\ )_\sigma$. We consider the induced representation $\mathrm{Ind}_P^{G_\infty}(\nu,\sigma)$ on the space of the classes of measurable functions $F : G_\infty \to \mathfrak{H}_\sigma$ such that

$$F(uamg) = e^{\nu(\log a)}\delta_P^{1/2}(a)\sigma(m)F(g),\ a \in A_P,\ m \in M_P^1,\ u \in U_{P,\infty},\ g \in G_\infty. \tag{5-4}$$

The functions $f \in C_c^\infty(G_\infty)$ act on $\mathrm{Ind}_P^{G_\infty}(\nu,\sigma)$ as bounded operators:

$$\mathrm{Ind}_{P(\mathbb{R})}^{G_\infty}(\nu,\sigma)(f).F(g) = \int_{G_\infty} f(h)F(gh)dh.$$

The induced representation $\mathrm{Ind}_P^{G_\infty}(\nu,\sigma)$ is unitary under the usual scalar product

$$(F_1,F_2) = \int_{K_\infty}(F_1(k),F_2(k))_\sigma\ dk$$

if $\nu \in \sqrt{-1}\mathfrak{a}_P^*$.

For a minimal parabolic subgroup P_∞, the Langlands decomposition is $P_\infty = A_\infty M_\infty N_\infty$ (fixed in Section 1). Now, letting $P = P_\infty$, (5-2) implies

$$\begin{aligned}
&\mathrm{Ind}_{P_\infty}^{G_\infty}(\nu,\sigma)(\varphi).F(g) = \int_{G_\infty} F(gh)\varphi(h)dh = \int_{G_\infty}\varphi(g^{-1}h)F(h)dh \\
&= \int_{N_\infty}\int_{A_\infty}\int_{M_\infty}\int_{K_\infty} e^{\nu(\log a)}\delta_{P_\infty}^{-1/2}(a)\varphi(g^{-1}uamk)\sigma(m)F(k)du\ da\ dm\ dk \\
&= \int_{A_\infty}\int_{M_\infty}\int_{K_\infty} e^{\nu(\log a)}\delta_{P_\infty}^{-1/2}(a)\left(\int_{N_\infty(\mathbb{R})}\varphi(g^{-1}uamk)du\right)\sigma(m)F(k)da\ dm\ dk = 0.
\end{aligned}$$

Hence

$$\mathrm{Ind}_{P_\infty}^{G_\infty}(\nu,\sigma)(\varphi) = 0,\quad \nu \in \mathfrak{a}_\infty^* \otimes_{\mathbb{R}} \mathbb{C}. \tag{5-5}$$

Next, we show that $\mathbf{tr}(\pi(\varphi)) = 0$ for every irreducible admissible representation π of G_∞. Indeed, for an appropriate $\nu \in \mathfrak{a}_\infty^* \otimes_{\mathbb{R}} \mathbb{C}$ and $\sigma \in \hat{M}_\infty$, π is infinitesimally equivalent to a closed irreducible subquotient Π of $\mathrm{Ind}_{P_\infty}^{G_\infty}(\nu,\sigma)$. But (5-5) implies $\Pi(\varphi) = 0$. Hence, we obtain

$$\mathbf{tr}(\pi(\varphi)) = \mathbf{tr}(\Pi(\varphi)) = 0,$$

since irreducible infinitesimally equivalent representations have equal characters.

Now, we apply the Plancherel theorem [16]. Let $\mathcal{M}$ be the set of G_∞–classes of Levi subgroups M (including G_∞) such that M^1 has representations in the discrete series. We identify $\mathcal{M}$ with the set of representatives taken among Levi subgroups of standard parabolic subgroups. In other words, we identify $\mathcal{M} - \{G_\infty\}$ with the set $\mathcal{P}$ of representatives of the set of all standard parabolic subgroups of G_∞ under the association. If $\sigma \in \hat{M_P^1}$ is a representation in the discrete series, we write $d(\sigma)$ for its formal degree. Now, we write the Plancherel theorem. We can fix measures on $\sqrt{-1}\mathfrak{a}_P^*$ and on the unitary dual $\hat{M_P^1}$ of M_P^1 such that

$$\varphi(1) = \sum_{\substack{\pi \text{ is in the discrete} \\ \text{series for } G_\infty}} d(\pi)\cdot\mathbf{tr}(\pi(\varphi)) + \sum_{P\in\mathcal{P}}\int_{\sqrt{-1}\mathfrak{a}_P^*}\int_{\hat{M_P^1}}\mathbf{tr}(\mathrm{Ind}_P^{G_\infty}(\nu,\sigma)(\varphi))d\nu\ d\sigma. \tag{5-6}$$

Since $\mathbf{tr}(\pi(\varphi)) = 0$ for every irreducible admissible representation π of G_∞, (5-6) implies $\varphi(1) = 0$. Finally, we observe that for $g_0 \in G_\infty$, then we can apply the above consideration to $r_{g_0}\varphi$, where $r_{g_0}\varphi(g) = \varphi(gg_0)$. Hence $r_{g_0}\varphi(1) = \varphi(g_0) = 0$, for all $g_0 \in G_\infty$. This proves the proposition. □

6. Spectral Decomposition of Adelic Poincaré Series

In this section we study the spectral decomposition of the Poincaré series defined by Theorem 3-2. We decompose $L^2_{cusp}(G(k)\setminus G(\mathbb{A}))$ into irreducible subspaces:

$$L^2_{cusp}(G(k)\setminus G(\mathbb{A})) = \hat{\oplus}_j \mathfrak{H}^j. \tag{6-1}$$

Let $K = K_\infty \times \prod_{v\in V_f} K_v$ be a maximal compact subgroup of $G(\mathbb{A})$, where $K_v = G(\mathcal{O}_v)$ for almost all v. For each j, we find an unitary irreducible representation $(\hat{\pi}^j, \mathfrak{V}^j)$ of $G(\mathbb{A})$ which is unitary equivalent to $\mathfrak{H}^j$ and factorizable

$$\mathfrak{V}^j = \mathfrak{V}^j_\infty \hat{\otimes}_{v\in V_f} \mathfrak{V}^j_v$$

into a restricted tensor product of local irreducible unitary representations $(\hat{\pi}^j_\infty, \mathfrak{V}^j_\infty)$ of G_∞ and $(\hat{\pi}^j_v, \mathfrak{V}^j_v)$ of $G(k_v)$ $(v\in V_f)$.

The space of K–finite vectors $(\mathfrak{H}^j)_K$ in $\mathfrak{H}^j$ is isomorphic to the usual restricted tensor product $\pi^j = \pi^j_\infty \otimes_{v\in V_f} \pi^j_v$, where π^j_v (resp., π^j_∞) is a representation of $G(k_v)$ (resp., a $(\mathfrak{g}_\infty, K_\infty)$–module) on the space of K_v–finite (resp., K_∞–finite) vectors $(\mathfrak{V}^j_v)_K$ (resp., $(\mathfrak{V}^j_\infty)_K$) in $\mathfrak{V}^j_v$ (resp., $\mathfrak{V}^j_\infty$). Let χ_j be the infinitesimal character of π^j_∞.

The main result of this section is the following theorem:

Theorem 6-2. *Let S be a finite set of places of k containing all infinite places such that G is defined over $\mathcal{O}_v$ for $v\notin S$. For each $v\in S-V_\infty$, let $\mathfrak{M}_v$ be a Bernstein's equivalence class represented by $(M(k_v),\rho_v)$, where M is a Levi subgroup of G defined over k_v and ρ_v is a supercuspidal representation of $M(k_v)$. Further, for each $v\in S-V_\infty$, we fix $f_v\in C_c^\infty(G(k_v))(\mathfrak{M}_v)$ such that $f_v(1)\neq 0$. We let $f_v = char_{G(\mathcal{O}_v)}$ for $v\notin S$. For $v\in S-V_f$, we choose an open–compact subgroup L_v such that f_v is right–invariant under L_v. We define the open compact subgroup L of $G(\mathbb{A}_f)$ as follows: $L=\prod_{v\in S-V_\infty} L_v\times\prod_{v\notin S} G(\mathcal{O}_v)$. Assume that $\delta\in\hat{K}_\infty$ appears in the closure of the image of the map (3-4) (see Theorem 3-2). Let $f_\infty\in C_c^\infty(G_\infty)$ such that Theorem 3-2 (i)–(iv) hold. Next, we decompose:*

$$\textit{the orthogonal projection of } P(f) \textit{ to } L^2_{cusp}(G(k)\setminus G(\mathbb{A})) = \sum_j \psi_j, \quad \psi_j\in\mathfrak{H}^j. \tag{6-3}$$

Then we have the following:

- (i) *For all j, $\psi_j\in\mathcal{A}_{cusp}(G(k)\setminus G(\mathbb{A}))$ is right–invariant under L and transforms according to δ i.e., $E_\delta(\psi_j)=\psi_j$.* [2]
- (ii) *Assume $\psi_j\neq 0$. Then π^j_v belongs to the Bernstein's class of $(M(k_v),\rho_v)$ for all $v\in S-V_\infty$.*
- (iii) *Assume that $P(f)$ is cuspidal. Then the number of indices j in (6-3) such that $\psi_j\neq 0$ is infinite. Moreover, let χ be an infinitesimal character. Then there are only finitely many indices j such that $\psi_j\neq 0$ and $\chi_j=\chi$.*
- (iv) *Assume that $P(f)$ is cuspidal. Then there exists infinitely many irreducible unitary representations of G_∞ which contain δ and belong to $L^2_{cusp}(\Gamma_L\setminus G_\infty)$; their $(\mathfrak{g}_\infty, K_\infty)$–modules are among the modules π^j_∞. More precisely, a $(\mathfrak{g}_\infty,K_\infty)$–module X is a K_∞–finite part of a such representation if and only if there exists j such that $\psi_j|_{G_\infty}\neq 0$ and $X\simeq\pi^j_\infty$.*

Proof. First, Theorem 3-2 (iii) implies $E_\delta(P(f)) = P(f)$ and $P(f)$ is right–invariant under L. Hence the same is true for the orthogonal projection of $P(f)$ to $L^2_{cusp}(G(k)\setminus G(\mathbb{A}))$. As it has a

[2]Obviously, $z.\psi_j = \chi_j(z)\psi_j$, for all $z\in\mathcal{Z}(\mathfrak{g}_\infty)$.

unique spectral decomposition, we obtain $E_\delta(\psi_j)=\psi_j$ and that ψ_j is right–invariant under L. It remains to show that $\psi_j \in \mathcal{A}_{cusp}(G(k)\setminus G(\mathbb{A}))$. But ψ_j is K_∞–finite and L–invariant, hence it belongs to K–finite part of $\mathfrak{V}_j$. In particular, the discussion before the statement of the theorem shows that ψ_j is also $\mathcal{Z}(\mathfrak{g}_\infty)$–finite. Now, the claim follows from ([9], 4.3 (ii)).

We prove (ii). One triviality is seen form (6-3). Namely, for all j such that $\psi_j \neq 0$ we have the following:

$$\text{(6-4)} \qquad \int_{G(k)\setminus G(\mathbb{A})} P(f)(g)\overline{\psi_j(g)}dg = \int_{G(k)\setminus G(\mathbb{A})} \psi_j(g)\overline{\psi_j(g)}dg > 0.$$

Now, we unfold the integral on the left–hand–side of (6-4)

$$\text{(6-5)} \quad 0 < \int_{G(k)\setminus G(\mathbb{A})} P(f)(g)\overline{\psi_j(g)}dg = \int_{G(k)\setminus G(\mathbb{A})} \left(\sum_{\gamma\in G(k)} f(\gamma\cdot g)\right)\overline{\psi_j(g)}dg =$$

$$\int_{G(k)\setminus G(\mathbb{A})} \left(\sum_{\gamma\in G(k)} f(\gamma\cdot g)\overline{\psi_j(\gamma\cdot g)}\right)dg = \int_{G(\mathbb{A})} \overline{\psi_j(g)}f(g)dg.$$

We remind the reader that $F \in C_c^\infty(G(\mathbb{A}))$ acts on a closed $G(\mathbb{A})$–invariant subspace $\mathfrak{H}$ of $L^2(G(k)\setminus G(\mathbb{A}))$ by the following formula:

$$F.\psi(g) = \int_{G(\mathbb{A})} \psi(gh)F(h)dh, \quad \psi \in \mathfrak{H}.$$

Also, the space $\overline{\mathfrak{H}}$, consisting of all $\overline{\psi}$, $\psi \in \mathfrak{H}$, is $G(\mathbb{A})$–invariant and closed. It is clear that $\mathfrak{H}$ is irreducible if and only if $\overline{\mathfrak{H}}$ is irreducible. It is a contragredient representation of $\mathfrak{H}$. Below, we denote by $\widetilde{\pi}$ the contragredient representation of π.

Next, we observe that $\overline{\psi_j} \in C^\infty(G(k)\setminus G(\mathbb{A}))$ since ψ_j is an automorphic form by (i). Hence $f.\overline{\psi_j}(g) = \int_{G(\mathbb{A})} \overline{\psi_j}(gh)f(h)dh$ again belongs to $C^\infty(G(k)\setminus G(\mathbb{A}))$. Hence, the inequality in (6-4) implies that $f.\overline{\psi_j}$ is not identically zero. Hence, using the notation introduced before the statement of the theorem, we obtain

$$0 \neq \widetilde{\widetilde{\pi}}^j(f) = \widetilde{\widetilde{\pi}}^j_\infty(f_\infty)\hat{\otimes}_{v\in V_f}\widetilde{\widetilde{\pi}}^j_v(f_v).$$

This implies $\widetilde{\widetilde{\pi}}^j_v(f_v) \neq 0$ for all $v \in V_f$. Hence $\widetilde{\pi}^j_v(f_v) \neq 0$ for all $v \in V_f$. Now, (ii) follows from Lemma 4-2 (ii).

We prove (iii). Assume that $P(f)$ is cuspidal. Then, it is equal to its orthogonal projection to $L^2_{cusp}(G(k)\setminus G(\mathbb{A}))$. If the sum in (6-3) is finite, we would obtain that $P(f) \in \mathcal{A}_{cusp}(G(k)\setminus G(\mathbb{A}))$. Hence, it is $\mathcal{Z}(\mathfrak{g}_\infty)$–finite and K_∞–finite. The same is true for its restriction to G_∞ which is a non–zero compactly supported Poincaré series for Γ_L (see (1-7) for a definition) by Theorem 3-2 and Propositiom 2-2. Hence $P(f)|_{G_\infty}$ is real analytic, but its support is contained in the set of the form described by Theorem 3-2 (iv). This is easy to see that this is a contradiction applying the argument from the proof given in the very last part of ([19], Section 3). Finally, by a theorem of Harish–Chandra ([9], 4.3 (i)), the space of all automorphic forms on $G(\mathbb{A})$ which are right–invariant under L, transforms according to δ and have infinitesimal character χ is finite–dimensional. Since non–zero functions among ψ_j are linearly independent, there must exist only finitely many indices j such that $\psi_j \neq 0$ and $\chi_j = \chi$. This completes the proof of (iii).

Finally, we prove (iv). First, Propositiom 2-2 shows that $P(f)|_{G_\infty}$ is Γ_L–cuspidal. Clearly, $E_\delta(P(f)|_{G_\infty}) = P(f)|_{G_\infty}$. Now, Theorem 3-2 (iv) and the proof given in the very last part of ([19], Section 3) imply that there exists infinitely many irreducible unitary representations of G_∞

which contain δ and belong to $L^2_{cusp}(\Gamma_L \setminus G_\infty)$. Next, we describe a relation between the spectral decomposition of $P(f)$ when cuspidal and that of $P(f)|_{G_\infty} \in L^2_{cusp}(\Gamma_L \setminus G_\infty)$. First, we recall some statements that are contained in [9] implicitly. Let $C \subset G(\mathbb{A}_f)$ be the minimal set such that $G(\mathbb{A}) = G(k) \cdot C \cdot G_\infty \cdot L$. Such C always exists [6]. We may assume that $1 \in C$. The minimality of C implies that the classes $G(k) \cdot c \cdot G_\infty \cdot L$ $(c \in C)$ are disjoint. One can easily show that they are open and closed in $G(\mathbb{A})$. Let $\varphi \in C_c(G(k) \setminus G(\mathbb{A})$ be supported in $G(k) \cdot c \cdot G_\infty \cdot L$ and right–invariant under L, then one can show the following integration formula:

$$\int_{G(k)\setminus G(\mathbb{A})} \varphi(g)dg = vol_{G(\mathbb{A}_f)}(L) \cdot \int_{\Gamma_{cLc^{-1}}\setminus G_\infty} \varphi(g_\infty, c)dg_\infty$$

arguing as in the proof of Lemma 2-3. We remaind the reader that $\Gamma_{cLc^{-1}}$ is a congruence subgroup attached to the open compact subgroup $cLc^{-1} \subset G(\mathbb{A}_f)$ (see (1-7)). This implies that the map

$$L^2(G(k) \setminus G(\mathbb{A}))^L \to \oplus_{c\in C} L^2(\Gamma_{cLc^{-1}} \setminus G_\infty),$$

defined by

$$\varphi \mapsto \oplus_{c\in C} \varphi|_{G_\infty \times \{c\}}$$

is a unitary equivalence of (unitary) representations of G_∞. In particular, the projection to a component $L^2(\Gamma_{cLc^{-1}} \setminus G_\infty)$ is a continuous G_∞–map. Next, it is indicated in [9] (and easy to check) that we have the following isomorphism using the same map:

$$\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))^L \simeq \oplus_{c\in C} \mathcal{A}_{cusp}(\Gamma_{cLc^{-1}} \setminus G_\infty),$$

which is now an equivalence of $(\mathfrak{g}_\infty, K_\infty)$–modules. In the same way we obtain the unitary equivalence

$$L^2_{cusp}(G(k) \setminus G(\mathbb{A}))^L \simeq \oplus_{c\in C} L^2_{cusp}(\Gamma_{cLc^{-1}} \setminus G_\infty). \tag{6-6}$$

(Actually, the cuspidality in both cases can be treated using the methods of Lemma 2-3. We leave the details to the reader.)

Since $P(f)$ is cuspidal, $P(f) = \sum_j \psi_j$ is a decomposition in $L^2_{cusp}(G(k)\setminus G(\mathbb{A}))^L$. Thus, the corresponding decomposition in $L^2_{cusp}(\Gamma_{cLc^{-1}} \setminus G_\infty)$ is the following one: $P(f)|_{G_\infty\times\{c\}} = \sum_j \psi_j|_{G_\infty\times\{c\}}$, for all $c \in C$. Above discussion shows that $\psi_j|_{G_\infty\times\{c\}} \in \mathcal{A}_{cusp}(\Gamma_{cLc^{-1}} \setminus G_\infty)$. In particular, we have the following:

$$P(f)|_{G_\infty} = \sum_j \psi_j|_{G_\infty}, \quad \psi_j|_{G_\infty} \in \mathcal{A}_{cusp}(\Gamma_L \setminus G_\infty). \tag{6-7}$$

Finally, assume $\psi_j|_{G_\infty} \neq 0$. Then the closed G_∞–invariant subspace of $L^2(G(k) \setminus G(\mathbb{A}))^L$ generated by ψ_j is a direct sum of copies of $\hat{\pi}^j_\infty$ (see the begining of this section for the notation). (The number of copies is finite since it must be finite in each $L^2_{cusp}(\Gamma_{cLc^{-1}} \setminus G_\infty)$ (see (6-6)). Since the projection to $L^2_{cusp}(\Gamma_L \setminus G_\infty)$ in (6-6) is a bounded G_∞–map which is the restriction to G_∞, it follows that $\psi_j|_{G_\infty}$ generates a closed G_∞–invariant subspace of $L^2_{cusp}(\Gamma_L \setminus G_\infty)$ which is isomorphic to the direct sum of finitely many copies of $\mathfrak{V}^j_\infty$. Because of (6-7), only such unitary representations of G_∞–contribute to the spectral decomposition of $P(f)|_{G_\infty}$. Now, a well–known equivalence between irreducible unitary representations of G_∞ and unitarizable $(\mathfrak{g}_\infty, K_\infty)$–modules proves (iv). This completes the proof of the theorem. □

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