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RECONSTRUCTION OF THE TRANSMISSION COEFFICIENT FOR STEPLIKE FINITE-GAP BACKGROUNDS

IRYNA EGOROVA AND GERALD TESCHL

ABSTRACT. We consider scattering theory for one-dimensional Jacobi operators with respect to steplike quasi-periodic finite-gap backgrounds and show how the transmission coefficient can be reconstructed from minimal scattering data. This generalizes the Poisson–Jensen formula for the classical constant background case.

1. INTRODUCTION

In classical one-dimensional scattering theory the transmission coefficient can be reconstructed from the reflection coefficient via the well-known Poisson–Jensen formula. This formula plays a crucial role in inverse scattering theory since it shows how to compute the left scattering data from the right one and vice versa. Moreover, it is also one of the key ingredients for deriving the associated sum rules which have attracted an enormous amount of interest recently (see e.g. [17], [21], [23], [24], [31]). Furthermore, these sum rules are intimately connected with conserved quantities of the associated completely integrable lattices (see [26], [27]). Finally, the reconstruction formula can be viewed as the solution of a scalar Riemann–Hilbert factorization problem which arises in the nonlinear steepest decent method [5] when deriving the long-time asymptotics (see [13], [18], respectively [19]).

Moreover, the same is true in case of scattering with respect to a finite-gap background [6], [30]. In this situation the analogous formula was given in [28] including the associated sum rules (see also [7], [9], [22]). Again there is a close relation with the solution of a scalar Riemann–Hilbert factorization problem on the underlying Riemann surface which arises in the nonlinear steepest decent method [14], [15], [16], and [20].

However, while scattering theory with a steplike constant background is classical and goes back to the early sixties [4] (see [2] for the most recent results), even in this case the reconstruction formula is unknown to the best of our knowledge except for the case when the two spectra overlap. This might be related to the fact that the case where the two spectra do not overlap will not be solved in terms of elementary methods, but will already require tools from the theory of elliptic surfaces, as we will see below. In case of steplike finite-gap backgrounds, scattering theory is again well-understood by now [8], [10], however, the reconstruction formula is again unknown to the best of our knowledge (except for the case of two finite-gap backgrounds in the same isospectral class [8]). The main purpose of our paper is to fill this gap

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and provide a reconstruction formula for the left/right transmission coefficient in terms of the left/right scattering data.

2. Notation

We begin by introducing some required background from the theory of hyperelliptic curves to be used in the remainder of this article. For further information and proofs we refer for instance to [1], [3], [11], [12], or [27].

Let $\mathbb{M}$ be the Riemann surface associated with the function $P^{1/2}(z)$, where

$$P(z) = \prod_{j=0}^{2g+1} (z - E_j), \qquad E_0 < E_1 < \cdots < E_{2g+1}, \tag{2.1}$$

$g \in \mathbb{N}$. $\mathbb{M}$ is a compact, hyperelliptic Riemann surface of genus g. We will choose $P^{1/2}(z)$ as the fixed branch

$$P^{1/2}(z) = -\prod_{j=0}^{2g+1} \sqrt{z - E_j}, \tag{2.2}$$

where $\sqrt{.}$ is the standard root with branch cut along $(-\infty, 0)$.

A point on $\mathbb{M}$ is denoted by $p = (z, \pm P^{1/2}(z)) = (z, \pm)$, $z \in \mathbb{C}$. The two points at infinity are denoted by $p = \infty_\pm$. We use $\pi(p) = z$ for the projection onto the extended complex plane $\mathbb{C} \cup \{\infty\}$. The points $\{(E_j, 0), 0 \le j \le 2g+1\} \subseteq \mathbb{M}$ are called branch points and the sets

$$\Pi_\pm = \{(z, \pm P^{1/2}(z)) \mid z \in \mathbb{C}\backslash\Sigma\} \subset \mathbb{M}, \qquad \Sigma = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}], \tag{2.3}$$

are called upper and lower sheet, respectively. Note that the boundary of $\Pi_\pm$ consists of two copies of Σ corresponding to the two limits from the upper and lower half plane.

Let $\{a_j, b_j\}_{j=1}^g$ be loops on the Riemann surface $\mathbb{M}$ representing the canonical generators of the fundamental group $\pi_1(\mathbb{M})$. We require a_j to surround the points E_{2j-1}, E_{2j} (thereby changing sheets twice) and b_j to surround E_0, E_{2j-1} counterclockwise on the upper sheet, with pairwise intersection indices given by

$$a_j \circ a_k = b_j \circ b_k = 0, \qquad a_j \circ b_k = \delta_{jk}, \qquad 1 \le j, k \le g. \tag{2.4}$$

The corresponding canonical basis $\{\zeta_j\}_{j=1}^g$ for the space of holomorphic differentials can be constructed by

$$\underline{\zeta} = \sum_{j=1}^{g} \underline{c}(j) \frac{\pi^{j-1} d\pi}{P^{1/2}}, \tag{2.5}$$

where the constants $\underline{c}(.)$ are given by

$$c_j(k) = C_{jk}^{-1}, \qquad C_{jk} = \int_{a_k} \frac{\pi^{j-1} d\pi}{P^{1/2}} = 2 \int_{E_{2k-1}}^{E_{2k}} \frac{z^{j-1} dz}{P^{1/2}(z)} \in \mathbb{R}.$$

The differentials fulfill

$$\int_{a_j} \zeta_k = \delta_{j,k}, \qquad \int_{b_j} \zeta_k = \tau_{j,k}, \qquad \tau_{j,k} = \tau_{k,j}, \qquad 1 \le j, k \le g. \tag{2.6}$$

For further information we refer to [11], [27, App. A].

In addition, we will need Green's function (in the potential theoretic sense) of the upper sheet Π_+:

Lemma 2.1 ([28]). *The Green function of Π_+ with pole at z_0 is given by*

$$g(z,z_0) = -\mathrm{Re} \int_{E_0}^{p} \omega_{p_0 \tilde{p}_0}, \quad p=(z,+),\ p_0=(z_0,+), \tag{2.7}$$

where $\tilde{p}_0 = \overline{p_0}^*$ (i.e., the complex conjugate on the other sheet) and ω_{pq} is the normalized Abelian differential of the third kind with poles at p and q.

Clearly, we can extend $g(z,z_0)$ to a holomorphic function on $\mathbb{M}\backslash\{p_0\}$ by dropping the real part. By abuse of notation we will denote this function by $g(p,p_0)$ as well. However, note that $g(p,p_0)$ will be multivalued with jumps in the imaginary part across b-cycles. We will choose the path of integration in $\mathbb{C}\backslash[E_0,E_{2g+1}]$ to guarantee a single-valued function.

From the Green function we obtain the Blaschke factor (cf. [28])

$$B(p,\rho) = \exp\Big(g(p,\rho)\Big) = \exp\Big(\int_{E_0}^{p} \omega_{\rho\rho^*}\Big), \qquad \pi(\rho)\in\mathbb{R}, \tag{2.8}$$

which has the following properties:

Lemma 2.2. *The Blaschke factor satisfies*

$$B(E_0,\rho)=1, \quad \text{and} \quad B(p^*,\rho)=B(p,\rho^*)=B(p,\rho)^{-1}; \tag{2.9}$$

it is real-valued for $\pi(p)\in(-\infty,E_0)$.

Moreover,

$$|B(p,\rho)|=1, \quad p\in\Sigma, \qquad \arg(B(p,\rho))=\delta_j(\rho), \qquad \pi(p)\in[E_{2j-1},E_{2j}], \tag{2.10}$$

where we set $E_{-1}=-\infty$, $E_{2g+2}=\infty$ and

$$\delta_j(\rho) = \begin{cases} 0, & j=0,\\ \frac{1}{2}\int_{b_j}\omega_{\rho\rho^*}, & j=1,\dots,g,\\ 0, & j=g+1. \end{cases} \tag{2.11}$$

Proof. The first part including the fact that $|B(p,\rho)|=1$, $p\in\Sigma$, is proven in [28, Lem. 3.3]. To see the formula for the argument first observe that $\omega_{\rho\rho^*}$ is real-valued on $\pi^{-1}(\mathbb{R}\backslash\Sigma)$ and purely imaginary on $\pi^{-1}(\Sigma)$. This can be seen from the explicit expression (2.13) for $\omega_{\rho\rho^*}$ given below. Hence, taking the path of integration along the lift of the real axis, we see that the integral is real for $p=(\lambda,\pm)$ with $\lambda<E_0$ or $\lambda>E_{2g+1}$. For $p=(\lambda,\pm)$ with $\lambda\in[E_{2j-1},E_{2j}]$ the imaginary part is constant and given by half the b_j period. □

The above Abelian differential is explicitly given by

$$\omega_{pq} = \left(\frac{P^{1/2}+P^{1/2}(p)}{2(\pi-\pi(p))} - \frac{P^{1/2}+P^{1/2}(q)}{2(\pi-\pi(q))} + P_{pq}(\pi)\right)\frac{d\pi}{P^{1/2}}, \tag{2.12}$$

where $P_{pq}(z)$ is a polynomial of degree $g-1$ which has to be determined from the normalization $\int_{a_\ell}\omega_{pp^*}=0$. In particular,

$$\omega_{pp^*} = \left(\frac{P^{1/2}(p)}{\pi-\pi(p)} + P_{pp^*}(\pi)\right)\frac{d\pi}{P^{1/2}}. \tag{2.13}$$

3. Reconstructing the transmission coefficient

Let $H_q^\pm$ be two quasi-periodic finite-band Jacobi operators,[1]

$$H_q^\pm f(n) = a_q^\pm(n) f(n+1) + a_q^\pm(n-1) f(n-1) + b_q^\pm(n) f(n), \quad f \in \ell^2(\mathbb{Z}), \tag{3.1}$$

associated with the hyperelliptic Riemann surface of the square root

$$P_\pm^{1/2}(z) = -\prod_{j=0}^{2g_\pm+1} \sqrt{z - E_j^\pm}, \qquad E_0^\pm < E_1^\pm < \cdots < E_{2g_\pm+1}^\pm, \tag{3.2}$$

where $g_\pm \in \mathbb{N}$ and $\sqrt{\cdot}$ is the standard root with branch cut along $(-\infty, 0)$. In fact, $H_q^\pm$ are uniquely determined by fixing a Dirichlet divisor $\sum_{j=1}^{g^\pm} \hat\mu_j^\pm$, where $\hat\mu_j^\pm = (\mu_j^\pm, \sigma_j^\pm)$ with $\mu_j^\pm \in [E_{2j-1}^\pm, E_{2j}^\pm]$ and $\sigma_j^\pm \in \{-1, 1\}$. The spectra of $H_q^\pm$ consist of $g_\pm + 1$ bands

$$\sigma_\pm := \sigma(H_q^\pm) = \bigcup_{j=0}^{g_\pm} [E_{2j}^\pm, E_{2j+1}^\pm]. \tag{3.3}$$

We are interested in scattering theory for the operator

$$H f(n) = a(n-1) f(n-1) + b(n) f(n) + a(n) f(n+1), \quad n \in \mathbb{Z}, \tag{3.4}$$

whose coefficients are asymptotically close to the coefficients of $H_q^\pm$ on the corresponding half-axes:

$$\sum_{n=0}^{\pm\infty} |n| \Big(|a(n) - a_q^\pm(n)| + |b(n) - b_q^\pm(n)| \Big) < \infty. \tag{3.5}$$

Let $\psi_q^\pm(z,n)$ be the Floquet solutions of the spectral equations

$$H_q^\pm \psi(n) = z \psi(n), \qquad z \in \mathbb{C}, \tag{3.6}$$

that decay for $z \in \mathbb{C} \setminus \sigma_\pm$ as $n \to \pm\infty$. They are uniquely defined by the condition $\psi_q^\pm(z,0) = 1$, $\psi_q^\pm(z,\cdot) \in \ell^2(\mathbb{Z}_\pm)$. The solution $\psi_q^+(z,n)$ (resp. $\psi_q^-(z,n)$) coincides with the upper (resp. lower) branch of the Baker–Akhiezer functions of H_q^+ (resp. H_q^-), see [27].

The two solutions $\psi_\pm(z,n)$ of the spectral equation

$$H\psi = z\psi, \quad z \in \mathbb{C}, \tag{3.7}$$

which are asymptotically close to the Floquet solutions $\psi_q^\pm(z,n)$ of the background equations (3.6) as $n \to \pm\infty$, are called Jost solutions.

Next, we introduce the sets

$$\sigma^{(2)} = \sigma_+ \cap \sigma_-, \quad \sigma_\pm^{(1)} = \overline{\sigma_\pm \setminus \sigma^{(2)}}, \quad \sigma = \sigma_+ \cup \sigma_-, \tag{3.8}$$

where σ is the (absolutely) continuous spectrum of H and $\sigma_+^{(1)} \cup \sigma_-^{(1)}$, $\sigma^{(2)}$ are the parts which are of multiplicity one, two, respectively.

In addition to the continuous part, H has a finite number of eigenvalues situated in the gaps, $\sigma_d = \{\lambda_1, ..., \lambda_s\} \subset \mathbb{R} \setminus \sigma$ (see, e.g., [25]). For every eigenvalue we introduce the corresponding norming constants

$$\gamma_{\pm,k}^{-1} = \sum_{n \in \mathbb{Z}} |\psi_\pm(\lambda_k, n)|^2, \quad 1 \le k \le s. \tag{3.9}$$

[1] Everywhere in this paper the sub or super index "+" (resp. "−") refers to the background on the right (resp. left) half-axis.

Note that this definition has to be slightly modified in the unlikely event that $\psi_q^{\pm}(z,n)$ and hence $\psi_{\pm}(z,n)$ has a pole at $z=\lambda_k$ (see [10] for details). The transmission and reflection coefficients are defined as usual via the scattering relations

$$T_{\mp}(\lambda)\psi_{\pm}(\lambda,n)=\overline{\psi_{\mp}(\lambda,n)}+R_{\mp}(\lambda)\psi_{\mp}(\lambda,n),\quad \lambda\in\sigma_{\mp}. \tag{3.10}$$

Here the values of $\psi_{\pm}(\lambda,n)$ for $\lambda\in\sigma_{\pm}$ are to be understood as limits from above $\psi_{\pm}(\lambda,n)=\lim_{\varepsilon\downarrow 0}\psi_{\pm}(\lambda+\mathrm{i}\varepsilon,n)$ (the corresponding limits from below just give the complex conjugate values $\overline{\psi_{\pm}(\lambda,n)}=\lim_{\varepsilon\downarrow 0}\psi_{\pm}(\lambda-\mathrm{i}\varepsilon,n)$).

The following result is an immediate consequence of [10, Lem. 5.1].

Theorem 3.1 ([10])**.** *Suppose $a(n)$, $b(n)$ satisfy* (3.4)*, then $a(n)$, $b(n)$ are uniquely determined by one of the sets of its "partial" scattering data $\mathcal{S}_+$ or $\mathcal{S}_-$, where*

$$\begin{aligned}\mathcal{S}_{\pm}=\Big\{&R_{\pm}(\lambda),\ \lambda\in\sigma_{\pm};\ |T_{\pm}(\lambda)|^2,\ \lambda\in\sigma_{\mp}^{(1)};\\ &\lambda_1,\dots,\lambda_s\in\mathbb{R}\setminus(\sigma_+\cup\sigma_-),\ \gamma_{\pm,1},\dots,\gamma_{\pm,s}\in\mathbb{R}_+\Big\}.\end{aligned} \tag{3.11}$$

This leads to the natural question if there is a simple way to compute $\mathcal{S}_+$ from $\mathcal{S}_-$ and vice versa (i.e., without solving the inverse scattering problem). It turns out that this question reduces to the reconstruction of the transmission coefficient $T_{\pm}(z)$ from $\mathcal{S}_{\pm}$. In fact, this follows from the following lemma.

Lemma 3.2 ([10])**.** *The transmission coefficients $T_{\pm}(z)$ admit a meromorphic extension to $\mathbb{C}\backslash\sigma$. In general they have simple poles at the eigenvalues λ_k of H. In addition, there are simple poles at $\mu_j^{\pm}\in\mathbb{R}\backslash\sigma_{\pm}$ which are not poles of $\psi_q^{\pm}(z,1)$ (i.e., $\sigma_j^{\pm}=\mp 1$) and simple zeros at $\mu_j^{\mp}\in\mathbb{R}\backslash\sigma_{\mp}$ which are poles of $\psi_q^{\mp}(z,1)$ (i.e., $\sigma_j^{\mp}=\mp 1$). A pole at $\mu_j^{\pm}$ could cancel with a zero at $\mu_j^{\mp}$ or could give a second order pole if $\mu_j^{\pm}=\lambda_k$.*

Moreover, the entries of the scattering matrix have the following properties:

$$\begin{array}{lll}\textbf{(a)} & \rho_+(z)T_+(z)=\rho_-(z)T_-(z), & \\ \textbf{(b)} & \dfrac{T_{\pm}(\lambda)}{\overline{T_{\pm}(\lambda)}}=R_{\pm}(\lambda), & \lambda\in\sigma_{\pm}^{(1)},\\ \textbf{(c)} & 1-|R_{\pm}(\lambda)|^2=\dfrac{\rho_{\pm}(\lambda)}{\rho_{\mp}(\lambda)}\,|T_{\pm}(\lambda)|^2, & \lambda\in\sigma^{(2)},\\ \textbf{(d)} & \overline{R_{\pm}(\lambda)}T_{\pm}(\lambda)+R_{\mp}(\lambda)\overline{T_{\pm}(\lambda)}=0, & \lambda\in\sigma^{(2)},\end{array}$$

where

$$\rho_{\pm}(z)=\frac{\prod_{j=1}^{g_{\pm}}(z-\mu_j^{\pm})}{P_{\pm}^{1/2}(z)}. \tag{3.12}$$

Hence, the problem is to reconstruct the meromorphic function $T_+(z)$, $z\in\mathbb{C}\backslash\sigma$ from its boundary values

$$\begin{cases}|T_+(\lambda)|^2, & \lambda\in\sigma_-^{(1)},\\ |T_+(\lambda)|^2=\frac{\rho_-(\lambda)}{\rho_+(\lambda)}(1-|R_+(\lambda)|^2), & \lambda\in\sigma^{(2)},\\ \dfrac{T_+(\lambda)}{\overline{T_+(\lambda)}}=R_+(\lambda), & \lambda\in\sigma_+^{(1)}.\end{cases} \tag{3.13}$$

That is, we know its absolute value on σ_- and its argument on the rest $\sigma_+^{(1)}$. There will be three Riemann surfaces involved, the one corresponding to $\sigma=\sigma_+\cup\sigma_-$ and the ones corresponding to $\sigma_{\pm}$. All objects corresponding to σ will be denoted as in Section 2, while the objects associated with $\sigma_\pm$ will have an additional $\pm$ sub/supscript.

Theorem 3.3. *The transmission coefficient $T_+(z)$ can be reconstructed from the reflection coefficient $R_+(z)$ and the eigenvalues λ_j via*

$$T_+(z) = \Bigg(\prod_{\mu_j^-\in M^-} B_-(z,\mu_j^-)\Bigg)\Bigg(\prod_{\mu_j^+\in M^+} B_-(z,\mu_j^+)^{-1}\Bigg)\Bigg(\prod_{k=1}^{s} B_-(z,\lambda_k)^{-1}\Bigg)\times$$

$$\exp\Bigg(\frac{Q(z)^{-1}}{\pi\mathrm{i}}\int_{\sigma_-^{(1)}} Q\log(|T_+|)\omega_{zz^*} \tag{3.14}$$

$$+\frac{Q(z)^{-1}}{2\pi\mathrm{i}}\int_{\sigma^{(2)}} Q\,\Big(\log\big(\frac{\rho_-}{\rho_+}\big)+\log(1-|R_+|^2)\Big)\omega_{zz^*}$$

$$+\frac{Q(z)^{-1}}{2\pi}\int_{\sigma_+^{(1)}} Q\,\Big(\arg(R_+)+\delta^-\Big)\omega_{zz^*}\Bigg), \tag{3.15}$$

where the integrals are taken over the lift of the indicated spectra to the upper sheet Π_u (of the Riemann surface associated with σ). Moreover, we use the convention that we identify z with $(z,+)$, and similarly for λ_k, $\mu_j^\pm$, whenever used in the argument of a function defined on a Riemann surface. Here

$$M^\pm = \{\mu_j^\pm \,|\, \mu_j^\pm\in\mathbb{R}\setminus\sigma \text{ and } \sigma_j^\pm = -1\}, \tag{3.16}$$

$$Q(z) = \prod_j \sqrt{z-e_j}, \quad \text{where } e_j \text{ are defined via } \bigcup_j [e_{2j}, e_{2j+1}] = \sigma_+^{(1)}, \tag{3.17}$$

and

$$\delta^-(\lambda) = \sum_\ell \delta_\ell^- \chi_{[E_{2\ell-1}^-, E_{2\ell}^-]}(\lambda) \tag{3.18}$$

with (cf. Lemma 2.2)

$$\delta_\ell^- = -\sum_{\mu_j^-\in M^-}\delta_\ell^-(\mu_j^-) + \sum_{\mu_j^+\in M^+}\delta_\ell^-(\mu_j^+) + \sum_{k=1}^{s}\delta_\ell^-(\lambda_k). \tag{3.19}$$

Proof. We start by considering the multivalued function

$$t_+(z) = \Bigg(\prod_{\mu_j^-\in M^-} B_-(z,\mu_j^-)^{-1}\Bigg)\Bigg(\prod_{\mu_j^+\in M^+} B_-(z,\mu_j^+)\Bigg)\Bigg(\prod_{k=1}^{s} B_-(z,\lambda_k)\Bigg)T_+(z) \tag{3.20}$$

which has neither zeros nor poles on Π_u and satisfies

$$\begin{cases} |t_+(\lambda)|^2 = |T_+(\lambda)|^2, & \lambda\in\sigma_-,\\ \arg(t_+(\lambda)) = \arg(T_+(\lambda)) + \delta_\ell^-, & \lambda\in\sigma_+^{(1)}\cap[E_{2\ell-1}^-, E_{2\ell}^-]. \end{cases} \tag{3.21}$$

Moreover, the absolute value of $t_+(z)$ is single-valued and hence its logarithm is a harmonic function on Π_u which can be reconstructed from its boundary values. To accommodate the fact that we know its absolute value on σ_- and its argument on $\sigma_+^{(1)}$ we consider

$$Q(z)\log(t_+(z)). \tag{3.22}$$

Note that since $t_+(z)$ might still have zeros and poles on σ, the function $\log(t_+(z))$ might have logarithmic singularities on σ.

Since $Q(\lambda)$ is real-valued for $\lambda\in\mathbb{R}\backslash\sigma_+^{(1)}$ and purely imaginary for $\lambda\in\sigma_+^{(1)}$, we infer that the real part of $Q(z)\log(t_+(z))$ is harmonic on Π_u and can be reconstructed from its boundary values

$$\mathrm{Re}\big(Q(\lambda)\log(t_+(\lambda))\big)=\begin{cases} Q(\lambda)\log(|T_+(\lambda)|), & \lambda\in\sigma_-,\\ \mathrm{i}Q(\lambda)\big(\arg(T_+(\lambda))+\delta_\ell^-\big), & \lambda\in\sigma_+^{(1)}\cap[E_{2\ell-1}^-,E_{2\ell}^-],\end{cases} \tag{3.23}$$

using Green's function:

$$\begin{aligned}\mathrm{Re}\big(Q(z)\log(t_+(z))\big)=&\mathrm{Re}\Bigg(\frac{1}{\pi\mathrm{i}}\int_{\sigma_-}Q\log(|T_+|)\omega_{zz^*}\\ &+\frac{1}{\pi}\int_{\sigma_+^{(1)}}Q\big(\arg(T_+)+\delta^-\big)\omega_{zz^*}\Bigg).\end{aligned} \tag{3.24}$$

Dropping the real part we get

$$\begin{aligned}T_+(z)=&\Bigg(\prod_{\mu_j^-\in M^-}B_-(z,\mu_j^-)\Bigg)\Bigg(\prod_{\mu_j^+\in M^+}B_-(z,\mu_j^+)^{-1}\Bigg)\Bigg(\prod_{k=1}^{s}B_-(z,\lambda_k)^{-1}\Bigg)\times\\ &\exp\Bigg(\frac{Q(z)^{-1}}{\pi\mathrm{i}}\int_{\sigma_-}Q\log(|T_+|)\omega_{zz^*}+\frac{Q(z)^{-1}}{\pi}\int_{\sigma_+^{(1)}}Q\big(\arg(T_+)+\delta^-\big)\omega_{zz^*}\Bigg).\end{aligned} \tag{3.25}$$

In fact, by [29, Thm. 1] both the left-hand and the right-hand side have the same absolute value and hence can only differ by a constant with absolute value one (in particular, the right-hand side is single-valued since the left-hand side is). This constant must be one since both sides are real-valued for real-valued z to the left of σ. □

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