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Vienna, Preprint ESI 1979 (2007) November 21, 2007

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at

On the large N expansion in hyperbolic sigma-models

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Abstract. Invariant correlation functions for $\mathrm{SO}(1, N)$ hyperbolic sigma-models are investigated. The existence of a large N asymptotic expansion is proven on finite lattices of dimension $d \geq 2$. The unique saddle point configuration is characterized by a negative gap vanishing at least like $1/V$ with the volume. Technical difficulties compared to the compact case are bypassed using horospherical coordinates and the matrix-tree theorem.

Mathematics Subject Classification (2000) 41A60, 82B80.

Keywords hyperbolic sigma-models, large N.

1. Introduction

Noncompact sigma-models differ in several non-manifest ways from compact ones. Among the differences is the fact that in a large N expansion of the $d \geq 2$ dimensional lattice systems the dynamically generated gap is negative and vanishes in the limit of infinite lattice size [4, 7]. The termwise defined infinite volume limits of invariant correlation functions also do not show exponential clustering [7, 5]. The justification of the large N expansion in the noncompact models likewise has to proceed differently as the dualization procedure familiar from the compact models is ill-defined.

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The goal of this letter is to present a solid justification of the $1/N$ expansion for $\mathrm{SO}(1,N)$ invariant nonlinear sigma-models on a finite lattice. The main result is a proof that the $1/N$ expansion of invariant correlation functions is asymptotic to all orders in any finite volume $V = L^d$, $d \geq 2$. The 'dual' action used to generate the expansion arises by performing Gaussian integrals in horospherical coordinates, thereby reducing the number of dynamical degrees of freedom per site from N to 1. This dual action was used in [7] to relate the large N coefficients of the $\mathrm{SO}(1,N)$ model to those of its compact $\mathrm{SO}(N+1)$ invariant counterpart. A heuristic derivation based on a dualization procedure was outlined in Appendix C of [5] using [4], where also the approach to the $N \to \infty$ limit was checked numerically. In [7] we verified algebraically that due to a sign reversal (compared to the compact model) in the 1-loop polarization function the relevant 'saddle point' does indeed represent a *local* minimum of the dual action.

The relevant saddle point is in fact a *global* minimum of the dual action, as one can probe numerically. This fact is proven here by a convexity argument based on Kirchhoff's matrix-tree theorem; a side result is that this saddle point is the *only* critical point of the dual action in the domain of integration. The main theorem then readily follows.

What we cannot show so far is uniform asymptoticity of the expansion in the volume, which would then show that the termwise thermodynamic limit yields the correct asymptotic expansion of the model in infinite volume. Kupiainen [6] managed to show the corresponding result for the compact $\mathrm{O}(N)$ models for the region of high temperature (higher than the critical temperature of the limiting spherical model), but his proof relies in an essential way on features absent in the hyperbolic models: in the $\mathrm{O}(N)$ models the large N saddle point has a mass gap and exponential decay as long as one is in the high temperature regime. As emphasized before, this is not the case in the non-compact models. Direct computation suggests nevertheless the existence of a termwise thermodynamic limit [5], whose asymptoticity we have to leave open for now.

2. Invariant correlators via horospherical coordinates

We consider the $\mathrm{SO}(1,N)$ hyperbolic sigma-models with standard lattice action, defined on a hypercubic lattice $\Lambda \subset \mathbb{Z}^d$ of volume $V = |\Lambda| = L^d$. The dynamical variables ("spins") will be denoted by n_x^a, $x \in \Lambda$, $a = 0, \ldots, N$, and periodic boundary conditions are assumed throughout $n_{x+L\hat{\mu}} = n_x$. The target manifold is the upper half of the two-sheeted N-dimensional hyperboloid, i.e.

$$\begin{aligned} \mathbb{H}_N &= \{n \in \mathbb{R}^{1,N} \,|\, n \cdot n = 1,\, n^0 > 0\}\,, \\ a \cdot b &= a^c \eta_{cd} b^d = a^0 b^0 - a^1 b^1 - \ldots - a^N b^N = a^0 b^0 - \vec{a} \cdot \vec{b}\,. \end{aligned} \tag{2.1}$$

As indicated we shall also use the notation $\vec{a} = (a^1, \ldots, a^N)$ for vectors in $\mathbb{R}^N$. The isometry group of $\mathbb{H}_N$ is $\mathrm{SO}_0(1,N)$.

In terms of the hyperbolic spins the lattice action reads

$$S = \beta \sum_{x,\mu} (n_x \cdot n_{x+\hat\mu} - 1) = -\frac{\beta}{2} \sum_x n_x \cdot (\Delta n)_x \geq 0\,, \tag{2.2}$$

where $\beta > 0$ and $\Delta_{xy} = -\sum_\mu [2\delta_{x,y} - \delta_{x,y+\hat\mu} - \delta_{x,y-\hat\mu}]$, as usual. We write

$$d\Omega(n) = 2d^{N+1}n\, \delta(n\cdot n - 1)\theta(n^0)\,, \tag{2.3}$$

for the invariant measure on $\mathbb{H}_N$.

The goal in the following is to describe the invariant correlation functions $\langle n_{x_1} \cdot n_{y_1} \ldots n_{x_r} \cdot n_{y_r}\rangle$ for the lattice statistical field theory with dynamical variables n_x, $x \in \Lambda$, and action (2.2). This is conveniently done in terms of a generating functional. Since the invariance group $\mathrm{SO}_0(1,N)$ of the action (2.2) has infinite volume, the usual generating functional for connected invariant correlation functions is ill defined. A technically convenient way to gauge fix is to hold one spin, say n_{x_0}, $x_0 \in \Lambda$, fixed. Then no Faddeev-Popov determinant arises and only the complications coming from the superficial lack of translation invariance have to be dealt with. We therefore consider the following generating functional

$$\exp W[H] = \mathcal{N} \int \prod_x d\Omega(n_x) \delta(n_{x_0}, n^\uparrow)\, \exp\Big\{ - S + \frac{1}{2}\sum_{x,y} H_{xy}(n_x \cdot n_y - 1)\Big\}\,. \tag{2.4}$$

Here $\delta(n,n')$ is the invariant point measure on H^N, $n^\uparrow = (1,0,\ldots,0)$, and sources $H_{xy} = H_{yx} < 0$, $H_{xx} = 0$, give damping exponentials. The normalization $\mathcal{N} = \mathcal{N}[H]$ is chosen such that $\exp W[0] = 1$. Connected $2r$ point functions are defined by

$$W[H] = \sum_{r\geq 1} \frac{1}{r!\,2^r}\, W_r(x_1,y_1;\ldots;x_r,y_r)\, H_{x_1y_1}\ldots H_{x_ry_r}\,,$$

$$W_r(x_1,y_1;\ldots;x_r,y_r) := h_{x_1y_1}\ldots h_{x_ry_r} W[H]\Big|_{H=0}\,, \quad h_{xy} := \frac{\delta}{\delta H_{xy}}\,. \tag{2.5}$$

In particular $W_1(x,y) = \langle n_x \cdot n_y\rangle - 1$, $W_2(x_1,y_1;x_2,y_2) := \langle n_{x_1}\cdot n_{y_1} n_{x_2}\cdot n_{y_2}\rangle - \langle n_{x_1}\cdot n_{y_1}\rangle\langle n_{x_2}\cdot n_{y_2}\rangle$, where $\langle\ \rangle$ are the functional averages with respect to $\mathcal{N}^{-1}e^{-S}$. Note that $W_r(\ldots;x,x;\ldots) = 0$.

In the above we tacitly assumed that $W[H]$ and the correlation functions computed from it do not depend on the site x_0 of the frozen spin and are translation invariant. If we momentarily indicate the dependence on the site as W_{x_0} one has trivially

$$W_{x_0}[\tau_a H] = W_{x_0+a}[H]\,, \quad (\tau_a H)_{xy} = H_{x+a,y+a}\,. \tag{2.6}$$

Thus, if W_{x_0} is independent of x_0 it is also translation invariant. The Bolzmann factor f in (2.4) can be viewed as a function on the group via $F(g_0, \ldots, g_n) = f(g_0 n^{\uparrow}, \ldots, g_n n^{\uparrow})$, where we picked some ordering of the sites x_i, $i = 0, 1, \ldots, s := V-1$, and identified n_{x_i} with $g_i n^{\uparrow}$. Then W_{x_i} is of the form

$$\int \prod_j d\Omega(n_j)\, \delta(n_i, n^{\uparrow}) f(n_1, \ldots, n_s) = \text{const}\, U_i \,,$$

$$U_i := \int \prod_{j \neq i} dg_j\, F(g_1, \ldots, g_{i-1}, e, g_{i+1}, \ldots, g_s)\,. \tag{2.7}$$

Using the invariance of F under $g_i \mapsto h^{-1} g_i$ and the unimodulariy of the measure dg one verifies: $U_i = U_0$ for all i.

The hyperboloid $\mathbb{H}_N$ admits an alternative parameterization in terms of so-called horospherial coordinates. These arise naturally from the Iwasawa decomposition of $\mathrm{SO}_0(1, N)$. Here it suffices to note the relation to the hyperbolic spins

$$n^0 = \text{ch}\theta + \frac{1}{2} t^2 e^{-\theta}\,, \quad n^1 = \text{sh}\theta + \frac{1}{2} t^2 e^{-\theta}\,, \quad n^i = e^{-\theta} t_{i-1}, \ \ i = 2, \ldots, N\,, \tag{2.8}$$

and that $\mathbb{H}_N \ni n \mapsto (\theta, t_1, \ldots, t_{N-1}) \in \mathbb{R}^N$ is a bijection. It is convenient to write $\vec{t} = (t_1, \ldots, t_N)$ and $\vec{t} \cdot \vec{t}' = t_1 t_1' + \ldots t_N t_N'$. For the dot product of two spins n_x, $n_y \in \mathbb{H}_N$ this gives

$$n_x \cdot n_y = \text{ch}(\theta_x - \theta_y) + \frac{1}{2} (\vec{t_x} - \vec{t_y})^2 e^{-\theta_x - \theta_y}\,, \tag{2.9}$$

and for the measure (2.3)

$$d\Omega(n) = e^{-\theta(N-1)} d\theta\, dt_1 \ldots dt_{N-1} = e^{-\theta(N-1)} d\theta\, d\vec{t}. \tag{2.10}$$

The key advantage of horospherical coordinates is manifest from (2.9), (2.10): for a quadratic action of the form (2.2) the integrations over the $\vec{t}$ variables are Gaussian and can be performed without approximations. The result is summarized in the

PROPOSITION 2.1. *The generating functional (2.4) can be rewritten as*

$$\exp W[H] = \exp\Big\{ -\frac{1}{2} \sum_{x,y} H_{xy} \Big\} \mathcal{N} \int_{\mathcal{D}(H)} \prod_{x \neq x_0} da_x$$

$$\times \exp\Big\{ -\frac{N+1}{2} \text{Tr} \ln \widehat{A} - \frac{\beta}{2} \sum_{x \neq x_0} a_x + \frac{\beta}{2} (\widetilde{A}^{-1})^{-1}_{x_0 x_0} \Big\}\,. \tag{2.11}$$

Here

$$A_{xy} = -\Delta_{xy} + \frac{1}{\beta} H_{xy} + \delta_{xy} a_x = \widetilde{A}_{xy} + a_{x_0} \delta_{x x_0} \delta_{xy}\,, \tag{2.12}$$

and $\mathcal{D}(H)$ is an open set given by

$$\mathcal{D}(H) = \{\ a \in (2d,\infty]^{V-1}\ |\ \ \widehat{A}\ \textit{positive definite}\ \}\,. \tag{2.13}$$

Remarks. (i) Compared to (2.4) the number of dynamical variables per site has been reduced from N to 1.

(ii) The H-dependence of the domain $\mathcal{D}(H)$ will produce extra contributions in the variations with respect to H defining the multipoint functions. Their direct computation is cumbersome but their form can be inferred by first varying (2.34) and then changing variables as before. For example

$$\begin{aligned}&\frac{\partial}{\partial H_{xy}} \exp\Big\{W[H] + \frac{1}{2}\sum_{xy} H_{xy}\Big\} \qquad\qquad (2.14)\\ &= \Big\langle -\frac{N-1}{2\beta}\Big[2(\widehat{A}^{-1})_{xy} - (\widehat{A}^{-1})_{xx}\frac{r_y}{r_x} - (\widehat{A}^{-1})_{yy}\frac{r_x}{r_y}\Big] + \frac{1}{2}\Big(\frac{r_x}{r_y}+\frac{r_y}{r_x}\Big)\Big\rangle\,,\end{aligned}$$

where $r_x = r_x(a,H)$ is given by (2.22) below. This is to be compared with the right hand side arising by varying (2.11), i.e. $\langle -\lambda(\widehat{A}^{-1})_{xy} + r_x r_y + \text{boundary terms}\rangle$. As we shall see below in a large N expansion the boundary terms do not contribute and (2.11) is a convenient starting point for such an expansion.

Underlying the Proposition is a nonlocal change of variables for which we prepare the

LEMMA 2.2. *(a) Defining a_{x_0} via (2.12) the condition* $\det A = 0$ *is equivalent to*

$$a_{x_0} = -\frac{\det \widetilde{A}}{\det \widehat{A}} = -\frac{1}{(\widetilde{A}^{-1})_{x_0x_0}}\,, \tag{2.15}$$

thereby determining a_{x_0} as a function of a_x, $x \neq x_0$.
(b) The map

$$\begin{aligned}&\chi : \mathbb{R}^{V-1} \to \mathcal{D}(H)\,, && \theta_x \mapsto a_x \quad x \neq x_0\,,\\ &a_x := \frac{1}{r_x}[(\Delta - \beta^{-1}H)r]_x\,, && r_x = e^{-\theta_x}\,,\ \ x\neq x_0\,,\ \ r_{x_0} = 1\,,\end{aligned} \tag{2.16}$$

with $\mathcal{D}(H)$ as in (2.13) is a diffeomorphism.

Proof. (a) Laplace expansion with respect to the x_0-th row gives $\det A = (2d + a_{x_0})\det\widehat{A} + R$, where R is the contribution from the columns $x \neq x_0$. Similarly $\det\widetilde{A} = 2d\det\widehat{A} + R$, with the same R. Eliminating R gives $\det A - \det\widetilde{A} = a_{x_0}\det\widehat{A}$, and using $\det\widetilde{A} = \det\widehat{A}/(\widetilde{A}^{-1})_{x_0x_0}$ one finds (2.15).

(b) We define

$$\begin{aligned} \mathcal{A}_{xy} &:= \mathcal{M}_{xy} + \frac{1}{\beta}H_{xy} - \frac{1}{\beta}\delta_{xy}\sum_z e^{\theta_x-\theta_z}H_{xz}\,, \\ \mathcal{M}_{xy} &:= -\Delta_{xy} + \delta_{xy}\sum_\mu (e^{\theta_x-\theta_{x+\hat\mu}} + e^{\theta_x-\theta_{x-\hat\mu}} - 2)\,, \end{aligned} \tag{2.17}$$

and write $\widehat{\mathcal{A}}$ for the matrix obtained from $\mathcal{A}$ by deleting the x_0-th row and column. Then $\mathcal{A}$ has a null eigenvector

$$\sum_y \mathcal{A}_{xy}\, e^{-\theta_y} = 0\,, \quad \det \mathcal{A} = 0\,, \tag{2.18}$$

but $\widehat{\mathcal{A}}$ has maximal rank and is is positive definite. To see the latter it suffices to note that

$$(r\vec{t}, \mathcal{A}\, r\vec{t}) = \sum_{\langle xy\rangle}(\vec{t}_x - \vec{t}_y)^2 r_x r_y - \frac{1}{2\beta}\sum_{x,y}(\vec{t}_x - \vec{t}_y)^2 H_{xy} r_x r_y \geq 0\,, \tag{2.19}$$

is non-negative for $H_{xy} \leq 0$ and vanishes if and only if all $\vec{t}_x$ are equal.

Further, by (a)

$$\mathcal{A}\circ\chi = A\,, \tag{2.20}$$

provided a_{x_0} is determined according to (2.15). Since

$$\frac{\partial a_x}{\partial\theta_y} = e^{\theta_x}\mathcal{A}_{xy}e^{-\theta_y}\,, \quad \det\left(\frac{\partial a_x}{\partial\theta_y}\right)_{x,y\neq x_0} = \det\widehat{\mathcal{A}} > 0\,, \tag{2.21}$$

the change of variables (2.16) is locally invertible. Global invertibility is best seen from the inversion formula

$$r_x(a,H) = -\sum_{y\neq x_0}(\widehat{A}^{-1})_{xy}A_{yx_0} = \frac{(\widetilde{A}^{-1})_{xx_0}}{(\widetilde{A}^{-1})_{x_0x_0}}\,, \quad x\neq x_0\,. \tag{2.22}$$

The first equation follows from (2.18), i.e. $\sum_{y\neq x_0}\widehat{A}_{xy}r_y = -A_{xx_0}r_{x_0}$, assuming $r_{x_0} = 1$. To derive the second expression in (2.22) we extend the relation $A_{xy} = -\Delta_{xy} + \beta^{-1}H_{xy} + \delta_{xy}a_x$, to $x, y = x_0$, and momentarily choose a_{x_0} not as in (2.15), but such that $\det A \neq 0$. Then (2.27) below is applicable and gives $r_x = (A^{-1})_{xx_0}/(A^{-1})_{x_0x_0}$. On the other hand $\widetilde{A}_{xy} := A_{xy} - a_{x_0}\delta_{xx_0}\delta_{xy}$ is manifestly independent of a_{x_0} and is nondegenerate. By (2.30) below r_x equals $(\widetilde{A}^{-1})_{xx_0}/(\widetilde{A}^{-1})_{x_0x_0}$, where one is free to adjust a_{x_0} such that $\det A = 0$, as required by (2.18), (2.20). One can also insert (2.22) into (2.18) and finds

$$\sum_y A_{xy}r_y(a,H) = \delta_{xx_0}[a_{x_0} + (\widetilde{A}^{-1})^{-1}_{x_0x_0}]\,, \tag{2.23}$$

consistent with (a).

So far $\mathcal{D}(H)$ entered as the image of $\mathbb{R}^{V-1}$ under χ. By definition of $r_x = \exp(-\theta_x)$ the domain $\mathcal{D}(H)$ is characterized by the condition $r_x(a,H) = (\widetilde{A}^{-1})_{xx_0}/(\widetilde{A}^{-1})_{x_0x_0} > 0$. We verify that this is also equivalent to the positive definiteness of the matrix $\widehat{A}$: First assume that all $r_x > 0$. Then by (2.19), (2.20) $\widehat{A}$ is positive definite. Conversely, assume that $\widehat{A}$ is positive definite, but that there is a y_0 such that $r_{y_0} < 0$. Remembering $r_{x_0} = 1$, choose $\vec{t}_x = \vec{s} \neq \vec{0}$ for all x satisfying $r_x > 0$ and $\vec{t}_x = \vec{0}$ for all other x. Then

$$(r\vec{t}, A\, r\vec{t}) = \sum_{\langle xy\rangle : r_x r_y < 0} \vec{s}^{\,2} r_x r_y - \frac{1}{2\beta} \sum_{x,y: r_x r_y < 0} \vec{s}^{\,2} H_{xy} r_x r_y < 0 \,. \tag{2.24}$$

This is a contradiction, so $a \in \mathcal{D}(H)$ if and only if $\widehat{A}$ is positive definite. By the Hurwitz (or Sylvester) criterion this is equivalent to

$$\mathcal{D}(H) = \{ a \in R^{V-1} \,|\, \det A_k > 0\,,\ \forall k = 1, \cdots, V-1\,,\ \ A_k = (A_{x_i x_j})_{1 \le i,j \le k} \}\,, \tag{2.25}$$

where we picked an arbitrary ordering of the lattice sites $x_0, x_1, \cdots, x_{V-1}$. For $k = 1$ one gets in particular $a_x > -2d$ for all $x \in \Lambda$ (recall $H_{xx} = 0$) and (2.13) follows. □

Before turning to the proof of the Proposition we prepare some simple auxiliary results. Let $A = (A_{xy})_{x,y\in\Lambda}$ be a symmetric invertible matrix such that the matrix $\widehat{A}$ arising from A by deleting its x_0-th row and column is positive definite. Then

$$\begin{aligned} &\int \prod_x d\phi_x \delta(\phi_{x_0}) \, \exp\Big\{ -\frac{1}{2} \sum_{x,y} \phi_x A_{xy} \phi_y + \sum_x J_x \phi_x \Big\} \\ &= (2\pi)^{\frac{V-1}{2}} (\det \widehat{A})^{-1/2} \exp\Big\{ \frac{1}{2} \sum_{x,y} J_x (\widehat{A}^{-1})_{xy} J_y \Big\}\,, \end{aligned} \tag{2.26}$$

for a real field ϕ_x, $x \in \Lambda$. The inverse of $\widehat{A}$ can be expressed in terms of the inverse of A via

$$(\widehat{A}^{-1})_{xy} = (A^{-1})_{xy} - \frac{(A^{-1})_{xx_0}(A^{-1})_{yx_0}}{(A^{-1})_{x_0x_0}}\,. \tag{2.27}$$

The determinant of $\widehat{A}$ is related to that of A by

$$\det A = \frac{\det \widehat{A}}{(A^{-1})_{x_0x_0}}\,. \tag{2.28}$$

Often a term in the x_0-th matrix element on the diagonal of A has to be split off according to $A_{xy} = \widetilde{A}_{xy} - c\delta_{xy}\delta_{x_0x}$. In this case the inverse of A is related to the inverse of $\widetilde{A}$ by

$$(A^{-1})_{xy} = (\widetilde{A}^{-1})_{xy} + \frac{c}{1 - c(\widetilde{A}^{-1})_{x_0x_0}} (\widetilde{A}^{-1})_{xx_0} (\widetilde{A}^{-1})_{yx_0}\,. \tag{2.29}$$

In particular $A_{x_0x_0} - (A^{-1})^{-1}_{x_0x_0} = \widetilde{A}_{x_0x_0} - (\widetilde{A}^{-1})^{-1}_{x_0x_0}$ and

$$\frac{1}{(A^{-1})_{x_0x_0}} = -c + \frac{1}{(\widetilde{A}^{-1})_{x_0x_0}}, \qquad \frac{(A^{-1})_{xx_0}}{(A^{-1})_{x_0x_0}} = \frac{(\widetilde{A}^{-1})_{xx_0}}{(\widetilde{A}^{-1})_{x_0x_0}}. \tag{2.30}$$

For the determinants one has

$$\det A = \det \widetilde{A} - c \det \widehat{A}. \tag{2.31}$$

Proof of the Proposition. We rewrite the action as

$$\begin{aligned} S &= \beta \sum_{x,\mu} \Big[\mathrm{ch}(\theta_x - \theta_{x+\hat{\mu}}) + \frac{1}{2}(\vec{t}_x - \vec{t}_{x+\hat{\mu}})^2 e^{-\theta_x - \theta_{x+\hat{\mu}}} - 1 \Big] \\ &= \beta \sum_{x,\mu} \mathrm{ch}(\theta_x - \theta_{x+\hat{\mu}}) + \frac{\beta}{2} \sum_{x,y} e^{-\theta_x - \theta_y} \mathcal{M}_{xy} \vec{t}_x \cdot \vec{t}_y - \beta d V, \end{aligned} \tag{2.32}$$

with $\mathcal{M}$ as in (2.17). The source term in (2.4) can be rewritten similarly and using also (2.10) one finds in a first step

$$\begin{aligned} \exp W[H] = \mathcal{N} \int \prod_x e^{-(N-1)\theta_x} d\theta_x \; e^{(N-1)\theta_{x_0}} \delta(\theta_{x_0}) \\ \times \exp\Big\{ -\beta \sum_{x,\mu} \mathrm{ch}(\theta_x - \theta_{x+\hat{\mu}}) + \frac{1}{2} \sum_{x,y} H_{xy} [\mathrm{ch}(\theta_x - \theta_y) - 1] \Big\} \\ \times \int \prod_x d\vec{t}_x \delta(\vec{t}_{x_0}) \exp\Big\{ -\frac{\beta}{2} \sum_{x,y} e^{-\theta_x} \vec{t}_x \cdot \mathcal{A}_{xy} \, e^{-\theta_y} \vec{t}_y \Big\}, \end{aligned} \tag{2.33}$$

with $\mathcal{A}_{xy}$ as in (2.17). After the rescaling $\vec{t}_x \mapsto e^{\theta_x} \vec{t}_x$ the Gaussians are of the form (2.26) and one obtains

$$\begin{aligned} \exp W[H] = \mathcal{N} \int \prod_x d\theta_x \delta(\theta_{x_0}) \; \exp\Big\{ -\frac{N-1}{2} \mathrm{Tr} \ln \widehat{\mathcal{A}} \Big\} \\ \times \exp\Big\{ -\beta \sum_{x,\mu} \mathrm{ch}(\theta_x - \theta_{x+\hat{\mu}}) + \frac{1}{2} \sum_{x,y} H_{xy} [\mathrm{ch}(\theta_x - \theta_y) - 1] \Big\}, \end{aligned} \tag{2.34}$$

with a redefined $\mathcal{N}$. Next one observes that the integration variables θ_x only occur through the combination (2.16). Indeed, $\mathcal{A}_{xy} = -\Delta_{xy} + \beta^{-1} H_{xy} + \delta_{xy} a_x(\theta)$, $\sum_{x,\mu} \mathrm{ch}(\theta_x - \theta_{x+\hat{\mu}}) = dV + \frac{1}{2} \sum_x r_x^{-1} (\Delta r)_x$, and $\sum_{x,y} H_{xy} \mathrm{ch}(\theta_x - \theta_y) = \sum_{x,y} r_x^{-1} H_{xy} r_y$. This suggests to change variables in (2.34) from θ_x, $x \neq x_0$ to a_x, $x \neq x_0$. The change of variables has been prepared in Lemma 2.2. Combining (2.34), (2.20), (2.21), (2.15) one arrives at (2.11). $\square$

3. Large N expansion for $W[H]$

Connected invariant correlation functions are defined via the moments of $W[H]$. In a large N expansion $\lambda := (N+1)/\beta$ is kept fixed and we write

$$W_r \sim \frac{\lambda^r}{(N+1)^{r-1}} \sum_{s\geq 0} \frac{1}{(N+1)^s} W_r^{(s)} \,. \tag{3.1}$$

The algorithm to compute the $W_r^{(s)}$ is described in [7]. The rationale for it is provided by the

THEOREM 3.1. The correlation functions W_r admit an asymptotic expansion of the form (3.1) whose coefficients coincide with those defined by the Laplace expansion of (2.11) where $\mathcal{D}(H)$ has been replaced by $\mathbb{R}^{V-1}$.

Remarks. (i) The core fact underlying the theorem is that the 'dual' action

$$S[a, H] = \frac{1}{2}\mathrm{Tr}\ln \widehat{A} + \frac{1}{2\lambda} \sum_{x\neq x_0} a_x - \frac{1}{2\lambda} (\widetilde{A}^{-1})^{-1}_{x_0 x_0} \,, \tag{3.2}$$

with A and $\widetilde{A}$ as in (2.12) has a unique minimum in the domain $\mathcal{D}(H)$. In generating (3.1) one sets

$$a_x = \omega_x + \frac{u_x}{\sqrt{N+1}} \,, \quad u_x \in \mathbb{R} \,, \tag{3.3}$$

and adjusts the ω_x to the values defining the minimum of S.

(ii) Substituting $a_x = 2i\lambda\alpha_x$ in (3.2) gives an effective action that can formally be obtained by mimicking the dualization procedure in the compact model, see Appendix C of [5]. The flip $\alpha_x \mapsto -\alpha_x$, $\lambda \mapsto -\lambda$, then relates it to the dual action of the compact model, see [7] for the relation between both large N expansions.

Proof. We establish consecutively: (a) uniqueness, (b) existence of a minimum in $\mathcal{D}(0)$, and (c) the fact that the asymptotic expansion (3.1) is unaffected by the replacement of $\mathcal{D}(H)$ with $\mathbb{R}^{V-1}$.

(a) Since the H-dependent terms in A_{xy} are $O(1/(N+1)$ it suffices show that $S[a, 0]$ has a unique minimum in $\mathcal{D}(0)$. We show that $S[a, 0]$ has at most one extremum in $\mathcal{D}(0)$, which if it exists must be a minimum. To this end we consider the preimage of $S[a, 0]$ under χ and show that it is a strictly convex function on $\mathbb{R}^{V-1}$. Thus we set

$$F(\theta) := 2S[a(\theta), 0] = \mathrm{Tr}\ln \widehat{\mathcal{M}} + \frac{1}{\lambda} \sum_x e^{\theta_x} (\Delta e^{-\theta})_x \,, \tag{3.4}$$

where $a_x(\theta) = e^{\theta_x}(\Delta e^{-\theta})_x$ and $\mathcal{M}$ is as before. To establish strict convexity of F it suffices to show that both terms in F are separately strictly convex.† For the second term this is manifest: shifting $\theta_x \mapsto \theta_x + \epsilon_x$ the term quadratic in ϵ_x is nonnegative as $-\Delta$ is positive semi definite.

To show convexity of $\mathrm{Tr}\ln\widehat{\mathcal{M}}$ we define $W = (w_{xy})_{x,y\in\Lambda}$, by

$$w_{xy} := e^{-\theta_x}\mathcal{M}_{xy}e^{-\theta_y}\,, \tag{3.5}$$

which obeys $\sum_y w_{xy} = 0$ and has matrix elements

$$w_{xy} = \begin{cases} -e^{-\theta_x-\theta_{x\pm\hat\mu}}\,, & y = x\pm\hat\mu\,,\\ e^{-\theta_x}\sum_\mu(e^{-\theta_{x+\hat\mu}} + e^{-\theta_{x-\hat\mu}})\,, & x = y\,,\\ 0 & \text{otherwise}\,. \end{cases} \tag{3.6}$$

Trivially $\ln\det\widehat{\mathcal{M}} = 2\sum_x\theta_x + \ln\det W$, so that strict convexity of $\ln\det W$ implies that of F. W has the form that makes the so-called matrix-tree theorem (see e.g. [2, 3, 1]) applicable. The matrix-tree theorem then entails

$$\det\widehat{W} = \sum_T w_T\,, \tag{3.7}$$

where the sum runs over all spanning trees built from nearest neighbor pairs, i.e. walks through the lattice Λ visiting every point of Λ once and

$$w_T = \prod_{(x,x\pm\hat\mu)\in T} w_{x,x\pm\hat\mu}\,. \tag{3.8}$$

The point of this representation is that it expresses $\det\widehat{W}$ as a sum of exponentials in the θ variables; the (strict) convexity of $\mathrm{Tr}\ln\widehat{W}$ follows from the well-known fact: if $Z(\theta) := \sum_i c_i e^{a_i\cdot\theta_i}$, $a_i,\,\theta_i\in\mathbb{R}^n$, $c_i\geq 0$, then $\ln Z(\theta)$ is convex.

(b) Here we proceed in two steps. In a first step we rewrite the stationarity conditions for $S[a,0]$ in a more transparent form. In a second step we present a solution for them in $\mathcal{D}(0)$.

For the first step we define the matrices M, $\widehat{M}$, $\widetilde{M}$ as $\mathcal{M}$, $\widehat{\mathcal{M}}$, $\widetilde{\mathcal{M}}$ expressed in the coordinates a_x and with the critical point parameters ω_x of (3.3) inserted, i.e.

$$M := \chi\circ\mathcal{M}\Big|_{a_x\to\omega_x} = -\Delta_{xy} + \delta_{xy}\omega_x\,, \tag{3.9}$$

†Convexity of a similar function was established by Spencer and Zirnbauer [8] using a different technique.

and similarly for $\widehat{M}$, $\widetilde{M}$. Note that $\det M = 0$ by Lemma 2.2 and (2.18). We are looking for a critical point of $S[\omega, 0] = \frac{1}{2} \ln \det \widehat{M} + \frac{1}{2\lambda} \sum_x \omega_x$ under the condition $\det M = 0$. Introducing a Lagrange multiplier μ for the latter we consider

$$\tilde{F}(\omega, \mu) := \ln \det \widehat{M} + \frac{1}{\lambda} \sum_x \omega_x + \mu \det M \,. \tag{3.10}$$

The conditions for a critical point ('saddle point equations') of $\tilde{F}$ are:

$$\det M = 0 \,, \tag{3.11a}$$

$$\mu = -\frac{1}{\lambda \det \widehat{M}} \,, \tag{3.11b}$$

$$\lambda \widehat{M}^{\text{co}}_{xx} + \det \widehat{M} - M^{\text{co}}_{xx} = 0 \,, \quad x \neq x_0 \,, \tag{3.11c}$$

where we denoted the cofactor matrix of M, $\widehat{M}$ by M^{co}, $\widehat{M}^{\text{co}}$, respectively.

The conditions (3.11) simplify when expressed in term of

$$D^{-1}_{xy} := M_{xy} - \lambda \delta_{xx_0} \delta_{xy} \,. \tag{3.12}$$

Indeed, using (2.31) for the cofactors one finds

$$(D^{-1})^{\text{co}}_{xx} = \begin{cases} M^{\text{co}}_{x_0 x_0} = \det \widehat{M} \,, & x = x_0 \,, \\ -\lambda \widehat{M}^{\text{co}}_{xx} + M^{\text{co}}_{xx} \,, & x \neq x_0 \,. \end{cases} \tag{3.13}$$

By (3.11c) also the $x \neq x_0$ cofactors reduce to $\det \widehat{M}$. Using (2.31) once more for $\det D = -\det M + \lambda \det \widehat{M}$ one sees that the saddle point equations (3.11) are equivalent to

$$-\lambda D_{xx} = 1 \,, \quad \forall x \,, \tag{3.14}$$

where the $x = x_0$ equation implements (3.11a).

Also the conditions characterizing $\mathcal{D}(0)$ can be expressed in terms of D. From the proof of Lemma 2.2 we know that $r_x(\omega, 0) = (\widetilde{M}^{-1})_{xx_0} / (\widetilde{M}^{-1})_{x_0 x_0} > 0$ characterizes $\mathcal{D}(0)$. On the other hand writing $(D^{-1})_{xy} = \widetilde{M}_{xy} + (\omega_{x_0} - \lambda) \delta_{xy} \delta_{xx_0}$, and applying (2.29) one has

$$D_{xy} = (\widetilde{M}^{-1})_{xy} + \frac{\lambda - \omega_{x_0}}{1 - (\lambda - \omega_{x_0})(\widetilde{M}^{-1})_{x_0 x_0}} (\widetilde{M}^{-1})_{xx_0} (\widetilde{M}^{-1})_{yx_0} \,. \tag{3.15}$$

Taking into account that $(\widetilde{M}^{-1})_{x_0 x_0} = -1/\omega_{x_0}$ one arrives at the following characterization:

$$\{\omega_x, \, x \neq x_0\} \in \mathcal{D}(0) \quad \text{if and only if} \quad -\lambda D_{xx_0} > 0 \,. \tag{3.16}$$

In a second step we now search for a the solution of Eq. (3.14) satisfying $-\lambda D_{xx_0} > 0$. Eq. (3.14) is a system of $V-1$ algebraic equations for the $V-1$ critical point parameters $\omega_x\,, x \neq x_0$, and difficult to tackle analytically. But the translation invariant form of the equation suggests the translation invariant ansatz

$$D^{-1}_{xy} = -\Delta_{xy} + \omega\delta_{xy}\,, \tag{3.17}$$

i.e.

$$\omega_x = \omega + \lambda\delta_{xx_0}\,. \tag{3.18}$$

The saddle point equations (3.14) then reduce to a single almost conventional gap equation for ω

$$D_{xx} = \frac{1}{V}\sum_p \frac{1}{E_p + \omega} = -\frac{1}{\lambda}\,, \tag{3.19}$$

where the sum is over all $p = \frac{2\pi}{L}(n_1, \ldots, n_d)$, $n_i = 0, 1, \ldots, L-1$, and $E_p := 2d - 2\sum_\mu \cos(p \cdot \hat{\mu})$. From (3.19) it is clear that all solutions ω must be negative. As shown in [7] there is a unique root $\omega = \omega_-(\lambda, V)$ of (3.19) characterized by the following two equivalent conditions:

$$-\frac{4}{2d+1}\sin^2\frac{\pi}{L} < \omega_- < 0\,, \tag{3.20a}$$

$$-\lambda D_{xy}\Big|_{\omega=\omega_-} \geq 1\,, \quad \text{for all } \; x, y\,. \tag{3.20b}$$

Since $-\lambda D_{xx_0}|_{\omega=\omega_-} > 1$ for this solution it lies in $\mathcal{D}(0)$.

(c) This can be seen from the following simple fact about saddle point expansions: Let $f \in C^\infty(\mathbb{R}^n)$ be such that $\exp(Nf(x))$ is integrable for all N and obeys

$$\mathrm{grad} f(0) = f(0) = 0\,, \qquad f(x) < -\delta \;\text{ for }\; |x| > \epsilon\,. \tag{3.21}$$

Then the integral has a saddle point expansion of the form

$$\int dx\, \exp(Nf(x)) \sim \sum_{n\geq 0} \frac{a_n}{N^n}\,, \tag{3.22}$$

and the expansion coefficients are insensitive to changes of the integrand bounded away from the saddle point: If $q \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, with $q(x) = 1$ for $|x| < \epsilon$, then

$$\int dx\, q(x)\, \exp(Nf(x)) \sim \sum_{n\geq 0} \frac{a_n}{N^n}\,. \tag{3.23}$$

This completes the proof of the theorem. □

Remarks. (i) Eq. (3.14) can be viewed as the normalization condition, $-\lambda D_{xx} = 1$, of the leading order two-point function. In fact [7]

$$\langle n_x \cdot n_y \rangle \Big|_{N=\infty} = -\lambda \widehat{D}_{xy} + \frac{\widetilde{D}_{xx_0} \widetilde{D}_{yx_0}}{\widetilde{D}^2_{x_0x_0}} = -\lambda D_{xy} \,, \tag{3.24}$$

where $\widetilde{D} = \widetilde{M}^{-1}$, $\widehat{D} = \widehat{M}^{-1}$. The first equality is obtained by evaluating $W[H]$ to leading order in $1/(N+1)$, the second equality follows by using (2.27) in (3.15).

(ii) The number of terms in (3.7) is given by $\det(-\widehat{\Delta})$, which is sizeable even for small lattices (but less than the naive $(V-1)!$ number of terms), e.g. for $d = 2$, $L = 3$ there are 11664 spanning trees.

(iii) In making the ansatz (3.18) we took the consistency with $\omega_{x_0} = -1/(\widetilde{M}^{-1})_{x_0x_0}$ (Eq. 2.15) for granted. Here $(\widetilde{M}^{-1})_{x_0x_0}$ is a ratio of polynomials (Toeplitz determinants) of degree $V-1$ in ω. Its direct computation is cumbersome but by assuming $\omega_{x_0} = \omega + \lambda$ and eliminating λ via (3.19) one sees that

$$\omega_{x_0}(\omega) = \omega - \Big[\frac{1}{V} \sum_p \frac{1}{E_p + \omega} \Big]^{-1} \,, \tag{3.25}$$

on the solutions of (3.19). Equivalently (3.19) is such that $M_{xy} = -\Delta_{xy} + \omega\delta_{xy} + \lambda\delta_{xy}\delta_{xx_0}$ has zero determinant.

(iv) We remark that the large volume asymptotics of ω_- is given by [7]

$$-V\omega_-(\lambda, V) = \begin{cases} \dfrac{4\pi}{\ln V}\Big(1 + O(1/\ln V)\Big) & d = 2\,, \\ \Big(\dfrac{1}{\lambda} + C_d\Big)^{-1} + O(V^{-\frac{d-2}{d}}) & d \geq 3\,, \end{cases} \tag{3.26}$$

where $C_d = \int_0^{2\pi} \frac{d^d p}{(2\pi)^d} \frac{1}{E(p)}$. In particular the gap $\omega_-(\lambda, V)$ vanishes in the infinite volume limit, in sharp contrast to the compact model.

The theorem and its proof have a number of interesting corollaries.

COROLLARY 3.2. *All solutions of Eqs. (3.14) satisfying $-\lambda D_{xx_0} > 0$ are constant: $\omega_x = \omega$, $x \neq x_0$.*

By inspection of examples one sees that the inequalities are essential for the validity of the result: nonconstant solutions outside the domain $\mathcal{D}(0)$ can easily be found. Since (3.14) is a system of $V-1$ algebraic equations for $V-1$ unknowns a direct proof of Corollary 3.2 seems difficult.

COROLLARY 3.3. *All solutions of (3.19) other than ω_- do not lie in $\mathcal{D}(0)$. The solution $\omega = \omega_-$ lies in $\mathcal{D}(0)$ and thus implies the positive definiteness of $\widehat{M}|_{\omega=\omega_-}$.*

We recall from [7] the form of the Hessian of the action (3.2) at the extremum $\omega_x = \omega_-$, $x \neq x_0$

$$S_2[u,H] = -\frac{1}{4}\sum_{x,y\neq x_0} u_x u_y [D_-(x-y)^2 - \lambda^2 D_-(x-x_0)^2 D_-(y-x_0)^2] + \frac{\lambda}{2}\sum_{x,y} H_{xy} D_-(x-y)\,. \tag{3.27}$$

Here $D_-(x-y) := D_{xy}|_{\omega=\omega_-}$ and the variables u_x are those of (3.3).

One can show that all the matrix elements in square brackets in Eq. (3.27) are negative. On account of the theorem we have

COROLLARY 3.4. $\quad S_2[u,0] \geq 0\,.$

More directly than here it has been shown in Appendix A of [7] that

$$S_2[u,H] \geq 0\,, \tag{3.28}$$

for all $H_{xy} \leq 0$ and all u configurations,

COROLLARY 3.5. *The minimum of $S[a,0]$ cannot lie at the boundary of $\mathcal{D}(0)$. More generally one has*

$$S[a,H] \to +\infty \quad \textit{as} \quad a \to \partial\mathcal{D}(H)\,, \tag{3.29}$$

where $\partial D(H)$ is the boundary of $\mathcal{D}(H)$.

To show (3.29) this it suffices to establish that $\det\widetilde{A} = 2d\det\widehat{A} - R$, where R is bounded from below by a (a_x independent) positive constant $\#C$. Indeed, using $a_x \leq -2d$ for all x, it then follows

$$S[a,H] \geq \frac{1}{2}\ln\det\widehat{A} + \frac{1}{2\lambda}\frac{\#C}{\det\widehat{A}} - 2dV\,, \tag{3.30}$$

and the positive second term dominates as a approaches the boundary of $\mathcal{D}(0)$ in (2.13b). Slightly more generally one has :

If $A_{xy} = -\Delta_{xy} + a_x\delta_{xy} + \beta^{-1}H_{xy}$, with $H_{xx} = 0$, $H_{xy} \leq 0$, is a positive semidefinite matrix on a hypercubic lattice of linear size L, and $R := (2d + a_{x_0})\det\widehat{A} - \det A$, then

$$R \geq \sum_{\text{cycles } C}\ \prod_{\langle xy\rangle\in C}(1-\beta^{-1}H_{xy}) \geq \#\text{ cycles on }\ \Lambda\,. \tag{3.31}$$

Here C is the set of cycles, i.e. closed oriented paths which connect only nearest neighbors and which visit each lattice point exactly once. On a torus of dimension d the number of these cycles is at least $2d$. We omit the proof.

Acknowledgment: E.S. is grateful to D. Brydges for giving him free instruction about the matrix-tree theorem.

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