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DIFFERENTIAL TWISTED K-THEORY AND APPLICATIONS

ALAN L. CAREY, JOUKO MICKELSSON, AND BAI-LING WANG

Abstract. In this paper, we develop differential characters in twisted K-theory and use them to define a twisted Chern character. In the usual formalism the 'twist' is given by a degree three Čech class while we work with differential twisted K-theory with twisting given by a degree 3 Deligne class. This resolves an unsatisfactory dependence on choices of representatives of differential forms in the definition of the Chern character map for twisted K-theory in the current literature. Twisted eta forms and twisted spin^c structures are also defined. To show the efficacy of our point of view we use our approach to study D-brane charges on a compact Lie group with non-trivial twisting by a Deligne class.

Contents

1. Introduction

Generalized differential cohomology theories have recently excited considerable interest. For example, Cheeger-Simons differential characters play a role in index theory. In [28], Lott developed $\mathbb{R}/\mathbb{Z}$-valued index theory for Dirac operators coupled to virtual complex vector bundles with trivial Chern character, extending Atiyah-Patodi-Singer's reduced

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eta-invariants for flat vector bundles. The resulting K-theory is called K-theory with $\mathbb{R}/\mathbb{Z}$ coefficients. It is a special version of what is now called 'differential K-theory' as proposed by Freed [19], Hopkins and Singer [25]. The latter paper was partly inspired by mathematical questions arising from string theory which also provides the motivation for our work.

We are interested in the notion of a twisted Chern character map with values in twisted differential cohomology. Recall that twisted K-theory $K^*(X,\sigma)$ (we refer to [27] for a history and to [10], [2] for the point of view motivated by string theory) of a compact differentiable manifold X depends on an integral degree 3 Čech class σ. A corresponding twisted de Rham cohomology theory $H^*(X, d-H)$ where H is the image of σ in de Rham cohomology with real coefficients was introduced in [10] and a proposal made for a twisted Chern character

$$ch_H : K^*(X,\sigma) \longrightarrow H^*(X, d-H).$$

This notation is plausible as the twisted cohomology group depends on the choice of H. As H is closed, locally on an open contractible set U_j we can always write $H = dB_j$, and the resulting collection of local 2-forms indexed by an open cover is called the B-field. It was argued that under a global change of B-field by an exact form $d\omega$, one would have (Cf. [3] [10] [29])

$$ch_{H+d\omega} = exp(\omega) ch_H.$$

There is a problem arising from this formulation, when ω is a closed differential form which doesn't represent a class in the image of $H^2_{dR}(X,\mathbb{Z})$ in $H^2_{dR}(X,\mathbb{R})$, as one would have

$$ch_H = exp(\omega) ch_H,$$

which would imply that ch_H is not well-defined as a twisted Chern character

$$ch_H : K^*(X,\sigma) \longrightarrow H^*(X, d-H).$$

One objective of this paper is to rectify this defect.

To simplify our notation, throughout this paper, we adopt the following convention. The Čech cohomology of X with values in the constant A-valued sheaf is denoted by $H^*(X; A)$, and the Čech cohomology of X with values in the continuous A-valued sheaf is denoted by $H^*(X; \underline{A})$. For a compact differentiable manifold X, the Čech cohomology of X is isomorphic to the singular cohomology with coefficients in A. There is the well known exact sequence of Čech cohomologies of X

$$(1.1) \qquad \to H^{k-1}(X, U(1)) \longrightarrow H^k(X,\mathbb{Z}) \longrightarrow H^k(X,\mathbb{R}) \longrightarrow H^k(X, U(1)) \to .$$

The Čech-de Rham isomorphism means that we may also use $H^k(X,\mathbb{R})$ to denote the de Rham cohomology of X with real coefficients which allows us to write the following commutative diagram relating Deligne cohomology $H^*_{\mathcal{D}}(X)$ of X and de Rham cohomology of X

$$\begin{array}{ccc} H^k_{\mathcal{D}}(X) & \xrightarrow{\;c\;} & H^k(X,\mathbb{Z}) \\ {\scriptstyle curv}\big\downarrow & & \big\downarrow{\scriptstyle \iota} \\ \Omega^k_{\mathbb{Z}}(X) & \longrightarrow & H^k(X,\mathbb{R}) \end{array} \tag{1.2}$$

where $\Omega^k_{\mathbb{Z}}(X)$ is the space of closed degree k differential forms representing cohomology classes in the image of the middle map $\iota : H^k(X,\mathbb{Z}) \longrightarrow H^k(X,\mathbb{R})$ in (1.1). With these notations understood, we have the following description of the third Deligne cohomology $H^3_{\mathcal{D}}(X)$with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & H^2(X,\mathbb{R}) & & \\
 & & & & \big\downarrow & & \big\downarrow & & \\
 & & & & H^2(X,U(1)) & \longrightarrow & H^2(X,U(1)) & & \\
 & & & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \dfrac{\Omega^2(X)}{\Omega^2_{\mathbb{Z}}(X)} & \longrightarrow & H^3_{\mathcal{D}}(X) & \xrightarrow{\;c\;} & H^3(X,\mathbb{Z}) & \longrightarrow & 0 \\
 & & & & \big\downarrow{\scriptstyle curv} & & \big\downarrow{\scriptstyle \iota} & & \\
 & & & & \Omega^3_{\mathbb{Z}}(X) & \longrightarrow & H^3(X,\mathbb{R}) & & \\
 & & & & \big\downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Returning now to our discussion, our modification of the twisted Chern character is defined on a refined version of twisted K-theory which we call differential twisted K-theory $K^*(X,\check{\sigma})$ with twisting given by a Deligne cohomology class $\check{\sigma} \in H^3_{\mathcal{D}}(X)$. Recall (Cf.(1.2)) that associated to a Deligne cohomology class $\check{\sigma}$, there is a uniquely defined characteristic class $\sigma \in H^3(X,\mathbb{Z})$ and a uniquely defined curvature form $curv(\check{\sigma}) = H$ in $\Omega^3_{\mathbb{Z}}(X)$. We will construct a twisted Chern character

$$ch_{\check{\sigma}} : K^*(X,\check{\sigma}) \longrightarrow H^*(X, d - H),$$

which resolves the above problem of a dependence of the usual twisted Chern character map on choices of representative differential forms for twisted K-theory $K^*(X,\sigma)$ with twisting given by $\sigma \in H^3(X,\mathbb{Z})$. We show that there is a canonical isomorphism from $K^*(X,\check{\sigma})$ to $K^*(X,\sigma)$, but the inverse is not canonical which explains the unsatisfactory features of the existing twisted Chern character on $K^*(X,\sigma)$.

Before we explain further the mathematical results in this paper we want to give the motivation from string geometry. Those unfamiliar with the terminology can skip this paragraph. Differential K-theory has previously been used to study anomaly cancellation problems for action principles in the presence of D-branes in string theory and M-theory.

For Type II superstring theory with non-trivial B-field on X, it is believed that Ramond-Ramond charges lie in twisted K groups (Cf.[39] [40]), $K^*(X, \sigma)$ where σ is the characteristic class of the B-field. The cohomological forms of Ramond-Ramond charges are twisted cohomology classes in the corresponding twisted cohomology $H^*(X, d-H)$, where H is a closed differential form representing the image of σ in $H^3(X, \mathbb{R})$.

In this paper, we will apply this differential twisted K-theory in a fashion that is compatible with the proposal by Freed [20] for twisted Chern character forms. A spin off of our approach is that we can generalise the notion of Spin^c structure. Recall that a real vector bundle V over X admits a Spin^c structure if and only if the third integral Stieffel-Whitney class

$$W_3(V) = 0.$$

Associated to $c \in H^2(X, \mathbb{Z})$ with $c \equiv w_2(V)(mod\ 2)$, there is a unique Spin^c structure such that the first Chern class of the determinant bundle is c. We introduce a $\check{\sigma}$-twisted Spin^c structure on a real vector bundle V associated to (B, b) where the twisting is given by a Deligne class $\check{\sigma} \in H^3_{\mathcal{D}}(X)$ such that

$$c(\check{\sigma}) = W_3(V), \qquad curv(\check{\sigma}) = dB$$

and $\check{\sigma} + w_2(V) = [B] + [b]$ for some $b \in H^2(X, \mathbb{R})$. When $W_3(V) = 0$, we can choose $\check{\sigma} = 0$, this recovers the (untwisted) Spin^c structure on V. We apply this notion of twisted Spin^c structure to establish the Riemann-Roch theorem (Theorems 12 and 14) in twisted K-theory which generalizes the Atiyah-Hirzebruch version of the Riemann-Roch theorem for K-oriented maps.

For each odd twisted K-class, we construct a finite rank vector bundle over each double intersections which provides a geometric model of odd differential twisted K-classes. Using these finite rank vector bundles and the twisted Čech-de Rham double complex, we construct a canonical twisted differential Chern character form associated to a twisted family of self-adjoint Fredholm operators representing a odd twisted K-class. We also consider twisted eta forms for those twisted K-classes that are torsion in twisted K-groups. This is useful in studying nonequivariant twisted K-theory of a compact Lie group. We discover that these twisted eta forms can be used to distinguish different twisted K-classes. Explicit computations are done for the Lie group $SU(2)$ and $SU(3)$.

2. Preliminary notions

In this section, we review differential K-theory which motivates our study of differential twisted K-theory, and geometry of gerbes which are crucial in our definition of differential twisted K-theory and twisted Chern character forms.

2.1. **Differential K-theory.**

Differential K-theory has its origins in K-theory with $\mathbb{R}/\mathbb{Z}$-coefficients introduced by Atiyah-Patodi-Singer ([1]) and Karoubi ([26]). It turns out that K-groups with $\mathbb{R}/\mathbb{Z}$-coefficients are subgroups of differential K-groups with vanishing curvature. We review differential K-theory, the main reference is [25] and [28], see also [7].

First, we recall the Bockstein exact sequence for K-theory with coefficients in $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ ([1] [26]):

$$\begin{array}{ccccc} K^0(X,\mathbb{Z}) & \xrightarrow{ch_0} & H^{ev}(X,\mathbb{R}) & \longrightarrow & K^0(X,\mathbb{R}/\mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K^1(X,\mathbb{R}/\mathbb{Z}) & \longleftarrow & H^{odd}(X,\mathbb{R}) & \xleftarrow[ch_1]{} & K^1(X,\mathbb{Z}) \end{array} \tag{2.1}$$

where $H^{ev}(X,\mathbb{R}) = \oplus_i H^{2i}(X,\mathbb{R})$, $H^{odd}(X,\mathbb{R}) = \oplus_i H^{2i+1}(X,\mathbb{R})$ and $ch_* : K^*(X) \to H^*(X,\mathbb{R})$ is the Chern charcater map.

Denote by $\Omega^{ev}(X)$ and $\Omega^{odd}(X)$ the space of differential forms of even and odd degree respectively. Let $\Omega_0^{ev}(X)$ and $\Omega_0^{odd}(X)$ be the set of closed differential forms whose cohomology class lies in the image of the Chern character map ch_0 and ch_1 respectively. Then we have the following exact sequence

$$\begin{array}{ccccc} \Omega_0^{ev}(X) & \longrightarrow & H^{ev}(X,\mathbb{R}) & \longrightarrow & \dfrac{\Omega^{ev}(X)}{\Omega_0^{ev}(X)} \\ d\uparrow & & & & \downarrow d \\ \dfrac{\Omega^{odd}(X)}{\Omega_0^{odd}(X)} & \longleftarrow & H^{odd}(X,\mathbb{R}) & \longleftarrow & \Omega_0^{odd}(X) \end{array}$$

whose proof follows from a direct computation.

Let (E, h^E) be a complex vector bundle over X with a Hermitian metric h^E. Given a pair of Hermitian connections ∇_1^E and ∇_2^E on (E, h^E), there is a well-defined Chern-Simons form (modulo exact forms), see section 2 of [28],

$$CS(\nabla_1^E, \nabla_2^E) \in \Omega^{odd}(X)/Im(d)$$

such that

$$dCS(\nabla_1^E, \nabla_2^E) = ch(E, \nabla_1^E) - ch(E, \nabla_2^E)$$

where $ch(E, \nabla_i^E)$ is the Chern-Weil form of (E, ∇_i^E). For a short exact sequence of Hermitian vector bundles

$$0 \to E_1 \longrightarrow E_2 \longrightarrow E_3 \to 0,$$

each E_i is equipped with a Hermitian connection ∇^{E_i}, there is a well-defined

$$CS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) \in \Omega^{odd}(X)/Im(d)$$

such that

$$dCS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) = ch(E_2, \nabla^{E_2}) - ch(E_1, \nabla^{E_1}) - ch(E_3, \nabla^{E_3}).$$

Definition 1. *A differential K-cocycle of X is a quadruple*

$$\check{E} = (E, h^E, \nabla^E, \omega)$$

where E is a complex vector bundle over X with a Hermitian metric h^E and a Hermitian connection ∇^E and $\omega \in \Omega^{odd}(X)/Im(d)$. A K-relation among three differential K-cocycles

$$\check{E}_1 = (E_1, h^{E_1}, \nabla^{E_1}, \omega_1), \check{E}_2 = (E_2, h^{E_2}, \nabla^{E_2}, \omega_2)\ \check{E}_3 = (E_3, h^{E_3}, \nabla^{E_3}, \omega_3)$$

is given by a short exact sequence of Hermitian vector bundles

$$0 \to E_1 \longrightarrow E_2 \longrightarrow E_3 \to 0,$$

and

$$CS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) \in \Omega^{odd}(X)/Im(d)$$

such that $\omega_2 = \omega_1 + \omega_3 + CS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3})$.

Definition 2. *The differential K-theory, denoted by $\check{K}^0(X)$, is the quotient of the free abelian group generated by differential K-cocycles of X, by the relation*

$$[\check{E}_2] = [\check{E}_1] + [\check{E}_3]$$

whenever there is a K-relation amongst $\check{E}_1$, $\check{E}_2$ and $\check{E}_3$.

There exist two natural homomorphisms

$$\check{K}^0(X) \longrightarrow K^0(X)$$

given by the forgetful map $[(E, h^E, \nabla^E, \omega)] \mapsto [E]$, and

$$\check{ch} : \check{K}^0(X) \longrightarrow \Omega^{ev}_0(X)$$

given by $[(E, h^E, \nabla^E, \omega)] \mapsto ch(E, \nabla^E) - d\omega$.

The following theorem is known to many experts in the field and envisaged in [25] and [19]. As there is no detailed proof for this theorem, we include one here for the benefit of readers.

Theorem 3. *There exist the following two exact sequences relating the differential K-theory $\check{K}^0(X)$ to the ordinary K-theory $K^0(X)$ and $K^1(X, \mathbb{R}/\mathbb{Z})$ respectively:*

$$0 \longrightarrow \frac{\Omega^{odd}(X)}{\Omega^{odd}_0(X)} \xrightarrow{i_1} \check{K}^0(X) \longrightarrow K^0(X) \longrightarrow 0,$$

$$0 \longrightarrow K^1(X, \mathbb{R}/\mathbb{Z}) \xrightarrow{j_1} \check{K}^0(X) \xrightarrow{\check{ch}} \Omega^{ev}_0(X) \longrightarrow 0$$

such that these two exact sequences fit into the following commutative diagram

$$\begin{array}{ccccccc} & & K^1(X, \mathbb{R}/\mathbb{Z}) & \xrightarrow{Id} & K^1(X, \mathbb{R}/\mathbb{Z}) & & \\ & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} & & \\ 0 \to \frac{\Omega^{odd}(X)}{\Omega^{odd}_0(X)} & \xrightarrow{i_1} & \check{K}^0(X) & \longrightarrow & K^0(X) & \to 0 & \\ & & \downarrow{\scriptstyle \check{ch}} & & \downarrow{\scriptstyle ch} & & \\ & & \Omega^{ev}_0(X) & \xrightarrow{i_2} & H^{ev}(X, \mathbb{R}) & & \end{array} \quad (2.2)$$

Proof. **Step 1.** We define the homomorphism

$$i_1 : \frac{\Omega^{odd}(X)}{\Omega_0^{odd}(X)} \longrightarrow \check{K}^0(X)$$

$$i_1([\omega]) = [(\underline{\mathbb{C}^n}, h_0, \nabla_0, \omega)] - [(\underline{\mathbb{C}^n}, h_0, \nabla_0, 0)]$$

where $(\underline{\mathbb{C}^n}, h_0, \nabla_0)$ is the trivial Hermitian vector bundle of rank n with a standard Hermitian metric h_0 and the trivial connection ∇_0. One can check that the map i_1 is well-defined (the map i_1 doesn't depend on the various choices). From the K-relation, we have

$$[(\underline{\mathbb{C}^n}, h_0, \nabla_0, \omega)] - [(\underline{\mathbb{C}^n}, h_0, \nabla_0, 0)] = [(E, h, \nabla, \omega)] - [(E, h, \nabla, 0)]$$

for any Hermitian vector bundle (E, h) with a connection ∇. If $\omega \in \Omega_0^{odd}(X)$, we can find $\phi : X \to U(n)$ for some n defining an isomorphism

$$(\underline{\mathbb{C}^n}, h_0) \longrightarrow (\underline{\mathbb{C}^n}, h_0)$$

and

$$\omega = CS(\nabla_0, \phi^*\nabla_0) = ch(\phi) \in \Omega_0^{odd}(X)/Im(d)$$

where ∇_0 is the trivial connection on $(\underline{\mathbb{C}^n}, h_0)$, and

$$ch(\phi) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} Tr(\phi^{2k+1}) \in \Omega_0^{odd}(X)$$

is the odd Chern character from of ϕ. Then

$$[(\underline{\mathbb{C}^n}, h_0, \nabla_0, \omega)] = [(\underline{\mathbb{C}^n}, h_0, \nabla_0, 0)]$$

which implies that $i_1([\omega]) = 0$.

Step 2. Assume that

$$[(\underline{\mathbb{C}^n}, h_0, \nabla_0, \omega)] = [(\underline{\mathbb{C}^n}, h_0, \nabla_0, 0)]$$

then there exists a gauge transformation $\phi : X \to U(n)$ such that

$$\omega = CS(\nabla_0, \phi^*\nabla_0) = ch(\phi) \in \Omega_0^{odd}(X)/Im(d).$$

We have $[\omega] = 0$ in $\dfrac{\Omega^{odd}(X)}{\Omega_0^{odd}(X)}$. Hence, the map i_1 is injective. Of course, the forgetful map from $\check{K}^0(X)$ to $K^0(X)$ is surjective.

Step 3. We show that the first sequence is exact at $\check{K}^0(X)$. Suppose that

$$[\check{E}_1] - [\check{E}_2] = [(E_1, h^{E_1}, \nabla^{E_1}, \omega_1)] - [(E_2, h^{E_2}, \nabla^{E_2}, \omega_2)]$$

is mapped to zero under the forgetful map

$$[E_1] - [E_2] = 0.$$

Then E_1 and E_2 are stably equivalent, so there is an isomorphism

$$\phi : E_1 \oplus \underline{\mathbb{C}^n} \cong E_2 \oplus \underline{\mathbb{C}^n}$$

for some trivial vector bundle $\underline{\mathbb{C}^n}$. So we have

$$\begin{aligned} & dCS(\nabla^{E_1}\oplus\nabla_0, \phi^*(\nabla^{E_2}\oplus\nabla_0)) \\ = \; & ch(E_1,\nabla^{E_1}) - ch(E_1,\nabla^{E_1}) \\ = \; & d\eta_0 \end{aligned}$$

for some $\eta_0\in\Omega^{odd}(X)/Im(d)$, as $ch(E_1)=ch(E_2)$. Therefore, we have

$$[(E_1\oplus\underline{\mathbb{C}^n}, h^{E_1}\oplus h_0, \nabla^{E_1}\oplus\nabla_0, 0)] = [(E_2\oplus\underline{\mathbb{C}^n}, h^{E_2}\oplus h_0, \nabla^{E_2}\oplus\nabla_0, \eta_0)]$$

Applying the K-relations, we get

$$\begin{aligned} & [\check{E}_1]-[\check{E}_2] \\ = \; & [(E_1\oplus\underline{\mathbb{C}^n}, h^{E_1}\oplus h_0, \nabla^{E_1}\oplus\nabla_0, \omega_1)] - [(E_2\oplus\underline{\mathbb{C}^n}, h^{E_2}\oplus h_0, \nabla^{E_2}\oplus\nabla_0, \omega_2)] \\ = \; & [(E_2\oplus\underline{\mathbb{C}^n}, h^{E_2}\oplus h_0, \nabla^{E_2}\oplus\nabla_0, \omega_1+\eta_0)] - [(E_2\oplus\underline{\mathbb{C}^n}, h^{E_2}\oplus h_0, \nabla^{E_2}\oplus\nabla_0, \omega_2)] \\ = \; & i_1(\omega_1+\eta_0-\omega_2). \end{aligned}$$

This proves the first exact sequence.

Step 4. We prove the second exact sequence. Recall that $K^1(X,\mathbb{R}/\mathbb{Z})$ consists of elements (Cf. [28])

$$[\check{E}_1]-[\check{E}_2] = [(E_1,h^{E_1},\nabla^{E_1},\omega_1)] - [(E_2,h^{E_2},\nabla^{E_2},\omega_2)]$$

in $\check{K}^0(X)$ such that

- $rank E_1 = rank E_2$.
- $ch(E_i,\nabla^{E_i}) = d\omega_i - rank E_i$.

Define the homomorphism j_1 to be the inclusion map, as $K^1(X,\mathbb{R}/\mathbb{Z})$ is a subgroup of $\check{K}^0(X)$. It is easy to see that $\check{ch}$ is a surjective map and $\check{ch}\circ j_1 = 0$ implies that $Ker(\check{ch})\subset Im(j_1)$.

Step 5. We show that $Im(j_1)\subset Ker(\check{ch})$. Given an element

$$[\check{E}_1]-[\check{E}_2] = [(E_1,h^{E_1},\nabla^{E_1},\omega_1)] - [(E_2,h^{E_2},\nabla^{E_2},\omega_2)]$$

satisfying $\check{ch}([\check{E}_1]-[\check{E}_2]) = ch(E_1,\nabla^{E_1}) - ch(E_,\nabla^{E_2}) - d\omega_1 + d\omega_2 = 0$. We have

$$ch([E_1]-[E_2]) = ch([E_1]) - ch([E_2]) = 0.$$

By the exact sequence (2.1),

$$K^1(X,\mathbb{R}/\mathbb{Z}) \xrightarrow{j_2} K^0(X) \xrightarrow{ch} H^{ev}(X),$$

we know that there exists an element

$$[\check{F}_1]-[\check{F}_2] = [(F_1,h^{F_1},\nabla^{F_1},\eta_1)] - [(F_2,h^{F_2},\nabla^{F_2},\eta_2)]$$

in $K^1(X,\mathbb{R}/\mathbb{Z})$, such that

$$j_2([\check{F}_1]-[\check{F}_2]) = [E_1]-[E_2].$$

This implies that $E_1\oplus F_1$ and $E_2\oplus F_2$ are stably equivalent. By the first exact sequence, we obtain

$$[\check{E}_1]-[\check{E}_2] = [\check{F}_1]-[\check{F}_2] + i_1([\omega]), \tag{2.3}$$

for some $\omega \in \Omega^{odd}(X)/\Omega_0^{odd}(X)$. Apply the Chern character map to (2.3), we know that $d\omega = 0$, which ensures that

$$i_1([\omega]) = [(\underline{\mathbb{C}}^n, h_0, \nabla_0, \omega)] - [(\underline{\mathbb{C}}^n, h_0, \nabla_0, 0)]$$

is an element in $K^1(X, \mathbb{R}/\mathbb{Z})$. Hence,

$$[\check{E}_1] - [\check{E}_2] = [\check{F}_1] - [\check{F}_2] + i_1([\omega]) \in K^1(X, \mathbb{R}/\mathbb{Z}).$$

This shows $Im(j_1) \subset Ker(\check{ch})$. This completes the proof of the second exact sequences. One can check that the diagram (2.2) is commutative. □

2.2. **Quillen's universal Chern character form.** In this subsection, we will recall Quillen's model of classifying spaces for K-theory, and his universal Chern character form of the canonical K-theory class (following [35]). Quillen's models are the infinite dimensional restricted Grassmannians and unitary groups relative to the space of Schatten class operators, (for both graded and ungraded Hilbert spaces). Later we will use this model to study differential characters in twisted K-theory.

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Let $\mathfrak{L}^p(\mathcal{H})$ denote the Schatten class, the Banach space of operators with the norm $\|T\|_p = (Tr(T^*T)^{p/2})^{1/p} < \infty$. Note that $\mathfrak{L}^2(\mathcal{H})$ is the Hilbert space of Hilbert-Schmidt operators. Let $\mathbf{U}^p$, $(p \in [1, \infty])$ be the subgroup of unitaries in $U(\mathcal{H})$, which are congruent to 1 modulo the space $\mathfrak{L}^p(\mathcal{H})$. The Schatten ideal for $p = \infty$ is by definition the space of compact operators with the operator norm. It was shown that the manifolds $\mathbf{U}^p$ are Banach manifolds and are smoothly homotopy equivalent to $Fred_*^{sa}(\mathcal{H})$ (see section 7 in [35]). Hence, the manifold $\mathbf{U}^p$ is a classifying space for odd K-theory

$$K^1(X) = [X, \mathbf{U}^p]$$

for any finite dimensional Cw complex X.

Now suppose that $\hat{\mathcal{H}}$ is a $\mathbb{Z}_2$ graded Hilbert space, i.e., a Hilbert space with an involution ϵ on $\hat{\mathcal{H}}$ whose eigenspaces are infinite dimensional. (A $\mathbb{Z}_2$ graded Hilbert space can always be formed from an ungraded Hilbert space $\mathcal{H}$ by setting $\hat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$, $\epsilon = 1 \oplus (-1)$.) Let

$$\mathbf{G}r^p = \mathbf{G}r^p(\hat{\mathcal{H}}, \epsilon) \qquad (p \in [1, \infty])$$

denote the Grassmannian relative to the Schatten class $\mathfrak{L}^p(\hat{\mathcal{H}})$, that is, $\mathbf{G}r^p$ is the set of involutions congruent to ϵ modulo the space $\mathfrak{L}^p(\hat{\mathcal{H}})$. Then the manifolds $\mathbf{G}r^p$ are Banach manifolds and are smoothly homotopy equivalent to $Fred(\hat{\mathcal{H}})$ (Cf. Proposition 7.9 and Proposition 7.17 in [35]). Hence, $\mathbf{G}r^p$ for any $p \in [1, \infty]$ is a classifying space for even K-theory.

The manifolds $\mathbf{G}r^p$ and $\mathbf{U}^p$ are Banach manifolds and smoothly homotopy equivalent to the Hilbert manifolds $\mathbf{G}r^2$ and $\mathbf{U}^2$. Thus the de Rham cohomology is well-defined using the complex of smooth differential forms, and there is a canonical isomorphism $H^*_{DR} \to H^*$ from the de Rham cohomology to singular cohomology with complex coefficients.

As classifying spaces for even and odd K-theory, there are cohomology classes

$$ch_{2n} \in H^{2n}(\mathbf{G}r^p), \qquad ch_{2n-1} \in H^{2n-1}(\mathbf{U}^p)$$

representing the $2n$-th and $(2n-1)$-th Chern character of the canonical K-theory classes. These characteristic classes are represented by differential forms Φ_n using Quillen's superconnections, see section 8 in [35]. Denote by

$$\Phi_{ev} = \sum_{n=0}^{\infty} \Phi_{2n}, \qquad \Phi_{odd} = \sum_{n=0}^{\infty} \Phi_{2n+1} \tag{2.4}$$

the even and odd degree differential forms on $\mathbf{G}r^p$ and $\mathbf{U}^p$ respectively. Φ_{ev} and Φ_{odd} are called universal Chern character forms, representing the Chern character of the canonical K-theory classes.

2.3. **Twisted K-theory and Twisted K-classes.** Given a class $\sigma \in H^3_{\check{C}ech}(X, \mathbb{Z})$ for a locally compact, metrizable and separable space X, there is a unique isomorphism class of principal $PU(\mathcal{H})$-bundles $\mathcal{P}_\sigma$ over X, where $PU(\mathcal{H})$ denotes the projective unitary group (equipped with the norm topology) of an infinite dimensional separable complex Hilbert space $\mathcal{H}$. There is a well-defined conjugation action of $PU(\mathcal{H})$ on the space $Fred(\mathcal{H})$ of bounded Fredholm operators. Let

$$Fred(\mathcal{P}_\sigma) = \mathcal{P}_\sigma \times_{PU(\mathcal{H})} Fred(\mathcal{H})$$

be the associated bundle of Fredholm operators.

If X is compact, the twisted K-group $K^0(X, \sigma)$ is defined to be the space of homotopy classes of sections of $Fred(\mathcal{P}_\sigma)$ (see [36]). If X is locally compact, then the twisted K-group $K^0(X, \sigma)$ is defined to be the space of homotopy classes of 'compactly supported sections' of $Fred(\mathcal{P}_\sigma)$, in the sense that these sections are invertible away from a compact set. If X_0 is a closed subset of X, we may define in the obvious way the relative twisted K-group

$$K^0(X, X_0; \sigma) = K^0(X - X_0, \sigma).$$

Replacing $Fred(\mathcal{H})$ by the space $Fred^{sa}_*$ of self-adjoint Fredholm operators with both positive and negative essential spectrum, we can similarly define the twisted K-group $K^1(X, \sigma)$.

Remarks on unbounded Fredholm operators. *In the applications we will need to deal with unbounded self adjoint Fredholm operators on a possibly graded Hilbert space. We will consider families of self adjoint Fredholm operators of the form $D + A$ where D is a fixed unbounded self adjoint operator and A is bounded and varies. These are encompassed by the above discussion provided we utilise the topology on the unbounded self adjoint Fredholm operators induced by the map $D+A \to F_{D+A} = (D+A)(1+(D+A)^2)^{-1/2}$* [13]*. This is justified in the discussion in the appendices to* [13] *where the estimate*

$$||F_{D+A} - F_D|| < C||A||$$

for C a universal constant, is proved. This estimate implies that if A varies smoothly in the uniform norm then so does F_{D+A} and hence in this sense so does $D + A$.

Just as in ordinary K-theory, we can define twisted K-groups $K^i(X, \sigma)$ for all $i \in \mathbb{Z}$ such that the twisted K-theory is a generalized cohomology theory with period 2 on the category of topological spaces equipped with a principal $PU(\mathcal{H})$-bundle. Twisted K-theory satisfies the following basic properties [2][15].

(1) For any proper continuous map $f : X \to Y$ there exists a natural pull-back map

$$f^* : K^i(Y,\sigma) \longrightarrow K^i(X, f^*\sigma), \tag{2.5}$$

for any $\sigma \in H^3(Y,\mathbb{Z})$.

(2) For any differentiable map $f : X \to Y$ between two smooth manifolds X and Y, there is a natural push-forward map

$$f_!^K : K^i\big(X, f^*\sigma + W_3(f)\big) \longrightarrow K^{i+d(f)}(Y,\sigma), \tag{2.6}$$

for any $\sigma \in H^3(Y,\mathbb{Z})$, $d(f) = dim(X) - dim(Y)\ mod(2)$, and $W_3(f) \in H^3(X,\mathbb{Z})$ is the image of $w_2(X) - f^*(w_2(Y)) \in H^2(X,\mathbb{Z}_2)$ under the Bockstein homomorphism

$$H^2(X,\mathbb{Z}_2) \longrightarrow H^3(X,\mathbb{Z}).$$

(3) If X is covered by two open subsets U_1 and U_2, there is a Mayer-Vietoris exact sequence

$$\begin{array}{ccccc} K^0(X,\sigma) & \longrightarrow & K^1(U_1\cap U_2,\sigma_{12}) & \longrightarrow & K^1(U_1,\sigma_1)\oplus K^1(U_2,\sigma_2) \\ \uparrow & & & & \downarrow \\ K^0(U_1,\sigma_1)\oplus K^0(U_2,\sigma_2) & \longleftarrow & K^0(U_1\cap U_2,\sigma_{12}) & \longleftarrow & K^1(X,\sigma) \end{array}$$

where σ_1, σ_2 and σ_{12} are the restrictions of $\sigma \in H^3(X,\mathbb{Z})$ to U_1, U_2 and $U_1 \cap U_2$ respectively.

Given a principal $PU(\mathcal{H})$-bundle $\mathcal{P}_\sigma$ over X, we can choose a local trivialization of $\mathcal{P}_\sigma$ with respect to a good open cover $X = \bigcup_i U_i$ with transition functions given by

$$g_{ij} : U_i \cap U_j \longrightarrow PU(\mathcal{H}).$$

Note that transition functions $\{g_{ij}\}$ together with the natural central extension (2.9)

$$1 \to U(1) \longrightarrow U(\mathcal{H}) \longrightarrow PU(\mathcal{H}) \to 1$$

determine a Čech representative $\{\sigma_{jk}\}$ of σ via the following isomorphisms of sheaf cohomologies

$$H^1(X, \underline{PU(\mathcal{H})}) \cong H^2(X, \underline{U(1)}) \cong H^3(X,\mathbb{Z}).$$

As we can choose lifts $\hat{g}_{ij} : U_{ij} = U_i \cap U_j \longrightarrow U(\mathcal{H})$, then we have

$$\sigma_{ijk} \cdot Id = \hat{g}_{ij}\hat{g}_{jk}\hat{g}_{ki} \tag{2.7}$$

for a $U(1)$-valued Čech cocycle $\sigma_{ijk} : U_{ijk} \to U(1)$, where $U_{ijk} = U_i \cap U_j \cap U_k$ and Id is the identity operator in $\mathcal{H}$.

Then an element in $K^0(X,\sigma)$ can be represented by a twisted family of Fredholm operators:

$$\psi_i : U_i \longrightarrow Fred(\mathcal{H}),$$

satisfying $\psi_i = \hat{g}_{ij}\psi_j\hat{g}_{ij}^{-1}$. Similarly, an element in $K^1(X,\sigma)$ can be represented by a twisted family of self-adjoint Fredholm operators:

$$\psi_i : U_i \longrightarrow Fred_*^{sa}(\mathcal{H}),$$

satisfying $\psi_i = \hat{g}_{ij}\psi_j\hat{g}_{ij}^{-1}$.

We have another equivalent description of the twisted K-class in $K^0(X,\sigma)$ using Quillen's choice of classifying spaces for even and odd K-theory (Cf. Section 7 in [35]).

The inclusion $PU(\mathcal{H}) \subset PU(\hat{\mathcal{H}},\epsilon)$ defines a continuous action of $PU(\mathcal{H})$ on $\mathbf{G}r^p$, hence, we can associate to $\mathcal{P}_\sigma$ a bundle of Grassmannians

$$\mathbf{G}r^p(\mathcal{P}_\sigma) = \mathcal{P}_\sigma \times_{PU(\mathcal{H})} \mathbf{G}r^p.$$

Then we the following isomorphism

$$K^0(X,\sigma) \cong [X, \mathbf{G}r^p(\mathcal{P}_\sigma)] \cong [\mathcal{P}_\sigma, \mathbf{G}r^p]^{PU(\mathcal{H})},$$

where $[X, \mathbf{G}r^p(\mathcal{P}_\sigma)]$ denotes the space of homotopy classes of sections on $Gr^p(\mathcal{P}_\sigma)$, and $[\mathcal{P}_\sigma, \mathbf{G}r^p]^{PU(\mathcal{H})}$ denotes the space of homotopy classes of $PU(\mathcal{H})$ equivariant maps from $\mathcal{P}_\sigma$ to $\mathbf{G}r^p$.

Similarly, we have the following isomorphism for the odd twisted K-theory

$$K^1(X,\sigma) \cong [X, \mathcal{P}_\sigma \times_{PU(\mathcal{H})} \mathbf{U}^p] \cong [\mathcal{P}_\sigma, \mathbf{U}_p]^{PU(\mathcal{H})},$$

where $[X, \mathcal{P}_\sigma \times_{PU(\mathcal{H})} \mathbf{U}^p]$ denotes the space of homotopy classes of sections on the associated bundle

$$\mathbf{U}^p(\mathcal{P}_\sigma) = \mathcal{P}_\sigma \times_{PU(\mathcal{H})} \mathbf{U}^p,$$

and $[\mathcal{P}_\sigma, \mathbf{U}^p]^{PU(\mathcal{H})}$ denotes the space of homotopy classes of $PU(\mathcal{H})$ equivariant maps from $\mathcal{P}_\sigma$ to $\mathbf{U}_p$. Therefore we can represent a twisted K^1-class in $K^1(X,\sigma)$ by a twisted family of unitary operators in U^p:

$$\psi_i : U_i \longrightarrow \mathbf{U}^p,$$

satisfying $\psi_i = \hat{g}_{ij}\psi_j\hat{g}_{ij}^{-1}$.

2.4. **Geometry of bundle gerbes.** In this subsection, we recall some basics of bundle gerbes and their connections and curvings from [33]. Let $\mathcal{G}_\sigma$ be the lifting bundle gerbe [31]

(2.8)
$$\begin{array}{ccc}
\mathcal{G}_\sigma & & \\
\downarrow & & \\
\mathcal{P}_\sigma^{[2]} & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & \mathcal{P}_\sigma \\
 & & \downarrow \\
 & & X
\end{array}$$

over X associated to $\mathcal{P}_\sigma$ and the central extension

(2.9)
$$1 \to U(1) \longrightarrow U(\mathcal{H}) \longrightarrow PU(\mathcal{H}) \to 1.$$

Note that $\mathcal{G}_\sigma$ is the natural groupoid $U(1)$-extension of the groupoid $\mathcal{P}_\sigma^{[2]} = \mathcal{P}_\sigma \times_X \mathcal{P}_\sigma$ with the source map given by $\pi_1 : (y_1, y_2) \mapsto y_1$ and the target map given by $\pi_2 : (y_1, y_2) \mapsto y_2$. The source map and the target map of $\mathcal{G}_\sigma$ are denoted by s and t respectively. Equipped with a bundle gerbe connection and a curving the lifting bundle gerbe provides a differential geometric realisation of a degree 3 Deligne cohomology class $\check{\sigma} \in H^3_{\mathcal{D}}(X)$. (Recall that Deligne cohomology is the hypercohomology group of the complex of sheaves on X:

$$\underline{U(1)} \xrightarrow{d\log} \Omega^1_X \xrightarrow{d} \Omega^2_X$$

where $\underline{U(1)}$ is the sheaf of smooth $U(1)$-valued functions on M and Ω^p_X is the sheaf of imaginary-valued differential p-forms on M.) The third Deligne cohomology group $H^3_{\mathcal{D}}(X)$ classifies bundle gerbes with connection and curving, up to an equivalence relation called stable isomorphism [33].

With a good cover $X = \bigcup_i U_i$ and associated trivialisation of $\mathcal{P}_\sigma$ with transition functions $\{g_{ij}\}$, we can represent the Deligne class $\check{\sigma}$ by a degree 2 Deligne cocycle

$$(\sigma_{ijk}, A_{ij}, B_i),$$

where σ_{ijk} is a $U(1)$-valued Čech cocycle representing $\sigma \in H^3(X, \mathbb{Z})$, A_{ij} is the bundle gerbe connection 1-form on U_{ij} under the chosen trivialization and satisfies

$$A_{ij} + A_{jk} + A_{ki} = d\log(\sigma_{ijk}),$$

and B_i is a gerbe curving satisfying

$$B_j - B_i = \frac{\sqrt{-1}}{2\pi} dA_{ij}.$$

The curvature of a Deligne cohomology class $\check{\sigma}$ is a closed 3-form $H = curv(\check{\sigma})$, satisfying

$$H|_{U_i} = dB_i,$$

which represents the image of σ in $H^3(X, \mathbb{R})$. Denote by $\Omega^k_{\mathbb{Z}}(X)$ the space of those closed k-forms representing classes in the image of the map $\iota : H^3(X, \mathbb{Z}) \to H^3(X, \mathbb{R})$.

In summary, the third Deligne cohomology $H^3_{\mathcal{D}}(X)$ is characterized by the following commutative diagram whose rows and columns are exact

(2.10)
$$\begin{array}{ccccccccc}
 & & & & 0 & & H^2(X,\mathbb{R}) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H^2(X,U(1)) & \longrightarrow & H^2(X,U(1)) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dfrac{\Omega^2(X)}{\Omega^2_{\mathbb{Z}}(X)} & \longrightarrow & H^3_{\mathcal{D}}(X) & \xrightarrow{c} & H^3(X,\mathbb{Z}) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle curv} & & \downarrow{\scriptstyle \iota} & & \\
 & & & & \Omega^3_{\mathbb{Z}}(X) & \longrightarrow & H^3(X,\mathbb{R}) & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

From this diagram we can read off the fact that a Deligne cohomology class $\check{\sigma}$ has a vanishing curvature and a vanishing characteristic class,

$$curv(\check{\sigma}) = 0, \qquad \sigma = 0,$$

if and only if $\check{\sigma}$ is a class from

$$\frac{\Omega^2_{cl}(X)}{\Omega^2_{\mathbb{Z}}(X)} \cong \frac{H^2(X,\mathbb{R})}{\overline{H}^2(X,\mathbb{Z})} \hookrightarrow H^2(X,U(1)) \hookrightarrow H^3_{\mathcal{D}}(X),$$

where $\Omega^2_{cl}(X)$ denotes the space of closed 2-forms on X and $\overline{H}^2(X,\mathbb{Z})$ denotes the image of the map $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})$.

Conversely, given a Deligne class $\check{\sigma} \in H^3_{\mathcal{D}}(X)$, we choose a geometric realization given by a lifting bundle gerbe $\mathcal{G}_\sigma$ with a bundle gerbe connection θ and a curving B. We denote this triple by $\mathcal{G}_{\check{\sigma}} = (\mathcal{G}_\sigma, \theta, B)$ and diagrammatically by:

(2.11)
$$\begin{array}{ccc}
(\mathcal{G}_\sigma, \theta) & & \\
\downarrow & & \\
\mathcal{P}^{[2]}_\sigma & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & (\mathcal{P}_\sigma, B) \\
 & & \downarrow \\
 & & X.
\end{array}$$

Note that the bundle gerbe connection θ is compatible with the bundle gerbe multiplication, the curving $B \in \Omega^2(\mathcal{P}_\sigma, i\mathbb{R})$ satisfies

$$(\pi_2^* - \pi_1^*)B = \frac{\sqrt{-1}}{2\pi} F_\theta, \tag{2.12}$$

where F_θ is the curvature of θ.

Equivalently, we can represent a Deligne class $\check{\sigma}$ by a principal $PU(\mathcal{H})$-bundle $\mathcal{P}_\sigma$. The transition functions for this bundle we write as

$$g_{ij} : U_i \cap U_j \longrightarrow PU(\mathcal{H}).$$

Over $PU(\mathcal{H})$ there is a canonical $U(1)$-bundle $U(\mathcal{H})$ with a connection 1-form θ such that

$$\hat{g}_{ij}^*\theta = A_{ij},$$

where $\hat{g}_{ij}$ is the lift: $U_i \cap U_j \longrightarrow U(\mathcal{H})$. We can choose a locally defined 2-form B_i on U_i such that

$$B_j - B_i = \frac{\sqrt{-1}}{2\pi} dA_{ij},$$

dB_i is a globally defined 3-from H, the curvature of $\check{\sigma}$.

There is yet another definition of twisted K-groups developed in [10] and [15] using the theory of bundle gerbes [31]. We summarise this point of view now. We consider Hilbert bundles W over $\mathcal{P}_\sigma$ with structure group $\mathbf{U}^1$ (the subgroup of $U(\mathcal{H})$ of unitary operators of the form 1+ trace class) and an action of the groupoid $\mathcal{G}_\sigma$, i.e., an isomorphism

$$\phi : \mathcal{G}_\sigma \times_{(s,\pi)} W \longrightarrow W$$

where $\mathcal{G}_\sigma \times_{(s,\pi)} W$ is the fiber product of the source $s : \mathcal{G}_\sigma \to \mathcal{P}_\sigma$ and $\pi : W \to \mathcal{P}_\sigma$ such that

(1) $\pi \circ \phi(g, v) = t(g)$ for $(g, v) \in \mathcal{G}_\sigma \times_{(s,\pi)} E$, and t is the target map $\mathcal{G}_\sigma \to \mathcal{P}_\sigma$.
(2) ϕ is compatible with the bundle gerbe multiplication $m : \mathcal{G}_\sigma \times_{(s,t)} \mathcal{G}_\sigma \to \mathcal{G}_\sigma$, which means

$$\phi \circ (id \times \phi) = \phi \circ (m \times id).$$

Denote by $\mathcal{E}(M, \mathcal{G}_\sigma)$ the additive category of $\mathcal{G}_\sigma$-modules with $\mathbf{U}^1$ structure group. In [10], it was shown that

$$K^0(X, \sigma) \cong [\mathcal{P}_\sigma, Fred(\mathcal{H})]^{PU(\mathcal{H})} \cong K(\mathcal{E}(M, \mathcal{G}_\sigma)), \tag{2.13}$$

where $[\mathcal{P}_\sigma, Fred(\mathcal{H})]^{PU(\mathcal{H})}$ denotes the space of homotopy classes of $PU(\mathcal{H})$-equivaraint maps from $\mathcal{P}_\sigma$ to $Fred(\mathcal{H})$, and $K(\mathcal{E}(M, \mathcal{G}_\sigma))$ denotes the Grothendieck group of the additive category $\mathcal{E}(M, \mathcal{G}_\sigma)$ of $\mathcal{G}_\sigma$-modules with structure group $\mathbf{U}^1$.

A twisted K^0-class is represented by a twisted family of Fredholm operators

$$\psi : X \longrightarrow Fred(\mathcal{P}_\sigma).$$

We can associate with this situation a twisted Fredholm complex

$$T_\psi : E_0 \longrightarrow E_1$$

over $\mathcal{P}_\sigma$, which means, each E_i is a $U(\mathcal{H})$-equivariant Hilbert bundle over $\mathcal{P}_\sigma$ with structure group $\mathbf{U}^1$, with $U(1) \subset U(\mathcal{H})$ acting by the scalar multiplication, and T_ψ is a $U(\mathcal{H})$-equivariant Fredholm bundle map. As in [20], we call

$$(E_0, E_1; T_\psi) \tag{2.14}$$

a geometric representative of the twisted K-class in $K^0(X, \sigma)$ defined by the section ψ of $Fred(\mathcal{P}_\sigma)$.

3. Differential twisted K-theory

3.1. **Definition of differential twisted K-groups.** In this subsection, we will apply Quillen's models of classifying spaces for K-theory to define differential twisted K-cocycles.

The twisted K-theory is defined as $[X, \mathbf{G}r^p(\mathcal{P}_\sigma)]$, the space of homotopy classes of sections on $Gr^p(\mathcal{P}_\sigma)$. We equip the lifting bundle gerbe $\mathcal{G}_\sigma$ with a bundle gerbe connection and a curving and hence obtain a degree 3 Deligne cohomology class $\check{\sigma}$, represented by

$$(\sigma_{ijk}, A_{ij}, B_i),$$

with respect to a good cover $\{U_i\}$ of X which trivializes $\mathcal{P}_\sigma$. The curvature of $\check{\sigma}$ is denoted by $H = curv(\check{\sigma})$, and $c(\check{\sigma}) = \sigma$.

Given a smooth section ψ of $Gr^p(\mathcal{P}_\sigma)$, with respect to a good cover $\{U_i\}$ of X which trivializes $\mathcal{P}_\sigma$, ψ can be represented by $\{\psi_i\}$ where ψ_i is a smooth map from U_i to $\mathbf{G}r^p$ such that

$$\psi_i = Ad(\hat{g}_{ij}) \cdot \psi_j.$$

Then $\psi_i^*\Phi_{ev}$ is a well-defined smooth differential form on U_i. In fact, using the Deligne cocycle $(\sigma_{ijk}, A_{ij}, B_i)$, one can show that

$$exp(B_i) \cdot \psi_i^*\Phi_{ev}$$

is a globally defined differential form in $\Omega^{even}(X)$ satisfying

$$(d - H)\big(exp(B_i) \cdot \psi_i^*\Phi_{ev}\big) = 0.$$

Definition 4. *A differential character in twisted K^0-theory with the twisting $\check{\sigma}$, sometimes called a differential K^0-cocycle, is an equivalence class of pairs (ψ, η) where ψ is a section of $\mathbf{G}r^p(\mathcal{P}_\sigma)$, and*

$$\eta \in \Omega^{odd}(X)/Im(d - H),$$

where $d - H$ is the twisted differential operator

$$d - H : \Omega^{odd}(X) \longrightarrow \Omega^{even}(X).$$

Two pairs (ψ_0, η_0) and (ψ_0, η_1) are called equivalent if ψ_0 is homotopic to ψ_1 via a family of smooth sections ψ_t for $t \in [0, 1]$, and there exists $\omega \in \Omega^{even}(X)/Im(d - H)$ such that

$$\eta_0 - \eta_1 + (d - H)\omega = \pi_*\big(exp(B_i) \cdot (\psi_t^i)^*\Phi_{ev}\big).$$

Here $\pi : X \times [0, 1] \to X$ is the projection.

Given a differential K^0-cocycle (ψ, η) where ψ is given by $\{\psi_i\}$, a smooth map from U_i to $\mathbf{G}r^p$, we can define the differential form

$$ch_{\check{\sigma}}(\psi, \eta) = exp(B_i) \cdot \psi_i^*\Phi_{ev} + (d - H)\eta.$$

It is a straight forward calculation to show that

(1) $(d - H)ch_{\check{\sigma}}(\psi_0, \eta_0) = 0$;
(2) $ch_{\check{\sigma}}(\psi_0, \eta_0) = ch_{\check{\sigma}}(\psi_1, \eta_1)$ for any two equivalent pairs (ψ_0, η_0) and (ψ_0, η_1).

Definition 5. *The differential twisted K-theory* $\check{K}^0(X, \check{\sigma})$ *with a twisting* $\check{\sigma} \in H^3_{\mathcal{D}}(X)$ *is the space of homotopy equivalence classes of differential twisted* K^0*-cocycles. The Chern character form of a differential twisted* K^0*-cocycle* (ψ, η) *is given by* $ch_{\check{\sigma}}(\psi, \eta)$*. This defines the Chern character map*

$$ch_{\check{\sigma}} : \check{K}^0(X, \check{\sigma}) \longrightarrow \Omega^{even}(X),$$

whose image consists of even degree differential forms on X *which are closed under* $d - H$*.*

Analogously, a differential character in twisted K^1-theory with the twisting $\check{\sigma}$, sometimes called a differential K^1-cocycle, is an equivalence class of pairs (ψ, η) where ψ is a section of $\mathbf{U}^p(\mathcal{P}_\sigma)$, and

$$\eta \in \Omega^{even}(X)/Im(d - H),$$

where $d - H$ is the twisted differential operator

$$d - H : \Omega^{even}(X) \longrightarrow \Omega^{odd}(X).$$

Two pairs (ψ_0, η_0) and (ψ_0, η_1) are called equivalent if ψ_0 is homotopic to ψ_1 via a family of smooth sections ψ_t for $t \in [0, 1]$, and there exists $\omega \in \Omega^{odd}(X)/Im(d - H)$ such that

$$\eta_0 - \eta_1 + (d - H)\omega = \pi_*\big(exp(B_i) \cdot (\psi^i_t)^*\Phi_{odd}\big).$$

Here $\pi : X \times [0, 1] \to X$ is the projection.

Given a differential K^1-cocycle (ψ, η) where ψ is represented by $\{\psi_i\}$, a smooth map from U_i to $\mathbf{U}^p$, we can define the differential form

$$ch_{\check{\sigma}}(\psi, \eta) = exp(B_i) \cdot \psi_i^*\Phi_{odd} + (d - H)\eta.$$

It is a straight forward calculation to show that

(1) $(d - H)ch_{\check{\sigma}}(\psi_0, \eta_0) = 0$;
(2) $ch_{\check{\sigma}}(\psi_0, \eta_0) = ch_{\check{\sigma}}(\psi_1, \eta_1)$ for any two equivalent pairs (ψ_0, η_0) and (ψ_0, η_1).

Definition 6. *The differential twisted K-theory* $\check{K}^1(X, \check{\sigma})$ *with a twisting* $\check{\sigma} \in H^3_{\mathcal{D}}(X)$ *is the space of homotopy equivalence classes of differential twisted* K^1*-cocycles. The Chern character form of a differential twisted* K^1*-cocycle* (ψ, η) *is given by* $ch_{\check{\sigma}}(\psi, \eta)$*, this defines the Chern character form map*

$$ch_{\check{\sigma}} : \check{K}^1(X, \check{\sigma}) \longrightarrow \Omega^{odd}(X, d - H),$$

whose image consists of odd degree differential forms on X *which are closed under* $d - H$*.*

3.2. **Two examples.** In this subsection, we give two examples to illustrate why we need to define a differential twisted K-theory with twisting given by a degree 3 Deligne class.

Assume that a class $\check{\sigma} \in H^3_{\mathcal{D}}(X)$ comes from $H^2(X, U(1)$, whose characteristic class and curvature are trivial. From (2.10), we know that $\check{\sigma}$ uniquely defines a class in

$$\frac{\Omega^2_{cl}(X)}{\Omega^2_{\mathbb{Z}}(X)} \cong \frac{H^2(X, \mathbb{R})}{\overline{H}^2(X, \mathbb{Z})} \hookrightarrow H^2(X, U(1)) \hookrightarrow H^3_{\mathcal{D}}(X).$$

In the first example, we assume that $\check{\sigma}$ satisfies $\check{\sigma} \equiv c_1$, $(\mod n)$ in $H^2(X, \mathbb{Z}_n) \subset H^2(X, U(1))$ for a class $c_1 \in H^2(X, \mathbb{Z})$. Hence it satisfies $c(\check{\sigma}) = 0$, as follows from the exact sequence

$$H^2(X, \mathbb{Z}) \xrightarrow{mod\ n} H^2(X, \mathbb{Z}_n) \longrightarrow H^3(X, \mathbb{Z}).$$

The following exact sequences in Čech cohomologies

$$\begin{array}{ccccccc}
H^1(X, \underline{U(1)}) & \xrightarrow{(\cdot)^n} & H^1(X, \underline{U(1)}) & \longrightarrow & H^2(X, \mathbb{Z}_n) & \longrightarrow & H^1(X, \underline{U(1)}) \\
\downarrow \cong & & \downarrow \cong & & \downarrow = & & \downarrow \cong \\
H^2(X, \mathbb{Z}) & \xrightarrow{\times n} & H^2(X, \mathbb{Z}) & \xrightarrow{mod\ n} & H^2(X, \mathbb{Z}_n) & \xrightarrow{\beta} & H^3(X, \mathbb{Z})
\end{array}$$

here $\underline{U(1)}$ denotes the sheaf of continuous $U(1)$-valued functions and $U(1)$ is the sheaf of locally constant $U(1)$-valued functions, shows that $\check{\sigma}$ can be represented by a closed differential 2-form ω on X such that

$$[\omega] = \frac{c_1}{n} \qquad \text{in } H^2(X, \mathbb{R}).$$

Proposition 7. *Let $\check{\sigma} \in H^3_{\mathcal{D}}(X)$ be a Deligne class determined by $c_1 \in H^2(X, \mathbb{Z})$, i.e., $\check{\sigma} \equiv c_1$ (mod n) in $H^2(X, \mathbb{Z}_n) \subset H^2(X, U(1)) \subset H^3_{\mathcal{D}}(X)$, then there is an isomorphism*

$$\phi : K^0(X) \longrightarrow K^0(X, \check{\sigma}),$$

such that the twisted Chern character map on $K^0(X, \check{\sigma})$:

$$ch_{\check{\sigma}} : K^0(X, \check{\sigma}) \longrightarrow H^*(X),$$

is given by $ch_{\check{\sigma}}(\phi([E])) = exp(\frac{c_1}{n})ch([E])$, for a K-class $[E] \in K^0(X)$.

Proof. Note that the characteristic class of $\check{\sigma}$ is given by the image of c_1 $(mod\ n)$ under the Bockstein map $H^2(X, \mathbb{Z}_n) \to H^3(X, \mathbb{Z})$, which is zero, and the curvature of $\check{\sigma}$ is also zero. As discussed above, there is a closed differential 2-form ω on X such that $exp(\omega)$ represents $exp(\frac{1}{n}c_1)$ in $H^*(X, \mathbb{R})$.

Choose a good open cover $X = \bigcup_i U_i$ such that $\omega|_{U_i} = d\rho_i$. We can employ the local bundle gerbe [31] with a connection θ and a curving ω

$$\begin{array}{ccc}
(\mathcal{G}_{\check{\sigma}}, \theta) & & \\
\downarrow & & \\
\bigsqcup_{i,j} U_{ij} & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & \bigsqcup_i (U_i, d\rho_i) \\
& & \downarrow \pi \\
& & X
\end{array}$$

where $U_{ij} = U_i \cap U_j$. The bundle gerbe multiplication is given by a Čech cocycle defined by $\check{\sigma} \in H^2(X, U(1))$. Equip $\mathcal{G}_{\check{\sigma}}$ with a trivial connection θ satisfying

$$\mathcal{G}_{\check{\sigma}} \times_{U(1)} \mathbb{C} \cong \pi_2^*(L_i, \rho_i) \otimes \pi_1^*(L_i^*, -\rho_i),$$

here (L_i, ρ_j) is a complex line bundle $L_i = U_i \times \mathbb{C}$ with a connection $d + \frac{1}{2\pi i}\rho_i$.

Then there is a category equivalence from the category $\mathcal{E}(X)$ of complex vector bundles with a connection and the category $\mathcal{E}(X, \mathcal{G}_{\check{\sigma}})$ of $\mathcal{G}_{\check{\sigma}}$-modules with a $\check{\sigma}$-connection:

$$\begin{array}{ccc} \mathcal{E}(X) & \longrightarrow & \mathcal{E}(X, \mathcal{G}_{\check{\sigma}}) \\ (E, \nabla_E) & \mapsto & (\pi^* E \otimes (\bigsqcup_i L_i), \nabla_E), \end{array}$$

which induces an isomorphism

$$\phi : K^0(X) \longrightarrow K^0(X, \check{\sigma}).$$

Then our definition of twisted Chern character form in the last section tells us that the twisted Chern character is represented by the differential form $exp(\omega)Ch(E, \nabla_E)$, where $Ch(E, \nabla_E)$ is the Chern character form of (E, ∇_E), hence, the twisted Chern character is given by

$$ch_{\check{\sigma}}(\phi([E])) = exp(\frac{c_1}{n})ch([E]).$$

□

In the second example, we explain in our terms the Chern character constructed by Baum-Connes in their study of the K-theory of a Γ-manifold twisted by a group 2-cocycle of a discrete group Γ. For simplicity, we consider only free and proper actions of discrete groups. Let X be a free and proper Γ-manifold. Let γ be a $U(1)$-valued group 2-cocycle of Γ. Hence, γ defines a chomology class in

$$H^2(\Gamma, U(1)) \cong H^2(B\Gamma, U(1)),$$

where $B\Gamma$ is a classifying space of Γ for proper actions.

In [5], a (Γ, γ)-vector bundle over X is a complex vector bundle E over X together with with a smooth map $E \times \Gamma \to E$ such that (with $\pi : E \to X$ the projection):

(1) $\pi(\xi \cdot g) = \pi(\xi) \circ g$ for each $\xi \in E$, $g \in \Gamma$;
(2) $\xi \cdot (g_1 g_2) = \gamma(g_1, g_2)(\xi \cdot g_1) \cdot g_2$ for each $g_1, g_2 \in \Gamma$.

Let $\mathcal{E}_{(\Gamma,\gamma)}(X)$ be the collection of triples (E_0, E_1, T) where E_0, E_1 are (Γ, γ)-vector bundles over X, T is a smooth morphism of vector bundles such that T is Γ-equivariant, and the support of T (points in X where T is not an isomorphism) is Γ-compact.

The K-group $K^0_{(\Gamma,\gamma)}(X)$ is defined to be the Grothendieck group of the category $\mathcal{E}_{(\Gamma,\gamma)}(X)$.

Note that the $U(1)$-valued group 2-cocycle σ defines a lifting bundle gerbe

$$\begin{array}{ccc} \mathcal{G}_\gamma & & \\ \downarrow & & \\ X \times \Gamma & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & X \\ & & \downarrow \pi_0 \\ & & X/\Gamma, \end{array}$$

a central extension of the natural groupoid structure on $X \times \Gamma$. Under the exact sequence of group cohomologies,

$$H^2(\Gamma, \mathbb{Z}) \longrightarrow H^2(\Gamma, \mathbb{R}) \longrightarrow H^2(\Gamma, U(1)) \xrightarrow{\delta} H^3(\Gamma, \mathbb{Z}),$$

a group cocycle $\gamma \in H^2(\Gamma, U(1)) = H^2(B\Gamma, U(1))$ defines a Deligne class

$$\check{\sigma} = f^*\gamma \in H^2(X/\Gamma, U(1)) \subset H^3_{\mathcal{D}}(X/\Gamma),$$

where $f : X/\Gamma \to B\Gamma$ the classifying map. The characteristic class of $\check{\sigma}$ is given by $f^*(\delta\gamma)$, a torsion element in $H^3(X/\Gamma, \mathbb{Z})$, and the curvature is zero. It is straight forward to see that a (Γ, γ)-vector bundle over X is actually a $\mathcal{G}_\gamma$-module. Hence, there is a category equivalence between $\mathcal{E}_{(\Gamma,\gamma)}(X)$ and the category of $\mathcal{G}_\gamma$-modules. This induces a natural isomorphism

$$\phi : K^0_{(\Gamma,\gamma)}(X) \longrightarrow K^0(X/\Gamma, f^*\gamma),$$

where $f^*\gamma \in H^2(X/\Gamma, U(1)) \subset H^3_{\mathcal{D}}(X/\Gamma)$. Composing with our twisted Chern character map

$$ch_{f^*\gamma} : K^0(X/\Gamma, f^*\gamma) \longrightarrow H^*(X/\Gamma, \mathbb{R}),$$

we get Baum-Connes's Chern character as in section 8 of [5]

$$ch_\gamma : K^0_{(\Gamma,\gamma)}(X) \longrightarrow H^*(X/\Gamma, \mathbb{R}).$$

3.3. **Properties of differential twisted K-groups.** Let X be a compact manifold with a Deligne cohomology class $\check{\sigma}$ whose Dixmier-Douady class is σ and whose curvature is H. Let $\Omega^{ev}_0(X, H)$ and $\Omega^{odd}_0(X, H)$ denote the image of the twisted Chern character form map

$$ch_{\check{\sigma}} : \check{K}^0(X, \check{\sigma}) \longrightarrow \Omega^{ev}(X),$$

and

$$ch_{\check{\sigma}} : \check{K}^1(X, \check{\sigma}) \longrightarrow \Omega^{odd}(X),$$

and let $H^{ev}(X, H)$ and $H^{odd}(X, H)$ be the even and odd degree twisted cohomology of X with respect to the twisted differential operator $d - H$ respectively.

There are two natural forgetful maps for $i = 0$ and 1

$$\check{K}^i(X, \check{\sigma}) \longrightarrow K^i(X, \sigma) \tag{3.1}$$

with σ the Dixmier-Douady class of $\check{\sigma}$ under the map $H^3_{\mathcal{D}}(X) \longrightarrow H^3(X, \mathbb{Z})$. Then the twisted Chern character form map

$$ch_{\check{\sigma}} : \check{K}^0(X, \check{\sigma}) \longrightarrow \Omega^{ev}_0(X, d - H) \qquad \check{K}^1(X, \check{\sigma}) \longrightarrow \Omega^{odd}_0(X, d - H)$$

induces the twisted Chern character maps

$$ch_{\check{\sigma}} : K^0(X, \sigma) \longrightarrow H^{ev}(X, H)$$

and

$$ch_{\check{\sigma}} : K^1(X, \sigma) \longrightarrow H^{odd}_0(X, H)$$

such that the following two diagrams commute

$$\begin{array}{ccc} \check{K}^0(X, \check{\sigma}) & \longrightarrow & K^0(X, \sigma) \\ \downarrow ch_{\check{\sigma}} & & \downarrow ch_{\check{\sigma}} \\ \Omega^{ev}_0(X, H) & \longrightarrow & H^{ev}(X, H) \end{array}$$

$$\begin{array}{ccc} \check{K}^1(X,\check{\sigma}) & \longrightarrow & K^1(X,\sigma) \\ \downarrow{\scriptstyle ch_{\check{\sigma}}} & & \downarrow{\scriptstyle ch_{\check{\sigma}}} \\ \Omega_0^{odd}(X,H) & \longrightarrow & H^{odd}(X,H). \end{array}$$

Remark 8. *Note that our twisted Chern character map*

$$ch_{\check{\sigma}} : K^0(X,\sigma) \longrightarrow H^{ev}(X,H)$$

depends on the choice of the Deligne cohomology class $\check{\sigma}$ and, as mentioned in the introduction, this enables us to resolve a difficulty with the usual definition of the twisted Chern character.

Denote by $K^1(X,\sigma,\mathbb{R}/\mathbb{Z})$ the kernel of the differential Chern character form map

$$ch_{\check{\sigma}} : \check{K}^0(X,\check{\sigma}) \longrightarrow \Omega_0^{ev}(X,H),$$

then we have the following exact sequence similar to the second exact sequence in Theorem 3:

$$0 \longrightarrow K^1(X,\sigma,\mathbb{R}/\mathbb{Z}) \longrightarrow \check{K}^0(X,\check{\sigma}) \xrightarrow{ch_{\check{\sigma}}} \Omega_0^{ev}(X,H) \longrightarrow 0.$$

Given an element in $K^1(X,\sigma,\mathbb{R}/\mathbb{Z})$, we can represent it by a differential K^0-cocycle (ψ,η) satisfying

$$ch_{\check{\sigma}}(\psi,\eta) = exp(B_i)\cdot\psi_i^*\Phi_{ev} + (d-H)\eta = 0,$$

with $\{\psi_i\}$ given by a family of maps from U_i to $\mathbf{G}r^p$. Then under the forgetful map (3.1), we know

$$ch_{\check{\sigma}}([\psi]) = 0.$$

On the other hand, given $[\psi] \in K^0(X,\sigma)$ with $ch_{\check{\sigma}}([\psi]) = 0$, we can find an element $\eta \in \Omega^1(X)$ such that

$$exp(B_i)\cdot\psi_i^*\Phi_{ev} = (d-H)\eta.$$

It is easy to see that

$$[(\psi,-\eta)] \in K^1(X,\check{\sigma},\mathbb{R}/\mathbb{Z}).$$

Hence, we have established an exact sequence

$$K^1(X,\check{\sigma},\mathbb{R}/\mathbb{Z}) \longrightarrow K^0(X,\sigma) \longrightarrow H^{ev}(X,H).$$

In fact, the proof of Theorem 3 can be adapted to establish the following theorem which relates our differential twisted K-theory with twisted K-theory.

Theorem 9. *There exist the following two exact sequences relating the differential twisted K-theory $\check{K}^0(X,\check{\sigma})$ to the twisted K-theory $K^0(X,\sigma)$ and $K^1(X,\sigma,\mathbb{R}/\mathbb{Z})$ respectively:*

$$0 \longrightarrow \frac{\Omega^{odd}(X)}{\Omega_0^{odd}(X,H)} \longrightarrow \check{K}^0(X,\check{\sigma}) \longrightarrow K^0(X,\sigma) \longrightarrow 0,$$

$$0 \longrightarrow K^1(X,\check{\sigma},\mathbb{R}/\mathbb{Z}) \longrightarrow \check{K}^0(X,\check{\sigma}) \xrightarrow{ch_{\check{\sigma}}} \Omega_0^{ev}(X,H) \longrightarrow 0$$

such that these two exact sequences fit into the following commutative diagram

(3.2)
$$\begin{array}{ccccc}
 & & K^1(X,\check{\sigma},\mathbb{R}/\mathbb{Z}) & \xrightarrow{Id} & K^1(X,\check{\sigma},\mathbb{R}/\mathbb{Z}) \\
 & & \downarrow & & \downarrow \\
0 \to \dfrac{\Omega^{odd}(X)}{\Omega_0^{odd}(X,H)} & \longrightarrow & \check{K}^0(X,\check{\sigma}) & \longrightarrow & K^0(X,\sigma) \to 0 \\
 & & \downarrow {\scriptstyle ch_{\check{\sigma}}} & & \downarrow {\scriptstyle ch_{\check{\sigma}}} \\
 & & \Omega_0^{ev}(X,H) & \longrightarrow & H^{ev}(X,H).
\end{array}$$

This commutative diagram of exact sequences, together with the push-forward maps in twisted K-theory and cohomology theory determines a canonical push-forward map in differential twisted K-theory.

3.4. **Riemann-Roch theorem in twisted K-theory.** In [15], we showed that there is a natural push-forward map (2.6) associated to any differentiable map $f : X \to Y$ and any $\sigma \in H^3(Y,\mathbb{Z})$

$$f_!^K : K^i\big(X, f^*\sigma + W_3(f)\big) \longrightarrow K^{i+d(f)}(Y,\sigma),$$

where

(1) $d(f) = dim(X) - dim(Y)\ mod(2)$;
(2) $W_3(f) \in H^3(X,\mathbb{Z})$ is the image of $w_2(X) - f^*(w_2(Y)) \in H^2(X,\mathbb{Z}_2)$ under the Bockstein homomorphism

$$H^2(X,\mathbb{Z}_2) \longrightarrow H^3(X,\mathbb{Z}).$$

Choose a Deligne cohomology class $\check{\sigma} \in H^3_{\mathcal{D}}(Y)$ whose Dixmier-Douady class is σ and whose curvature is H. Then $f^*\check{\sigma} \in H^3_{\mathcal{D}}(X)$ has Dixmier-Douady class $f^*\sigma$ and curvature f^*H. Let $w_2(f)$ denote the Deligne cohomology class

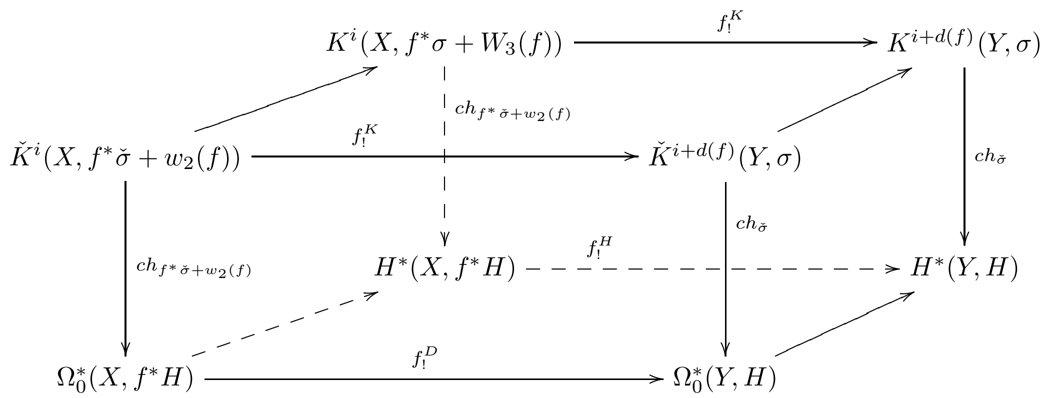
$$w_2(X) - f^*(w_2(Y)) \in H^2(X,\mathbb{Z}_2) \subset H^2(X,U(1)) \subset H^3_{\mathcal{D}}(X),$$

together with the middle vertical exact sequence in the commutative diagram (2.10), we obtain a Deligne cohomology class

$$f^*\check{\sigma} + w_2(f) \in H^3_{\mathcal{D}}(X)$$

whose Dixmier-Douady class is $f^*\sigma + W_3(f)$ and whose curvature is f^*H.

The Riemann-Roch theorem in differential twisted K-theory can be summarized by the following Chern character defect diagram. (Cf.[11] for the Riemann-Roch theorem in twisted K-theory for K-oriented maps, i.e, $W_3(f) = 0$). This diagram is not quite commutative, the non-commutative part is prescribed by the Chern character defect.



Here $f_!^H$ denotes the Gysin homomorphism in cohomology and $f_!^D$ denotes the push-forward map on differential forms.

We will discuss two special cases of the Riemann-Roch theorem in differential twisted K-theory which are enough for applications in this paper. Assume the map $f : X \to Y$ is an embedding and either

(1) $f^*\sigma + W_3(f) = 0$ or,
(2) let $\sigma \in H^3(Y, \mathbb{Z})$ be such that $f^*\sigma + W_3(f) = W_3(X)$.

Case 1. As $f^*\sigma + W_3(f) = 0$, we have

$$K^i(X, f^*\sigma + W_3(f)) \cong K^i(X).$$

We investigate the Chern character defect in the following diagram

$$\begin{CD} K^i(X) @>{f_!^K}>> K^{i+d(f)}(Y,\sigma) \\ @V{ch}VV @VV{ch_{\check{\sigma}}}V \\ H^*(X) @>{f_!}>> H^*(Y,H). \end{CD}$$

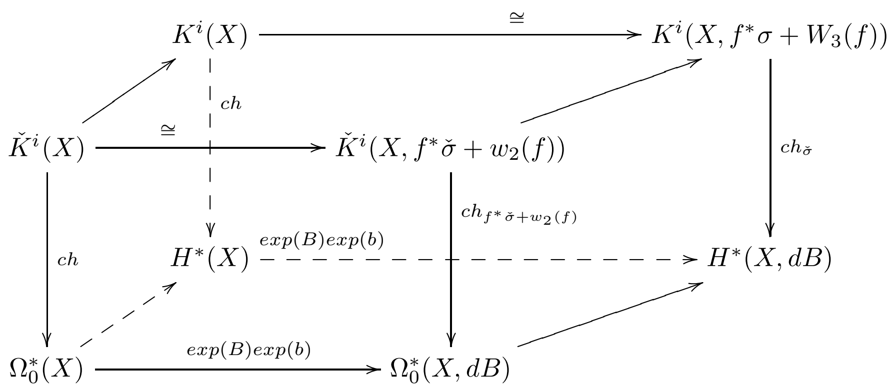
The twisted Chern character $ch_{\check{\sigma}+w_2(f)}$ on $K^i(X, f^*\sigma + W_3(f))$ is different to the Chern character on $K^i(X)$ due to the role of the Deligne cohomology class in the twisted Chern character. The Deligne class $\check{\sigma} + w_2(f)$ is non-zero with its curvature given by $f^*H = dB$ for some global 2-form B on X. Note that B defines a Deligne class according to (2.10)

$$[B] = [(1, 0, B)],$$

such that $c(\check{\sigma} + w_2(f) - [B]) = 0$ and $curv(\check{\sigma} + w_2(f) - [B]) = 0$. So as a class in $H^2(X, U(1))$, $\check{\sigma} + w_2(f) - [B]$ is the image of $b \in H^2(X, \mathbb{R})$ under the exponential map

$$H^2(X, \mathbb{R}) \longrightarrow H^2(X, U(1)).$$

Then we have the following commutative diagram



whose proof is similar to the proof of Proposition 7. Notice that the map $exp(B) \cdot exp(b)$ doesn't depend on the choices of B and b.

Definition 10. *Let V be a real vector bundle over a compact manifold X. Let B be a 2-form on X and $b \in H^2(X, \mathbb{R})$. Let $\sigma_X \in H^3(X, \mathbb{Z})$. A* **$\sigma_X$-twisted** *Spin*^{c} **structure** *on V associated to (B, b) is a degree 3 Deligne cohomology class $\check{\sigma}_X \in H^3_{\mathcal{D}}(X)$ such that*

1. *$c(\check{\sigma}_X) = \sigma_X$ and $curv(\check{\sigma}_X) = dB$.*
2. *$\check{\sigma}_X + w_2(V) = [B] + [b]$ in $H^3_{\mathcal{D}}(X)$.*

where $[B] = [(1, 0, B)]$ and $[b]$ are Deligne cohomology classes in $H^3_{\mathcal{D}}(X)$ under the maps

$$\Omega^2(X) \longrightarrow \frac{\Omega^2(X)}{\Omega^2_0(X)} \subset H^3_{\mathcal{D}}(X) \text{ and } H^2(X, \mathbb{R}) \longrightarrow H^2(X, U(1)) \subset H^3_{\mathcal{D}}(X)$$

respectively (Cf. (2.10)).

The identity $\check{\sigma}_X + w_2(V) = [B] + [b]$ implies that

$$\sigma_X + W_3(V) = 0.$$

So our σ_X-twisted *Spin*^{c} structure on V associated to (B, b) can be thought of as a $c(\check{\sigma}_X)$-twisted *Spin*^{c} structure as defined by Douglas [18] with a trivialization given by (B, b). Two trivializations (B_1, b_1) and (B_2, b_2) are called equivalent if there exists an $\omega \in \Omega^2_0(X)$ such that

$$B_2 = B_1 + \omega, \qquad \text{and} \qquad b_2 = b_1 - [\omega]$$

where $[\omega]$ is the class in $H^2(X, \mathbb{R})$ represented by ω. The standard Mayer-Vietoris sequence argument and the Riemann-Roch Theorem for *Spin*^{c} vector bundles lead to the following proposition.

Proposition 11. *Let $\pi : V \to X$ be a real vector bundle of even rank over a compact manifold X with a σ_X-twisted Spin^c structure $\check{\sigma}_X$ on V associated to (B, b). Let ϕ_K and ϕ_H be Thom isomorphisms in twisted K-theory and cohomology theory. Then the Chern* *character defect in the following diagram*

$$\begin{array}{ccccc} K^i(X) & \xrightarrow[\cong]{\phi} & K^i(X, \sigma_X + W_3(V)) & \xrightarrow{\phi_K} & K^i(V, \pi^*\sigma_X) \\ \Big\downarrow ch & & \Big\downarrow ch_{\check{\sigma}_X + w_2(V)} & & \Big\downarrow ch_{\pi^*\check{\sigma}_X} \\ H^*(X) & \xrightarrow{exp(B)exp(b)} & H^*(X, dB) & \xrightarrow{\phi_H} & H^*(V, d\pi^*B) \end{array}$$

is given by the relation

$$ch_{\pi^*\check{\sigma}_X}\big(\phi_K \circ \phi([E])\big) = \phi_H\big(exp(B)exp(b)\hat{A}(V)^{-1}ch([E])\big),$$

for each element $[E] \in K^i(X)$ *and* $\hat{A}(V)$ *is the A-hat genus of* V

$$\hat{A}(V) = \Pi_i \frac{x_i/2}{\sinh(x_i/2)}$$

where the Pontrjagen classes of V *are the elementary symmetric functions of* $\{x_i^2\}$.

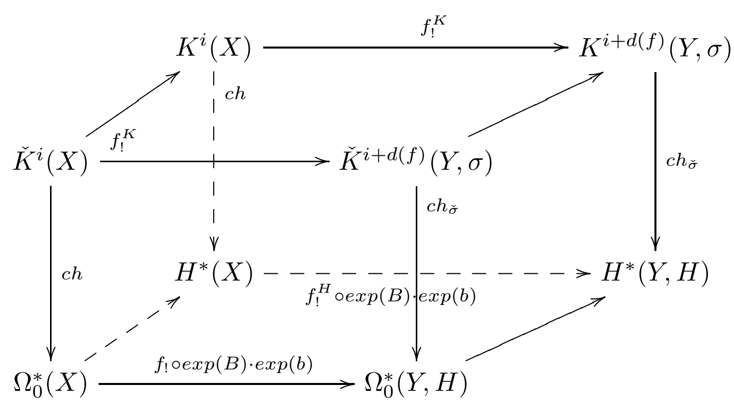
Theorem 12. *Let* $f : X \to Y$ *be an embedding and* $\sigma \in H^3(Y, \mathbb{Z})$ *such that*

$$f^*\sigma + W_3(f) = 0.$$

Let $\check{\sigma}$ *be a Deligne class in* $H^3_{\mathcal{D}}(Y)$ *with* $c(\check{\sigma}) = \sigma$ *and* $curv(\check{\sigma}) = H$. *Let* (B, b) *consist of elements* $B \in \Omega^2(X)$ *and* $b \in H^2(X, \mathbb{R})$ *such that*

$$f^*\check{\sigma} + w_2(f) = [B] + [b]$$

in $H^3_{\mathcal{D}}(X)$. *Then the following diagram*



is not commutative and the failure is measured by the Chern character defect described by the relation

$$ch_{\check{\sigma}}\big(f_!^K([E])\big) = f_!^H\big(exp(B) \cdot exp(b)\hat{A}(\nu_f)ch([E])\big), \tag{3.3}$$

for each element $[E] \in K^i(X)$ *where* $\hat{A}(\nu_f) = f^*(\hat{A}(TY))^{-1}\hat{A}(TX)$ *is the defect.*

Proof. We assume that $d(f) = dim(Y) - dim(X) = 0(\ mod 2)$, the odd $d(f)$ case can be obtained by the similar arguments. Denote by ν_f the normal bundle of X in Y under f, then ν_f is equipped with a $f^*(\sigma + W_3(Y))$-twisted $Spin^c$ structure $f^*(\check{\sigma} + w_2(Y))$. We represent ν_f as a tubular neighbourhood of X in Y, then Proposition 11 gives rise to the Chern character defect

$$ch_{\check{\sigma}}\big(\phi_K([E])\big) = \phi_H\big(exp(B)exp(b)\hat{A}(\nu_f)^{-1}ch([E])\big),$$

for each element $[E] \in K^i(X)$ in the following diagram

$$\begin{array}{ccc} K^i(X) & \xrightarrow[\cong]{\phi_K} & K^i(\nu_f, \sigma) \\ \Big\downarrow ch & & \Big\downarrow ch_{\check\sigma} \\ H^*(X) & \xrightarrow{[U]\cdot exp(\pi^*B)exp(\pi^*b)} & H^*(\nu_f, d\pi^*B) \end{array}$$

here U is the Thom form of ν_f with compact support such that

$$\phi_H(\omega) = [U]\cdot \pi^*\omega \qquad \text{for } \omega \in H^*(X)$$

is the Thom isomorphism in cohomology theory. This Chern character defect can be lifted to the differential K-group

$$\begin{array}{ccc} \check K^i(X) & \xrightarrow[\cong]{\phi_K} & \check K^i(\nu_f, \check\sigma) \\ \Big\downarrow ch & & \Big\downarrow ch_{\check\sigma} \\ \Omega^*(X) & \xrightarrow{f_!\circ exp(B)exp(b)} & \Omega^*(\nu_f, d\pi^*B) \end{array}$$

such that

$$\begin{aligned} ch_{\check\sigma}\big(\phi_K([\check E])\big) &= f_! \circ exp(B)exp(b)(\hat A(\nu_f)^{-1}ch([\check E])) \\ &= U\cdot \pi^*\big(exp(B)exp(b)\hat A(\nu_f)^{-1}ch([\check E])\big) \end{aligned}$$

for each element $[\check E] \in \check K^i(X)$, here we need to choose a connection on ν_f to get a differential form representing $\hat A(\nu_f)^{-1}$.

As ν_f is an open neighbourhood of X in Y, then the diagram

$$\begin{array}{ccc} K^i(v_f, \sigma) & \xrightarrow{f_!^K} & K^i(Y, \sigma) \\ ch_{\check\sigma}\Big\downarrow & & \Big\downarrow ch_{\check\sigma} \\ H^*(\nu_f, H) & \xrightarrow{f_!^H} & H^*(Y, H) \end{array}$$

is commutative and the corresponding diagram in differential twisted K-theory is also commutative. Hence, we have the Chern character defect (3.3) as in Theorem for the following diagram

$$\begin{array}{ccc} K^i(X) & \xrightarrow{f_!^K} & K^{i+d(f)}(Y, \sigma) \\ \Big\downarrow ch & & \Big\downarrow ch_{\check\sigma} \\ H^*(X) & \xrightarrow{f_!^H\circ exp(B)exp(b)} & H^*(Y, H) \end{array}$$

and the corresponding defect for differential twisted K-groups. □

Remark 13. *When the twisting $\check\sigma = 0$ in Theorem 12, then we have $W_3(f) = 0$, that is, f is K-oriented. The choice of b defines a Spin^c structure on the normal bundle of f. Then Theorem 12 gives rise to the Atiyah-Hirzebruch version of the Riemann-Roch theorem (Cf. Theorem V.4.17 in* [26]*.*

Case 2. We now investigate the Chern character defect in the following diagram

$$\begin{array}{ccc} K^i(X, f^*\sigma + W_3(f)) & \xrightarrow{f_!^K} & K^{i+d(f)}(Y,\sigma) \\ \downarrow{\scriptstyle ch_{f^*\check{\sigma}+w_2(f)}} & & \downarrow{\scriptstyle ch_{\check{\sigma}}} \\ H^*(X, f^*H) & \xrightarrow{f_!} & H^*(Y,H). \end{array}$$

for an embedding $f: X \to Y$ with $f^*\sigma + W_3(f) = W_3(X)$. In this case we have

$$K^i(X, f^*\sigma + W_3(f)) \cong K^i(X, W_3(X)).$$

As before, we have $f^*H = dB$ and

$$f^*\check{\sigma} + w_2(f) - w_2(X) - [B]$$

defines an element $b \in H^2(X,\mathbb{R})$ such that

$$f^*\check{\sigma} + w_2(f) = w_2(X) + [B] + [b].$$

Therefore, we have the following commutative diagram

$$\begin{array}{ccc} K^i(X, W_3(X)) & \xrightarrow{\cong} & K^i(X, f^*\sigma + W_3(f)) \\ \downarrow{\scriptstyle ch_{w_2(X)}} & & \downarrow{\scriptstyle ch_{f^*\check{\sigma}+w_2(f)}} \\ H^*(X) & \xrightarrow{exp(B)\cdot exp(b)\cdot} & H^*(X, dB) \end{array} .$$

On the other hand, elements in $K^0(X, W_3(X))$ can be represented by Clifford bundles when X is equipped with a Riemannian metric. Going through the definition of our differential Chern character, we know that, for a Clifford bundle E over X representing $[E] \in K^0(X, W_3(X))$,

$$ch_{w_2(X)}([E]) = Ch(E/S),$$

where $Ch(E/S)$ is the relative Chern character of the Clifford bundle E in [8] (See also [32]). We state the Riemann-Roch theorem in this case as follows whose proof is similar to the proof of Theorem 12.

Theorem 14. *Let $f: X \to Y$ be an embedding and $\sigma \in H^3(Y,\mathbb{Z})$ such that*

$$f^*\sigma + W_3(f) = W_3(X).$$

Let $\check{\sigma}$ be a Deligne class in $H^3_{\mathcal{D}}(Y)$ with $c(\check{\sigma}) = \sigma$ and $curv(\check{\sigma}) = H$. Let (B, b) be the pair of elements $B \in \Omega^2(X)$ and $b \in H^2(X,\mathbb{R})$ such that

$$f^*\check{\sigma} + w_2(f) = w_2(X) + [B] + [b]$$

in $H^3_{\mathcal{D}}(X)$. Then the following diagram

$$\begin{array}{ccc} K^i(X, W_3(X)) & \xrightarrow{f_!^K} & K^{i+d(f)}(Y,\sigma) \\ \downarrow{\scriptstyle ch_{w_2(X)}} & & \downarrow{\scriptstyle ch_{\check{\sigma}}} \\ H^*(X) & \xrightarrow{f_!^H \circ exp(B)\cdot exp(b)} & H^*(Y,H) \end{array}$$

not commutative. The Chern character defect can be described by the relation

$$ch_{\check{\sigma}}\big(f_!^K([E])\big) = f_!^H\big(exp(B)\cdot exp(b)\hat{A}(\nu_f)ch_{w_2(X)}(E)\big),$$

for each element $[E]\in K^i(X, W_3(X))$ *represented by a Clifford bundle* E*, where* $\hat{A}(\nu_f) = f^*(\hat{A}(TY))^{-1}\hat{A}(TX)$.

Corollary 15. *Given a Clifford bundle* E *over a compact oriented even dimensional manifold* X*, then the Atiyah-Singer index of the Dirac operator* $\not{D}$ *is given by*

$$Index(\not{D}) = \int_X \hat{A}(X)ch_{w_2(X)}(E).$$

4. Geometric model of differential twisted K-theory

In this subsection, we will construct finite rank vector bundles over each double intersection $U_i\cap U_j$ with a connection, and apply this geometric model to represent differential twisted K-classes. Here we shall discuss in detail the case of odd differential twisted classes, geared to the examples to be discussed later. The even case can be handled in a similar way except that there is a grading operator Γ anticommuting with the family of self-adjoint operators defining the K-classes. The resulting local vector bundles will be accordingly graded vector bundles. There is a related construction in the even case in [24].

Let X be a compact manifold and $\mathcal{P}_\sigma$ a fixed principal $PU(\mathcal{H})$ bundle over M whose Dixmier-Douady class is σ. Let $f:\mathcal{P}_\sigma\to Fred_*$ be a $PU(\mathcal{H})$ equivariant map to self-adjoint Fredholm operators with both positive and negative essential spectrum, i.e., the homotopy class of f is an element of $K^1(X,\mathcal{P}_\sigma)$. We make the additional assumption that the operators $f(x)$ have discrete spectrum (which is the case in many physics examples). This is no real limitation, since by [4] the space $Fred_*$ is homotopy equivalent to the subspace $\mathcal{F}_*$ of operators of norm less than or equal to one and with essential spectrum ± 1. Moreover the map from unbounded to bounded Fredholms $D\to D(1+D^2)^{-1/2}$ introduced earlier maps the unbounded self adjoint operators with discrete spectrum into the space $\mathcal{F}_*$.

Next choose an open cover $\{U_i\}$, $i=1,2,\ldots,n$, of X such that on each U_i there is a local section $\phi_i: U_i\to\mathcal{P}_\sigma$ and for each i there is a real number λ_i not in the spectrum of the operators $f(\phi_i(x))$. Furthermore, we can require that in each interval (λ_i,λ_j) there are only finite number of eigenvalues and each with finite multiciplicit using the model $\mathcal{F}_*$ above and selecting the λ_i's in the open interval $(-1,1)$. Then over each $U_{ij}=U_i\cap U_j$ we have a finite rank vector bundle E_{ij} spanned by the eigenvectors of $f(\phi_j(x))$ with eigevalues in the open interval (λ_i,λ_j), with a chosen ordering $\lambda_i<\lambda_j$ for $i<j$.

Let the transition functions be denoted $g_{ji}: U_{ji}\to PU(\mathcal{H})$ and let $\phi_i(x)=\phi_j(x)g_{ji}(x)$. If $\hat{g}_{ij}$ is a lift of g_{ij} to $U(\mathcal{H})$ then

$$\hat{g}_{ij}\hat{g}_{jk}\hat{g}_{ki} = \sigma_{ijk}$$

on triple intersections with σ_{ijk} taking values in $U(1)$.

Next we define linear maps $\phi_{ijk} : E_{ij} \to E_{ik}$ for $\lambda_i < \lambda_j < \lambda_k$ as follows. First act by $\hat{g}_{jk}$ on E_{ij} and then use the inclusion to E_{ik}. The vector bundle $\phi_{ijk}(E_{ij})$ can be identified as the tensor product $L_{jk} \otimes E_{ij}$ where the complex line bundle L_{jk} over U_{jk} comes from the lifting bundle gerbe, i.e., from the pull-back of the central extension of $PU(\mathcal{H})$ with respect to the map $g_{jk} : U_{jk} \to PU(\mathcal{H})$. Thus we can identify

$$L_{jk} \otimes E_{ij} \oplus E_{jk} = E_{ik}. \tag{4.1}$$

In the untwisted case we can identify E_{ji} as the virtual bundle bundle $-E_{ij}$. In the twisted case we have to remember that the vector bundles are defined using the local sections attached to the second index. Therefore we identify E_{ji} with $-L_{ji} \otimes E_{ij}$ the twist coming again from the transition function g_{ji} relating the local sections on open sets U_i, U_j.

Since the vector bundles E_{ij} are defined via projections P_{ij} onto finite dimensional subspaces $P_{ij}H$, they come equipped with a natural connection ∇_{ij} and we can extend the above equality (4.1) to

$$(L_{kj}, A_{kj}) \otimes (E_{ij}, \nabla_{ij}) \oplus (E_{jk}, \nabla_{jk}) = (E_{ik}, \nabla_{ik}) \tag{4.2}$$

for the Deligne cocycle $\check{\sigma} = (\sigma_{ijk}, A_{ij}, B_i)$ of the gerbe. For simplicity, we normalize B_i such that the first Chern class c_{ij} of L_{ij} is represented by $B_i - B_j$.

Remark 16. *In the case of a trivial $PU(\mathcal{H})$ bundle one has $\sigma_{ijk} = 1$ and actually one has a global family of Fredholm operators parametrized by points on X. In the untwisted case, the spectral subspaces E_{ij} are directly parametrized by points in X and we have $E_{ij} \oplus E_{jk} = E_{ik}$.*

A pair E_{ij}, F_{ij} of families of local vector bundles relative to the same open cover are *equivalent* if there are vector bundles E_i over the open sets U_i such that

$$F_{ij} = \hat{g}_{ij} E_i^{-1} \oplus E_{ij} \oplus E_j$$

on U_{ij}. This happens for example when we consider a pair of spectral cuts $\lambda_i < \mu_i$ and set E_i equal to the spectral subspace for the family of operators $f(\phi_i(x))$ defined by the open interval (λ_i, μ_i) in the spectrum. In particular, F_{ij} is said to be a trivial cocycle of finite rank vector bundles if one has $E_{ij} = 0$ in the above formula.

If the families E_{ij} and F_{ij} are defined with respect to a pair of open covers U_i, V_i we say that they are equivalent if their restrictions to a common refinement of the covers are equivalent. We can now prove the homotopy invariance:

Proposition 17. *Let $f : P_\sigma \times [0,1] \to Fred_*$ be a homotopy connecting a twisted K-theory element $f_0 = f(\cdot, 0)$ to $f_1 = f(\cdot, 1)$. Then the cocycle $\{E_{ij}\}$ of local vector bundles on U_{ij} defined with the help of spectral cuts λ_i for f_0 is equivalent to the cocycle F_{ij} of local vector bundles defined by spectral cuts μ_i for f_1.*

Proof. For a given value of the homotopy parameter t we have an open cover $U_i(t)$ of X and spectral cuts $\lambda_i(t)$ defining a family of vector bundles $E_{ij}(t)$. Because of spectral flow, neither the cuts $\lambda_i(t)$ nor the vector bundles are everywhere continuous as a function of

t. However, by the continuity of the family f_t and by the discreteness of the spectrum (no accumulation points) we know that in some open neighborhood $D(t_0) \subset [0,1]$ of a point t_0 the operators $f(x,t)$ for $(x,t) \in U_i(t_0) \times D(t_0)$ have the spectral cut at $\lambda(t_0)$. Since the interval $[0,1]$ is compact, we have a finite family of open covers $\{U_i(t_a)\}$ parametrized by the points $t = t_0 < t_1 < .. < t_n = 1$ on the interval. Each open cover is finite, so we may pass to the union of these open covers. To show the local families at t_0 and t_n are equivalent it is sufficient to show that any consequtive families at t_i and t_{i+1} become equivalent on the common refinement. Taking the common refinement as the set of all intersections $V_{jk} = U_j(t_i) \cap U_k(t_{i+1})$ we have a pair of vector bundles $E^1_{jk;lm}$ and $E^2_{jk;lm}$ defined over the open sets $V_{jk;lm} = V_{jk} \cap V_{lm}$. The former is obtained from $E_{jl}(t_i)$ over $U_{jl}(t_i)$ as restriction to $V_{jk;lm}$ whereas the latter is the restriction of $E_{km}(t_{i+1})$. Define the local vector bundles F_{jk} through the spectral cut $(\lambda_j(t_i), \lambda_k(t_{i+1}))$ and the local section $\phi_k(t_{i+1})$ when $\lambda_j(t_i) < \lambda_k(t_{i+1})$. If the bounds are in the reverse order we take as F_{jk} the dual of the bundle with the same spectral cut.

We have now

$$E^2_{jk;lm} = E^1_{jk;lm} \oplus g_{km}(t_{i+1}) \cdot F^{-1}_{jk} \oplus F_{lm}$$

where $g_{km}(t_{i+1})$ is the transition function defined by $\phi_k(t) g_{km}(t) = \phi_m(t)$. This means that the local systems E^1, E^2 are equivalent on the common refinement. This is true for $i = 0, 1, \ldots n-1$ which completes the proof of homotopy invariance. □

Let ℓ_{ij} be the top exterior power of E_{ij} and n_{ij} the rank of E_{ij}. Then the collection of integers $\{n_{ij}\}$ is an integral Čech cocycle. Furthermore,

$$L^{n_{ij}}_{jk} \otimes \ell_{ij} \otimes \ell_{jk} = \ell_{ik}. \tag{4.3}$$

As noted in [30] in the case when n_{ij} is trivial, i.e., $n_{ij} = n_i - n_j$ for some locally constant integer valued functions n_i, one can define $\ell'_{ij} = \ell_{ij} \otimes L^{n_j}_{ij}$ and one has

$$\ell'_{ij} \otimes \ell'_{jk} = \ell'_{ik}.$$

This gerbe is however defined only modulo integer powers of L since one can always shift $n_i \mapsto n_i + n$ for a constant n.

In any case, we get from (4.2) the cocycle relation for the Chern character forms of the vector bundles involved,

$$e^{B_j - B_k} \omega_{ij} + \omega_{jk} = \omega_{ik}, \tag{4.4}$$

where ω_{ij} is the Chern character of E_{ij}. More generally (recalling that $B_i - B_j = c_{ij}$), we can define a twisted Čech coboundary operator δ_c acting on the sheaf of differential forms on the open sets $U_{i_1 \ldots i_p}$ by setting

$$(\delta_c \omega)_{i_1 i_2 \ldots i_{p+1}} = \sum_{a=1}^{a=p} (-1)^{a+1} \omega_{i_1 \ldots \hat{i}_a \ldots i_{p+1}} + (-1)^p e^{c_{i_p i_{p+1}}} \wedge \omega_{i_1 \ldots i_p},$$

satisfying $\delta_c^2 = 0$. Then $\{\omega_{ij}\}$ is a twisted Čech cocycle. Note that the twisted cocycles are not antisymmetric in all indices. Instead, one has the following braiding relations:

(1) The cocycles $\omega_{i_0 i_1 \ldots i_p}$ are antisymmetric in the first p indices.

(2) The following braiding relation

$$\omega_{i_1\dots i_p i_0} = (-1)^p e^{c_{i_p i_0}} \omega_{i_0 i_1 \dots i_p}.$$

Remark 18. *In the even case $K^0(X, P_\sigma)$, the local Chern characters ω_{ij} are odd differential forms. Concretely, denoting by $F_{ij}(x)$ the sign of the operators $f(\phi_i(x))$ in the spectral subspace $\lambda_i < f(\phi_i(x))^2 < \lambda_j$ the odd forms are given up to normalization as*

$$\omega_{ij}^{[2p+1]} = tr\Gamma F(dF)^{2p+1}.$$

The twisted Čech cohomology above is compatible with the Mayer-Vietoris argument in K-theory. Since $K^1(U_i) = 0$ on each open set U_i a twisted K-theory class on X comes from a cocycle of even K-theory classes on the intersections U_{ij} (concretely given in terms of the vector bundles E_{ij}) but the cocycle of even classes maps to zero in $K^1(X, P_\sigma)$ if it comes as a Čech coboundary of K-theory classes on the open sets U_i, given as vector bundles E_i, or formal differences of vector bundles, such that

$$E_j = L_{ij} \otimes E_i \oplus E_{ij}$$

which implies for the Chern characters

$$ch(E_j) = e^{c_{ij}} ch(E_i) + ch(E_{ij}),$$

i.e., the local Chern characters $ch(E_{ij})$ form a twisted coboundary of the characters $ch(E_i)$.

There is an inverse map from the twisted cocycle of local vector bundles to a global object in $K^1(X, P_\sigma)$. To make the construction simple, we use the alternative classifying space $U_1(H)$ of unitary operators which differ from the unit by trace-class operators. Any vector bundle E_{ij} can be defined using a projection valued map $P_{ij} : U_{ij} \to L(H)$. We assume that the projections $P_{ij}(x)$ commute if the second indices are equal. (This is automatically the case in when the projections are defined from a twisted K-theory element as above.) indices

On the overlap U_{ij} we require

$$g_{jk}^{-1} P_{ij} g_{jk} + P_{jk} = P_{ik}$$

be the cocycle property of vector bundles E_{ij}.

We put

$$g_i(x) = e^{2\pi i \sum_j \rho_j P_{ji}((x)}$$

where $\sum \rho_i(x) = 1$ is a partition of unity subordinate to the open cover $\{U_i\}$. The function g_i is defined on U_i.

From this we obtain

$$g_{ji}^{-1}(\log g_j) g_{ji} = 2\pi i\, g_{ji}^{-1} \sum_k \rho_k P_{kj} g_{ji} = \log(g_i) - 2\pi i P_{ji}$$

so that

$$g_{ji}^{-1} g_j g_{ji} = g_i e^{2\pi i P_{ji}} = g_i$$

on the overlap U_{ij}.

Denote by $\bar{\omega}_{ij}$ the even differential form $e^{B_j}\omega_{ij}$. We have then

$$\delta(\bar{\omega}_{ij}) = \bar{\omega}_{jk} - \bar{\omega}_{ik} + \bar{\omega}_{ij} = 0, \qquad (d-H)\bar{\omega}_{ij} = 0.$$

Applying the tic-tac-toe argument for the following twisted Čech-de Rham double complex,

(4.5)

$$\begin{array}{ccccccc}
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
\Omega^{even}(\{U_{ijk}\}) & \xrightarrow{d-H} & \Omega^{odd}(\{U_{ijk}\}) & \xrightarrow{d-H} & \Omega^{even}(\{U_{ijk}\}) & \xrightarrow{d-H} & \\
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
\Omega^{even}(\{U_{ij}\}) & \xrightarrow{d-H} & \Omega^{odd}(\{U_{ij}\}) & \xrightarrow{d-H} & \Omega^{even}(\{U_{ij}\}) & \xrightarrow{d-H} & \\
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
\Omega^{even}(\{U_{i}\}) & \xrightarrow{d-H} & \Omega^{odd}(\{U_{i}\}) & \xrightarrow{d-H} & \Omega^{even}(\{U_{i}\}) & \xrightarrow{d-H} & \\
\uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
\Omega^{even}(X) & \xrightarrow{d-H} & \Omega^{odd}(X) & \xrightarrow{d-H} & \Omega^{even}(X) & \xrightarrow{d-H} &
\end{array}$$

we obtain an odd-degree $(d-H)$-closed differential form Θ via the following diagram chase:

(4.6)

$$\begin{array}{ccccc}
0 & & & & \\
\uparrow\delta & & & & \\
\bar{\omega}_{ij} & \xrightarrow{d-H} & 0 & & \\
\uparrow\delta & & \uparrow\delta & & \\
\eta_i & \longrightarrow & (d-H)\eta_i & \longrightarrow & 0 \\
 & & \uparrow\delta & & \uparrow\delta \\
 & & \Theta & \xrightarrow{d-H} & 0
\end{array}$$

with respect to the good cover $\{U_i\}$. We can choose

$$\eta_i = \sum_k \rho_k \bar{\omega}_{ki}$$

satisfying $\delta(\eta_i) = \bar{\omega}_{ij}$ by direct calculation. Applying $d-H$ to η_i and diagram chasing, we have a locally defined odd differential form $(d-H)\eta_i$ on each U_i such that $(d-H)\eta_i = (d-H)\eta_j$ over U_{ij}, hence we obtain a globally defined odd differential form Θ such that

$$\Theta|_{U_i} = (d-H)\eta_i, \qquad (d-H)\Theta = 0.$$

It is easy to see that Θ is the twisted differential Chern character form associated to an element of $K^1(X, P_\sigma)$ we started with, i.e., a $PU(\mathcal{H})$ equivariant map $f : P_\sigma \to Fred_*$ to self-adjoint Fredholm operators with both positive and negative essential spectrum.

5. Applications to $SU(2)$ and $SU(3)$

We start with the case of $X = SU(2) = S^3$ and the $PU(\mathcal{H})$ bundle over $SU(2)$ with Dixmier-Douady form equal to $k \in \mathbb{Z} \cong H^3(SU(2), \mathbb{Z})$ where k is an integer. In this case it is sufficient to consider an open cover U_0, U_1 consisting of slightly extended hemispheres with an intersection homotopic to S^2. The twisted K-theory in this case was already computed in [Rosenberg] and found to be

$$K^1(SU(2), [k]) \cong \mathbb{Z}_k.$$

In this section we want to explain how this fits to the discussion in the previous sections.

In the twisted cocycle of vector bundles we have now only a single element

$$(E_{01}, \nabla_{01}) \longrightarrow U_{01}$$

with no condition on the vector bundle. The only topological information is the rank n_{01} of the vector bundle (which could be negative for formal differences of vector bundles) and the total degree on U_{01}, which corresponds to an integer m times the Chern class of the basic complex line bundle on S^2, represented by the first Chern form

$$\omega_{01}^{[2]} = c_1(E_{01}, \nabla_{01}).$$

The Chern character form of (E_{01}, ∇_{01}) modulo the Chern character of

$$(E_1, \nabla_1) - (E_0, \nabla_0) \otimes L_{01},$$

where L_{01} is a complex line bundle on the intersection of degree k, is then equal to $\mathbb{Z}_k$ since both E_0 and E_1 is trivial and they are characterized by the ranks n_0, n_1. This is actually just the standard argument using the Mayer-Vietoris sequence for two open sets.

Choosing a partition of unity ρ_0, ρ_1 subordinate to the open cover U_i and forms B_i with

$$dB_i = H = kH_0, \qquad B_0 - B_1 = c_{01}$$

on the overlap $U_{01} = U_0 \cap U_1$, where H_0 is the normalized volume form on $SU(2) = S^3$, we obtain the the $(d-H)$-exact form on $SU(2)$ representing the twisted Chern differential character form Θ,

$$\begin{aligned} \Theta_0 &= (d-H)[\rho_1 n_{10} + \rho_1(\omega_{10}^{[2]} + B_0 n_{10})] \\ &= n_{10} d\rho_1 + d\rho_1 \wedge \omega_{10}^{[2]} + n_{10} d\rho_1 \wedge B_0. \end{aligned}$$

on U_0, and

$$\begin{aligned} \Theta_1 &= (d-H)[\rho_0 n_{01} + \rho_0(\omega_{01}^{[2]} + B_1 n_{01})] \\ &= n_{01} d\rho_0 + d\rho_0 \wedge \omega_{01}^{[2]} + n_{01} d\rho_0 \wedge B_1. \end{aligned}$$

on U_1. Note that $n_{01} = -n_{10}$, $d\rho_0 + d\rho_1 = 0$ and $\omega_{10}^{[2]} + B_0 n_{10} = -(\omega_{01}^{[2]} + B_1 n_{01})$ imply that $\Theta_0 = \Theta_1$ on U_{01}. As $H^{odd}(X, d-H) = 0$, we know that

$$\Theta = (d-H)(\eta_{[0]} + \eta_{[2]})$$

for a globally defined even form $(\eta_{[0]}+\eta_{[2]})$, called a twisted eta potential. We will show that these twisted eta potentials are localized to certain conjugacy classes (called D-branes) such that

$$\int_{SU(2)} \eta_{[0]} H_0 = m/k$$

using the Riemann-Roch theorem in twisted K-theory (Theorem 12).

A Fredholm operator realization of the twisted K-theory classes in $K^1(SU(2),[k])$ is obtained from the family of hamiltonians in a supersymmetric Wess-Zumino-Witten model, [Mi]. Actually in this case all the classes can be realized as equivariant twisted K-theory classes, equivariant under the conjugation action of $SU(2)$ on itself. We recall some basic properties of the Fredholm family from [Mi, MiPe, FHT].

We have self-adjoint Fredholm operators Q_A in a fixed Hilbert space H parametrized by $SU(2)$ vector potentials A on the unit circle S^1. These transforms equivariantly under gauge transformations,

$$\hat{g}^{-1} Q_A \hat{g} = Q_{A^g}$$

where g is an element of the loop group LG, $\hat{g}$ is a projective representation of the loop group in H, i.e., a representation of the standard central extension of level k, and $A^g = g^{-1}Ag + g^{-1}dg$ is the gauge transformed vector potential.

Let $p:\mathcal{A}\to G$ be the canonical projection from the space of vector potentials on the circle to the group of holonomies around the circle; the fiber of this projection is the group of based loops $\Omega G\subset LG$. The spectrum of Q_A depends only on the projection $p(A)\in G$, by the equivariantness property.

The zero modes of the operators Q_A are all localized at a conjugacy class $\mathcal{C}_j$, the so called ‘ D-brane’, in G. The conjugacy class is diffeomorphic to the sphere S^2 given by

$$\mathcal{C}_j = \{ghg^{-1} | g\in G\}$$

where $h = e^{i\pi\frac{2j+1}{k}\sigma_3}$ with $\sigma_3 = diag(1,-1)$ and $2j = 0,1,2,\dots k-2$. The zero modes form a complex line bundle L_j over $\mathcal{C}_j$ with Chern class represented by $2j+1$ times the basic 2-form on $S^2\cong\mathcal{C}_j$.

For the later example it is worth noticing here that under the map $p:\mathcal{A}\to G$, the conjugacy classes correspond to coadjoint orbits in the Lie algebra of the central extension of the loop algebra; the space of vector potentials on the circle is the (non-centrally extended) loop algebra, the coadjoint action corresponds to the gauge action on $\mathcal{A}$.

If we consider the eigenvalue λ of Q_A instead of the zero eigenvalue we can still conclude from the continuity of the family Q_A as a function of A that for sufficiently small values of $|\lambda|$ the eigenvectors corresponding to this eigenvalue are localized at a 2-sphere close to the 2-sphere $\mathcal{C}_j$ defined by the zero modes and the eigenvectors corresponding to λ form a complex line bundle of winding number $2j+1$. Thus if we fix a small real number $0<\epsilon$ the spectral subspace $-\epsilon < Q_A < \epsilon$ is a complex line in a tubular neighborhood $\tilde{\mathcal{C}}_j$ of $\mathcal{C}_j$.

Setting

$$U_\pm = \{g\in G | \pm\epsilon\notin Spec(Q_A), A\in p^{-1}(g)\},$$

the intersection $U_- \cap U_+$ consists of $\tilde{\mathcal{C}}_j$ and two contractible components $D_\pm$ (upper and lower hemispheres in S^3). In this case we have only one spectral vector bundle E_{-+} which is the extension of L_j to the tubular neighborhood $\tilde{\mathcal{C}}_j$ and a trivial line bundle on the remaining components $D_\pm$ of $U_- \cap U_+$. The constraint $0 < 2j+1 < k$ is compatible with the known result $K^1(SU(2),k) = \mathbb{Z}/k\mathbb{Z}$. The construction gives a realization for all elements except the neutral element in $K^1(SU(2),k)$. The neutral element can then be obtained for example as the sum of classes corresponding to $2j+1=1$ and $2j+1=k-1$.

Let $\check{\sigma}$ be a Deligne cocycle in $H^3_{\mathcal{D}}(SU(2))$ such that $c(\check{\sigma}) = k \in H^3(SU(2),\mathbb{Z})$ and $curv(\check{\sigma}) = H = kH_0$. As the twisted cohomology $H^{odd}(SU(2),kH_0) = 0$, instead we study the differential twisted Chern character form by applying the Riemann-Roch theory (Theorem 12).

Let $\iota : \mathcal{C}_j \to SU(2)$ be the inclusion map, then $\iota^*\check{\sigma}$ can be represented by a differential 2-form $B \in \Omega^2(\mathcal{C}_j)$. Here we can set b to be zero as we have (Cf.(2.10))

$$H^3_{\mathcal{D}}(\mathcal{C}_j) \cong H^2(\mathcal{C}_j, U(1)) \cong \frac{\Omega^2(\mathcal{C}_j)}{\Omega^2_{\mathbb{Z}}(\mathcal{C}_j)} \cong U(1).$$

Let L_j be the zero modes line bundle over $\mathcal{C}_j$. By Theorem 12, the differential twisted Chern character corresponding to $[-L_j, \mathcal{C}_j]$ is given by

$$U \cdot (1 - c_1(L_j, \nabla_j))(1+B) = (d-H)(\eta_{[0]} + \eta_{[2]}),$$

where $c_1(L_j, \nabla_j)$ is the first Chern class form for L_j (equipped with a connection) and U is the Thom form on $\mathbb{R}$ with support contained in $(-\epsilon, \epsilon)$, $\int_{\mathbb{R}} U = 1$. This gives

$$d\eta_{[2]} = \eta_{[0]} H + U \cdot (-c_1(L_j, \nabla_j) + B).$$

Integrating over $SU(2)$, we have

$$\begin{aligned}
\int_{SU(2)} \eta_{[0]} H &= \int_{SU(2)} U \cdot (c_1(L_j, \nabla_j) - B) \\
&= \int_{\mathcal{C}_j} (c_1(L_j, \nabla_j) - B) \\
&= (2j+1) - \int_{\mathcal{C}_j} B.
\end{aligned}$$

By Stokes formula

$$\int_{C_j} B mod\ k = \int_D dB mod\ k = \int_D H mod\ k$$

where D is a 3-ball bounded by C_j. There are only two topologically distinct choices of such D, the results differ by $\int_{S^3} H = k$. Hence, mod k, it depends only on H and C_j. Thus up to a normalization shift by $\int_{\mathcal{C}_j} B$, we have, as $H = kH_0$,

$$\int_{SU(2)} \eta_{[0]} H_0 = \frac{2j+1}{k}.$$

Hence, the twisted eta potential distinguishes the twisted K-classes in $K^1(SU(2),k)$.

In a similar way, one has $K^1(SO(3),k) = \mathbb{Z}_k$, for $k \in H^3(SO(3),\mathbb{Z}) = \mathbb{Z}$. Here actually one could have an additional twisting by an element of $H^1(SO(3),\mathbb{Z}_2) = \mathbb{Z}_2$ which has been studied in [12].

As the final example we study the odd K-group $K^1(SU(3), kH_0)$.

Here we need a representation of the twisted affine Lie algebra $A_2^{(2)}$. Here the twist refers to an outer automorphism τ of $\mathbf{su(3)}$ with $\tau^2 = 1$. Then this algebra is defined as a central extension of the subalgebra $L_\tau G$ consisting of smooth maps $g : [0, \pi] \to \mathbf{su(3)}$ such that $g(\pi) = \tau(g(0))$. In the case of $SU(3)$ we can take τ to be given by complex conjugation of matrices. This automorphism of the Lie algebra integrates to an automorphism of the group $SU(3)$, again given by complex conjugation.

The loop algebra of $SO(3)$ is clearly a subalgebra of $L_\tau G$ and the central extension of the former on level k is obtained as a restriction of the central extension of level k of the latter.

The gauge conjugation action sending $Q_A \to \hat{g}^{-1} Q_A \hat{g} = Q_{A^g}$ is now given by the coadjoint right action $A \mapsto A^g$ where the 'vector potential' A is an element in the dual $L_\tau \mathbf{su(3)}^*$. The coadjoint orbits have been analyzed in [38] and are shown to be equal to theso called *twisted conjugacy classes* in $SU(3)$. Recall that in this terminology a twisted conjugacy class $C(h)$ corresponding to $h \in SU(3)$ is defined as

$$C(h) = \{gh\tau(g)^{-1} | g \in SU(3)\}.$$

The general formula for the (twisted) conjugacy class corresponding to the zero modes of the operators Q_A is $h = \exp\left[-2\pi(\lambda^\vee + \rho^\vee)/k\right]$ where ρ is half the sum of positive roots of the Lie algebra of constant gauge transformations (which in this case is $\mathbf{so(3)}$) and λ is the highest weight of an irreducible $\mathbf{so(3)}$ representation, [21]. Here $\lambda^\vee \in \mathbf{h}$ is the dual of a weight $\lambda \in \mathbf{h}^*$, the duality is determined by the Killing form. Note also that the level $k \geq \kappa$, where κ is the dual Coxeter number (in the case of $SU(n)$ this number is equal to n). The reason for this is that in the Wess-Zumino-Witten model construction $k = k' + \kappa$ where k' is the level of an arbitrary irreducible loop group representation and the shift κ comes from a loop group representation constructed from the Clifford algebra of the loop group.

As shown in [37] the twisted conjugacy classes C_x defined by $h(x) = \exp 2x\rho^\vee$ define a foliation of $SU(3)$ when $0 \leq x \leq \pi/4$. Concretely, as a 3×3 matrix,

$$h(x) = \begin{pmatrix} \cos(2x) & \sin(2x) & 0 \\ -\sin(2x) & \cos(2x) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

When $x = 0$ the twisted conjugacy class can be identified as the 5-dimensional space $M_5 = SU(3)/SO(3)$, when $x = \pi/4$ it is $S^5 = SU(3)/SU(2)$ whereas for $0 < x < \pi/4$ we obtain the 'seven brane' $M_7 = SU(3)/SO(2)$.

The class defined by the zero modes of Q_A is the generic orbit M_7 corresponding to the parameter value $x = (2j+1)/k$, where $2j = 0, 1, 2 \ldots k-3$, the shift by 3 coming from the dual Coxeter number of $SU(3)$. In the case of the twisted loop algebra $A_2^{(2)}$ we have an additional constraint: The spin j is integer when $k-3$ is even and j is a half-integer when $k-3$ is odd.

The degree of the zero mode line bundle is computed as in the $SU(2)$ case. The zero modes are localized in the finite-dimensional vacuum subspace of the Hilbert space $H_{j,k}$ which carries a representation $j \otimes 1/2$ of $\mathbf{so(3)}$. (In general, the zero mode bundle has rank equal to the multiplicity of the representation ρ inside of the spin representation of the Clifford algebra of the group of constant loops; here however the Clifford algebra representation is two dimensional and at the same time the irreducible fundamental representation of the Lie algebra $\mathbf{so(3)}$).) When $x = (2j+1)/k$ the complex line is spanned by the highest weight vector $v_{j+1/2}$ of $\mathbf{so(3)}$ weight $j+1/2$, [21]. The zero mode bundle is then the associated line bundle $L = SU(3) \times_{2j+1} \mathbb{C}$, defined by the principal bundle $SO(2) \to SU(3) \to M_7$ through the one dimensional representation of $SO(2)$ with character $2j+1$. In particular, for even k the spin j is a half-integer and thus the degree of the zero mode bundle is even. This is in accordance with the known result $K^1(SU(3), k) = \mathbb{Z}/(k/2)\mathbb{Z}$ for even k. Twisted K^1 is a rank one module over the untwisted $K^0(SU(3))$, tensoring with any of the elements in $K^1(SU(3), k)$ gives only even elements in $\mathbb{Z}_k$.

In the case when k is odd, the degree of the zero mode bundle is also odd and the tensor product operation $K^0(SU(3)) \times K^1(SU(3), k) \to K^1(SU(3), k)$, gives both the even and odd elements in $\mathbb{Z}_k$.

The eta forms detect the twisted K-theory classes in a similar way as in the case of $SU(2)$. In the present setting we have nonzero eta forms in even degrees up to form degree six. However, they do not contain independent information since for any fixed level k the only parameter is the twisting of the line bundle L which is given by the integer $2j+1$. To determine this integer from the differential data one proceeds as in the case of $SU(2)$ above. Consider the subgroup $SU(2) \subset SU(3)$ given by the the matrices with $+1$ on the diagonal in the lower right corner. The intersection of $SU(2) = S^3$ with the orbit M_7 is a union of two spheres S^2_a and S^2_b. The reason for this is that the points $h(x)$ and $h(-x)$ are conjugate to each others by the twisted action of $g = diag(1, -1, -1)$ which lies outside of $SU(2)$; one can see easily that $h(\pm x)$ are not conjugate by elements of $SU(2)$ and their union is $M_7 \cap SU(2)$. Taking into account the higher eta forms we have from Theorem 12,

$$U \cdot \exp\left(-c_1(L_j, \nabla_j)\right) \exp B = (d - H)\eta,$$

where $\eta = \eta_{[0]} + \ldots \eta_{[6]}$. Picking up the component of form degree 3 we get

$$\begin{aligned} \int_{SU(2)} \eta_{[0]} H &= \int_{SU(2)} U \cdot (c_1(L_j, \nabla_j) - B) \\ &= 2(2j+1) - \int_{SU(2)\cap M_7} B \end{aligned}$$

and thus again, up to a shift with the j independent constant $\int_{S^2} B$ we get

$$\int_{SU(2)} \eta_{[0]} H_0 = 2\frac{2j+1}{k},$$

the factor 2 coming from the integration of c_1 over the two distinct 2-spheres.

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