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DISCRETE SPECTRUM OF A THREE PARTICLE SCHRÖDINGER OPERATOR WITH A HOMOGENEOUS MAGNETIC FIELD.

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Abstract. We study the discrete spectrum of the Schrödinger operator for a system of three identical particles with short-range interactions in a homogeneous magnetic field.All the two-particle subsystems are supposed to be unstable. Finiteness of the discrete spectrum is established under some assumptions on the solutions of the corresponding two-particle Schrödinger equation .

1 Introduction

In this paper we continue studying the discrete spectrum of the Schrödinger operator H_0 for a system of three identical particles in a homogeneous magnetic field (see[1,2]).We assume that the center of mass motion has been separated and the potentials of interaction are short-range and spherically symmetric.

It is known that without any magnetic field a system of three identical particles with short-range interactions in the case when two-particle hamiltonians have discrete spectrum, may have only finitely many eigenvalues. On the contrary, in a homogeneous magnetic field such a system may have either finite or infinite discrete spectrum [1]. For the operator H_0 there are no results on finiteness or infinitude of the discrete spectrum in the case when two-particle hamiltonians have no discrete eigenvalues. In the present paper we establish that in the absence of resonanses at the bottom of the essential spectrum of two-particle hamiltonians(virtual levels), the discrete spectrum of the operator H_0 is finite. This assumption means that the two-particle Schrödinger equation for the value of the spectral parameter corresponding to the bottom of the essential spectrum has no solutions in some Hilbert space (for details see [3]). The question of the finiteness or infinitude for a system with two-particle virtual levels is open.

2 Main definitions and results

Let $Z_0 = \{1, 2, 3\}$ be a system of three identical particles with masses $M = 1$ and charges $e = -1$ in a homogeneous magnetic field B, and let $r_j = (r_j^1, r_j^2, r_j^3)$ be the position vector

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of the j-th particle. For an arbitrary pair of particles $\alpha = i, j$ we introduce the Jacobi coordinates

$$q^{\alpha} = (q_1^{\alpha}, q_2^{\alpha}, q_3^{\alpha}) = r_j - r_i,$$

and

$$\xi^{\alpha} = (\xi_1^{\alpha}, \xi_2^{\alpha}, \xi_3^{\alpha}) = \frac{1}{2}(r_j + r_i) - r_p, \qquad p \neq i, j.$$

After separation of the center of mass motion, the Schrödinger operator H_0 takes the following form :

$$H_0 = (i\nabla_{q^{\alpha}} + A_{q^{\alpha}})^2 + \frac{3}{4}(i\nabla_{\xi^{\alpha}} + A_{\xi^{\alpha}})^2 + \sum_{i<j} V(|r_{ij}|), \tag{2.1}$$

where

$$A_{\xi^{\alpha}} = \frac{B}{3}\{-\xi_2^{\alpha}, \xi_1^{\alpha}, 0\}; \qquad A_{q^{\alpha}} = \frac{B}{4}\{-q_2^{\alpha}, q_1^{\alpha}, 0\};$$

$B > 0$ being constant.

We assume that the potential $V(|r|)$ has the following properties :

i) $|V(|r|)| \leq c_0(1 + |r|)^{-2-\delta_0}, r \in R^3$ *for some* $\delta_0 > 0$ *and* $c_0 > 0$*;*

ii) there exists a constant $a > 0$ *such that* $V(|r|) \in C(\{|r| > a\})$ *and* $V(|r|) > 0$ *for* $|r| \geq a$*.*

Note that the assumption of positivity of $V(|r|)$ for large $|r|$ is not very restrictive, because if $V(|r|)$ is negative for large $|r|$, the two-particle hamiltonias have discrete spectrum. It is just the situation considered in [1].

For an arbitrary pair $\alpha = (i, j)$ denote

$$h_{\alpha} = (i\nabla_{q^{\alpha}} + A_{q^{\alpha}})^2 + V(|r_{ij}|). \tag{2.2}$$

It is known [1] that for the operator H_0 the theorem of the HVZ type holds. This means that

$$\sigma_{ess}(H_0) = [\mu, \infty),$$

where $\mu = inf\sigma(h_{\alpha}) + B$. Here σ and σ_{ess} stand for spectrum and essential spectrum respectively.

In the present paper we study the operator H_0 in the case when h_{α} has no discrete spectrum. This implies that

$$inf\sigma_{ess}(H_0) = inf\sigma(h_{\alpha}) + B = inf\sigma_{ess}(h_{\alpha}) + B = \tag{2.3}$$

$$inf\sigma\{(i\nabla_{q^{\alpha}} + A_{q^{\alpha}})^2\} + B = 2B.$$

Definition . *We shall say that the operator* h_{α} *such that* $\sigma_{disc}(h_{\alpha}) = \emptyset$ *has no resonanses at the bottom of its essential spectrum (virtual levels) if one can find a number* $\varepsilon_0 > 0$ *such that*

$$inf\sigma\{(i\nabla_{q^{\alpha}} + A_{q^{\alpha}})^2 + (1 + \varepsilon)V(|q_{\alpha}|)\} = inf\sigma(h_{\alpha}) = B \tag{2.4}$$

for all $0 \leq \varepsilon < \varepsilon_0$.

Properties of resonances have been studied in [3]. Let us mention now only that if the operator h_α has no virtual levels, the equation

$$h_\alpha \psi = B\psi$$

has no solutions in the Hilbert space with the inner product

$$[\psi, \varphi] = 2([(i\nabla_{q^\alpha} + A_{q^\alpha})^2 - B]^{\frac{1}{2}}\psi, [(i\nabla_{q^\alpha} + A_{q^\alpha})^2 - B]^{\frac{1}{2}}\varphi) + (V\psi, \varphi).$$

The main result of this paper is the following

Theorem . *Let the potential V satisfy the conditions i), ii) and the hamiltonians of all two-particle subsystems h_α have neither discrete spectrum nor virtual levels. Then the discrete spectrum of the operator H_0 is finite .*

3 Proof of theorem

In this section we prove the main theorem. Some properties of two-particle hamiltonians used in the proof will be obtained in section 4.

To prove theorem it is sufficient to find a finite dimentional subspace $\mathcal{M}$ such that

$$(H_0\psi, \psi) \geq 2B\|\psi\|^2 \tag{3.1}$$

for all $\psi \in C_0^2, \psi - \mathcal{M}$.

To this end we shall first construct a suitable partition of unity. As the first step let us separate the region in which all the interparticle distances are small.

Precisely, let α be a fixed pair of particles and a_1 be a positive number. Define

$$S(a_1) = \{(q^\alpha, \xi^\alpha) : |q^\alpha|^2 + |\xi^\alpha|^2 \leq a_1\} \tag{3.2}$$

$u(t), v(t) \in C_0^2(R_+^1), u^2 + v^2 = 1, u = 1$ for $t \leq a_1, u = 0$ for $t \geq 2a_1$ and $\lim v'^2(t)(1 - v^2)^{-1} = 0$, as $t \to 2a_1 - 0$.

Following [4], it is easy to show that for any $\psi \in C_0^2(R^6)$

$$(H_0\psi, \psi) \geq L_1[\psi u(|q^\alpha|^2 + |\xi^\alpha|^2)] + L_2[\psi v(|q^\alpha|^2 + |\xi^\alpha|^2)], \tag{3.3}$$

where

$$L_1[\varphi] = (H_0\varphi, \varphi) - C\|\varphi\|^2, \tag{3.4}$$

$$L_2[\varphi] = (H_0\varphi, \varphi) - \varepsilon \int_{supp\{\nabla v\}} |\varphi|^2 dq^\alpha d\xi^\alpha, \tag{3.5}$$

and the constant $\varepsilon > 0$ may be chosen small for large $C > 0$.

The function ψu is compactly supported. Using the discreteness of the spectrum of the Laplacian with the Dirichlet boundary conditions one can easily construct a finite - dimentional subspace $\mathcal{M}$ such that for all $\psi - \mathcal{M}$

$$(H_0\psi u, \psi u) \geq (c + 2B)\|\psi u\|^2 \tag{3.6}$$

Let us prove that for a_1 large enough and small $\varepsilon > 0$, the lower bound

$$L_2[\psi v] \geq 2B\|\psi v\|^2 \tag{3.7}$$

also holds.

Let $a_0 > a$ (see condition ii) for definition of a) be a fixed number and let

$$\mathcal{V}(|r|) = V(|r|)\chi(|r| \leq a_0), \quad r \in R^3,$$

where χ stands for the characteristic function of the interval $[0, a_0]$. Let us define

$$\mathcal{H}_0 = (i\nabla_{q^\alpha} + A_{q^\alpha})^2 + \frac{3}{4}(i\nabla_{\xi^\alpha} + A_{\xi^\alpha})^2 + \sum_{i<j}\mathcal{V}(|r_{ij}|). \tag{3.8}$$

It is clear from ii) that

$$V(|r_{ij}|)(1 - \chi(|r_{ij}| \leq a_0)) \geq 0.$$

Moreover, if a_1 is large enough, at each point of $supp\{\psi v\}$ the inequality $|r_{ij}| \geq a$ holds for at least two pairs of particles. Therefore for large a_1 on the support of ψv we have

$$\sum_{i<j} V(|r_{ij}|)(1 - \chi(|r_{ij}| \leq a_0)) > 0.$$

For the region $a_1 \leq |q^\alpha|^2 + |\xi^\alpha|^2 \leq 2a_1$ is compact, one can find such a number $\kappa_0 > 0$ that

$$\sum_{i<j} V(|r_{ij}|)(1 - \chi(|r_{ij}| \leq a_0)) > \kappa_0,$$

for $(q^\alpha, \xi^\alpha) \in supp\{|\nabla v|\}$. Let us pick the constant ε in (3.5) less then κ_0. Then

$$L_2[\psi v] \geq (\mathcal{H}_0\psi v, \psi v). \tag{3.9}$$

The next step of the proof is a new partition of the configuration space. Its aim is to separate regions in which the distance from one particle to the others is large along the direction of the magnetic field .

Let

$$\mathcal{K}_2^\alpha = \{(q^\alpha, \xi^\alpha) : |q_3^\alpha| \leq b_2|\xi_3^\alpha|, \quad |\xi_3^\alpha| \geq a_2\}. \tag{3.10}$$

It is easy to see that if b_2 is small and $a_2 \neq 0$, regions $\mathcal{K}_2^\alpha$ do not overlap for different α. We denote by $\mathcal{K}_3$ and $\mathcal{K}_1$ the following regions :

$$\mathcal{K}_3(a_3, b_2, a_2) = \{(q^\alpha, \xi^\alpha) : |q_3^\alpha|^2 + |\xi_3^\alpha|^2 \geq a_3, \quad (q^\alpha, \xi^\alpha) \notin \bigcup_\beta \mathcal{K}_2^\beta(b_2, a_2)\}, \tag{3.11}$$

$$\mathcal{K}_1 = \{(q^\alpha, \xi^\alpha) : (q^\alpha, \xi^\alpha) \notin \bigcup_\beta \mathcal{K}_2^\beta(b_2, a_2) \bigcup \mathcal{K}_3\}. \tag{3.12}$$

Let $\chi_1, \chi_{2\alpha}, \chi_3$ be characteristic functions of $\mathcal{K}_1, \mathcal{K}_2^\alpha, \mathcal{K}_3$ respectively. It is clear that

$$\chi_1 + \sum_\alpha \chi_{2\alpha} + \chi_3 = 1$$

and

$$(\mathcal{H}_0\psi v, \psi v) = L_{1\chi_1}[\psi v] + \sum_\alpha L_{1\chi_{2,\alpha}}[\psi v] + L_{1\chi_3}[\psi v], \tag{3.13}$$

where

$$L_{1\chi_j}[\varphi] = \int_{supp\{\chi_j\}} |(i\nabla_{q^\alpha} + A_{q^\alpha})\varphi|^2 dq^\alpha d\xi^\alpha + \frac{3}{4}\int_{supp\{\chi_j\}} |(i\nabla_{\xi^\alpha} + A_{\xi^\alpha})\varphi|^2 dq^\alpha d\xi^\alpha + \tag{3.14}$$

$$\int \sum_{s<t} \mathcal{V}(|r_{st}|)|\varphi|^2 \chi_j dq^\alpha d\xi^\alpha.$$

Notice that on the support of χ_3 for large a_3 all the potentials $\mathcal{V}(|r_{ij}|)$ vanish and

$$L_{1\chi_3}[\psi v] \geq 2B\|\psi v \chi_3\|^2. \tag{3.15}$$

On the support of $\chi_{2\alpha}$ for large a_2 only the potential $\mathcal{V}(|q^\alpha|)$ does not vanish. Thus we have

$$L_{1\chi_{2\alpha}} = \int_{supp\{\chi_{2\alpha}\}} |(i\nabla_{q^\alpha} + A_{q^\alpha})\psi v|^2 dq^\alpha d\xi^\alpha + \int_{supp\{\chi_{2\alpha}\}} \mathcal{V}(|q^\alpha|)|\psi v|^2 dq^\alpha d\xi^\alpha + \tag{3.16}$$

$$+\frac{3}{4}\int_{supp\{\chi_{2\alpha}\}} |(i\nabla_{\xi^\alpha} + A_{\xi^\alpha})\psi v|^2 dq^\alpha d\xi^\alpha \geq B\|\psi v \chi_{2\alpha}\|^2 +$$

$$+\int_{supp\{\chi_{2\alpha}\}} |(i\nabla_{q^\alpha} + A_{q^\alpha})\psi v|^2 dq^\alpha d\xi^\alpha + \int_{supp\{\chi_{2\alpha}\}} \mathcal{V}(|q^\alpha|)|\psi v|^2 dq^\alpha d\xi^\alpha.$$

If we choose the number a_0 large ,then in vew of theorem 4.3 the operator

$$h = (i\nabla_{q^\alpha} + A_{q^\alpha})^2 + \mathcal{V}(|q^\alpha|)$$

has no virtual levels and therefore for large a_2 the last two terms in (3.16) are greater then $B\|\psi v \chi_{2\alpha}\|^2$.

It remains to show that

$$L_{1\chi_1}[\psi v] \geq 2B\|\psi v\|^2. \tag{3.17}$$

Recall that the function $\psi v \chi_1$ is supported in the region $|q_3^\alpha|^2 + |\xi_3^\alpha|^2 \leq a_3$. In this region all the distances along the direction of the magnetic field are bounded. We shall divide this region into subregions such that in each subregion there is at least one particle for which the distance to other particles in the plane orthogonal to the direction of the magnetic field is large. Let $u_1(t) \in C_0^2(R_+^1), u_1(t) = 1$ for $t \leq 1, u_1(t) = 0$ for $t \geq 2$. For fixed numbers $\gamma > 0, \gamma_0 > 0$ let us define the functions u_α and u_0 as

$$u_0(q^\alpha, \xi^\alpha) = u_1(\gamma_0 \sum_{i=1,2;\beta} \{|q_i^\beta|^2 + |\xi_i^\beta|^2\}), \tag{3.18}$$

$$u_\alpha(q^\alpha, \xi^\alpha) = u_1(\gamma^{-1}\{|q_1^\alpha|^2 + |q_2^\alpha|^2\}\{|\xi_1^\alpha|^2 + |\xi_2^\alpha|^2\}^{-1}). \tag{3.19}$$

It is easy to see that for small γ and γ_0 the supports of the functions $\psi v \chi_1 u_\alpha$ do not overlap for different α . Moreover, for fixed γ , γ_0 and fixed numbers a_2, a_3 (see definition of the function χ_1) one can choose a large number a_1 such that :

1. Only one potential ($\mathcal{V}(|q^\alpha|)$) does not vanish on the support of $\psi v \chi_1 u_\alpha$;

2. All the potentials vanish on the supports of the functions $\psi v \chi_1 u_0$ and $\psi v \chi_1 (1 - \sum_\alpha u_\alpha^2)^{\frac{1}{2}} (1 - u_0^2)^{\frac{1}{2}}$.

Let us denote by $\mathcal{P}_{0,0}$ the orthogonal projector onto the eigensubspace of the operator

$$T_{-2} = (i\frac{\partial}{\partial q_1^\alpha} - \frac{B}{4} q_2^\alpha)^2 + (i\frac{\partial}{\partial q_2^\alpha} + \frac{B}{4} q_1^\alpha)^2 +$$

$$\frac{3}{4}(i\frac{\partial}{\partial \xi_1^\alpha} - \frac{B}{3}\xi_2^\alpha)^2 + \frac{3}{4}(i\frac{\partial}{\partial \xi_2^\alpha} + \frac{B}{3}\xi_1^\alpha)^2,$$

corresponding to its lowest eigenvalue 2B. Let $g_2 = (I - \mathcal{P}_{0,0})\psi v \chi_1$, $g_{2\alpha} = g_2 u_\alpha, g_{20} = g_2 u_0, g_{21} = g_2(1 - \sum_\alpha u_\alpha^2 - u_0^2)^{\frac{1}{2}}$. We denote by ε_0 such a number that

$$(1 - \varepsilon_0)[(i\nabla_{q^\alpha} + A_{q^\alpha})^2 - B] + \mathcal{V}(|q^\alpha|) \geq 0.$$

Taking into account that for any $\psi \in C_0^2(R^6)$

$$\int_{supp\{\chi_1\}} [|(i\nabla_{q^\alpha} + A_{q^\alpha})\psi|^2 - B|\psi|^2] dq^\alpha d\xi^\alpha \geq 0$$

and

$$\int_{supp\{\chi_1\}} [\frac{3}{4}|(i\nabla_{\xi^\alpha} + A_{\xi^\alpha})\psi|^2 - B|\psi|^2] dq^\alpha d\xi^\alpha \geq 0,$$

one can verify the following lower bound for the the kinetic part of the functional $L_{1\chi_1}$

$$\int_{supp\{\chi_1\}} |(i\nabla_{q^\alpha} + A_{q^\alpha})\psi v|^2 dq^\alpha d\xi^\alpha + \frac{3}{4}\int_{supp\{\chi_1\}} |(i\nabla_{\xi^\alpha} + A_{\xi^\alpha})\psi v|^2 dq^\alpha d\xi^\alpha \tag{3.20}$$

$$-2B\int_{supp\{\chi_1\}} |\psi v|^2 dq^\alpha d\xi^\alpha \geq \frac{\varepsilon_0}{2}\{\int_{supp\{\chi_1\}} |\frac{\partial \psi v}{\partial q_3^\alpha}|^2 dq^\alpha d\xi^\alpha + \frac{3}{4}\int_{supp\{\chi_1\}} |\frac{\partial \psi v}{\partial \xi_3^\alpha}|^2 dq^\alpha d\xi^\alpha +$$

$$B\int_{supp\{\chi_1\}} |g_2|^2 dq^\alpha d\xi^\alpha\} - \int_{supp\{\chi_1\}} \{\sum_\alpha \{|\nabla u_\alpha|^2 + |\nabla(1 - u_0^2 + \sum_\alpha u_\alpha^2)^{\frac{1}{2}}|^2\}|g_2|^2 dq^\alpha d\xi^\alpha +$$

$$(1-\frac{\varepsilon_0}{2})\{\sum_\alpha\int_{supp\{\chi_1\}}|\frac{\partial\psi v}{\partial q_3^\alpha}|^2u_\alpha^2dq^\alpha d\xi^\alpha+\int_{supp\{\chi_1\}}\{|(i\nabla_{q^\alpha}+A_{q^\alpha})g_{2\alpha}|^2-B|g_{2\alpha}|^2\}dq^\alpha d\xi^\alpha\}.$$

Moreover, if γ,γ_0 are small, then

$$\sum_\alpha\{|\nabla u_\alpha|^2+|\nabla(1-u_0^2+\sum_\alpha u_\alpha^2)^{\frac{1}{2}}|^2\leq\frac{\varepsilon_0B}{4}$$

and

$$\int_{supp\{\chi_1\}}\{\sum_\alpha\{|\nabla u_\alpha|^2+|\nabla(1-u_0^2+\sum_\alpha u_\alpha^2)^{\frac{1}{2}}|^2\}|g_2|^2dq^\alpha d\xi^\alpha\leq\frac{\varepsilon_0B}{4}\|g\chi_1\|^2. \qquad (3.21)$$

Denote

$$L_\alpha[\psi v]=\int_{supp\{\chi_1\}}[(1-\frac{\varepsilon_0}{2})\{|\frac{\partial\psi v}{\partial q_3^\alpha}|^2u_\alpha^2+|(i\nabla_{q^\alpha}+A_{q^\alpha})g_{2\alpha}|^2- \qquad (3.22)$$
$$B|g_{2\alpha}|^2\}+\mathcal{V}(|q^\alpha|)|\psi v|^2]dq^\alpha d\xi^\alpha.$$

It follows from (3.20),(3.21) that the inequality

$$L_\alpha[\psi v]\geq-\frac{\varepsilon_0B}{4}\|g_2\|^2,\quad\forall\alpha,$$

entails (3.17). To establish this lower bound we consider the integral over q^α in (3.22) for fixed ξ^α. If $|\xi_1^\alpha|^2+|\xi_2^\alpha|^2$ is not large $\psi v\chi_1=0$ and this integral is nonnegative. So we must estimate it only for large values of $|\xi_1^\alpha|^2+|\xi_2^\alpha|^2$. If ξ_3^α is fixed, q_α^3 takes values from $-c$ to c ,where c is some positive number that may be chosen large for suitable chosen parameters of the region $\mathcal{K}_1$. By theorem 4.4 for fixed ξ_α

$$J(\xi^\alpha)=\int_{|q_\alpha^3|\leq c}[(1-\frac{\varepsilon_0}{2})\{|\frac{\partial\psi v}{\partial q_3^\alpha}|^2u_\alpha^2+|(i\nabla_{q^\alpha}+A_{q^\alpha})g_{2\alpha}|^2-B|g_{2\alpha}|^2\}+\mathcal{V}(|q^\alpha|)|\psi v|^2]dq^\alpha$$
$$\geq-\delta\int|g_{2\alpha}(q^\alpha,\xi^\alpha)|^2dq^\alpha,$$

where the number δ may be chosen small for large c and $|\xi_1^\alpha|^2+|\xi_2^\alpha|^2$. Let us take $\delta\leq\frac{\varepsilon_0B}{4}$. Then the inequality (3.17) follows, which completes the proof of Theorem.

4 Spectral properties of the two-particle hamiltonians h_α

In this section we prove some elementary facts on two-particle Hamiltonians with a magnetic field. Let

$$h=(i\nabla_r+A_r)^2+V(|r|), \qquad (4.1)$$

$$r = (r_1, r_2, r_3), \quad A_r = \frac{B}{4}\{-r_2, r_1, 0\},$$

and the potential V satisfy the conditions $i)$ and $ii)$ (see section 2). Let us consider the $SO(2)$ group of rotations in R^3 around the r_3 axis (the direction of the magnetic field). By P^m we denote the projector in $L_2(R^3)$ onto the subspace of functions possesing the symmetry of the weight m with respect to this group; $h^m = P^m h$.

Theorem 4.1 . *For any $\gamma > 0$ one can find a number $m_0 > 0$ such that for all $m, |m| > m_0$, the operator*

$$h^m(\gamma) = P^m[\gamma(i\nabla_r + A_r)^2 + V(|r|)] \tag{4.2}$$

with the potential $V(r)$ satisfying conditions $i)$ and $ii)$, has no discrete spectrum .

Proof . The bottom of the essential spectrum for the operator $h^m(\gamma)$ is $\gamma B(|m| - m + 1)$. Consequently to prove the theorem it suffices to show that for $|m| > m_0$

$$(h^m(\gamma)\psi, \psi) \geq \gamma B(|m| - m + 1)\|\psi\|^2 \tag{4.3}$$

for all $\psi \in C_0^2(R^3), P^m\psi = \psi$.

Due to the condition $ii)$ one can find such positive constants $b, \varepsilon_1, \varepsilon_2$ that

1. $V(|r|) \geq 0$ for all $|r| > b_0$,
2. $V(|r|) \geq \varepsilon_1$ for $b_0 - \varepsilon_2 \leq |r| \leq b_0 + \varepsilon_2$.

Let us take such real functions $u(t), v(t) \in C_2(R^1_+)$ $u^2 + v^2 = 1, u(t) = 1$ for $t \leq b_0 - \varepsilon_2, u(t) = 1$ for $t \geq b_0 + \varepsilon_2, (v')^2(1 - v^2)^{-1} \to 0$ *as* $t \to b_0 + \varepsilon_2 - 0$.

Following [5] for any fixed $\varepsilon \in (0, \varepsilon_1)$ one can find such a constant $C(\varepsilon) > 0$,that

$$(h^m(\gamma)\psi, \psi) \geq (h^m(\gamma)\psi u, \psi u) - C(\varepsilon)\|\psi u\|^2 + \tag{4.4}$$

$$(h^m(\gamma)\psi v, \psi v) - \varepsilon \int_{supp|\nabla v|} |\psi v|^2 dr$$

The function ψu is supported in the region $|r| \leq b_0 + \varepsilon_2$. In this region the operator $h^m(\gamma)$ with the Dirichlet boundary conditions has poor discrete spectrum and for $|m| \to \infty$ its lowest eigenvalue tends to ∞ as $|m|^2$. Hence for large $|m|$ the inequality

$$(h^m(\gamma)\psi u, \psi u) - C(\varepsilon)\|\psi u\|^2 \geq \gamma B(|m| - m + 1)\|\psi u\|^2 \tag{4.5}$$

holds. Further,

$$(h^m(\gamma)\psi v, \psi v) - \varepsilon \int_{supp|\nabla v|} |\psi v|^2 dr \geq \gamma B(|m| - m + 1)\|\psi v\|^2 \tag{4.6}$$

because $V(|r|) - \varepsilon > 0$ on the support of $|\nabla v|$. The inequality (3.3) follows from the inequalities (3.4)-(3.6). The theorem is proved.

Theorem 4.2 . *Let V satisfy the conditions i)and ii). Suppose that the operator h has no discrete spectrum and virtual levels. Then one can find a large number a_0 such that the operator h with the potential*

$$\mathcal{V} = V(|r|)\chi(|r| \leq a_0)$$

also has no discrete spectrum and virtual levels .

Remark. This Theorem is not trivial though it may seem so because of the condition $V(r) \to 0, |r| \to \infty$. The point is that any small negative perturbation of the magnetic operator

$$h = (i\nabla_r + A_r)^2 + V_1(|r|) \tag{4.7}$$

may produce an infinite number of discrete eigenvalues.

Proof of the theorem 4.2 . Establish the folowing simple test of the absence of discrete spectrum for the Schrödinger operator in one-dimensional case.

Lemma 4.1 . *Let $V(x)$, $x \in R^1$ be a real-valued function such that $V(x) = V(-x)$, $\int_{-\infty}^{\infty} V(x)dx > 0$ and for some positive constants $C_0, \delta_0, |V(x)| \leq C_0(1+|x|)^{-2-\delta_0}$. Then if the inequality*

$$V(x) - \kappa|V(x)| \geq \frac{1}{4}(1+|x|)^{-2}$$

holds for

$$\kappa = (\int_{-\infty}^{\infty} |V(x)|dx)(\int_{-\infty}^{\infty} V(x)dx)^{-1}, \tag{4.8}$$

then the discrete spectrum of the operator

$$h_0 = -\frac{\partial^2}{\partial x^2} + V(x) \tag{4.9}$$

is empty.

Proof. Let us check that $(h_0\psi, \psi) \geq 0$ for any $\psi(x) \in C_0^2(R^1)$. Let $\psi_0 = \psi(x) - \psi(0)$, then

$$(h_0\psi, \psi) = \int_{-\infty}^{\infty} |\psi_0'|^2 dx + \int_{-\infty}^{\infty} |\psi_0'|^2 V(x)dx \tag{4.10}$$

$$+2\psi(0)\int_{-\infty}^{\infty} V(x)\psi_0(x)dx + |\psi(0)|^2 \int_{-\infty}^{\infty} V(x)dx$$

For a fixed ψ_0 the right hand side of (4.10) may be considered as a function of the parameter $\psi(0)$. It takes its minimal value for

$$\psi(0) = -(\int_{-\infty}^{\infty} V(x)\psi_0(x)dx)(\int_{-\infty}^{\infty} V(x)dx)^{-1}, \tag{4.11}$$

and this value is

$$\int_{-\infty}^{\infty} |\psi_0'|^2 dx + \int_{-\infty}^{\infty} |\psi_0'|^2 V(x)dx - (\int_{-\infty}^{\infty} V(x)\psi_0(x)dx)^2(\int_{-\infty}^{\infty} V(x)dx)^{-1} \geq \tag{4.12}$$

$$\int_{-\infty}^{\infty} |\psi_0'|^2 dx + \int_{-\infty}^{\infty} (V(x) - \kappa |V(x)|) |\psi_0|^2 dx$$

Therefore

$$(h_0\psi, \psi) \geq \int_{-\infty}^{\infty} |\psi_0'|^2 dx + \int_{-\infty}^{\infty} (V(x) - \kappa |V(x)|) |\psi_0|^2 dx \geq \tag{4.13}$$

$$\int_{-\infty}^{\infty} |\psi_0'|^2 dx - \frac{1}{4} \int_{-\infty}^{\infty} (1 + x^2)^{-1} |\psi_0|^2 dx.$$

Recall that function $\psi_0(0) = 0$,so by the Hardy enequality that the right hand side of (4.13) is nonnegative. Lemma is proved.

Let us proceed to the proof of theorem 4.2 .

Let $h(U)$ be the operator (4.7) with the potential $U, h^m(U) = P^m h(U)$. It follows from theorem 4.1 that there are such numbers $\varepsilon_0 > 0$ and $m_0 > 0$ that

$$inf h^m((1 + \varepsilon_0)\mathcal{V}) \geq B(|m| - m + 1) \geq B, |m| > m_0. \tag{4.14}$$

Futhermore, if $m < 0$ for small $\varepsilon_0 > 0$, then

$$inf(h^m((1 + \varepsilon_0)V)) > B,$$

and consequently for large a_0

$$inf(h^m((1 + \varepsilon_0)\mathcal{V})) > B. \tag{4.15}$$

Hence to prove the theorem it suffices to show that (4.15) also holds for all $0 \leq m \leq m_0$ and some small $\varepsilon_0 > 0$. Let m_1 be fixed, $|m_1| \leq m_0$, ψ be an arbitrary function from $C_0^2(R^3)$ such that $P^{m_1}\psi = \psi$ and let

$$T = (i\nabla_r + A_r)^2 - B.$$

It follows from the absence of virtual levels that one can find such a positive constant ε that

$$(1 - \varepsilon)(T\psi, \psi) + (V\psi, \psi) \geq 0. \tag{4.16}$$

Let us take $\varepsilon_0 < (1 - \frac{\varepsilon}{2})^{-1} - 1$.Then (4.15) holds if

$$L[\psi] = (1 - \frac{\varepsilon}{2})(T\psi, \psi) + (\mathcal{V}\psi, \psi) \geq 0.$$

Thus we have

$$L[\psi] = (1 - \varepsilon)(T\psi, \psi) + (V\psi, \psi) + \frac{\varepsilon}{2}(T\psi, \psi) + \tag{4.17}$$

$$((\mathcal{V} - V)\psi, \psi) = (1 - \varepsilon)(T\psi, \psi) + (1 - \gamma)(V\psi, \psi) +$$

$$\frac{\varepsilon}{2}(T\psi, \psi) + ((\mathcal{V} - V)\psi, \psi) + \gamma(V\psi, \psi),$$

where $\gamma \in (0,1)$ will be chosen later. It is clear that

$$(1-\varepsilon)(T\psi,\psi)+(1-\gamma)(V\psi,\psi)\geq 0$$

and we must show that

$$L_1[\psi]=\frac{\varepsilon}{2}(T\psi,\psi)+\gamma(V\psi,\psi)+((\mathcal{V}-V)\psi,\psi)\geq 0 \tag{4.18}$$

for suitably chosen γ and a_0.

Let $\varphi_0^{m_1}$ be the eigenfunction of the operator $T_0^{m_1}=T_0P^{m_1}$, where

$$T_0=(i\frac{\partial}{\partial r_1}-\frac{B}{4}r_2)^2+(i\frac{\partial}{\partial r_2}-\frac{B}{4}r_1)^2$$

corresponding to its minimal eigenvalue B; $\mathcal{P}_0$ be a projector in $\mathcal{L}_2(R^3)$ on the corresponding subspace ,$\mathcal{P}_1=I-\mathcal{P}_0$. It is obvious that

$$\frac{\varepsilon}{2}(T\psi,\psi)=\frac{\varepsilon}{2}(T\mathcal{P}_0\psi,\mathcal{P}_0\psi)+\frac{\varepsilon}{2}(T\mathcal{P}_1\psi,\mathcal{P}_1\psi)\geq \tag{4.19}$$

$$\frac{\varepsilon B}{2}\|\frac{\partial}{\partial r_3}\mathcal{P}_0\psi\|^2+\frac{\varepsilon B}{2}\|\frac{\partial}{\partial r_3}\mathcal{P}_1\psi\|^2.$$

Further, for any $\varepsilon_1>0$

$$(V\psi,\psi)=\int V|\mathcal{P}_0\psi+\mathcal{P}_1\psi|^2dr\geq \tag{4.20}$$

$$\int|\mathcal{P}_0\psi|^2\{V(r)-\varepsilon_1|V(r)|)\}dr+\int|\mathcal{P}_1\psi|^2\{V(r)-\varepsilon_1^{-1}|V(r)|)\}dr,$$

and

$$((\mathcal{V}-V)\psi,\psi)\geq(\{(\mathcal{V}-V)-\varepsilon_1|\mathcal{V}-V|\}\mathcal{P}_0\psi,\mathcal{P}_0\psi)+ \tag{4.21}$$

$$(\{(\mathcal{V}-V)-\varepsilon_1^{-1}|\mathcal{V}-V|\}\mathcal{P}_1\psi,\mathcal{P}_1\psi).$$

The potential $V(r)$ is bounded and for small $\gamma>0$

$$\gamma|\int|\mathcal{P}_1\psi|^2\{V(r)-\varepsilon_1^{-1}|V(r)|)\}dr|\leq\frac{\varepsilon B}{4}\|\mathcal{P}_1\psi\|^2. \tag{4.22}$$

Moreover,

$$|\mathcal{V}-V|=V(r)\chi(|r|\geq a_0)\leq c_0(1+a_0)^{-2-\delta_0}$$

and for large a_0 one has

$$|(\{(\mathcal{V}-V)-\varepsilon_1^{-1}|\mathcal{V}-V|\}\mathcal{P}_1\psi,\mathcal{P}_1\psi)|\leq\frac{\varepsilon B}{4}\|\mathcal{P}_1\psi\|^2. \tag{4.23}$$

It follows from (4.19)-(4.23) that for small γ and large a_0

$$L_1[\psi]\geq\frac{\varepsilon}{2}\|\frac{\partial}{\partial r_3}\mathcal{P}_0\psi\|^2+\gamma(\{V(|r|)-\varepsilon_1|V(|r|)|\}\mathcal{P}_0\psi,\mathcal{P}_0\psi)+ \tag{4.24}$$

$$([(\mathcal{V}-V)-\varepsilon_1|\mathcal{V}-V|]\mathcal{P}_0\psi,\mathcal{P}_0\psi).$$

Let us note that$(\mathcal{V}-V)<0$ for large a_0 and

$$(\{(\mathcal{V}-V)-\varepsilon_1|\mathcal{V}-V|\}=(1+\varepsilon_1)(\mathcal{V}-V).$$

The function $\mathcal{P}_0\psi$ may be rewritten in the form

$$\mathcal{P}_0\psi=\varphi_0^{m_1}(r_1,r_2)f(r_3)$$

with some function f. Thus it follows from (4.24) that $L_1[\psi]\geq 0$ if the one dimensional operator

$$h^{(1)}=-\frac{\partial^2}{\partial {r_3}^2}+\mathcal{V}_1(r_3) \tag{4.25}$$

with the potential

$$\mathcal{V}_1(r_3)=2\varepsilon^{-1}\int\{\gamma[V(|r|)-\varepsilon_1|V(r)|]-(1+\varepsilon_1)(V-\mathcal{V})\}|\varphi_0^{m_1}(r_1,r_2)|^2dr_1dr_2 \tag{4.26}$$

has no discrete spectrum.

Now we can apply lemma 4.1 to this operator. Since virtual levels of h are absent, we have

$$\int V(|r|)|\varphi_0^{m_1}(r_1,r_2)|^2dr_1dr_2>0.$$

So if ε_1 is small and a_0 is large $\int\mathcal{V}_1(r_3)dr_3$ is also positive. Furthermore, it is clear that

$$|\mathcal{V}_1(r_3)|\leq c_0(1+|r_3|)^{-2-\delta_0} \tag{4.27}$$

The constant c_0 in (4.27) can be chosen an arbitrarily small for small $\gamma>0$ and large a_0. Due to lemma 4.1 to prove the theorem one has only to show that for suitable $\gamma>0,\varepsilon_1>0,a_0>0$ the inequality

$$\mathcal{V}_1(r_3)-\kappa|\mathcal{V}_1(r_3)|\geq-\frac{1}{4}(1+|r_3|)^{-2} \tag{4.28}$$

holds with

$$\kappa=(\int_{-\infty}^{\infty}|\mathcal{V}_1(r_3)|dr_3)(\int_{-\infty}^{\infty}\mathcal{V}_1(r_3)dr_3)^{-1}.$$

For fixed $\gamma>0$ one can take $\varepsilon_1>0$ so small and a_0 so large that

$$\varepsilon_1\int|V(|r|)||\varphi_0^{m_1}(r_1,r_2)|^2dr_1dr_2<\frac{1}{4}\int V(|r|)|\varphi_0^{m_1}(r_1,r_2)|^2dr_1dr_2,$$

and

$$(1+\varepsilon_1)\int(V(|r|)-\mathcal{V}(|r|))|\varphi_0^{m_1}(r_1,r_2)|^2dr_1dr_2<\frac{1}{4}\gamma\int|V(|r|)||\varphi_0^{m_1}(r_1,r_2)|^2dr_1dr_2.$$

In this case

$$\kappa < \kappa_0 \equiv 4(\int V(|r|)|\varphi_0^{m_1}(r_1,r_2)|^2 dr)^{-1}(\int |V(|r|)||\varphi_0^{m_1}(r_1,r_2)|^2 dr).$$

Let us pick $\gamma > 0$ small and a_0 large such that for the constant c_0 the inequality

$$c_0 < \frac{1}{4}(1+\kappa)^{-1}$$

holds. Then (4.28) follows from (4.27). The theorem is proved.

Theorem 4.3. *Let the operator h have no discrete spectrum and the potential $V(|r|)$ have a compact support. Then one can find such a positive number a that for any $\psi \in C_0^2(R^3)$*

$$L[\psi] \equiv \int_{|r_3|\leq a} \{|(i\nabla_r + A_r)\psi|^2 - B|\psi|^2 + V(|r|)|\psi|^2\} dr \geq 0. \tag{4.29}$$

Proof. The proof relies upon

Lemma 4.2 . *Let the potential V be as in Theorem 4.3. Let $\varphi_0^{m_1}$ be the eigenfunction of the operator $T_0^{m_1} = T_0 P^{m_1}$,*

$$T_0 = [(i\frac{\partial}{\partial r_1} - \frac{B}{4}r_2)^2 + (i\frac{\partial}{\partial r_2} + \frac{B}{4}r_1)^2] \tag{4.30}$$

corresponding to its minimal eigenvalue B. Let $g(r) \in C_0^2(R^3), g(r) - \varphi_0^{m_1}$for any fixed $r_3, f(r_3) \in C_0^2(R^1), g(r) = 0$ for $|r_3| \leq a$

$$\psi(r) = \varphi_0^{m_1}(r_1,r_2)f(r_3) + g(r), \tag{4.31}$$

and the enequality (4.29) does not hold. Then the operator h has discrete spectrum.

Proof of lemma 4.2 Define the sequence of functions

$$\psi_n = \varphi_0^{m_1}(r_1,r_2)f_n(r_3) + g \quad n = 2,3,...,$$

where $f_n(r_3) = f$ for $|r_3| \leq a, f_n = 0$ for $|r_3| \geq na, f_n = f(a)\frac{r_3 - na}{a(1-n)}$ for $r_3 \in [a, na], f_n = f(-a)\frac{r_3 - na}{-a(1-n)}$ for $r_3 \in [-na, -a]$. If the inequality (4.29) does not hold, then for large n

$$(h\psi_n, \psi_n) < B\|\psi_n\|^2 = inf\sigma_{ess}(h)\|\psi_n\|^2,$$

which proves the lemma.

Proof of theorem 4.3. Due to the absence of virtual levels one can find such a number $\gamma_0 > 0$ that

$$L_1[\psi_0] \equiv (1-\gamma_0)\{(T_0\psi_0,\psi_0) + \|\frac{\partial}{\partial r_3}\psi_0\|^2\} + (V\psi_0,\psi_0) \geq 0, \quad \forall\psi_0 \in C_0^2(R^3). \tag{4.32}$$

Let $u(t), v(t) \in C^2(R^1_+), u^2+v^2=1, u=0$ for $t>1, u=0$ for $t \le \frac{1}{2}$ and let $c_0 = max\{(u')^2+(v')^2\}$. By a_1 we denote such a number that $a_1 > 2[c_0 B^{-1}\gamma_0^{-1}]^{\frac{1}{2}}$ and $V(r)=0$ for $|r| \ge \frac{a_1}{2}$. We shall prove by contradiction that (4.29) holds for all $a > a_1$. Let us assume that (4.29) does not hold for some function ψ .Then one can find such a weight m_1 that it does not hold for the function $\psi_1 = P^{m_1}\psi$ also. The function ψ_1 has the form

$$\psi_1 = \varphi_0^{m_1}(r_1,r_2)f(r_3)+g. \tag{4.33}$$

where $g - \varphi_0^{m_1}$. There are two possibilities.

If $g(r) \equiv 0$ for $|r| \ge a_1$, then the enequality (4.29) results from lemma 4.2.

Suppose now that $g(r) \not\equiv 0$ for $|r| \ge a_1$. By $\psi_{1,0}$ we denote the function

$$\psi_{1,0}(r) = \varphi_0^{m_1}(r_1,r_2)f(r_3)+g(r)u(|r_3|a_1^{-1}). \tag{4.34}$$

It suffices to show that

$$L_1[\psi_{1,0}] \equiv \int_{|r_3|\le a_1}[(1-\gamma_0)\{|(i\nabla_r + A_r)\psi_{1,0}|^2 - B|\psi_{1,0}|^2\} + V(|r|)|\psi_{1,0}|^2]dr < 0. \tag{4.35}$$

Then the statement of the theorem will follow from this inequality and lemma 4.2.

Let us now prove (4.35). Since $V(|r|)v(|r_3|a_1^{-1})=0$ for large a_1, one has

$$L[\psi_1] \ge L_1[\psi_{1,0}] + \gamma_0 \int_{|r_3|\le a_1}\{|(i\nabla_r + A_r)\psi_1|^2 - B|\psi_1|^2\}dr- \tag{4.36}$$

$$a_1^{-2}(1-\gamma_0)\int_{|r_3|\le a_1}|g|^2[v'^2(|r_3|a_1^{-1}) + u'^2(|r_3|a_1^{-1})]dr.$$

The last term here is bounded by

$$a_1^{-2}(1-\gamma_0)\int_{|r_3|\le a_1}|g|^2[v'^2(|r_3|a_1^{-1})+u'^2(|r_3|a_1^{-1})]dr \le (1-\gamma_0)c_0a_1^{-2}\int_{|r_3|\le a_1}|g|^2dr \le \tag{4.37}$$

$$\frac{1}{2}(1-\gamma_0)\gamma_0 B\int_{|r_3|\le a_1}|g|^2dr \le \gamma_0\int_{|r_3|\le a_1}\{|(i\nabla_r + A_r)\psi_1|^2 - B|\psi_1|^2\}dr.$$

Taking into account that $L[\psi_1]<0$, we obtain from (4.36),(4.37)that

$$L_1[\psi_{1,0}] \le L[\psi_1] < 0,$$

which contradicts the absence of virtual levels by lemma 4.2. The theorem is proved.

Throughout the rest of the paper we study the operator h with the potential $V(|r|)$ such that $|V(|r|)|$ has a compact support and for some $a>0, V|r|) \in C(|r| \ge a), V(|r|) \ge 0$ for $|r| \ge a$ and $V(a) > 0$, whose properties are used in the final part of the proof of main theorem in Section3.

Let $u(t), v(t)$ be the same functions as in the proof of theorem 4.1, $\mathcal{P}$ be the projector in $\mathcal{L}_2(R^3)$ onto the eigensubspace of the operator T_0 (see proof of the theorem

4.2) corresponding to the eigenvalue B. In contrast to $\mathcal{P}_0, \mathcal{P}$ is the projector onto the infinite-dimensional subspace.

Theorem 4.4 . *Let the operator h have no discrete spectrum. Then for any $\delta > 0$ one can find such a number $b > 0$ that for all $\psi \in C_0^2(R^3)$*

$$L[\psi] \equiv \int_{|r_3|\le b} |\frac{\partial}{\partial r_3}\mathcal{P}\psi|^2 u^2([|r_1|^2+|r_2|^2]b^{-1})dr + \int_{|r_3|\le b} |\frac{\partial}{\partial r_3} g|^2 u^2([|r_1|^2+|r_2|^2]b^{-1})dr +$$

$$(V\psi,\psi) + ((T_0 - B)gu, gu) \ge -\delta\{\|\frac{\partial}{\partial r_3}\psi\|^2 + \|g\|^2\},$$

where $g = (I - \mathcal{P})\psi$.

Proof Theorem follows from the next two lemmas:

Lemma 4.3 . *For an arbitrary $\gamma > 0$ and $\gamma_1 > 0$ one can find such a number $m_0 = m_0(\gamma, \gamma_1)$ that for all $m_1, |m_1| > m_0$ and any function*

$$\psi(r_1, r_2) = \varphi_0^{m_1}(r_1, r_2) + g(r_1, r_2),$$

where $\mathcal{P}P^{m_1}\varphi_0^{m_1} = \varphi_0^{m_1}, P^{m_1}g = g$, the inequality

$$\gamma(T_0 g, g) + (V\psi, \psi) \ge \gamma(B - \gamma_1)\|g\|^2$$

holds.

Lemma 4.4 . *Let m_1 be a weight of the representation of the $SO(2)$ group and $\delta > 0$ be a fixed number. Then one can find such a constant $b > 0$ that for any $\psi \in C_0^2(R^3)$, $\quad P^{m_1}\psi = \psi, \psi = \varphi_0^{m_1}(r_1, r_2)f(r_3) + g, g - \varphi_0^{m_1}$ the inequality holds:*

$$L[\psi] \equiv \int_{|r_3|\le b} [|\frac{\partial f}{\partial r_3}|^2 |\varphi_0^{m_1}(r_1,r_2)|^2 u^2([|r_1|^2+|r_2|^2]b^{-1}) + |\frac{\partial g}{\partial r_3}|^2 u^2([|r_1|^2+|r_2|^2]b^{-1})]dr +$$

$$(V\psi,\psi) + \int_{|r_3|\le b} ((T_0 - B)gu([|r_1|^2+|r_2|^2]b^{-1}), gu([|r_1|^2+|r_2|^2]b^{-1}))_{\mathcal{L}_2(R^4)} dr_3 \tag{4.38}$$

$$\ge -\delta[\|\frac{\partial \psi}{\partial r_3}\|^2 + \|g\|^2].$$

Proof of lemma 4.3 . It is easy to see that for $\varepsilon < 1$ the potential $V_1(|r|) = V(|r|) - \varepsilon|V(|r|)|$ is nonnegative for $|r| \ge a$ and $V_1(a) > 0$ .For such a potential there exists a positive number m_{01} such that for all r_3 and all $m_1, |m_1| > m_{01}$ the inequality

$$(\varphi_0^{m_1} V_1(|r|), \varphi_0^{m_1}) > 0 \tag{4.39}$$

takes place. Further,

$$(V\psi,\psi) = (\varphi_0^{m_1} V(|r|), \varphi_0^{m_1}) + 2Re(\varphi_0^{m_1} V(|r|), g) + \tag{4.40}$$

$$(Vg,g) \geq ((V-\varepsilon|V|)\varphi_0^{m_1},\varphi_0^{m_1},)+((V-\varepsilon^{-1}|V|)g,g)$$

It follows from the last two relations that it is sufficient to show that

$$L_0[g] \equiv \gamma(T_0 g,g)+((V-\varepsilon^{-1}|V|)g,g) \geq \gamma(B-\gamma_1)\|g\|^2 \tag{4.41}$$

for large $|m_1|$. Let $u(t), v(t)$ be the functions defined in the beginning of this subsection, $u = u([|r_1|^2+|r_1|^2]b^{-1}), v = v([|r_1|^2+|r_1|^2]b^{-1})$.As in the proof of the theorem 4.1 we have:

$$L_0[g] \geq L_0[gu]-cb^{-2}\|gu\|^2+L_0[gv]-cb^{-2}\|gv\|^2 \tag{4.42}$$

for some positive constant c. The function gu has a compact support and therefore as in the proof of theorem 4.1 we get that

$$L_0[gu]-cb^{-2}\|gu\|^2 \geq \gamma B\|gu\|^2. \tag{4.43}$$

Moreover, if b is large, then

$$cb^{-2}\|gv\|^2 \leq \gamma\gamma_1\|gv\|^2.$$

Recall that $V(|r|)=0$ for $r \in suppv$. Along with the previous estimate this yields that

$$L_0[gv]-cb^{-2}\|gv\|^2 \geq (T_0 gv,gv)-$$
$$\gamma\gamma_1\|gv\|^2 \geq \gamma(B-\gamma_1)\|gv\|^2,$$

which completes the proof.

Proof of lemma 4.4 . Let $b>0$ be a large number such that $V(|r|)=0$ for $|r|>b$.Asuume temperarily that $f(r_3)=0$ and $g(r)=0$ for $|r_3|\geq b$. By ψ_1 we denote the function $\psi_1=\varphi_0^{m_1}f+gu$. Then

$$L[\psi]=((h-B)\psi_1,\psi_1)-\int|\frac{\partial f}{\partial r_3}|^2|\varphi_0^{m_1}(r_1,r_2)|^2(1-u^2([|r_1|^2+|r_2|^2]b^{-1}))dr- \tag{4.44}$$

$$2Re\int\frac{\partial f}{\partial r_3}\varphi_0^{m_1}(r_1,r_2)\frac{\partial g}{\partial r_3}udr.$$

Because of the decay properties of the function $\varphi_0^{m_1}(r_1,r_2)$,we have for large b

$$\|(1-u^2)^{\frac{1}{2}}\varphi_0^{m_1}\| \leq \frac{\delta}{4} \tag{4.45}$$

and

$$Re\int\frac{\partial f}{\partial r_3}\varphi_0^{m_1}(r_1,r_2)\frac{\partial g}{\partial r_3}udr=-Re\int\frac{\partial f}{\partial r_3}\varphi_0^{m_1}(r_1,r_2)\frac{\partial g}{\partial r_3}(1-u)dr \geq \tag{4.46}$$

$$-\frac{\delta}{2}\|\frac{\partial g}{\partial r_3}\|^2-\frac{2}{\delta}\|\frac{\partial f}{\partial r_3}\varphi_0^{m_1}(r_1,r_2)(1-u)\|^2 \geq -\frac{\delta}{2}\|\frac{\partial g}{\partial r_3}\|^2-\frac{\delta}{2}\|\frac{\partial f}{\partial r_3}\|^2.$$

For $((h-B)\psi_1, \psi_1) \geq 0$, it follows from (4.44)-(4.46) that if $g(r) = 0$ and $f(r_3) = 0$ for $|r_3| \geq b$, the inequality (4.38) holds and lemma is proved. Using the same argument as in the poof of thorem 4.3, it is easy to show that (4.38) holds also for functions that do not vanish for $|r_3| \geq b$. Lemma is proved.

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