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Boltzmanngasse 9
A-1090 Wien, Austria

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Hiroshi Maehara

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Reversing a polyhedral surface

Hiroshi Maehara
Ryukyu University, Okinawa, Japan

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Abstract

We introduce a new variety of flexatube, a *rhomboflexatube*. It is obtained from a cardboard rhombohedron by removing a pair of opposite faces (rhombi), and then subdividing the remaining four faces by pairs of diagonals. It is reversible, that is, it can be turned inside out by a series of folds, using edges and diagonals of the rhombi. To turn a rhomboflexatube inside out is quite a challenging puzzle. We also consider the reversibility of general polyhedral surfaces. We show that if an orientable polyhedral surface with boundary is reversible, then its genus is 0 and for every interior vertex, the sum of face angles at the vertex is at least 2π. After defining tube-attachment operation, we show that every polyhedral surface obtained from a rectangular tube by applying tube-attachment operations one after another can be subdivided so that it becomes reversible.

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1 Introduction

Deformation of geometric objects in a space has been studied by many researchers with great interest.

A deformation of a polygonal arc or polygonal cycle in the plane is a continuous motion of the arc or cycle such that during the motion, each edge remains a line segment of fixed length. The *carpenter's rule problem* asks whether every polygonal arc in the plane can be deformed, with *avoiding* *self-intersections*, into a polygonal arc lying on a straight line. Connelly *et al* [3] proved, among other things, that this is always possible.

Since a state (locations of the vertices) of a polygonal cycle with n vertices can be represented by a point in $2n$-space, all states obtained by deforming the polygonal cycle (with allowing self-intersections) determine a subset of $2n$-space. The 'space of shapes' (the *configuration space*) of the polygonal cycle is then obtained as the quotient space of this subset under the relation corresponding to 'congruence'. Havel [6] proved that the configuration space of an equilateral pentagon (that is, 5-vertex-polygonal cycle with equal edge-lengths) in the plane is a connected orientable closed 2-dimensional manifold of genus 4. Maehara [7] classified the configuration spaces for pentagons with edges of all different lengths in the plane.

A polyhedral surface M is a 2-dimensional manifold in R^3 obtained by attaching cardboard polygons along their edges. The cardboard polygons are supposed to be very thin, and the thickness is regarded to be 0. We assume that the dihedral angle of every pair of mutually adjacent polygons can vary from 0 to 2π unless the motion is blocked by other polygons. Each polygon of M is called simply a face of M. A subdivision of M is a polyhedral surface obtained by subdividing faces of M into small polygons.

A *deformation* of M is a continuous motion of M such that the motion of each face is a rigid motion, no two face cross each other during the motion, and yet the dihedral angles of adjacent faces change. A polyhedral surface that admits a deformation is called *flexible*, otherwise, it is called *rigid*.

Note that in our deformation, two distinct faces can touch, though they cannot cross each other. When the dihedral angle of a pair of adjacent faces becomes 0, the two faces become one sheet though they cannot go through each other.

Cauchy proved in 1813 that every closed convex polyhedral surface in R^3 is rigid, and Gluck [5] proved that almost all closed polyhedral surfaces of genus 0 in R^3 with all triangular faces are rigid. However, Connelly [1,2] found a flexible closed polyhedral surface of genus 0 in R^3 with all triangular faces.

For polyhedral surfaces with boundary, there are also interesting problems. If a polyhedral surface with boundary can be deformed so that all dihedral angles become π, then the surface is called *developable*. A *face-cycle* of a polyhedral surface is a cyclic sequence of (at least three) distinct faces in which each pair of consecutive faces share a common edge. Maehara [8] proved that the surface of a convex polyhedron cannot have a developable face-cycle.

Example 1 (Folding container). From a cubical box, remove a face and triangulate the remaining five faces as in Figure 1 left. This polyhedral surface can be folded flat into a square.

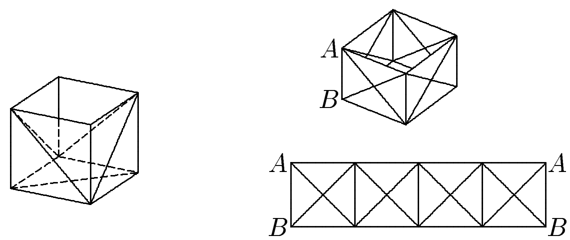

Figure 1: A folding container (left) and a flexatube (right)

Example 2 (Flexatube). From a cubical box, remove a pair of opposite faces, and triangulate the remaining four faces by pairs of diagonals, see Figure 1 right. The resulting polyhedral 'tube' consisting of 16 triangles is called a *flexatube* [4,9,10]. This tube is reversible!

What is meant by reversible? Paint the outside of a flexatube with red and the inside with blue. Then 'to reverse the flexatube' means to deform the flexatube so that its outside becomes blue. To reverse a flexatube is actually possible, though it is not easy. A flexatube is a variation of *flexagons* that were originally discovered in 1939 by Arthur H. Stone, see [8, p. 14].

Stimulated by the flexatube, I looked for other intriguing variations of flexatube. Fortunately, I could find one, which I named *rhomboflexatube*.

Example 3 (Rhomboflexatube). From a rhombohedron (a parallelepiped whose six faces are rhombi with interior angles 60° and 120°), remove a pair of opposite faces, and triangulate the remaining 4 faces by pairs of diagonals. The resulting polyhedral tube consisting of 16 triangles is the rhomboflexatube. Figure 2 shows how to make a paper model of rhomboflexatube. This tube is also reversible.

To reverse a rhomboflexatube is a challenging puzzle. A solution I found is exquisite and complicated. The condition that each rhombus has 60° angle is essential to the solution.

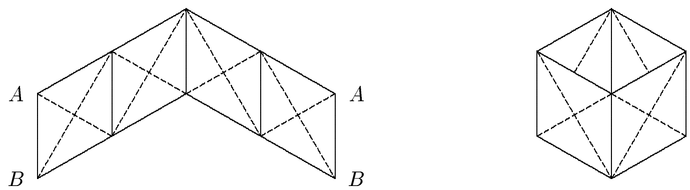

Figure 2: A rhomboflexatube

Motivated by the flexatube and rhomboflexatube, we also consider the reversibility for general polyhedral surfaces. We prove that if an orientable polyhedral surface M (with boundary) is reversible, then M has genus 0, and for any interior vertex p of M, the sum of face-angles at p is greater than or equal to 2π (Theorems 1, 2).

A *tube-attachment* operation is defined in the following way:

From a face of polyhedral surface M, cut out a rectangle, and attach a rectangular tube at the rectangular hole as shown in Figure 3. (If necessary, we subdivide the face with rectangular hole to make it the union of polygons.)

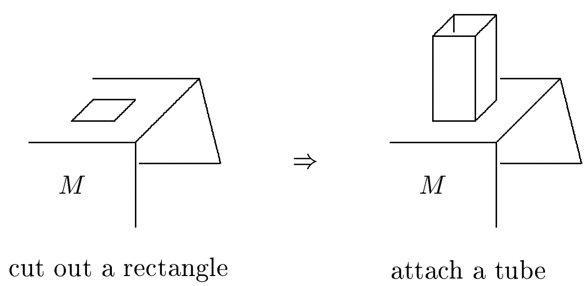

Figure 3: A tube-attachment

We prove that every polyhedral surface obtained from a polygonal tube by applying tube-attachment operations one after another can be subdivided so that it becomes reversible (Theorems 3, 4).

Example 4. The surface shown in Figure 4 can be subdivided so that it becomes reversible, since the surface can be obtained from a rectangular tube by applying tube-attachment operations one after another.

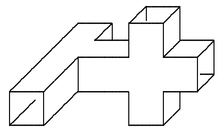

Figure 4: By a subdivision, this surface becomes reversible

2 Reversible and s-reversible surfaces

We state here a precise definition of the reversibility for general orientable polyhedral surface.

An orentable polyhedral surface M (with boundary) is called *reversible* if there is a deformation $f_t : M \to R^3$ $(0 \le t \le 1)$ such that the correspondence

$$M \ni x \mapsto f_1(x) \in f_1(M)$$

is an orientation-reversing congruence between M and $f_1(M)$, see Figure 5. If a subdivision M' of M is reversible, then M is called *subdivision-reversible* (shortly *s-reversible*).

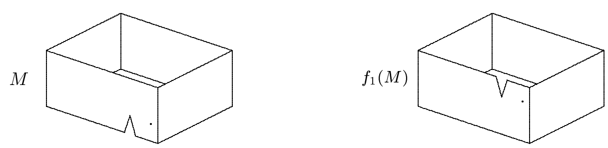

Figure 5: Congruent in orientation-reversing fashion

In this strict sense, a flexatube and a rhomboflexatube are reversible, and the surface obtained from a cubical box by removing a pair of opposite faces is s-reversible, since a flexatube is its subdivision. It is also obvious that a surface obtained as a part of an s-reversible surface is also s-reversible.

3 Genus and convex points

Let M denote a connected *orientable* polyhedral surface (with boundary) in R^3. The genus of M is the genus of the closed surface obtained by capping off each of the boundary components of M with a disk.

Theorem 1. *If M is s-reversible, then the genus of M is* 0.

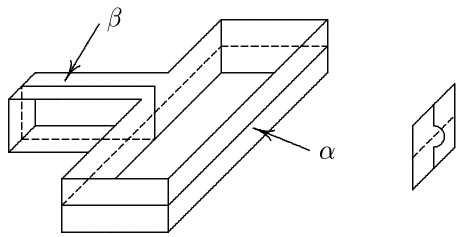

Figure 6: Non-splittable link (α, β)

Proof. Suppose that M has positive genus. Then, we can draw a simple closed curve α in one side of M, and draw another simple closed curve β in the other side of M, so that the two curves α and β form a pair of 'non-splittable link', see Figure 6 left. This figure shows the case of genus 1, but for any M of positive genus, we can draw such curves α, β, similarly. (In the strict sense, (α, β) does not form a link since the faces are supposed to have thickness 0, and then the two curves intersect. So, let us regard that a small part of β is lifted as in Figure 6 right and fixed rigidly on the face.) By a deformation, each face moves as a rigid body, and no two faces go through each other during a deformation, the movement of the link (α, β) induced by a deformation of M is regarded as an ambient isotopy. If M is s-reversible, then by reversing M the pair (α, β) goes to a splittable link, which is a contradiction. Hence, if M is s-reversible, then the genus of M is 0. □

A *convex point* of M is a vertex p of M such that it does not lie on the boundary of M and the sum of the face angles at p is less than 2π.

Theorem 2. *If M is s-reversible, then M has no convex point.*

To prove this, we use the following obvious fact.

Lemma 1. *It is impossible to bisect the surface area of a sphere by a closed curve that is shorter than the length of a great circle of the sphere.* □

Proof of Theorem 2. Suppose that a subdivision M' is reversible. The point p is also a convex point of M'. Paint one side of M with red, and the other side with blue. Let S be a sphere of sufficiently small radius centered at p. Let γ be the closed curve obtained as the intersection $S \cap M'$, see

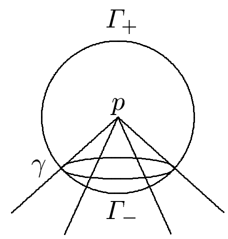

Figure 7: A neighborhood of p

Figure 7. Since p is a convex point, γ is shorter than the great circle of S. Among the two regions of S divided by γ, let Γ_+ be the region corresponding to the red-face-side of M, and Γ_- be the region corresponding to the blue-face-side of M. Suppose $|\Gamma_+| > |\Gamma_-|$ in M'. Then, by reversing M', we have $|\Gamma_+| < |\Gamma_-|$. Hence, in the midway of the deformation, it happens that $|\Gamma_+| = |\Gamma_-|$. However, since γ is shorter than the great circle, this is impossible by the above lemma. □

4 Some basic operations

Let us introduce here a few other operations related to a rectangular tube.

(1) Fold-in- and pull-out-operations.

By subdividing a rectangular tube suitably, we can 'fold in' a part of the tube as in the Figure 8.

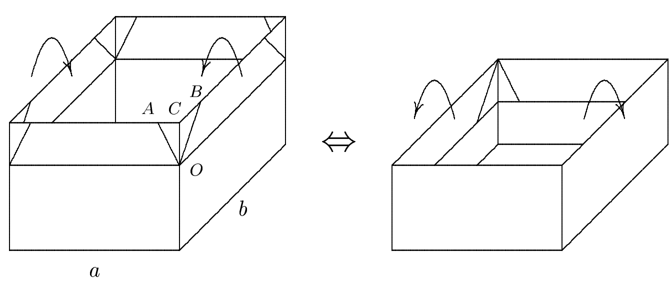

Figure 8: Fold-in and pull-out operations

Let us explain a little more. Put $OC = x$, $AC = BC = y$, and let $a \times b$ be the size of the base rectangle. In order to fold in as shown in Figure 8 right, the three vertices A, C, B need to become collinear in the midway of deformation. If $y/x > \sqrt{2}$, then one of A, B need to move out of the $a \times b$ rectangle in the midway. So, we assume that $y/x < \sqrt{2}$. Then A, B can remain within $a \times b$ rectangle. (This is important to introduce fold-out-operation.) If $y/x < \sqrt{2} - 1$, then A, B cannot go down to the level of O. Hence we also assume $y/x > \sqrt{2} - 1$. If we take x, y to satisfy $y < \min\{a/2, b/2\}$ and

$$\sqrt{2} - 1 < y/x < \sqrt{2}$$

then we can fold in (and pull out) the tube by length x keeping A, B within the $a \times b$ rectangle.

(2) Fold-out-operation.

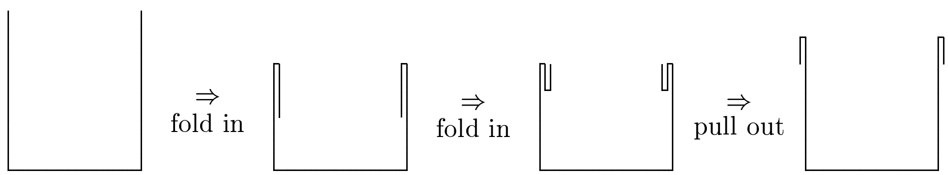

Figure 9: Fold-out-operation

Figure 9 shows how to 'fold out' a part of rectangular tube. Since the faces are supposed to have thickness 0, by subdividing suitably, we can do the pull-out-operation in Figure 9.

(3) Flattening- and raising-operations.

Figure 10 shows how to flatten and raise a short tube.

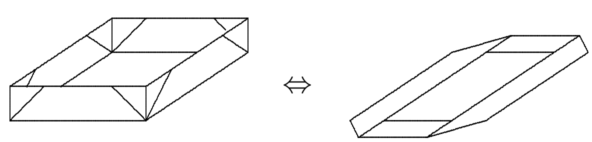

Figure 10: Flattening- and raising-operations

5 Application of the operations

Theorem 3. Every rectangular tube is s-reversible.

Proof. In the case of a very short tube, we can *subdivide and reverse* (shortly, *s-reverse*) it by a fold-in-operation. In case of a long tube, by repeating fold-out-operations, we make the tube very short, then s-reverse it, and then apply pull-out-operations, see Figure 11. □

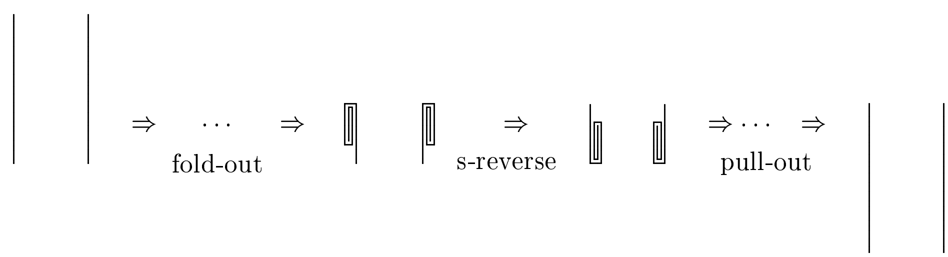

Figure 11: Reverse a long tube

Since every polygonal tube has a subdivision that can be deformed into a subdivision of rectangular tube, the next corollary follows.

Corollary 1. *Every polygonal tube is s-reversible.* □

Corollary 2. *From a pyramid, remove the bottom face and cut off the remaining convex point. Then the resulting polyhedral surface is s-reversible.*

Proof. By making many 'pleats', we can change the shape of the surface into a slender tube. By s-reversing this tube, and then by unfolding the pleats, we can s-reverse the original surface. □

There are s-reversible polyhedral surfaces that are not tube-like.

Example 5. From a box, remove a face and then cut off 4 convex points, see Figure 12 top-left. The resulting surface is s-reversible.

To reverse this surface, first subdivide the surface as in Figure 12 top-right. Then by repeating fold-out-operations, make the surface very short. Then, we can push down the 'ceiling'. Finally, by pull-out-operations, we get the surface reversed.

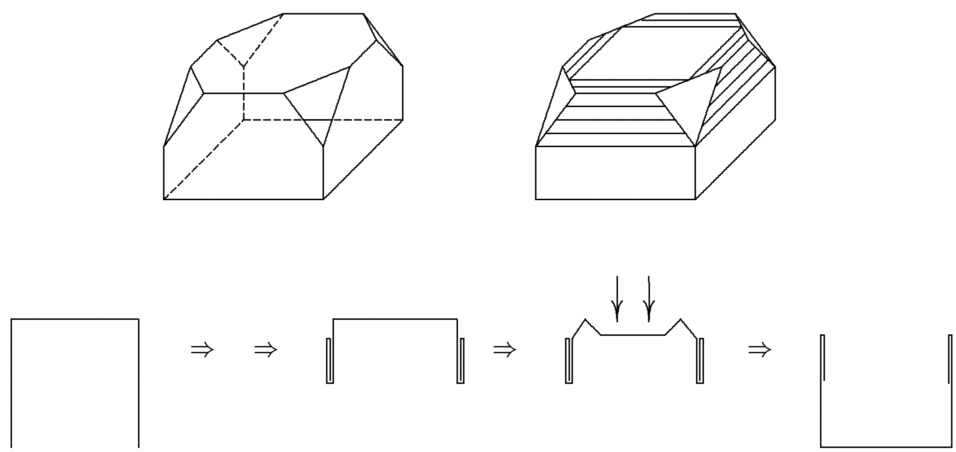

Figure 12: Proof of Example 5

Theorem 4. *The surface obtained from an s-reversible polyhedral surface M by applying a tube-attachment operation is also s-reversible.*

Proof. By repeating fold-out-operations, make the attached tube very short, and flatten it on the face, see Figure 13. Then the resulting surface is regarded as a part of M, and we can s-reverse it. Then, raise the short tube, and fold in it, and then pull out. □

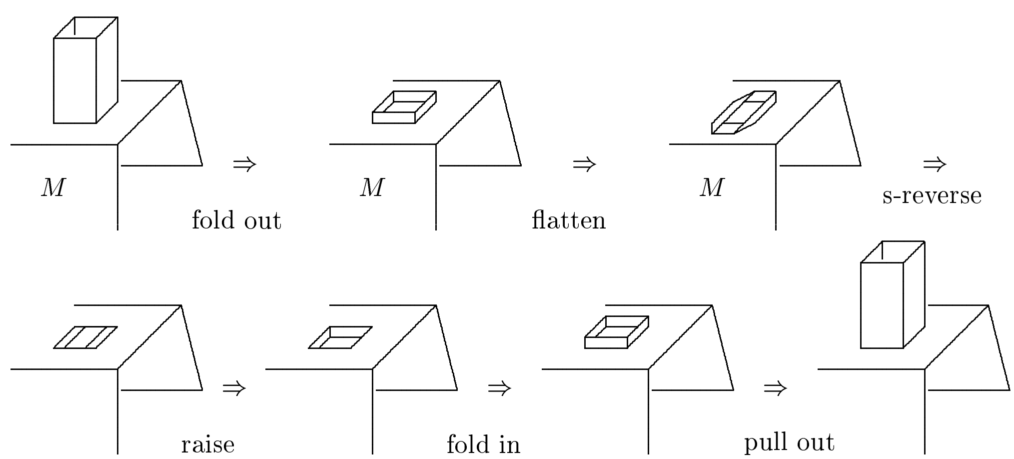

Figure 13: Proof of Theorem 4

Problem 1. Find a polyhedral surface of genus 0 with no convex point that is not s-reversible.

Problem 2. Is the surface in Figure 14 s-reversible? Notice that this is *not* a surface obtained from a tube by applying tube-attachment operations.

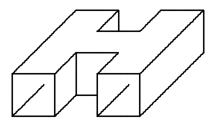

Figure 14: Is this s-reversible?

Conjecture. The surface obtained from a tetrahedron by cutting off the four convex points (see Figure 15) would not be s-reversible, provided that each cut off part is small.

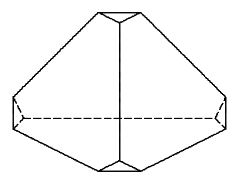

Figure 15: Is this not s-reversible?

Probably, it would be also true that no matter how finely you subdivide the surface of Figure 15, it cannot be flattened on a plane. Of course it is assumed that each cut off part is small.

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