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SYMMETRIC DESIGNS ON LIE ALGEBRAS AND INTERACTIONS OF HAMILTONIAN SYSTEMS

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ABSTRACT. Nonhamiltonian interaction of hamiltonian systems is considered. Dynamical equations are constructed by use of symmetric designs on Lie algebras. The results of analysis of these equations show that some class of symmetric designs on Lie algebras beyond Jordan ones may be useful for a description of almost periodic, asymptotically periodic, almost asymptotically periodic, and possibly, more chaotic systems. However, the behaviour of systems related to symmetric designs with additional identities is simpler than for general ones from different points of view. These facts confirm a general thesis that various algebraic structures beyond Lie algebras may be regarded as certain characteristics for a wide class of dynamical systems.

Many important classical hamiltonian dynamical systems are connected with Lie algebras [1,2] or their nonlinear generalizations [3]. An interaction of hamiltonian systems may be of different kinds. First, it may be hamiltonian, i.e. defined by a subsidiary term \mathcal{H}_{int} in the hamiltonian and, possibly, by a certain (maybe rather nonlinear) deformation of initial ("free") Poisson brackets. Second, it may be nonhamiltonian, i.e. with nonconservative (nonpotential) forces of interaction, however, one may suppose that it is still nondissipative, i.e. the total energy is conserved. To describe algebraic structures governing such interactions is an important unsolved problem of mathematical physics. There exist, at least, two approaches to the problem. The first approach associates the nonhamiltonian interaction with certain deformations or generalizations of the initial algebraic structures (Lie algebras) such as f.e. isotopic pairs [4] or general (nonlinear) I-pairs [5]. The second approach defines such interaction by use of subsidiary algebraic structures on Lie algebras.

Key words and phrases. Classical dynamics, Lie algebras, hamiltonian systems, nonhamiltonian interaction, triple systems.

Definition 1 [6]. A *design*¹ on a Lie algebra \mathfrak{g} is a \mathfrak{g} -equivariant algebraic structure on it.

In the paper [6] the Jordan designs on Lie algebras were considered, and the related nonhamiltonian dynamics was explored. the results confirmed the supposition that the formalism of Jordan designs may be a useful tool for an investigation of asymptotically almost periodic systems.

However, some generalizations of Jordan designs may be useful, too. At least, there are no any self-evident reasons to be restricted by a Jordan structure². So it is rather natural to try to verify other algebraic structures as possible "governing" objects for nonhamiltonian interactions, and to specify the class of dynamical systems, which is convenient to describe by them.

This paper is devoted to a certain class of designs on Lie algebras, which may be of an interest.

Definition 2. A *symmetric design* on a Lie algebra \mathfrak{g} with the commutator $[\cdot, \cdot]$ is a \mathfrak{g} -equivariant structure of a triple system on \mathfrak{g} with a trilinear operation $\langle \cdot, \cdot, \cdot \rangle$ such that $\langle X, A, Y \rangle = \langle Y, A, X \rangle$ ($\forall A, X, Y \in \mathfrak{g}$) and $[A, \langle X, A, X \rangle] + [X, \langle A, X, A \rangle] = 0$ ($\forall A, X \in \mathfrak{g}$).

Remark 1. If a Lie algebra \mathfrak{g} admits two symmetric designs with trilinear operations $\langle \cdot, \cdot, \cdot \rangle'$ and $\langle \cdot, \cdot, \cdot \rangle''$ then it admits an infinite family of symmetric designs with trilinear operations $\lambda \langle \cdot, \cdot, \cdot \rangle' + \mu \langle \cdot, \cdot, \cdot \rangle''$. It means that symmetric designs on a fixed Lie algebra form a linear space.

It seems that a linear space of all symmetric designs on a Lie algebra possesses a subsidiary hidden algebraic structure, its unravelling is an important problem, which is, unfortunately, out of a general line of the present paper.

Example 1. Let \mathfrak{A} be an associative algebra with an involution $*$. The Lie algebra $\mathfrak{g} = \{X \in \mathfrak{A} : X^* = -X\}$ is supplied by, at least, two structures of symmetric designs: $\langle X, Y, \cdot \rangle' = XYZ + ZYX$, $\langle X, Y, Z \rangle'' = \frac{1}{2}(XZY + ZXY + YXZ + YZX)$.

Example 2. A Lie algebra \mathfrak{g} with an invariant bilinear form $\langle \cdot, \cdot \rangle$ is supplied by, at least, two structures of symmetric designs: $\langle X, Y, Z \rangle' = \langle X, Y \rangle Z + \langle Z, Y \rangle X$ and $\langle X, Y, Z \rangle'' = \langle X, Z \rangle Y$.

Example 3. An arbitrary Lie algebra \mathfrak{g} possesses a symmetric design with the trilinear operation $\langle X, Y, Z \rangle = -([X, Y], Z) + [X, [Y, Z]]$. This symmetric design will be called *canonical*. Note that $\langle X, X, X \rangle = 0$ in a Lie algebra with the canonical symmetric design³.

Note that the trilinear operations $\langle \cdot, \cdot, \cdot \rangle'$ of examples 1 and 2 are Jordan triple products, considered in [6].

¹ An English word "design" is used here as a rough translation of an original Russian term "y3op". Probably, such translation is not the best because the English word "design" is reserved for another mathematical object. However, we shall use it but only as a translation of the Russian "y3op".

² This argument was formulated by Prof. D.V. Alekseevskii in a private talk (ESI, 30 Nov. 1994) with the author.

³ *Remark 2.* Unfortunately, the author is not acknowledged on an implicit axiomatic definition of the canonical symmetric triple product. Certainly, its construction does not claim a presence

Definition 3. Let \mathfrak{g} be a Lie algebra with a symmetric design $\langle \cdot, \cdot, \cdot \rangle$. Differential equations

$$\begin{cases} \dot{A}_t = \{\mathcal{H}, A_t\} + \alpha \langle B_t, A_t, B_t \rangle \\ \dot{B}_t = \{\mathcal{H}, B_t\} - \alpha \langle A_t, B_t, A_t \rangle \end{cases}$$

where $(A_t, B_t) \in \mathfrak{g} \oplus \mathfrak{g}$, $\{\cdot, \cdot\}$ is a Lie–Poisson bracket, $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_{\text{int}}$ of interaction, are called *the dynamical equations associated with the symmetric design on the Lie algebra \mathfrak{g}* .

These equations generalize a certain partial case of dynamical equations related to Jordan designs [6].

Note that the dynamical equations associated with the canonical symmetric design look like coupled Landau–Lifschitz equations (see [7]). However, a certain principal difference exist — our dynamical equations are linear by an internal state and quadratic an by an external field whereas the Landau–Lifschitz equations and their analogs are linear by an external field and quadratic by an internal state. Namely, such form of dynamical equations provides the conservative law for a whole system.

If \mathfrak{g} admits a nondegenerate invariant bilinear form $\langle \cdot, \cdot \rangle$, then it is reasonable to consider an interaction hamiltonian in the form $a \langle A, B \rangle + b \langle \langle A, B, A \rangle, B \rangle$. In this case the dynamical equations will be rewritten in the form:

$$\begin{cases} \dot{A}_t = \{\mathcal{H}_1, A_t\} + a[B_t, A_t] + \alpha \langle B_t, A_t, B_t \rangle + 2b \langle \langle B_t, B_t | A_t, A_t \rangle \rangle \\ \dot{B}_t = \{\mathcal{H}_2, B_t\} - a[B_t, A_t] - \alpha \langle A_t, B_t, A_t \rangle - 2b \langle \langle B_t, B_t | A_t, A_t \rangle \rangle \end{cases}$$

where

$$\begin{aligned} \langle \langle A, B | X, Y \rangle \rangle &= \frac{1}{2}([A, \langle X, B, Y \rangle] + [B, \langle X, A, Y \rangle]) = \\ &= -\frac{1}{2}([X, \langle A, Y, B \rangle] + [Y, \langle A, X, B \rangle]). \end{aligned}$$

Let's consider the simplest model example of a Lie algebra $\mathfrak{so}(3)$. In this case the linear space of all symmetric designs is two dimensional, and is spanned by designs of either exaple 1 or example 2. The hamiltonians \mathcal{H}_i will be linear hamiltonians of rotators $\mathcal{H}_1 = \langle \Omega_1, A \rangle$, $\mathcal{H}_2 = \langle \Omega_2, B \rangle$, moreover, $\Omega_1 = \Omega_2 = \Omega$. This is just a generalization of a situation described in [6].

A symmetric design on $\mathfrak{so}(3)$ will be considered as a deformation of the Jordan design of [6], i.e. $\langle X, Y, Z \rangle_\varepsilon = \langle X, Y \rangle Z + \langle Y, Z \rangle X + 2\varepsilon \langle X, Z \rangle Y$. Dynamical equations have the form (cf.[6]):

$$\begin{cases} \dot{A} = [\Omega, A] + a[B, A] + 2\alpha \langle A, B \rangle B + 2\alpha\varepsilon \langle B, B \rangle A + 2b \langle A, B \rangle [B, A] \\ \dot{B} = [\Omega, B] - a[B, A] - 1\alpha \langle A, B \rangle A - 2\alpha\varepsilon \langle A, A \rangle B - 2b \langle A, B \rangle [B, A] \end{cases}$$

It is convenient to put $\tau = \langle A, B \rangle$, $\rho = \langle A, A \rangle + \langle B, B \rangle$, $\sigma = \langle A, A \rangle - \langle B, B \rangle$. Then $\dot{\rho} = 0$ whereas

$$(1) \quad \begin{cases} \dot{\tau} = -2\alpha(1 + \varepsilon)\tau\sigma \\ \dot{\sigma} = 8\alpha\tau^2 + 2\alpha\varepsilon(\rho^2 - \sigma^2) \end{cases}$$

of a Lie algebra, one may use only a structure of a Lie triple system (intimately related to abstract symmetric spaces). Canonical symmetric designs systematically appear in descriptions of dissipative dynamics (see f.e. [7] and refs wherein).

Let's express τ via σ , $\tau = \frac{1}{2\sqrt{\alpha}}\sqrt{\frac{\sigma}{2} - \alpha\varepsilon(\rho^2 - \sigma^2)}$, and substitute the resulted expression into the equation for $\dot{\tau}$. A differential equation on σ is derived, namely,

$$(2) \quad \ddot{\sigma} + 4\alpha(1 + 2\varepsilon)\sigma\dot{\sigma} + 4\alpha^2\varepsilon(1 + \varepsilon)(\rho^2 - \sigma^2)\sigma = 0.$$

First, let's investigate a behaviour of the system of differential equations (1) qualitatively. Such behaviour essentially depends on the value of ε . For all values of ε the system has two boundary critical points $\tau = 0$ and $\sigma = \pm\rho$; if $-1, \varepsilon, 0$ it has also two proper critical points $\sigma = 0$ and $\tau = \pm\frac{\rho\zeta}{2}$ ($\zeta = \sqrt{-\varepsilon}$, $0 < \zeta, 1$). If $\varepsilon = -1$ there exists a critical curve $\rho^2 = \sigma^2 + 4\tau^2$.

The linearized system at the boundary critical point $\tau = 0$, $\sigma = \pm\rho$ has the form

$$\begin{cases} \dot{u} = \mp 2\alpha\rho(1 + \varepsilon)u \\ \dot{v} = \mp 4\alpha\varepsilon\rho v \end{cases}$$

($\tau = u$, $\sigma = \pm\rho \mp v$) whereas this system at the proper critical point $\sigma = 0$, $\tau = \pm\frac{\rho\zeta}{2}$ has the form

$$\begin{cases} \dot{u} = -\alpha\zeta(1 - \zeta^2)\rho v \\ \dot{v} = 8\alpha\zeta\rho u \end{cases}$$

($\tau = \pm(\frac{\rho\zeta}{2} + u)$, $\sigma = v$).

If proper critical points do not exist (i.e. $\varepsilon < -1$ or $\varepsilon > 0$) one of boundary critical points is attractive and another is repulsive; such situation is not interesting. On the contrary, if $-1 < \varepsilon < 0$ the boundary points are hyperbolic, whereas proper ones are elliptic. Therefore, only two cases ($\varepsilon = -1$ and $-1 < \varepsilon < 0$) will be considered.

Case 1 ($\varepsilon = -1$). The symmetric design $\langle \cdot, \cdot, \cdot \rangle_{-1}$ is just canonical one.

The system (2) has the form

$$\begin{cases} \dot{\tau} = 0 \\ \dot{\sigma} = 2\alpha(\sigma^2 + 4\tau^2 - \rho^2) \end{cases}$$

So $\tau = \tau_0$ whereas $\sigma = \varkappa \operatorname{cth} 2\varkappa\alpha t$ ($\varkappa = \sqrt{\rho^2 - 4\tau_0^2}$).

Complete dynamical equations have the form

$$\begin{cases} \dot{A} = [\Omega, A] + (a + 2b\tau_0)[B, A] - \alpha(\rho - \sigma)A + 2\alpha\tau_0 B \\ \dot{B} = [\Omega, B] - (a + 2b\tau_0)[B, A] - 2\alpha\tau_0 A + \alpha(\rho + \sigma)B \end{cases}$$

Put $C = (\rho - \varkappa)A + 2\tau_0 B$ then $C \xrightarrow[t \rightarrow \infty]{} 0$, and hence, $\dot{A} \sim [\Omega, A]$, $\dot{B} \sim [\Omega, B]$, so the dynamics is asymptotically periodic with period Ω independent on the initial conditions.

Case 2 ($-1 < \varepsilon < 0$). Let's put $\zeta = \sin\eta/2$ then the system (1) is written as

$$(3) \quad \begin{cases} \dot{\tau} = -2\alpha \cos^2 \frac{\eta}{2} \tau \sigma \\ \dot{\sigma} = 8\alpha\tau^2 - 2\alpha \sin^2 \frac{\eta}{2} (\rho^2 - \sigma^2) \end{cases}$$

whereas the equation (2) has the form

$$\ddot{\sigma} + 2\alpha \cos \eta \sigma \dot{\sigma} - \alpha^2 \sin \eta (\rho^2 - \sigma^2) \sigma = 0.$$

Note that the symmetric design $\langle \cdot, \cdot, \cdot \rangle$ is Jordan iff $\varepsilon = 0$ or $\varepsilon = -1/2$. The least case corresponds to $\eta = \frac{\pi}{2}$. The system (3) is transformed into

$$(5) \quad \begin{cases} \dot{\tau} = -\alpha \tau \sigma \\ \dot{\sigma} = 8\alpha \tau^2 - \alpha(\rho^2 - \sigma^2) \end{cases}$$

whereas the equation (4) is rewritten as

$$(6) \quad \ddot{\sigma} - \alpha^2(\rho^2 - \sigma^2)\sigma = 0.$$

It describes an evolution of an anharmonic oscillator, the solutions are expressed via elliptic functions (and hence, the dynamics (5) is periodic).

However, the detailed analysis of the complete dynamical equations seems to be beyond purely analytic methods and claims special numerical computations.

The situation for arbitrary $\eta \neq \frac{\pi}{2}$ is similar.

In spite of the incompleteness of the drawn picture the obtained (partially qualitative) results confirm an initial supposition of Prof. D.V. Alekseevskii that some class of symmetric designs on Lie algebras beyond Jordan ones may be also useful for a description of almost periodic, asymptotically periodic, asymptotically almost periodic, and possibly, more chaotic dynamical systems. Such designs, really, may form only a bounded domain in the whole linear space of all symmetric designs on a Lie algebra. Also some special points of this domain, really, may have a greater importance, if they are related to designs with additional identities for the trilinear operation (f.e. the Jordan designs). In such points the dynamics either possesses additional explicitly integrable components and asymptotically invariant tori or simplifies its behaviour qualitatively. This fact confirms the general thesis that various algebraic structures beyond Lie algebras may be regarded as certain implicit (hidden) internal characteristics for a wide class of dynamical systems.

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