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# Perelomov Problem and Inversion of the Segal–Bargmann Transform

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Vienna, Preprint ESI 1657 (2005) June 1, 2005

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at

# Perelomov problem and inversion of the Segal-Bargmann transform

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We reconstruct a function by values of its Segal-Bargmann transform at points of a lattice.

**1. Formulation of the result.** Fix $\tau > 0$. For a function $f \in L^2(\mathbb{R})$, we define the coefficients

$$\gamma_{m,k} = \int_{-\infty}^{\infty} e^{-ikx-\tau mx} f(x) e^{-x^2/4}\, dx$$

where $m$, $k$ range in $\mathbb{Z}$. We intend to reconstruct $f$ by $\gamma_{m,k}$. As Perelomov showed, this is impossible for $\tau > \pi$; for $\tau \leqslant \pi$, the problem is overdetermined (see [6]-[7], [2], more recent results in [5], [3]). There are many ways for reconstruction of $f$. We propose a formula that seems relatively simple and relatively closed.

Denote $q := e^{-2\pi\tau}$. Define the coefficicients

$$\mathcal{E}_m(\tau) = \frac{(-1)^m q^{m(m-1)/2}}{\prod_{l=1}^{\infty}(1-q^l)^3} \sum_{j\geqslant 0} (-1)^j q^{j(j+2m+1)/2} \tag{1}$$

Then

$$f(x) = e^{x^2/4} \sum_m \Big\{ \mathcal{E}_m(\tau) e^{m\tau x} \sum_k \gamma_{m,k} e^{ikx} \Big\}$$

The interior sum is an $L^2$-sum of a Fourier series, the exterior sum is a.s. convergent series.

**2. Preliminaries on $\theta$-functions.** Let $0 < q < 1$. Denote

$$R(z;q) := (1-z)\prod_{n=1}^{\infty}(1-q^n)(1-zq^n)(1-z^{-1}q^n) = \sum_{-\infty}^{\infty} (-1)^n z^n q^{n(n-1)/2}$$

(this is the Jacobi triple identity, see, for instance, [1]). Obviously,

$$R(qz;q) = -z^{-1} R(z;q)$$

Iterating this identity, we obtain

$$R(q^n z;q) = (-z)^{-n} q^{-n(n-1)/2} R(z;q) \tag{2}$$

The function

$$\eta(z) = \exp\Big\{ -\frac{1}{2\ln q} \ln^2|z| + \frac{1}{2}\ln q \ln|z| \Big\} \tag{3}$$

satisfies the requrence equation $\eta(qz) = |z|^{-1}\eta(z)$. Hence $|R(z;q)|$ can be represented in the form

$$|R(z;q)| = \eta(z)\psi(z); \qquad \text{where } \psi(qz) = \psi(z) \tag{4}$$

Obviously

$$R'(1;q) = \frac{d}{dx} R(x;q)\Big|_{x=1} = -\prod (1-q^n)^3$$

Differentiating (2) and subsituting $z = 1$, we obtain

$$R'(q^n;q) = (-1)^n q^{-n(n-1)/2} R'(1;q) \tag{5}$$

**3. Interpolation problem.** Denote $g(x) = f(x)e^{-x^2/4}$. By the Poisson summation formula

$$e^{m\tau x} \sum_{k=-\infty}^{\infty} \gamma_{m,k} e^{ikx} = \sum_{j=-\infty}^{\infty} g(x + 2\pi j) e^{-2\pi\tau m j}$$

Denote the right-hand side of this identity by $A_m$ Consider the function

$$G_x(z) := \sum_{j=-\infty}^{\infty} g(x + 2\pi j) z^j$$

defined in the domain $\mathbb{C} \setminus 0$,

$$G_x(q^m) = A_m$$

We obtain an interpolation problem for holomorphic functions, and solve it in a standard way (see [4]).

Denote

$$\widetilde{G}_x(z) = \sum_{n=-\infty}^{\infty} A_n \frac{R(z;q)}{(z-q^n)R'(q^n;q)} = \sum_{n=-\infty}^{\infty} A_n \frac{(-1)^{n+1} q^{n(n-1)/2}}{\prod(1-q^j)^3} \frac{R(z;q)}{(z-q^n)} \tag{6}$$

Obviously,

$$G_x(q^n) = \widetilde{G}_x(q^n) \tag{7}$$

Hence,

$$G_x(z) = \widetilde{G}_x(z) + R(z;q)\alpha(z) \tag{8}$$

for some function $\alpha(z)$ holomorphic in $\mathbb{C} \setminus 0$.

LEMMA. $G_x(z) = \widetilde{G}_x(z)$, i.e., $\alpha(z) = 0$.

Our final formula is a corollary of this lemma. Indeed, $g(x)$ is the Laurent coefficient of $G_x(z)$ in $z^0$; it remains to evaluate the Laurent expansion of

$$(z-q^n)^{-1} \sum_{l=-\infty}^{\infty} (-1)^l z^l q^{l(l-1)/2}$$

Assuming $|z| > q^n$, we obtain

$$(z^{-1} + z^{-2}q^n + z^{-3}q^{2n} + \dots) \cdot \sum_{l=-\infty}^{\infty} (-1)^l z^l q^{l(l-1)/2}$$

and we obtain (1) as a coefficient in the front of $z^0$.

**4. Proof of Lemma.** We represent the identity (8) in the form

$$G_x(z)/R(z;q) = \widetilde{G}_x(z)/R(z;q) + \alpha(z) \tag{9}$$

For a function $\Phi(z)$ we denote

$$\mathcal{M}_k[\Phi] := \max_{|z|=q^{k+1/2}} |\Phi(z)|$$

We intend to analize the behavior of these maxima for summands of (9) as $k \to \pm\infty$.

A) First,

$$\infty > \int_{\mathbb{R}} |f(x)|^2 dx = \int_0^{2\pi} \Big( \sum_{j=-\infty}^{\infty} |f(x+2\pi j)|^2 \Big)\, dx$$

Hence (by the Fubbini theorem) the value

$$V_x := \sum_{j=-\infty}^{\infty} |f(x+2\pi j)|^2$$

is finite for almost all $x$.

B) By the Schwartz inequality,

$$\begin{aligned} |G_x(z)| &= \Big| \sum f(x+2\pi j) e^{-(x+2\pi j)^2/4} z^j \Big| \leqslant \\ &\leqslant \Big( \sum |f(x+2\pi j)|^2 \Big)^{1/2} \Big( \sum e^{-(x+2\pi j)^2/2} |z|^{2j} \Big)^{1/2} = \\ &= V_x^{1/2} \cdot \Big[ e^{-x^2} R(-|z|^2 e^{-2\pi x - 2\pi^2}; e^{-4\pi^2}) \Big]^{1/2} \end{aligned}$$

Applying (3)-(4), we obtain for $|G_x(z)|$ an upper estimate of the form

$$|G_x(z)| \leqslant \exp\Big\{ \frac{1}{4\pi^2} \ln^2 |z| + O(\ln|z|) + O(1) \Big\} \tag{10}$$

In particular,

$$|A_m| = |G_x(q^m)| \leqslant \exp\Big\{ \frac{1}{4\pi^2} \ln^2 q \; m^2 + O(m) + O(1) \Big\}$$

By (5),

$$R'(q^m;q) = \exp\big\{ -m^2 \ln q/2 + O(m) + O(1) \big\}$$

Since $(-\ln q) = 2\pi\tau < 2\pi$, we obtain the following estimate

$$|A_m / R'(q^m;q)| \leqslant \exp\{-\varepsilon m^2\}$$

C) Consider the function (it is one of summands in (9)

$$\mathcal{M}_k\big[\widetilde{G}_x(z)/R(z;q)\big] \;=\; \mathcal{M}_k\Big[\sum_m \frac{A_m}{R'(q^m;q)}\cdot\frac{1}{z-q^m}\Big] \;\leqslant\; \sum \frac{e^{-\varepsilon m^2}}{|q^{k+1/2}-q^m|}$$

Next,

$$|q^{k+1/2}-q^m| = q^m|1-q^{-m+k+1/2}| \geqslant q^m(1-q^{1/2})$$

This implies the boundedness of the sequence $\mathcal{M}_k[\cdot]$.

Secondly,

$$|q^{k+1/2}-q^m| \geqslant q^{k+1}(1-q^{1/2})$$

Hence, $\mathcal{M}_k[\cdot]$ tends to 0 as $k\to-\infty$.

D) By (4)

$$\mathcal{M}_k\big[R(z)^{-1}\big] \sim \eta(z)^{-1}\Big|_{|z|=p^{k+1/2}}$$

By (3), (10)

$$\mathcal{M}_k\big[G_x(z)/R(z;q)\big] \to 0 \qquad \text{as } k\to\pm\infty$$

E) We have

$$\mathcal{M}_k\big[\alpha(z)\big] \leqslant \mathcal{M}_k\big[G_x(z)/R(z;q)\big] + \mathcal{M}_k\big[\widetilde{G}_x(z)/R(z;q)\big]$$

Thus $\mathcal{M}_k[\alpha(z)]$ tends to 0 as $k\to-\infty$; and remains bounded as $k\to+\infty$. Since $\alpha(z)$ is holomorphic in $\mathbb{C}\setminus 0$, we have $\alpha(z)=0$.

**Acknowledgements.** I am grateful to Yu. Lyubarskii who explained me various tricks related to entire functions. I also thank H.Feichtinger, A.M.Vershik, and K.Grochenig for discussion of the subject.

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