# Twisted WZW Branes from Twisted REA's

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## Abstract

Quantum geometry of twisted Wess–Zumino–Witten branes is formulated in the framework of twisted Reflection Equation Algebras. It is demonstrated how the representation theory of these algebras leads to the correct classification of branes. A semiclassical formula for quantised brane positions is derived and shown to be consistent with earlier string-theoretic analyses.

*Key words:* (twisted) D-branes, WZW models, quantum groups, (twisted) reflection equation algebras
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## 1   Introduction

Branes on group manifolds and quotients thereof have long been at the focus of research efforts aimed at understanding the deformation of classical geometry and gauge dynamics effected by string propagation in background fluxes [2].

While the branes naturally lead to the concept of a curved non-commutative space [2,3], they are still amenable to direct investigation using diverse methods such as the Lagrangian formalism of the associated WZW models [4,5] confining the branes to (twisted) conjugacy classes, effective field theory formulated in terms of the Dirac–Born–Infeld functional [6] proving their stability, matrix models [2,3] providing a semi-classical picture of the geometry and gauge dynamics, renormalisation group techniques [2,3,7,8] capturing brane condensation phenomena, K-theory [7,9,10] classifying their charges and Boundary Conformal Field Theory (BCFT) offering access to their microscopic structure via the boundary state construction.

In the latter approach, (twisted) branes are identified with states in the Hilbert space of the bulk (or closed string) theory implementing (twisted) gluing conditions for chiral currents of the bulk CFT ($a$ is an index of the adjoint representation of the horizontal Lie algebra $\mathfrak{g} \equiv LieG$ of the Kac–Moody algebra $\widehat{\mathfrak{g}}_\kappa$, $n$ enumerates Laurent modes and $B$ is a boundary state label):

$$\left( J_n^a \otimes \mathbb{I} + \omega(\mathbb{I} \otimes J_{-n}^a) \right) |B \gg^\omega = 0, \tag{1}$$

where $\omega$ is an outer automorphism of the current algebra $\widehat{\mathfrak{g}}_\kappa$ (see, e.g., [11,12]). Thus branes break the full chiral symmetry algebra $\widehat{\mathfrak{g}}_\kappa^L \times \widehat{\mathfrak{g}}_\kappa^R$ of the bulk WZW to the subalgebra spanned by annihilators of $|B \gg^\omega$, isomorphic to $\widehat{\mathfrak{g}}_\kappa$.

Non-commutative geometry entered the stage thus set in [2] where a matrix model of "fuzzy" physics of untwisted branes was explicitly derived in the large volume (or, equivalently, large level $\kappa$) limit. The twisted case was then examined at great length in [3], along similar lines. The semi-classical approach of [2] was later extended in [13] where an Ansatz for brane geometry and gauge dynamics at arbitrary level was advanced, based on the fundamental concept of quantum group symmetry, as suggested by the underlying (B)CFT, and the well-known correspondence between untwisted affine Lie algebras $\widehat{\mathfrak{g}}_\kappa$ and Drinfel'd–Jimbo quantum algebras $\mathcal{U}_q(\mathfrak{g})$ (see, e.g., [14]). The latter proposal was shown to successfully encode essential (untwisted) brane data such as tensions, localisations, the algebra of functions, internal gauge excitations and interbrane open string modes. It was also generalised in [15] to a class of orbifold backgrounds, known as simple current orbifolds $SU(N)/\mathbb{Z}_N$, whereby the basic structure of the associated matrix model, a so-called Reflection Equation Algebra[3] (REA) $REA_q(A_N)$, was examined extensively. The study revealed an attractive geometric picture behind the compact algebraic framework of the REA's, which was next exploited in an explicit construction of some new quantum geometries corresponding to (fractional) orbifold branes.

One particular aspect of the non-classical WZW geometry is quantisation of brane locations within $G$. It can be derived rather straightforwardly from

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[3] Cp [16–18], see also: the Appendix.

the relative-cohomological constraints on the background fluxes of the relevant Lagrangian boundary WZW model ascertaining well-definedness of the associated path integral [4]. We shall explicitly refer to some results of the cohomological analysis in what follows.

In this paper, we discuss an algebraic framework relevant to the analysis at arbitrary level of twisted branes on $SU(2n + 1)$ group manifolds. Accordingly, we specialise our exposition to the case $\widehat{\mathfrak{g}}_\kappa = A_{2n}^{(1)}$ (in which $\omega_c$ is the standard $\mathbb{Z}_2$-reflection of the Dynkin diagram). The exposition is centred on the CFT-inspired notion of twisted quantum group symmetry, as represented by so-called twisted Reflection Equation Algebras (tREA) $\text{tREA}_q(A_{2n})$. The latter are directly related to quantum algebras $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$, with a known representation theory [19–25]. The $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ are (coideal) subalgebras [4] of $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ - a quantum-algebraic counterpart of the classical subalgebra structure: $\mathfrak{so}_{2n+1} \hookrightarrow \mathfrak{su}_{2n+1}$ [28,29]. Using these facts, we provide evidence of an intricate relationship between twisted boundary states [12,30,31] and the representation theory of the $\omega_c$-invariant subalgebra $\mathfrak{so}_{2n+1} \cong (\mathfrak{su}_{2n+1})^{\omega_c}$, and subsequently reconcile our result with the structure of the representation theory of $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ at $q$ a root of unity [5], embedded in that of $\mathcal{U}_q(\mathfrak{su}_{2n+1})$. We also rederive the quantisation rule for twisted brane positions within the WZW group manifold of $SU(3)$ (originally obtained from cohomological analysis in [32]), whereby we establish - in direct analogy with the untwisted case - a simple geometric meaning of the Casimir operators of $\text{tREA}_q(A_{2n})$.

Let us now give an outline of the present paper. Section 2. is a warm-up presentation of the classical geometry of the twisted branes. Section 3. discusses chosen features of the twisted Reflection Equation Algebras. Section 4. contains the main results of this work: classification of the twisted branes through the representation theory of the tREA's and a semiclassical derivation of the quantisation rule for brane positions in the $SU(2n + 1)$ group manifold. In the appendices attached, we list further properties of Reflection Equations and the $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ algebras.

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[4] Structures of this kind have long been known to arise naturally in the related context of $(1 + 1)$-dimensional integrable models on a half-line, with involutively twisted gluing condition for chiral symmetry currents at the boundary, cp [26], see also [27].

[5] As dictated by the CFT.

## 2  Classical geometry of twisted WZW branes

At the classical level, stable branes of the WZW model in the Lie group target $G$ are described by (twisted) conjugacy classes of the form:

$$\mathcal{C}^\omega(t) = \left\{ ht\omega(h^{-1}) \mid h \in G \right\}, \tag{2}$$

with $t$ in the "symmetric" subgroup $T^\omega$ of the maximal torus $T \subset G$, i.e. $t \in T$ with $\omega(t) = t$, whence - in particular - the conjugacy classes are invariant under $\omega$. When $G = SU(2n+1)$ and $\omega = \omega_c$ (the case of interest) we may choose complex conjugation $\rho$ as a group-integrated representative of $\omega$, whereby the above reduces to ($T$ denotes transposition)

$$\mathcal{C}^\rho(t) = \left\{ hth^T \mid h \in SU(2n+1) \right\}. \tag{3}$$

Let $K_t = \{h \in G : hth^T = t\}$ be the stabiliser subgroup (in the twisted adjoint representation) of $t \in T^\omega$. For $t = \mathbb{I}$, the stabiliser $K_t$ coincides with the group $SO(2n+1)$. In the algebraic setup to be developed, we shall encounter a quantum deformation of this group (see Sec.3.). Clearly, $\mathcal{C}^\omega(t)$ can be viewed as a homogeneous space [6]:

$$\mathcal{C}^\omega(t) \cong G/K_t. \tag{4}$$

The twisted conjugacy classes are invariant under the twisted adjoint action of the vector subgroup $G \cong G_V \hookrightarrow G_L \times G_R$ of the group of symmetries of the target manifold,

$$G\mathcal{C}^\omega(t)\omega(G^{-1}) = \mathcal{C}^\omega(t). \tag{5}$$

This is a classical counterpart of the symmetry breaking pattern: $\widehat{\mathfrak{g}}_\kappa^L \times \widehat{\mathfrak{g}}_\kappa^R \to \widehat{\mathfrak{g}}_\kappa$ mentioned under (1). In this context, the distinguished character of the $\omega$-invariant subgroup derives from the fact that a given twisted conjugacy class contains full regular conjugacy classes [32] of all its elements relative the adjoint action of the subgroup $G^\omega \subset G$,

$$g \in \mathcal{C}^\omega(t) \implies G^\omega g(G^\omega)^{-1} \equiv G^\omega g\omega((G^\omega)^{-1}) \subset \mathcal{C}^\omega(t). \tag{6}$$

The remaining part of the original bulk symmetry, $G_L \times G_R$, translates - just as in the untwisted case - into covariance of the ensuing physical model under rigid one-sided translations of twisted conjugacy classes within $G$,

$$G_L\mathcal{C}^\omega(t)G_R = G_L\mathcal{C}^\omega(t)\omega(G_L^{-1})\omega(G_L)G_R = \mathcal{C}^\omega(t)G. \tag{7}$$

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[6] In this picture, the map: $G/K_t \to \mathcal{C}^\omega(t)$, $hK_t \mapsto ht\omega(h^{-1})$ is manifestly well-defined and bijective. Note that the left hand side is a one-sided (right) coset of $G$.

This reflects the residual freedom in the definition of the boundary state consisting in the choice of the inner automorphism twisting the gluing condition [11].

Upon specialising the above presentation to the case of $SU(3)$ for the sake of illustration and preparation for Sec.4.2, we obtain a classification of twisted branes in terms of twisted conjugacy classes in $SU(3)$. For the specific choice of the group-integrated representation of $\omega$ given by complex conjugation $\rho$, we can parametrise the latter as

$$
\mathcal{C}^{\rho}(\theta) = \left\{ h \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} h^{T} \;\middle|\; h \in SU(3) \;\wedge\; \theta \in \left[0, \frac{\pi}{2}\right] \right\}, \qquad (8)
$$

from which it transpires that there are two species of twisted branes in this background: a 5-dimensional twisted conjugacy class of the group unit, with a maximal stabiliser, $\mathcal{C}^{\rho}(0) \cong SU(3)/SO(3)$, and generic 7-dimensional twisted conjugacy classes which can be regarded as homogeneous spaces $SU(3)/SO(2)$. We shall make explicit use of the parametrisation (8) in the sequel.

# 3 Twisted Reflection Equations

In this section, we shall discuss (quantum) algebras relevant to the description of twisted branes. The arguments we invoke are of the kind presented in [13], i.e. they are based on the pattern of symmetry breaking induced by twisted branes (cp the discussion of the previous section).

Thus we propose to consider a twisted Reflection Equation (tRE):

$$
\text{tRE}^{-} \quad : \quad \mathbf{R}_{12}\mathbf{K}^{-}{}_{1}\mathbf{R}_{12}^{t_1}\mathbf{K}^{-}{}_{2} = \mathbf{K}^{-}{}_{2}\mathbf{R}_{12}^{t_1}\mathbf{K}^{-}{}_{1}\mathbf{R}_{12}, \qquad (9)
$$

in which $\mathbf{R}$ is a bi-fundamental realisation of the standard universal $\mathcal{R}$-matrix of the relevant quantum group $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ and $\mathbf{K}^{-}$ are operator-valued matrices of generators of the twisted Reflection Equation Algebra $\text{tREA}_q(A_{2n})$ (see: the Appendix).

Equations of this kind (parametrised by additional physical quantities) have long been known to describe couplings of bulk modes to the boundary in $(1+1)$-dimensional integrable models on a half-line, with involutively twisted gluing condition for chiral symmetry currents at the boundary (see [26,27], and the references within). Furthermore, the respective algebraic structures ensuing from (9) and its dynamical counterpart from the papers cited share

many essential features (coideal property, an intimate relation to the so-called symmetric pairs).

The twisted left-right (co)symmetries [13] of the tRE: $\mathbf{K}^- \mapsto \mathbf{t}^T \mathbf{K}^- \mathbf{s}$, realised in terms of $(\mathbf{t}, \mathbf{s}) \in \mathcal{G}_L \otimes_{\mathcal{R}} \mathcal{G}_R \equiv SU_q(2n + 1) \otimes_{\mathcal{R}} SU_q(2n + 1)$ (we have $q = e^{\pi i/(\kappa + 2n + 1)}$, as indicated by the underlying CFT), provide a quantum version of the classical left-right isometry of the group manifold, which should be a symmetry of the problem (to be broken by branes). There is another tRE with the same symmetry properties,

$$\text{tRE}^+ \quad : \quad \mathbf{R}_{21}\mathbf{K}^+{}_1 \mathbf{R}_{21}^{t_1} \mathbf{K}^+{}_2 = \mathbf{K}^+{}_2 \mathbf{R}_{21}^{t_1} \mathbf{K}^+{}_1 \mathbf{R}_{21}. \tag{10}$$

The transformation rule for $\mathbf{K}^+$ reads $\mathbf{K}^+ \mapsto (S\mathbf{t})\mathbf{K}^+(S\mathbf{s})^T$ ($S$ is the antipode of the Hopf algebra $SU_q(2n+1)$). As we shall discuss in App.A.2 and following [29], the two tRE's define the same quantum algebra $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ [20], a quantum deformation of $\mathfrak{so}_{2n+1}$. tRE$^\pm$ differ in the manner the algebra $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ is embedded in them. In view of the prominent rôle played by $SO(2n + 1)$ in the description of twisted $A_{2n}$ branes (see Sec.2.), the appearance of the latter algebra should be regarded as an encouraging fact.

As it turns out [28], we need both $\mathbf{K}^+$ *and* $\mathbf{K}^-$ to construct Casimir operators for this algebra[7] . They shall play an important part in our discussion of brane geometries (see Sec.4.2.). The Casimir operators can be cast in the form:

$$c_m := \mathrm{tr}\left(\mathbf{X}\left(\mathbf{D}\mathbf{X}\right)^{m-1}\right), \qquad m \in \overline{1, 2n - 1}, \tag{11}$$

where $\mathbf{X} := \mathbf{K}^-\mathbf{K}^+$ and $\mathbf{D} := \mathrm{diag}(q^{-2 \cdot 2n}, q^{-2 \cdot (2n-1)}, \ldots, 1)$, the latter being straightforwardly related to the antipode $S$ through

$$\mathbf{D}^{-1}\mathbf{s}\mathbf{D} = S^2\mathbf{s}. \tag{12}$$

In the spirit of the papers [13,15], we would like to identify branes with appropriately chosen irreducible representations of the tREA defined above. Further evidence in favour of such an assignment as well as the details of the identification shall be provided in Sect.4. For the present, though, we focus on a particular consequence of this idea: clearly, it should entail the existence of an algebraic counterpart of (5). And indeed, the vector part of the $\mathcal{G}_L \otimes_{\mathcal{R}} \mathcal{G}_R$ symmetry, realised as

$$\mathbf{K}^- \mapsto \mathbf{s}^T\mathbf{K}^-\mathbf{s} \quad , \quad \mathbf{K}^+ \mapsto (S\mathbf{s})\mathbf{K}^+(S\mathbf{s})^T \tag{13}$$

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[7] In the case at hand, i.e. for the deformation parameter $q$ a root of unity, there are - as usual - additional central elements in the algebra, originally discovered in [22]. They shall not be considered in this paper. In particular, for $A_2$ with our subsequent choice of the representation theory, they are known to carry no interesting information [24].

possesses the required properties. In addition to preserving the respective tRE's, it also leaves the values of all $c_m$'s unchanged. This follows from the fact that under the above transformations $\mathbf{X} \mapsto \mathbf{s}^T \mathbf{X}(S\mathbf{s})^T$, $\mathbf{XDX} \to \mathbf{s}^T \mathbf{X}(S\mathbf{s})^T \mathbf{D} \mathbf{s}^T \mathbf{X}(S\mathbf{s})^T$ etc. Upon applying (12), we then obtain $\mathbf{D}^{-1}(S\mathbf{s})^T \mathbf{D} \mathbf{s}^T = \mathbb{I}$ and so we readily verify $c_m \mapsto \mathrm{tr}\left(\mathbf{s}^T \mathbf{X}(\mathbf{DX})^{m-1}(S\mathbf{s})^T\right)$. That leads us directly to the conclusion.

Next, we turn to the representation theory of (9)-(10). Recall that $\mathrm{tREA}_q(A_{2n})$ is related to a particular deformation of $\mathfrak{so}_{2n+1}$ denoted by $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$. The representation theory of $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ is known in considerable detail (see, e.g., [20,23].). Here, we are interested only in the highest weight irreducible representations. For $q = e^{\pi i/(\kappa+2n+1)}$, these are of the classical type, with the corresponding highest weights truncated to a fundamental domain in a $(\kappa+2n+1)$-dependent way outlined below. We adopt labelling by signatures[8] : $\vec{m} = (m_1, m_2, \ldots, m_n) =: \sum_{i=1}^n m_i \vec{e}_i$ such that all $m_i$'s are integers or all are half-integers, subject to the dominance condition:

$$m_1 \geq m_2 \geq \ldots \geq m_n \geq 0. \tag{15}$$

The truncation scheme has not been worked out in all generality as of this writing. It is known [23] in the simplest case of $\mathcal{U}'_q(\mathfrak{so}_3)$,

$$2m_1 \leq \kappa + 2, \tag{16}$$

and inspection of the algebra $\mathcal{U}'_q(\mathfrak{so}_5)$ and its representations (cp [20]) reveals that the candidate formula is[9] $m_1 + m_2 \leq \kappa + 5$. Thus, it seems plausible that in the general case of irreducible representations of $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ highest weights are truncated as:

$$m_1 + m_n \leq \kappa + 2n + 1. \tag{17}$$

We shall return to this issue in the next section.

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[8] The signatures can readily be expressed in terms of the Dynkin labels of the corresponding weights:

$$2m_i = 2\sum_{j=i}^{n-1} \lambda_i + \lambda_n, \quad (i < n), \qquad 2m_n = \lambda_n. \tag{14}$$

[9] At the threshold, matrix elements of the generators of $\mathcal{U}'_q(\mathfrak{so}_5)$ develop poles. Analogous pathology occurs for $\mathcal{U}'_q(\mathfrak{so}_3)$ and extends to Clebsch–Gordan coefficients, as well as the associated $6j$-symbols.

# 4    Geometry of twisted branes from the tREA

In the present section, we unravel a number of features of the tREA's introduced, indicating towards an intimate relationship between the latter and twisted branes of the WZW models of type $A_{2n}$.

## 4.1    Algebraic truncation of twisted brane labels

Below, we address the issue of microscopic localisation of twisted branes from two vantage points: the BCFT one, based on the notion of a (twisted) boundary state, and that of a suitably truncated representation theory of the $\omega_c$-invariant subalgebra $\mathfrak{so}_{2n+1}$ which we consistently embed in the representation theory of $\text{tREA}_q(A_{2n})$. The identifications made shall then be tested in a semi-classical approximation in Sect.4.2.

Let us start by recalling that the non-classical geometry of a maximally symmetric WZW brane has been successfully encoded in the representation theory of $REA_q(\mathfrak{g})$ [13,15]. A crucial rôle in this approach has been played by the map $REA_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ given by $\mathbf{M} = \mathbf{L}^+ \mathbf{M}_0 S\mathbf{L}^-$, in which $\mathbf{L}^\pm$ are the familiar FRT operators of [18] (see: the Appendix). The map provides us with tools necessary to show that there is a one-to-one correspondence between highest weight irreducible representations of $REA_q(\mathfrak{g})$ and (untwisted) branes. Moreover, it gives geometrical information about branes in terms of Casimir operators.

For the tRE, there is a similar embedding of $\text{tREA}_q(A_{2n}) \cong \mathcal{U}'_q(\mathfrak{so}_{2n+1})$ in $\mathcal{U}_q(\mathfrak{su}_{2n+1})$,

$$
\mathbf{K}^- = \left(\mathbf{L}^+\right)^T \mathbf{C}^- \mathbf{L}^- \quad , \quad \mathbf{K}^+ = S\mathbf{L}^+ \mathbf{C}^+ \left(S\mathbf{L}^-\right)^T , \tag{18}
$$

with $\mathbf{C}$ - a constant ($c$-number-valued) matrix solution of tRE. In what follows, we take $\mathbf{C} := \text{diag}(c_1, c_2, \ldots, c_{2n+1})$ such that $\lim_{q \to 1} c_i = 1$, $i \in \overline{1, 2n+1}$. This choice guarantees that in the classical limit, $q \to 1$, (18) defines the embedding:

$$
\mathcal{U}(\mathfrak{so}_{2n+1}) \ni I_{i+1,i} \longmapsto F_i - E_i \in \mathcal{U}(\mathfrak{su}_{2n+1}), \tag{19}
$$

in which $I_{i+1,i}$ denote generators of $\mathcal{U}(\mathfrak{so}_{2n+1})$. The map (18) determines a branching of representations $R_\lambda$ of $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ ($\mathbf{L}^+, \mathbf{L}^- \in \mathcal{U}_q(\mathfrak{su}_{2n+1})$) into those of $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$,

$$
R_\lambda \longrightarrow \bigoplus_{\vec{m}} \tilde{b}_\lambda^{\vec{m}} R_{\vec{m}}. \tag{20}
$$

Analogously, the map (19) determines the classical counterpart of (20). Motivated by the analysis of the untwisted case, as well as by the considerations

of [3] and [32] we propose the following identification:

**Twisted branes correspond to those highest weight irreducible representations of $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ which show up on the right hand side of (20), with the branching coefficient $\tilde{b}^{\vec{m}}_\lambda$ determining the intersection of the untwisted brane described by $R_\lambda$ with the twisted one associated to $R_{\vec{m}}$.**

The rule has to be supplemented by a truncation of $\vec{m}$'s (denoted by a tilde in (20)), *stricter* than the one on the highest weight irreducible representations of $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$. The truncation is imposed on $R_{\vec{m}}$ as detailed below. Apart from the truncation, the branching follows the purely classical ($q = 1$) pattern. It appears that for $\kappa \in 2\mathbb{N}^*$ one can find a relatively easy algebraic prescription for the truncation [10] by demanding not only that the number of surviving irreducible representations agree with the number of admissible boundary states from the lattice of dominant fractional symmetric affine weights of $A_{2n}$ (cp [12]), but also that the ensuing distribution of $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$-representations over $P^\kappa_+(A_{2n})$ possess the $\mathbb{Z}_{2n+1}$ simple current symmetry of twisted conjugacy classes. It reads

$$2m_1 \leq \kappa \tag{21}$$

and is to be iteratively imposed on the representation theory of $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ which comes with a tensor product structure elucidated in [25].

Here is a description of the procedure leading to (20). As the input we use the known [25] fact: $\tilde{b}^{\vec{m}}_{\Lambda_1} = \delta^{\vec{e}_1}_{\vec{m}}$ ($R_{\Lambda_1}$ and $R_{\vec{e}_1}$ are the fundamental representations of $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ and $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$, respectively). The procedure is itarative. Let $R_\lambda = \bigoplus_{\vec{m}} b^{\vec{m}}_\lambda R_{\vec{m}}$ be known (we start with $R_{\Lambda_1}$). In a single step, we tensor $R_\lambda$ with $R_{\Lambda_1}$. On the $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ side, this yields $R_\lambda \otimes R_{\Lambda_1} = \bigoplus_{\mu \in P^\kappa_+(A_{2n})} \mathcal{N}_{\lambda, \Lambda_1}{}^\mu R_\mu$ ($\mathcal{N}_{\lambda, \Lambda_1}{}^\mu$ are multiplicities). On the $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ side, we get $\bigoplus_{\vec{m}} \tilde{b}^{\vec{m}}_\mu R_{\vec{m}} \otimes R_{\vec{e}_1}$. Luckily [25], tensor products of the kind $R_{\vec{m}} \otimes R_{\vec{e}_1}$ are well-defined [11] and can be decomposed into irreducible components. We may then derive the branching coefficients $\tilde{b}^{\vec{m}}_\mu$ for the irreducible simple summands $R_\mu$ upon imposing the truncation (21). Clearly, we can reconstruct the entire representation theory of $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ over $P^\kappa_+(A_{2n})$ in this way, hence we retrieve all the desired intersections.

Several comments are well due at this point. First of all, our usage of the quantum algebras should not obscure the fact that the truncation could just as well be imposed in the classical setup (i.e. for $\mathfrak{so}_{2n+1}$). The good news is that it can be reconciled with the specific structure of the representation theory of $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ for $q$ a root of unity. Indeed, in consequence of (15), the

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[10] The significance of the parity of $\kappa$ was emphasised already in [32].

[11] Due to the fact that $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ is a coideal (non-Hopf) subalgebra of $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ tensoring is problematic in general.

present truncation $2m_1 \leq \kappa$ implies $m_1 + m_n \leq 2m_1 \leq \kappa < \kappa + 2n + 1$ and hence it is more restrictive than (16). Finally, the representations admitted by (20) correspond to those representations of the algebra $\mathfrak{so}_{2n+1}$ which can be integrated to representations of the group $SO(2n + 1)$ [3]. The latter fact shall be of prime relevance to the discussion of the next section.

Let us also note another, rather astonishingly exact correspondence between (20) and BCFT. Namely, we can calculate[12] scalar products of a twisted boundary state $|\dot{\mu} \gg_C^{\omega_c}$ with all admissible untwisted boundary states $|\lambda \gg_C$, whereby we obtain

$$
{}_C^{\omega_c} \ll \dot{\mu} || \lambda \gg_C = (n_\lambda^{\omega_c})_{\Psi(0)}^{\dot{\mu}} \, , \qquad \Psi(0) := \frac{1}{2} E\left(\frac{\kappa}{2}\right) (\Lambda_n + \Lambda_{n+1}) \, . \tag{22}
$$

Here, $n_\lambda^{\omega_c}$ are the so-called twisted fusion rules of the CFT [31] and $E(x)$ denotes the integral part of $x$. It appears that for even $\kappa$ the branching coefficients of (20) coincide with the twisted fusion rules as

$$
\tilde{b}_\lambda^{\vec{m}} = (n_\lambda^{\omega_c})_{\Psi(0)}^{\Psi(\vec{m})} \, , \tag{23}
$$

with the identification between the truncated representation theory of $\mathrm{tREA}_q(A_{2n})$ and the set of twisted boundary labels given by the mapping:

$$
\Psi \, : \, \vec{m} \longmapsto \Psi(\vec{m}) := \frac{1}{2} \sum_{i=1}^{n-1} (m_{n-i} - m_{n-i+1})(\Lambda_i + \Lambda_{2n+1-i}) + \frac{\kappa - 2m_1}{4}(\Lambda_n + \Lambda_{n+1}),
$$
$$
\tag{24}
$$

originally proposed in [30] and further discussed in [3]. Thus (24) completes our translation of the BCFT data into the quantum-algebraic language of the tREA. Note that it actually associates (through (20) and (22)) the trivial representation, $R_{\vec{0}}$, with the dimensionally reduced twisted brane (the one wrapping the twisted conjugacy class of the group unit) as the unique one having a non-vanishing overlap with (i.e. containing) the pointlike untwisted branes localised at the $2n + 1$ points in $SU(2n + 1)$ corresponding to the elements of the centre $Z(SU(2n + 1)) \cong \mathbb{Z}_{2n+1}$. We shall come back to this point in the next section.

## 4.2  Brane localisation from Casimir eigenvalues

We are not aware of any natural embedding $\mathrm{tREA}_q(A_{2n}) \hookrightarrow \mathrm{REA}_q(A_{2n})$. Recall that - following [13] - we assign to the latter algebra the rôle of the quantised algebra of functions on the group manifold. Thus, the lack of such a map prevents us from giving a direct geometrical meaning to various quantities

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[12] Details of the relevant BCFT computation leading to (22) shall be presented in an upcoming paper.

associated with tREA's, e.g. to their Casimir operators. Luckily, the situation is not hopeless. We may employ (18) and the map $\mathrm{REA}_q(A_{2n}) \to \mathcal{U}_q(\mathfrak{su}_{2n+1})$, (A.8), to construct a map $\mathrm{tREA}_q(A_{2n}) \to \mathrm{REA}_q(A_{2n})$ order by order in the parameter $1/\kappa$, in a manner consistent with the $q \to 1$ limiting procedure described in [13]. Using the above expansion we shall express the quadratic Casimir operator $c_1$ of $\mathrm{tREA}_q(A_{2n})$ in terms of the **M**-variables, that is in terms of solutions to the (untwisted) RE (cp [13,15]). All approximate equalities below are up to terms of higher order in the expansion parameter. We also choose $\mathbf{C} := \mathbb{I}$.

First, note that $K_{ii}^{\pm} \approx \mathbb{I}$ for all $i \in \overline{1, 2n+1}$. Hence $c_1 \approx \sum_i \mathbb{I} + \sum_{i>j} K_{ij}^- K_{ji}^+$. Upon subtracting the trivial part, we then define

$$\tilde{c}_1 := \sum_{i>j=1}^{2n+1} K_{ij}^- K_{ji}^+. \tag{25}$$

We also have $K_{ij}^- \approx \sum_{j \leq k \leq i} L_{ki}^+ L_{kj}^-$ and $K_{ji}^+ \approx \sum_{j \leq k \leq i} SL_{ik}^+ SL_{jk}^-$. Using the results from App.D of [15] we list the relevant (leading) terms of the **L$^{\pm}$**-operators:

$$L_{ij}^+ \approx \lambda E_{ji} \qquad , \qquad SL_{ij}^+ \approx -\lambda E_{ji}, \quad i < j$$

$$L_{ij}^- \approx -\lambda E_{ji} \qquad , \qquad SL_{ij}^- \approx \lambda E_{ji}, \qquad j < i$$

$$L_{ii}^{\pm} \approx \mathbb{I} \qquad , \qquad SL_{ii}^{\pm} \approx \mathbb{I},$$

with $E_{ij}$ defined as in [15] (their explicit form is not relevant here). The above yield

$$K_{ij}^- \approx \lambda(E_{ij} - E_{ji}) \approx -K_{ji}^+, \qquad j < i \tag{26}$$

and - since $M_{ij} \approx \lambda E_{ji}$ for $i \neq j$ - we conclude that

$$K_{ij}^{\mp} \approx M_{ij} - M_{ji}. \tag{27}$$

Thus

$$\tilde{c}_1 \approx -\sum_{i>j=1}^{2n+1} (M_{ij} - M_{ji})^2 = \frac{1}{2} \mathrm{tr}(\mathbf{M} - \mathbf{M}^T)^2. \tag{28}$$

At this stage, we may already evaluate the Casimir operator on a particular irreducible representation $R_{\vec{m}}$ of $\mathrm{tREA}_q(A_{2n})$. Thus we rewrite the left hand side after [20,21] in terms of components of the signature vector $\vec{m}$ labelling the irreducible representation chosen, whereby we obtain

$$\tilde{c}_1 \Big|_{R_{\vec{m}}} = q^{2n-1} \lambda^2 \sum_{j=1}^{n} [m_{n+1-j}]_q [m_{n+1-j} + 2j - 1]_q. \tag{29}$$

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On the present level of generality, we may draw one encouraging conclusion: the Casimir operator clearly vanishes on the trivial representation of the tREA, $R_{\vec{0}}$, and with our choice of truncation of admissible irreducible representations, (21), it is also the unique [13] representation with this property. Thus for $\vec{m} = \vec{0}$ we obtain: $\mathrm{tr}(\mathbf{M} - \mathbf{M}^T)^2 \propto \tilde{c}_1 = 0$, which is solved by symmetric matrices $\mathbf{M}$. This conforms with the known results for the dimensionally reduced brane [9] to which we consequently associate the zero $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$-signature, consistently with our microscopic analysis. Equivalently, from the (co)isometry (13) of irreducible representations of $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ we conclude that the geometry defined by $R_{\vec{0}}$ is encoded in the twisted $SU_q(2n+1)$-comodule algebra: $\mathbf{C} \mapsto \mathbf{s}^T \mathbf{C} \mathbf{s}$ and therefore it describes the twisted (quantum) conjugacy class of the group unit.

It turns out that we may extract further information from the semiclassical result (28)-(29), whereby we gain some insight into its physical meaning. To these ends we specialise the formulæ to the simplest physically relevant [14] case: $n = 1$. Plugging into (28) the explicit classical parametrisation (8) of twisted conjugacy classes of $G = SU(3)$,

$$M_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{30}$$

and comparing with (29) we get the relation:

$$-8\sin^2\theta = 2\lambda^2[\lambda_1/2]_q[\lambda_1/2 + 1]_q, \tag{31}$$

where - as previously - $\lambda_1 = 2m_1 \in \mathbb{N}$ [20]. We can regard (31) as a quantisation condition for brane positions. For $1 \ll \lambda_1 \ll \kappa$ it yields

$$\theta \approx \frac{\lambda_1 \pi}{2\kappa}. \tag{32}$$

Clearly, the above rule retains its validity for $\lambda_1 = 0$, hence we may expect it to be generally applicable in the large $\kappa$ limit.

The significance of the classical limit (32) of our quantum-algebraic result follows from the fact that it is amenable to direct comparison with the data on twisted brane localisation which can be found in the literature [15]. Thus

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[13] Note that (16) does not guarantee the uniqueness.

[14] The classical $SU(2)$ has no non-trivial diagram automorphisms.

[15] As for exact BCFT data of, e.g., [9] it unavoidably becomes obscured by the conventions adopted in the original papers. They differ from ours in the choice of the representative of the class of automorphisms implementing the Dynkin diagram reflection on the group level.

we compare (32) with the relative-cohomological analysis of [32], using the same group-integrated representative of $\omega_c$ as the one quantised by the tRE's (9)-(10). The analysis yields a quantisation rule:

$$\theta = \frac{(2n - \kappa)\pi}{2\kappa}, \quad n \in \overline{E\left(\frac{\kappa}{2}\right), \kappa}, \tag{33}$$

which falls in perfect agreement with (32) (for even $\kappa$) and, consequently, lends support to our proposal. Indeed, upon restricting in (29) to integer-spin irreducible representations of $\mathcal{U}'_q(\mathfrak{so}_3)$, the two quantisation formulæ become fully equivalent. The latter representations, on the other hand, are precisely the ones that appear in (truncated) branchings of the irreducible representations of $\mathrm{REA}_q(A_2)$ used in [15] in the description of untwisted branes, as determined by (20).

## 5   Summary and conclusions

In the present paper, we have discussed a class of quantum algebras, the twisted Reflection Equation Algebras $\mathrm{tREA}_q(A_{2n})$, in reference to twisted boundary states of WZW models for the groups $SU(2n + 1)$ and the associated brane worldvolumes wrapping (classically) twisted conjugacy classes within the group manifolds. The framework, developed as a straightforward extension of the previous constructions for untwisted WZW branes, based on the untwisted Reflection Equation Algebras $\mathrm{REA}_q(A_{2n})$, is a novel proposal for a compact algebraic description of the twisted branes. Our study provides several arguments in favour of its profound relationship to the CFT of twisted boundary states: classical-type irreducible representations of $\mathrm{tREA}_q(A_{2n})$ enjoy a (co)symmetry that quantises the twisted adjoint symmetry of the boundary states (the starting point of the construction) and in so doing they realise a symmetry breaking scenario analogous to the BCFT one (cp the introductory remarks under (1) and (5)); the eigenvalues of the Casimir operators of $\mathrm{tREA}_q(A_{2n})$ returned by these irreducible representations admit a simple physical interpretation in terms of quantum localisation rules for twisted brane geometries, shown to reproduce the known result for the simplest case of $SU(3)$ in the semiclassical approximation allowing for an explicit embedding $\mathrm{tREA}_q(A_{2n}) \hookrightarrow \mathrm{REA}_q(A_{2n})$; the representation theory of $\mathrm{tREA}_q(A_{2n})$, endowed with a restricted tensor product structure remarked upon under (21), seems to reproduce microscopic twisted brane density distributions within the quantum manifolds of the $SU(2n + 1)$ upon truncating the set of admissible dominant signatures (labels of the irreducible representation of $\mathrm{tREA}_q(A_{2n})$); the truncation is identical with the one suggested in [30] in the BCFT context.

In conclusion, we believe that there are sound reasons to regard the tREA's

as natural building blocks of quantum-algebraic matrix models for twisted branes on the $SU(2n + 1)$ WZW manifolds. While encouraged by the results obtained hitherto, we are aware of numerous questions that our study leaves unanswered, such as the harmonic analysis on the associated geometries, and the gauge dynamics of twisted WZW branes that the algebras are claimed to describe. We intend to return to them in a future publication.

## A    (Twisted) Reflection Equations

In this appendix, we discuss chosen properties of three RE's:

$$\mathrm{tRE}^- \quad : \quad \mathbf{R}_{12}\mathbf{K}^-{}_1\mathbf{R}_{12}^{t_1}\mathbf{K}^-{}_2 = \mathbf{K}^-{}_2\mathbf{R}_{12}^{t_1}\mathbf{K}^-{}_1\mathbf{R}_{12}, \tag{A.1}$$

$$\mathrm{tRE}^+ \quad : \quad \mathbf{R}_{21}\mathbf{K}^+{}_1\mathbf{R}_{21}^{t_1}\mathbf{K}^+{}_2 = \mathbf{K}^+{}_2\mathbf{R}_{21}^{t_1}\mathbf{K}^+{}_1\mathbf{R}_{21}, \tag{A.2}$$

$$\mathrm{RE0} \quad : \quad \mathbf{R}_{12}\mathbf{M}_1\mathbf{R}_{21}\mathbf{M}_2 = \mathbf{M}_2\mathbf{R}_{12}\mathbf{M}_1\mathbf{R}_{21}, \tag{A.3}$$

appearing in the paper. In the formulæ above, $\mathbf{R}$ is a bi-fundamental realisation of the standard universal $\mathcal{R}$-matrix of the relevant quantum group $\mathcal{U}_q(\mathfrak{su}_{2n+1})$, $\mathbf{R} \equiv (R_V \otimes R_V)(\mathcal{R})$, satisfying the celebrated Quantum Yang–Baxter Equation (see, e.g., [14]). The operator-valued matrix $\mathbf{K}^\mp$ (resp. $\mathbf{M}$) generates the twisted (resp. untwisted) Reflection Equation Algebras, $\mathrm{tREA}_q(A_{2n})^\mp$ (resp. $\mathrm{REA}_q(A_{2n})$) whose quantum group comodule structure and relation to twisted (resp. untwisted) quantum algebra $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ (resp. $\mathcal{U}_q(\mathfrak{su}_{2n+1})$) shall be discussed in the sequel.

## A.1 Symmetries of the RE's and their relation to $\mathcal{U}_q(\mathfrak{su}_{2n+1})$

The three RE's of interest enjoy the following (twisted) left-right (co)symmetries which are crucial for their applicability in an effective description of branes in WZW models ($S$ is the antipode of the Hopf algebra $SU_q(2n+1)$):

$$\mathbf{K}^- \mapsto \mathbf{t}^T \mathbf{K}^- \mathbf{s} \quad , \quad \mathbf{K}^+ \mapsto (S\mathbf{t}) \mathbf{K}^+ (S\mathbf{s})^T, \tag{A.4}$$

$$\mathbf{M} \mapsto \mathbf{t} \mathbf{M} S \mathbf{s}, \tag{A.5}$$

where

$$\mathbf{R}_{12} \mathbf{s}_1 \mathbf{s}_2 = \mathbf{s}_2 \mathbf{s}_1 \mathbf{R}_{12}, \quad \mathbf{R}_{12} \mathbf{t}_1 \mathbf{t}_2 = \mathbf{t}_2 \mathbf{t}_1 \mathbf{R}_{12}, \quad \mathbf{R}_{12} \mathbf{t}_1 \mathbf{s}_2 = \mathbf{s}_2 \mathbf{t}_1 \mathbf{R}_{12} \tag{A.6}$$

are the defining relations of (two copies of) the quantum group $SU_q(2n+1)$ associated to the $\mathcal{R}$-matrix $\mathbf{R}$.

Solutions to the three RE's under study can straightforwardly be realised in terms of generators of the (extended) quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ through

$$\mathbf{K}^- = \left(\mathbf{L}^+\right)^T \mathbf{C}^- \mathbf{L}^- \quad , \quad \mathbf{K}^+ = \left(S\mathbf{L}^+\right) \mathbf{C}^+ \left(S\mathbf{L}^-\right)^T, \tag{A.7}$$

$$\mathbf{M} = \mathbf{L}^+ \mathbf{M}_0 S \mathbf{L}^-, \tag{A.8}$$

where $\mathbf{C}^\mp$ and $\mathbf{M}_0$ denote respective (arbitrary) constant solutions ($c$-number-valued matrices) and $\mathbf{L}^\pm$ are the familiar FRT-operators [18]. The existence of the homomorphisms thus defined enables us to use the well-known representation theory of the quantum algebra $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ to induce a representation theory of the (t)REA's. In particular, the relevant (specialised) representation theory of $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ has been studied at some length in [15].

## A.2 The two embeddings $\mathcal{U}'_q(\mathfrak{so}_{2n+1}) \hookrightarrow tREA_q(A_{2n})^\mp$

The twisted quantum orthogonal algebra $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$, considered originally by Gavrilik and Klimyk in [19], is defined by the following commutation relations:

$$[\Pi_i, \Pi_j] = 0 \quad \text{if} \quad |i - j| > 1, \tag{A.9}$$

$$\Pi_i^2 \Pi_j - [2]_q \Pi_i \Pi_j \Pi_i + \Pi_j \Pi_i^2 = -\Pi_j \quad \text{if} \quad |i - j| = 1, \tag{A.10}$$

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satisfied by its generators $\Pi_i$, $i \in \overline{1, 2n+1}$. In the classical limit, $q \to 1$, the above relations reproduce the standard defining relations of $\mathcal{U}(\mathfrak{so}_{2n+1})$. They differ, on the other hand, from the defining relations of the quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{so}_{2n+1})$ (of Drinfel'd and Jimbo) associated to the universal $\mathcal{R}$-matrix for $\mathfrak{so}_{2n+1}$ (e.g.[14]).

In addition to the above generators, we define after [28] the operators $\Pi_{ji}^{\mp}$, $1 \leq i < j \leq 2n+1$ through:

$$\Pi_{i+1,i}^{\mp} := \Pi_i,$$

(A.11)

$$\Pi_{ji}^{\mp} := \Pi_{jk}^{\mp}\Pi_{ki}^{\mp} - q^{\mp 1}\Pi_{ki}^{\mp}\Pi_{jk}^{\mp} \quad \text{for arbitrary} \quad i < k < j.$$

It is then a matter of straightforward algebra to verify that the elements of the two operator-valued solutions to (A.1)-(A.2) provide a realisation of the algebra of $\Pi_{ji}^{\mp}$'s. More precisely, we have the identification:

$$K_{ij}^{-} = \lambda q^{2n-j}\Pi_{ij}^{-} \quad , \quad K_{ij}^{+} = -\lambda q^{2n+1-j}\Pi_{ji}^{+},$$

(A.12)

establishing a homomorphism $\mathcal{U}'_q(\mathfrak{so}_{2n+1}) \hookrightarrow \mathrm{tREA}_q(A_{2n})^{\mp}$. This, together with the explicit mappings $\mathrm{tREA}_q(A_{2n})^{\mp} \to \mathcal{U}_q(\mathfrak{su}_{2n+1})$, (A.7), embeds $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$ in $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ as the so-called coideal subalgebra [29]. Its representation theory, both of classical and non-classical type, has been discussed in great detail in a series of papers [19,20,23,25], also in relation to the representation theory of $\mathcal{U}_q(\mathfrak{su}_{2n+1})$. An important conclusion following from that analysis is that we can effectively restrict to $\mathcal{U}'_q(\mathfrak{so}_{2n+1})$-irreducible representations of the classical type as long as we are dealing with classical-type irreducible representations of $\mathcal{U}_q(\mathfrak{su}_{2n+1})$ (branching into the former).

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