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Abstract

In this paper we study representations of conformal nets associated with positive definite even lattices and their orbifolds with respect to isometries of the lattices. Using previous general results on orbifolds, we give a list of all irreducible representations of the orbifolds, which generate a unitary modular tensor category. 2000MSC:81R15, 17B69.

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1 Introduction

Let $\mathcal{A}$ be a completely rational conformal net (cf. §2 and (2.3) following [16]). Let Γ be a finite group acting properly on $\mathcal{A}$ (cf. definition (2.1)). The starting point of this paper is Th. 2.4 proved in [25] which states that the fixed point subnet (the orbifold) $\mathcal{A}^\Gamma$ is also completely rational, and by [16] $\mathcal{A}^\Gamma$ has finitely many irreducible representations which are divided into two classes: the ones that are obtained from the restrictions of a representation of $\mathcal{A}$ to $\mathcal{A}^\Gamma$ which are called untwisted representations, and the ones which are twisted. It follows from Th. 2.4 that twisted representation of $\mathcal{A}^\Gamma$ always exists if $\mathcal{A}^\Gamma \neq \mathcal{A}$. The motivating question for this paper is how to construct these twisted representations of $\mathcal{A}^\Gamma$.

It turns out that all representations of $\mathcal{A}^\Gamma$ are closely related to the solitons of $\mathcal{A}$. In [15], we construct solitons for the affine and permutation orbifolds. In this paper, we give a construction of solitons for the case of conformal nets associated with positive definite even lattices and isomteries of the lattices. We note that conformal nets associated with special lattices have appeared before in §3 of [26] and more recently in [17]. Our solitons correspond to "twisted representations" of the corresponding vertex operator algebras (VOAs), and such twisted representations have been studied in [18], [10], [11], [7], [8] and more recently [2]. We will show that these twisted representations in the VOA sense indeed give rise to solitons (cf. Definition 4.24 and Prop. 4.25). Compared to the constructions of [15], the notable difference is that the twisted representations are not related to the untwisted representations in a simple way as in the affine and permutation orbifolds: in the later cases twisted representations are constructed on the same spaces as that of untwisted representations, only with a "twisted" action, while in the cases of lattices the spaces are different. Hence it is nontrivial to show that in the cases of lattices these twisted representations in the VOA sense indeed give rise to solitons. Such questions were already encountered in [24] when the lattice has rank one, and were solved by identifying the orbifolds as special cosets in [26]. However this method of [24] does not work for general lattices. Our solution to this question consists of two steps. First we show that if the inner product on the lattice has some integral property, the question can be solved (cf. Prop. 4.8) by using a covering homorphism (cf. Prop. 4.6). For general lattice Q, we choose a sublattice $P \subset Q$ with finite index such that the restriction of the inner product from Q to P has the desired integral property as in the first step. By exploiting the related group structures (cf. Lemma 4.23) and results from the first step, we then give Definition 4.24 and show that they have the right properties in Prop. 4.25.

To show that these solitons give rise to all irreducible representations of the orbifolds, we use the same strategy as in [15] which is to compute the index of solitons and use Th. 2.4. Here we make use of the large group of (local) automorphisms (cf. Definition 3.13) of the conformal nets associated with the lattices, and an exhaustion trick (cf. the paragraph before Th. 4.30) to prove Th. 4.30. Th. 4.30 is the main result of this paper: it gives a list of all irreducible representations of the orbifold from solitons, and they generate a unitary modular tensor category (cf. [22]). We expect that this result will have applications in many concrete examples, and we plan to address such applications in the future.

The rest of the paper is organized briefly as follows: after introducing basic notions such as conformal nets, their representations, complete rationality, orbifolds and solitons in §2, we consider conformal nets associated with positive definite even lattices in §3. The main result in §3 is Cor. 3.19. In §4 we consider the constructions of solitons. As mentioned above we do this in two steps: the first construction in a special case is given in Definition 4.7 and its properties are studied in Prop. 4.8, Prop. 4.10, Prop. 4.13, Prop. 4.16; the general case is considered in subsection 4.1 where Th. 4.30 is proved.

2 Preliminaries

In this section we review some basic concepts of conformal nets which will be used. See §2 and §3 of [15] for a more detailed review.

2.1 Conformal nets

We denote by $\mathcal{I}$ the family of proper open intervals of S^1. A *net* $\mathcal{A}$ of von Neumann algebras on S^1 is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

from $\mathcal{I}$ to von Neumann algebras on a fixed Hilbert space $\mathcal{H}$ that satisfies:

A. *Isotony.* If $I_1 \subset I_2$ belong to $\mathcal{I}$, then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2).$$

If $E \subset S^1$ is any region, we shall put $\mathcal{A}(E) \equiv \bigvee_{E \supset I \in \mathcal{I}} \mathcal{A}(I)$ with $\mathcal{A}(E) = \mathbb{C}$ if E has empty interior (the symbol $\vee$ denotes the von Neumann algebra generated).

The net $\mathcal{A}$ is called *local* if it satisfies:

B. *Locality.* If $I_1, I_2 \in \mathcal{I}$ and $I_1 \cap I_2 = \varnothing$ then

$$[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\},$$

where brackets denote the commutator.

The net $\mathcal{A}$ is called *Möbius covariant* if in addition satisfies the following properties **C,D,E,F**:

C. *Möbius covariance.* There exists a strongly continuous unitary representation U of the Möbius group $\mathbf{G}$ on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* \;=\; \mathcal{A}(gI), \quad g \in \mathbf{G},\; I \in \mathcal{I}.$$

Note that this implies $\mathcal{A}(\bar{I}) = \mathcal{A}(I)$, $I \in \mathcal{I}$ (consider a sequence of elements $g_n \in \mathbf{G}$ converging to the identity such that $g_n\bar{I} \nearrow I$).

D. *Positivity of the energy.* The generator of the one-parameter rotation subgroup of U (conformal Hamiltonian) is positive.

E. *Existence of the vacuum.* There exists a unit U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for the von Neumann algebra $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I)$.

By the Reeh-Schlieder theorem Ω is cyclic and separating for every fixed $\mathcal{A}(I)$. The modular objects associated with $(\mathcal{A}(I), \Omega)$ have a geometric meaning

$$\Delta_I^{it} = U(\Lambda_I(2\pi t), \qquad J_I = U(r_I) \ .$$

Here Λ_I is a canonical one-parameter subgroup of $\mathbf{G}$ and $U(r_I)$ is a antiunitary acting geometrically on $\mathcal{A}$ as a reflection r_I on S^1.

This imply *Haag duality*:

$$\mathcal{A}(I)' = \mathcal{A}(I'), \quad I \in \mathcal{I} \ ,$$

where I' is the interior of $S^1 \smallsetminus I$.

F. *Irreducibility.* $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$. Indeed $\mathcal{A}$ is irreducible iff Ω is the unique U-invariant vector (up to scalar multiples). Also $\mathcal{A}$ is irreducible iff the local von Neumann algebras $\mathcal{A}(I)$ are factors. In this case they are III_1-factors in Connes classification of type III factors (unless $\mathcal{A}(I) = \mathbb{C}$ identically).

By a *conformal net* (or diffeomorphism covariant net) $\mathcal{A}$ we shall mean a Möbius covariant net such that the following holds:

G. *Conformal covariance.* There exists a projective unitary representation U of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of $PSU(1,1)$ such that for all $I \in \mathcal{I}$ we have

$$\begin{aligned} U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in \mathrm{Diff}(S^1), \\ U(g)xU(g)^* &= x, \quad x \in \mathcal{A}(I),\ g \in \mathrm{Diff}(I'), \end{aligned}$$

where $\mathrm{Diff}(S^1)$ denotes the group of smooth, positively oriented diffeomorphism of S^1 and $\mathrm{Diff}(I)$ the subgroup of diffeomorphisms g such that $g(z) = z$ for all $z \in I'$.
A (DHR) representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I} \mapsto \pi_I$ that associates to each I a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_{\tilde{I}} \restriction \mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \subset \mathcal{I} \ .$$

π is said to be Möbius (resp. diffeomorphism) covariant with positive energy if there is a projective unitary representation U_π of $\mathbf{G}$ (resp. $\mathrm{Diff}^{(\infty)}(S^1)$, the infinite cover of $\mathrm{Diff}(S^1)$) with positive energy (the generator of the rotation subgroup S^1 has non-negative spectrum) on $\mathcal{H}$ such that

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$ and $g \in \mathbf{G}$ (resp. $g \in \mathrm{Diff}^{(\infty)}(S^1)$). Note that if π is irreducible and diffeomorphism covariant then U is indeed a projective unitary representation of $\mathrm{Diff}(S^1)$.

2.2 The orbifolds

Let $\mathcal{A}$ be an irreducible conformal net on a Hilbert space $\mathcal{H}$ and let Γ be a finite group. Let $V : \Gamma \to U(\mathcal{H})$ be a unitary representation of Γ on $\mathcal{H}$. If $V : \Gamma \to U(\mathcal{H})$ is not faithful, we set $\Gamma' := \Gamma/\ker V$.

Definition 2.1. *We say that Γ acts properly on $\mathcal{A}$ if the following conditions are satisfied:*

(1) For each fixed interval I and each $g \in \Gamma$, $\alpha_g(a) := V(g)aV(g^*) \in \mathcal{A}(I), \forall a \in \mathcal{A}(I)$;

(2) For each $g \in \Gamma$, $V(g)\Omega = \Omega, \forall g \in \Gamma$.

We note that if Γ acts properly, then Γ commutes with $\mathbf{G}$.

Define $\mathcal{A}^\Gamma(I) := \mathcal{A}(I)P_0$ on $\mathcal{H}_0$ where $\mathcal{H}_0 := \{x \in \mathcal{H} | V(g)x = x, \forall g \in \Gamma\}$ and P_0 is the projection from $\mathcal{H}$ to $\mathcal{H}_0$. The unitary representation U of $\mathbf{G}$ on $\mathcal{H}$ restricts to an unitary representation (still denoted by U) of $\mathbf{G}$ on $\mathcal{H}_0$. Then:

Proposition 2.2. *The map $I \in \mathcal{I} \to \mathcal{A}^G(I)$ on $\mathcal{H}_0$ together with the unitary representation (still denoted by U) of $\mathbf{G}$ on $\mathcal{H}_0$ is an irreducible Möbius covariant net.*

The irreducible Möbius covariant net in Prop. 2.2 will be denoted by $\mathcal{A}^\Gamma$ and will be called the *orbifold of* $\mathcal{A}$ with respect to Γ. We note that by definition $\mathcal{A}^\Gamma = \mathcal{A}^{\Gamma'}$.

2.3 Complete rationality

By an interval of the circle we mean an open connected proper subset of the circle. If I is such an interval then I' will denote the interior of the complement of I in the circle. We will denote by $\mathcal{I}$ the set of such intervals. Let $I_1, I_2 \in \mathcal{I}$. We say that I_1, I_2 are disjoint if $\bar{I}_1 \cap \bar{I}_2 = \varnothing$, where $\bar{I}$ is the closure of I in S^1. Denote by $\mathcal{I}_2$ the set of unions of disjoint 2 elements in $\mathcal{I}$. Let $\mathcal{A}$ be an irreducible conformal net as in §2.1. For $E = I_1 \cup I_2 \in \mathcal{I}_2$, let $I_3 \cup I_4$ be the interior of the complement of $I_1 \cup I_2$ in S^1 where I_3, I_4 are disjoint intervals. Let

$$\mathcal{A}(E) := A(I_1) \vee A(I_2), \hat{\mathcal{A}}(E) := (A(I_3) \vee A(I_4))'.$$

Note that $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. Recall that a net $\mathcal{A}$ is *split* if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ is naturally isomorphic to the tensor product of von Neumann algebras $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ for any disjoint intervals $I_1, I_2 \in \mathcal{I}$. $\mathcal{A}$ is *strongly additive* if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ where $I_1 \cup I_2$ is obtained by removing an interior point from I.

Definition 2.3. *[16] $\mathcal{A}$ is said to be completely rational, or μ-rational, if $\mathcal{A}$ is split, strongly additive, and the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is finite for some $E \in \mathcal{I}_2$. The value of the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ (it is independent of E by Prop. 5 of [16]) is denoted by $\mu_{\mathcal{A}}$ and is called the μ-index of $\mathcal{A}$. If the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is infinity for some $E \in \mathcal{I}_2$, we define the μ-index of $\mathcal{A}$ to be infinity.*

The following theorem is proved in [25]:

Theorem 2.4. *Let $\mathcal{A}$ be an irreducible conformal net and let Γ be a finite group acting properly on $\mathcal{A}$. Suppose that $\mathcal{A}$ is completely rational or μ-rational as in definition 2.2. Then:*

(1): $\mathcal{A}^\Gamma$ is completely rational or μ-rational and $\mu_{\mathcal{A}^\Gamma} = |\Gamma'|^2 \mu_{\mathcal{A}}$;

(2): There are only a finite number of irreducible covariant representations of $\mathcal{A}^\Gamma$, and they give rise to a unitary modular category as defined in II.5 of [22] by the construction as given in §1.7 of [27].

Suppose that $\mathcal{A}$ and Γ satisfy the assumptions of Th.2.4. Then $\mathcal{A}^\Gamma$ has only a finite number of irreducible representations $\dot\lambda$ and

$$\sum_{\dot\lambda} d(\lambda)^2 = \mu_{\mathcal{A}^\Gamma} = |\Gamma'|^2 \mu_{\mathcal{A}}$$

where we use $d(\lambda)$ to denote the statistical dimension or the square root of index (cf. [14] and [20]).

2.4 Solitons

Let $\xi \in S^1$, and identify $\mathbb{R}$ with $S^1 \setminus \{\xi\} \simeq (0,1)$. Denote by $\mathcal{I}_0$ the set of open, connected, non-empty, proper subsets of $\mathbb{R}$, thus $I \in \mathcal{I}_0$ iff I is an open interval or half-line (by an interval of $\mathbb{R}$ we shall always mean a non-empty open bounded interval of $\mathbb{R}$).

Given a net $\mathcal{A}$ on S^1 we shall denote by $\mathcal{A}_0$ its restriction to $\mathbb{R} = S^1 \setminus \{-1\}$. Thus $\mathcal{A}_0$ is an isotone map on $\mathcal{I}_0$, that we call a *net on* $\mathbb{R}$.

A representation π of $\mathcal{A}_0$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I}_0 \mapsto \pi_I$ that associates to each $I \in \mathcal{I}_0$ a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_{\tilde I} \restriction \mathcal{A}(I) = \pi_I, \quad I \subset \tilde I, \quad I, \tilde I \in \mathcal{I}_0 \ .$$

A representation π of $\mathcal{A}_0$ is also called a *soliton*. If we wish to emphasize on the dependence of $\xi \in S^1$, we will write π as $\pi^{(\xi)}$.

3 Conformal nets associated with a lattice and their representations

Let Q be a positive definite even lattice. That is, Q is a free abelian group of finite rank with a positive definite $\mathbb{Z}$-valued bilinear form $\langle \cdot \rangle$ such that $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for all $\alpha \in Q$. $Q^* = \{\beta \in \mathbb{R}Q | \langle \beta, Q \rangle \subset \mathbb{Z}\}$ is the dual lattice of Q. There exists a bimultiplicative function $\epsilon : Q \times Q \to \{\pm 1\}$ satisfying $\epsilon(\alpha,\alpha) = (-1)^{\langle \alpha,\alpha \rangle/2}, \alpha \in Q$. Then by bimultiplicativity $\epsilon(\alpha,\beta)\epsilon(\beta,\alpha) = (-1)^{\langle \alpha,\beta \rangle}, \alpha,\beta \in Q$. Note that such 2-cocycle ϵ is unique up to equivalence. We say ϵ is trivial if $\epsilon(\alpha,\beta) = 1, \forall \alpha,\beta \in Q$.

We note that one can always choose a trivial 2-cocycle (in the equivalence class) if $\langle\alpha,\beta\rangle \in 4\mathbb{Z}, \forall\alpha,\beta\in Q$. Let $T=\mathbb{R}Q/Q$ be the torus. We will represent elements of T by $e^{2\pi i h}$ for $h\in\mathbb{R}Q$. Note that $e^{2\pi i h}=1$ iff $h\in Q$.

Denote by $LT=C^\infty(S^1,T)$. Every element of LT can be written as $e^{2\pi i f}$, where $f=f(\theta): S^1\to\mathbb{R}Q$ with $0\le\theta\le 1$ and $f(\theta)=\Delta_f\theta+f_0+f_1(\theta)$. Here $\Delta_f\in Q$ is called the "winding number" of f, and $f_1(\theta)=\sum_{n\neq 0} a_n e^{2\pi i n\theta}$ for some a_n. f_0 is called the"zero mode" of f. The rotation group S^1 acts naturally on LT, and we denote the action by R_θ. Define $\int_{S^1}\langle f,g\rangle d\theta := \int_0^1\langle f,g\rangle d\theta$.

Let $\mathcal{L}T=LT\times S^1$, and define a multiplication on $\mathcal{L}T$ as follows:

Definition 3.1.

$$(e^{2\pi i f},x_1)(e^{2\pi i g},x_2)=(e^{2\pi i(f+g)},x_1x_2\epsilon(\Delta f,\Delta g)e^{\pi i[\int\langle f'|g\rangle d\theta-\langle f(1)|\Delta_g\rangle+\frac{1}{2}\langle\Delta_f|\Delta_g\rangle]})$$

Lemma 3.2. *$\mathcal{L}T$ with the above multiplication is a central extension of LT. Moreover, the action of rotation R_θ on LT lifts naturally to $\mathcal{L}T$ and we have $R_\theta(e^{2\pi i f})R_\theta(e^{2\pi i g})=R_\theta(e^{2\pi i f}e^{2\pi i g})$ in $\mathcal{L}T$.*

Proof. We note that the associativity of the multiplication follows from the properties of 2-cocycle ϵ, and the rest follows by using definitions. ■

Remark 3.3. *Our choice of multiplication rules in the above definition is different from that of [21] (cf. Chapter 4 of [21]) in the special case when Q is a root lattice, and such a choice makes the action of rotations simpler than that of [21].*

Proposition 3.4. *Let f,g be such that $\mathrm{supp} e^{2\pi i f}\cap\mathrm{supp} e^{2\pi i g}=\varnothing$ where $\mathrm{supp} e^{2\pi i f}$ is defined to be the support of $e^{2\pi i f}$ as an element in LT. Then*

$$e^{2\pi i f}e^{2\pi i g}=e^{2\pi i g}e^{2\pi i f}$$

as elements in $\mathcal{L}T$.

Proof. Assume that $f(\theta)=\alpha\theta+f_0+f_1(\theta)$, $g(\theta)=\beta\theta+g_0+g_1(\theta)$ and $g(0)=0$ and $g(1)=\beta$. Then by definition 3.1:

$$e^{2\pi i f}e^{2\pi i g}(e^{2\pi i f})^{-1}=e^{2\pi i g}e^{2\pi i(\frac{1}{2}\langle\alpha,\beta\rangle+\langle\alpha,g_0\rangle-\langle\beta,f_0\rangle-\int_{S^1}\langle g_1'|f_1\rangle d\theta)}.$$

Note that

$$\begin{aligned}\int_{S^1}\langle g_1'|f_1\rangle d\theta &= \int_{S^1}\langle g_1'|f-\alpha\theta-f_0\rangle d\theta\\ &=\int_{S^1}\langle g'|f\rangle d\theta-\langle\alpha,\beta\rangle+\int_{S^1}\langle g|\alpha\rangle d\theta-\langle\beta|f_0\rangle\\ &=\int_{S^1}\langle g'|f\rangle d\theta-\langle\alpha,\beta\rangle+\int_{S^1}\langle\beta\theta+g_0|\alpha\rangle d\theta-\langle\beta|f_0\rangle\\ &=\int_{S^1}\langle g'|f\rangle d\theta-\frac{1}{2}\langle\alpha,\beta\rangle+\int_{S^1}\langle g_0|\alpha\rangle d\theta-\langle\beta|f_0\rangle.\end{aligned}$$

Since $\mathrm{supp} e^{2\pi i f} \cap \mathrm{supp} e^{2\pi i g} = \varnothing$, $\int_{S^1} \langle g'|f\rangle d\theta$ is either 0 or $\langle\alpha|\beta\rangle$. It follows that

$$e^{2\pi i f} e^{2\pi i g} (e^{2\pi i f})^{-1} = e^{2\pi i g}.$$

■

The structure of $\mathcal{L}T$ is well known (cf. Page 191 of [21]). The identity component $(\mathcal{L}T)^o$ of $\mathcal{L}T$ is canonically a product $T \times \tilde{V}_Q$, where $\tilde{V}_Q$ is the Heisenberg group defined as follows: Let W_0 be the set of maps $f : S^1 \to \mathbb{R}Q$ with winding number and zero mode being zeros. Then $\tilde{V}_Q$ is equal to $W_0 \times S^1$ as sets and multiplication is determined by

$$(f_1, \lambda_1)(f_2, \lambda_2) = (f_1 + f_2, \lambda_1 \lambda_2 e^{\frac{i}{2}\int \langle f_1' | f_2\rangle d\theta}).$$

Let W be the set of maps $f : S^1 \to \mathbb{R}Q$ with winding number zero, and $\tilde{W} := \tilde{V}_Q \times \mathbb{R}Q$. The following is essentially Prop. 9.5.10 in [21]:

Lemma 3.5. *We have:*

(1) $\tilde{V}_Q$ has a unique irreducible representation with positive energy on a Hilbert space denoted by $S(V)$;

(2) All irreducible representations of $\tilde{W}$ with positive energy are of the form $S(V)_\alpha$ for $\alpha \in \mathbb{R}Q$ where $S(V)_\alpha$ is the same as $S(V)$ as a representation of $\tilde{V}$ and the center $(0, h)$ of $\tilde{W}$ acts on $S(V)_\alpha$ as a scalar $e^{2\pi i \langle h|\alpha\rangle}$;

(3) All irreducible representations of $\mathcal{L}T$ with positive energy are of the form $H_\lambda = \bigoplus_{\alpha\in\lambda+Q} S(V)_\alpha$ for $\lambda \in Q^*$ where $Q^* = \{\beta \in \mathbb{R}Q | \langle\beta, Q\rangle \subset \mathbb{Z}\}$ is the dual lattice of Q. Moreover $e^{2\pi i \beta\theta}$ maps $S(V)_\alpha$ to $S(V)_{\alpha+\beta}$.

Proof. See the proof of Proposition 9.5.10 in [21]. ■

Consider the representation $S(V)_\alpha$ of $\tilde{W}$. By Theorem 7.6 of [12] (although Theorem 7.6 of [12] is stated for semisimple Lie algebras, but the same argument applies to the case of Heisenberg algebra), there exists a map $\varphi \in \mathrm{Diff}(S^1) \mapsto \pi_\alpha(\sigma(\varphi)) \in U(S(V)_\alpha)$ which is a unitary cocycle representation of $\mathrm{Diff}(S^1)$ on $S(V)_\alpha$, and

$$\pi_\alpha(\sigma(\varphi))\pi_\alpha(f(\cdot))\pi_\alpha(\sigma(\varphi)^*) = \pi_\alpha(f(\varphi^{-1}(\cdot))).$$

Let $\mathcal{B}_Q$ be a net on $S(V)_0$ such that

$$\mathcal{B}_Q(I) = \{\pi_0(f) | f \in \tilde{W}, \mathrm{supp} f \subset I\}''$$

where π_0 denotes the representation of $\tilde{W}$ on $S(V)_0$.

Proposition 3.6. *We have*

(1) $\mathcal{B}_Q$ is a conformal net on $S(V)_0$;

(2) $\mathcal{B}_Q$ is strongly additive.

Proof. (1) is obvious and (2) follows from the same argument of Proposition 1.3.2 of [23]. ■

Definition 3.7. *Let $\mathcal{A}_Q$ be a net of von Neumann algebra on H_0 such that $\mathcal{A}_Q(I) = \{\pi_0(e^{2\pi i f}) | e^{2\pi i f} \in \mathcal{L}T, \mathrm{supp} f \subset I\}$ where π denotes the representation of $\mathcal{L}T$ on H_0.*

We note that by definition $\mathcal{A}_Q$ is independent of the choices of 2-cocycle ϵ.

By the statement before Proposition 3.6, we have a unitary cocycle representation of $\mathrm{Diff}(S^1)$ on $H_0 = \sum_{\alpha \in Q} S(V)_\alpha$ such that

$$\pi(\sigma(\varphi))\pi(f)\pi(\sigma(\varphi)^*) = \pi(f^\varphi)$$

for $f \in \tilde{W}$ where $f^\varphi(\theta) := f(\varphi^{-1}(\theta))$. We claim that

$$\pi(\sigma(\varphi))\pi(e^{2\pi i f})\pi(\sigma(\varphi)^*) = c(f,\varphi)\pi(e^{2\pi i f^\varphi})$$

for the some phase factor $c(f,\varphi) \in \mathbb{C}$. First we have

Lemma 3.8. *$\mathcal{A}_Q$ is a local net.*

Proof. This follows from Proposition 3.4. ■

Proposition 3.9. *If $\varphi \in \mathrm{Diff}(I)$, and $\mathrm{supp} e^{2\pi i f} \cap I = \varnothing$, then*

$$\pi(\sigma(\varphi))\pi(e^{2\pi i f})\pi(\sigma(\varphi)^*) = \pi(e^{2\pi i f^\varphi}).$$

Proof. By Proposition 3.6, $\mathcal{B}_Q$ is a conformal net, and it follows that $\pi_0(\sigma(\varphi)) \in \mathcal{B}_Q(I)$. Note that π is a representation of $\mathcal{B}_Q$ on $\bigoplus_{\alpha \in Q} S(V)_\alpha$, and restrict to an irreducible representation of $\mathcal{B}_Q$ on each $S(V)_\alpha$. It follows that

$$Ad\pi(\pi_0(\sigma(\varphi))) = Ad\pi(\sigma(\varphi)).$$

So we have

$$Ad\pi(\sigma(\varphi))(\pi(e^{2\pi i f})) = Ad\pi(\pi_0(\sigma(\varphi)))(\pi(e^{2\pi i f})) = \pi(e^{2\pi i f})$$

where we have used Lemma 3.8 in the last equality since

$$\pi_0(\sigma(\varphi)) \in \mathcal{B}_Q(I) \subset \mathcal{A}_Q(I).$$

■

Proposition 3.10. *We have for $f \in \mathcal{L}T$*

$$\pi(\sigma(\varphi))\pi(e^{2\pi i f})\pi(\sigma(\varphi)^*) = c(f,\varphi)\pi(e^{2\pi i f^\varphi})$$

where $c(f,\varphi) \in \mathbb{C}$.

Proof. Since $\mathrm{Diff}(S^1)$ is a simple group, it is generated by $\mathrm{Diff}(I)$. It is sufficient to prove the proposition for $\varphi \in \mathrm{Diff}(I)$. Note that

$$\pi(e^{2\pi i f}) = \pi(e^{2\pi i(\alpha\theta + f_0)})\pi(e^{2\pi i f_1(\theta)}) = \pi(e^{2\pi i \alpha\theta})\pi(e^{2\pi i(f_0+f_1)})e^{-\frac{i}{2}\langle\alpha|f_0\rangle}$$

where α is the winding number of f. Since

$$Ad\pi(\sigma(\varphi))(\pi(e^{2\pi i(f_0+f_1)})) = \pi(e^{2\pi i(f_0+f_1^\varphi)})),$$

it is enough to show the proposition for $e^{2\pi i f} = e^{2\pi i \alpha\theta}$. By the same argument it reduces to show the proposition for $e^{2\pi i f}$ so that f has winding number α and $\mathrm{supp} e^{2\pi i f} \cap I = \varnothing$, which follows from Proposition 3.9. ■

Proposition 3.11. *(1) $\mathcal{A}_Q$ is a conformal net.*

(2) $\mathcal{A}_Q$ is strongly additive and split.

Proof. (1) follows from Propositions 3.10 and 3.9. As for (2), let I_1, I_2 be two subintervals of I obtained from I by removing an interior point of I. By Proposition 3.6, $\mathcal{B}(I) = \mathcal{B}(I_1) \vee \mathcal{B}(I_2)$. Since $\mathcal{A}(I)$ is generated by $\mathcal{B}(I)$ and $\pi(e^{2\pi i f(\theta)})$ with supp$e^{2\pi i f(\theta)} \subset I_1$, it follows that $\mathcal{A}(I_1) \vee \mathcal{B}(I) = \mathcal{A}(I)$, and so $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$.

The character $\mathrm{Tr} q^{L_0}$ of H_0 (L_0 represents the generator of rotation group S^1) is well known to be

$$\mathrm{Tr} q^{L_0} = \frac{\theta_Q(q)}{\eta(q)^l}$$

where

$$\theta_Q(q) = \sum_{\alpha \in Q} q^{\frac{\langle \alpha | \alpha \rangle}{2}}$$

is the theta function of the lattice of Q and

$$\eta(q) = q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n)$$

is the eta function. Here l is the rank of Q and $q = e^{2\pi i \tau}$ for τ in the upper half plane. Hence q^{L_0} is of trace class, and it follows from that $\mathcal{A}_Q$ is split (cf. [5]). ■

Definition 3.12. *Let $\lambda(\theta) : [0,1] \to \mathbb{R}Q$ be a smooth map with $\lambda(0) = 0$, $\lambda(1) = \lambda \in Q^*$, and $\lambda^{(n)}(0) = \lambda^{(n)}(1) = 0$ for all positive n. Define an automorphism $Ad_{\lambda(\theta)}$ of $\mathcal{L}T$ by the following formula:*

$$Ad_{\lambda(\theta)}(e^{2\pi i f}, c) = (e^{2\pi i f}, c e^{2\pi i \int \langle \lambda' | f \rangle d\theta}).$$

Note that if $\lambda \in Q^*$, $e^{2\pi i \lambda}$ lies in the center of $\mathcal{L}T$. For any interval $I \subset S^1$ we choose an element $P_{\lambda,I} \in LT$ such that $P_{\lambda,I}(\theta) = e^{2\pi i \lambda(\theta)}$ if $\theta \in I$ where $\lambda(\theta)$ is as in definition 3.12.

Definition 3.13. *Let $\lambda \in Q^*$. Define an automorphism Ad_λ of $\mathcal{A}_Q$ by*

$$Ad_\lambda(y) = \pi(P_{\lambda,I}) y \pi(P_{\lambda,I})^*$$

for $y \in \mathcal{A}_Q(I)$.

Lemma 3.14. *Ad_λ in Definition 3.13 is independent of the choice of $P_{\lambda,I}$ and $Ad_\lambda \pi(\mathcal{L}_I T) = \pi(Ad_\lambda \mathcal{L}_I T)$ for any I.*

Proof. If $P'_{\lambda,I}$ is another choice, then $P_{\lambda,I} P'_{\lambda.I} \in \mathcal{L}_{I'} T$ and by locality

$$\pi(P'_{\lambda,I}) y \pi(P'_{\lambda,I})^* = \pi(P_{\lambda,I}) y \pi(P_{\lambda,I})^*$$

for $y \in \mathcal{A}_Q(I)$. The equality in the proposition can be checked directly by definitions. ■

Proposition 3.15. *(1) $Ad_\lambda, \lambda \in Q^*$ gives an irreducible DHR representation of $\mathcal{A}_Q$, and each such representation corresponds to an irreducible representation of $\mathcal{L}T$, labeled by $\lambda \in Q^*/Q$ (We identify λ with its image in the quotient map $Q^* \to Q^*/Q$) as in (3) of Lemma 3.5.*

(2) Let ψ be an irreducible representation of $\mathcal{A}_Q$ with positive energy on H. Then ψ is isomorphic to Ad_λ for some $\lambda \in Q^*/Q$.

Proof. (1) Note that $e^{2\pi i\lambda}$ is in the center of $\mathcal{L}T$, and by the same proof of Proposition 5.8 of [15], Ad_λ is an irreducible DHR representation of $\mathcal{A}_Q$. Such a representation of $\mathcal{A}_Q$ corresponds to representation $\pi(Ad_\lambda \mathcal{L}T)$ of $\mathcal{L}T$ by Lemma 3.14.

(2) We will use an idea for the rank one case given in §4 of [25].

Let $f = f_0 + f_1(\theta) : S^1 \to \mathbb{R}Q$ be a map with winding number 0. Let $\{I_1, ..., I_k\}$ be a finite open covering of S^1. Assume that $\{\varphi_i\}$ is a partition of unity such that supp$\varphi_i \subset I_i$. Then $f = \sum_{i=1}^k f\varphi_i$. Let $c(f,\varphi) \in S^1$ be the phase factor in the center of $\mathcal{L}T$ so that $(e^{2\pi i f}, 1) = \prod_{j=1}^k (e^{2\pi i f\varphi_j}, 1)c(f,\varphi)$ as an element of $\mathcal{L}T$. By using Isotony we claim that the following map

$$f \to \psi(f) = \prod_{j=1}^k \psi_{I_j}(\pi_{I_j}(e^{2\pi i f\varphi_j}))c(f,\varphi)$$

is independent of the choice of $\{I_1, ..., I_k\}$ and $\{\varphi_i\}$. Moreover,

$$\psi(f)\psi(g) = c(f,g)\psi(f+g).$$

In fact, if $\{J_1, ..., J_n\}$ is another open covering of S^1 and $\{\bar\varphi_j\}$ is another partition of unity with supp$\bar\varphi_j \subset J_j$ for $j = 1, ..., n$, so that $J_{j_1} \cup J_{j_2} \neq S^1$ for any $1 \leq j_1, j_2 \leq n$. We have by Isotony

$$\pi_{I_s}(e^{2\pi i f\varphi_s}) = \prod_{j=1}^n \pi_{I_s\cap J_j}(e^{2\pi i f\varphi_s\bar\varphi_j})x_i$$

where the phase factor $x_i \in S^1$ is determined by

$$(e^{2\pi i f\varphi_s}, 1) = \prod_{j=1}^n (e^{2\pi i f\varphi_s\bar\varphi_j}, 1)x_i,$$

and $\pi_{I_s\cap J_j}(e^{2\pi i f\varphi_s\bar\varphi_j}) \in \mathcal{A}_Q(I_s \cap J_j)$ by definition. It follows that

$$\begin{aligned}
&\prod_{s=1}^k \psi_{I_s}(\pi_{I_s}(e^{2\pi i f\varphi_s}))c(f,\varphi) \\
&= \prod_{s=1}^k \prod_{j=1}^n \psi_{I_s\cap J_j}(\pi_{I_s\cap J_j}(e^{2\pi i f\varphi_s\bar\varphi_j}))x
\end{aligned}$$

with $x \in S^1$. Similarly,

$$\prod_{j=1}^{n} \psi_{J_j}(\pi_{J_j}(e^{2\pi i f\bar{\varphi}_j}))c(f,\bar{\varphi})$$
$$= \prod_{s=1}^{k}\prod_{j=1}^{n} \psi_{I_s\cap J_j}(\pi_{I_s\cap J_j}(e^{2\pi i f\varphi_s\bar{\varphi}_j}))y$$

for some $y \in S^1$. Note that $J_{j_1} \cup J_{j_2} \neq S^1$ for any $1 \leq j_1, j_2 \leq n$. It follows that in permuting the factors

$$\psi_{I_s\cap J_j}(\pi_{I_s\cap J_j}(e^{2\pi i f\varphi_s\bar{\varphi}_j}))$$

above, the phase factors are determined by the group law of $\mathcal{L}T$. Since

$$\prod_{s=1}^{k}\prod_{j=1}^{n}(e^{2\pi i f\varphi_s\bar{\varphi}_j},1)x = (e^{2\pi i f},1) = \prod_{s=1}^{k}\prod_{j=1}^{n}(e^{2\pi i f\varphi_s\bar{\varphi}_j},1)y,$$

it follows that $x = y$ and

$$\prod_{s=1}^{k} \psi_{I_s}(\pi_{I_s}(e^{2\pi i f\varphi_s})c(f,\varphi) = \prod_{j=1}^{n} \psi_{J_j}(\pi_{J_j}(e^{2\pi i f\bar{\varphi}_j})c(f,\bar{\varphi}).$$

By the independence of $\psi(f)$ on $\{\varphi_i\}$ above, it is straightforward to check that $\psi(f)\psi(g) = c(f,g)\psi(f+g)$ where

$$(e^{2\pi i f},1)(e^{2\pi i g},1) = (e^{2\pi i(f+g)}, c(f,g))$$

and $\psi(R_\theta)\psi(f)\psi(R_\theta)^* = \psi(f^\theta)$ where R_θ is the rotation by angle $2\pi\theta$.

Now for $f = \alpha\theta + f_0 + f_1(\theta)$ where α is the winding number of f, let $g_I(\theta) = \alpha\theta + g_0 + g_1(\theta)$ be such that $\mathrm{supp} e^{2\pi i g_I} \subset I$ and define

$$\psi(f) = \psi_I(\pi_I(e^{2\pi i g_I}))\psi(f - g_I)c(f, g_I)$$

where $c(f, g_I)$ in the center of $\mathcal{L}T$ is determined by

$$(e^{2\pi i f},1) = (e^{2\pi i g_I},1)(e^{2\pi i(f-g_I)},1)c(f,g_I)$$

in $\mathcal{L}T$. Note that $f - g_I \in C^\infty(S^1, \mathbb{R}Q)$ and $\psi(f - g_I)$ is well-defined as in the previous paragraph. One checks that $\psi(f)$ is independent of I and the choice of g_I, and

$$\psi(f)\psi(g) = c(f,g)\psi(f+g),$$

$$\psi(R_\theta)\psi(f)\psi(R_\theta)^* = \psi(f^\theta).$$

To show that the map $e^{2\pi i f} \in \mathcal{L}T \to \psi(f)$ is well-defined we need to check that if $f = \alpha$, then $\psi(\alpha)$ is the identity operator. For any f we get

$$\psi(R_1)\psi(f)\psi(R_1)^{-1} = \psi(f - \alpha) = \psi(f)e^{2\pi i \frac{1}{2}\langle\alpha,\alpha\rangle}\psi(\alpha).$$

But $\psi(R_1)$ is a scalar operator as ψ is an irreducible representation of $\mathcal{A}_Q$. It follows that $\psi(f) = \psi(f)\psi(\alpha)$, and $\psi(\alpha)$ is the identity operator since $\psi(f)$ is unitary. So we get a well -defined irreducible representation

$$e^{2\pi i f} \in \mathcal{L}T \to \psi(f)$$

of $\mathcal{L}T$ with positive energy. By Lemma 3.5 and (1), (2) is proved. ∎

Remark 3.16. *There is a similar result in the theory of vertex operator algebra. Let V_Q be the vertex operator algebra associated to the lattice Q (cf. [4],[11]). It has been proved in [6] that $V_{Q+\lambda}$ for $\lambda \in Q^*/Q$ gives a complete list of irreducible V_Q-modules up to isomorphism.*

Lemma 3.17. *Let $\mathcal{A}$ be a net with split property. If $\mathcal{A}$ has only finitely many irreducible representations with positive energy up to isomorphism, then every irreducible representation of $\mathcal{A} \otimes \mathcal{A}$ with positive energy π is of the form $\pi_1 \otimes \pi_2$ where π_i are irreducible representation of $\mathcal{A}$ with positive energy.*

Proof. As in Lemma 2.7 of [16], it is enough to show that $\pi(\mathcal{A} \otimes 1)''$ is a type I factor. Since $\pi_1 = \pi \restriction_{\mathcal{A}\otimes 1}$ is also a representation of $\mathcal{A}$ with positive energy, and $\mathcal{A}$ is split, by Proposition 5.6 of [16] and Lemma 5.14 of [3], $\pi_1 = \int_X^{\oplus} \pi_\lambda d\mu(\lambda)$ where π_λ are irreducible representations of $\mathcal{A}$ with positive energy for almost all λ. Since $\mathcal{A}$ has only finitely many irreducible representations with positive energy, it follows that $\pi(\mathcal{A} \otimes 1)''$ is a type I factor. ∎

Theorem 3.18. *Let $\mathcal{A}$ be a conformal net, and assume that $\mathcal{A}$ is strongly additive and split. If $\mathcal{A}$ has only finitely many irreducible representations with positive energy up to isomorphism, and if each such representation has finite index, then $\mathcal{A}$ is completely rational, and $\mu_{\mathcal{A}} = \sum_\lambda d(\lambda)^2$, where the sum is over all irreducible representations of $\mathcal{A}$ with positive energy.*

Proof. The theorem and its proof are essentially contained in §4 of [19] except for the positive energy condition. We will give a proof with necessary modifications compared to [19].

It is sufficient to show that $\mu_{\mathcal{A}}$ is finite. Consider $(\mathcal{A} \otimes \mathcal{A})^{\mathbb{Z}_2} \subset \mathcal{A} \otimes \mathcal{A}$ where $(\mathcal{A}\otimes\mathcal{A})^{\mathbb{Z}_2}$ is the fixed point subset of $\mathcal{A}\otimes\mathcal{A}$ under flip. By Corollary 4.6 of [19], we have a representation of $(\mathcal{A} \otimes \mathcal{A})^{\mathbb{Z}_2}$ which is a direct sum of two irreducible representations with positive energy. Let τ be one of them. Note that $\mu_{\mathcal{A}}$ is finite if and only if τ has finite index.

By Corollary 3.3 of [19], α_{τ^2} is a DHR representation of $\mathcal{A} \otimes \mathcal{A}$. Since $\pi(\mathrm{Diff}(I)) \subset (\mathcal{A} \otimes \mathcal{A})^{\mathbb{Z}_2}(I)$, by Lemma 4 of [1], α_{τ^2} is Möbius invariant with representation $\pi(\mathbf{G}) \in \alpha_{\tau^2} \restriction_{(\mathcal{A}\otimes\mathcal{A})^{\mathbb{Z}_2}}$. Since $\alpha_{\tau^2}|_{(\mathcal{A}\otimes\mathcal{A})^{\mathbb{Z}_2}}$ is a direct sum of τ^2 and $\sigma\tau^2$ (cf. §3.2 of [19] for the definition of σ), and both have positive energy by Theorem 5.13 of [3], it follows that α_{τ^2} is a Möbius covariant representation of $\mathcal{A} \otimes A$ with positive energy. Hence $\alpha_{\tau^2} = \int_X^{\oplus} \lambda d\mu(\lambda)$ where almost all λ are irreducible representations of $\mathcal{A} \otimes \mathcal{A}$ with positive energy. By Lemma 3.17, it follows that α_{τ^2} is a direct sum of irreducible representations of $\mathcal{A} \otimes \mathcal{A}$ with finite index.

Let $[\alpha_\tau] = \sum_{i=1}^2 [X_i]$ with X_i irreducible as sectors. Since $[\alpha_{\tau^2}] = [\alpha_\tau][\alpha_\tau]$, it follows that $X_i X_j$ and $X_j X_i$ contain sectors with finite index. By Lemma 3.6 of [19], X_i must have finite index, and so does α_τ. This proves that $\mu_{\mathcal{A}}$ is finite. ■.

Corollary 3.19. *$\mathcal{A}_Q$ is completely rational and*

$$\mu_{\mathcal{A}_Q} = |Q^*/Q|.$$

Proof. The proof follows by Propositions 3.11,3.15 and Theorem 3.18. ■

Remark 3.20. *It will be interesting if one can give a direct proof of Corollary 3.19 without using Theorem 3.18.*

4 Orbifolds

Let $\mathcal{A}_Q$ be the conformal net on $H = \bigoplus_{\alpha \in Q} S(V)_\alpha$ as in Section 3. We will consider a finite automorphism group Γ of $\mathcal{A}_Q$ which arises from isometries of the lattice Q as follows: for each $\sigma \in \Gamma$, there is an isometry of Q defined by the same letter σ, and moreover the following map

$$Ad_\sigma(e^{2\pi i(\Delta_f \theta + f_0)} e^{2\pi i f_1(\theta)}, x) = (\eta(\sigma)^{-1} e^{2\pi i(\sigma(\Delta_f)\theta + \sigma(f_0))} e^{2\pi i \sigma(f_1(\theta))}, x)$$

gives an automorphism of $\mathcal{L}T$ with finite order. Here $\eta(\alpha) = \pm 1$. For such $\sigma \in \Gamma$, let $\pi(g)$ be the unique unitary operator on H_0 such that $\pi(g) \cdot \Omega = \Omega$ (Ω is the vacuum vector) and

$$\pi(\sigma)\pi(e^{2\pi i f})\pi(\sigma)^* = \pi(Ad_\sigma e^{2\pi i f}).$$

One check easily that such unitary operator exists and is unique.

Lemma 4.1. *The map $g \mapsto Ad\pi(g)(y)$, $y \in \mathcal{A}_Q(I)$ defines a proper action of Γ on $\mathcal{A}_Q$. The fixed point subset $\mathcal{A}_Q^\Gamma$ is a conformal net.*

Proof. It is enough to show that $\pi(g)\pi(\varphi)\pi(g)^* = \pi(\varphi)$ for all $\varphi \in \mathrm{Diff}(S^1)$. Since $\mathrm{Diff}(I)$ for any I generates $\mathrm{Diff}(S^1)$ and $\mathrm{Diff}(S^1)$ is a perfect group, it suffices to check that $\pi(g)\pi(\varphi)\pi(g)^*\pi(\varphi)^* \in \mathbb{C}1$ for all $\varphi \in \mathrm{Diff}(S^1)$. Note that $\pi(g)\pi(\varphi)\pi(g)^*\pi(\varphi)^* \in \mathcal{A}_Q(I)$ and by definition, $\pi(g)\pi(\varphi)\pi(g)^*\pi(\varphi)^* \in \mathcal{B}_Q(I)'$. But $\mathcal{B}_Q(I)' \cap \mathcal{A}_Q(I) = \mathbb{C}1$, we immediately have $\pi(g)\pi(\varphi)\pi(g)^*\pi(\varphi)^* \in \mathbb{C}1$, as desired. ■

Fix $\sigma \in \Gamma$ of order N. Let $P_0 := \frac{1}{N}\sum_{1 \le i \le N} \sigma^i$ be the projection on $\mathbb{R}Q$. For any $\delta \in \mathbb{R}Q$, let δ_* be the unique element in the orthogonal complement of $P_0(\mathbb{R}Q)$ such that $\delta = \delta_0 + (1-\sigma)\delta_*, \delta_0 \in P_0(\mathbb{R}Q)$. We will use $(Q^*/Q)^\sigma$ to denote those elements of Q^*/Q which is fixed by σ. The set $(Q^*/Q)^\sigma$ can be represented as follows: let $Q_\sigma := \{\delta \in Q^* | (1-\sigma)\delta \in Q\}$, then $(Q^*/Q)^\sigma = Q_\sigma/Q$. The following lemma follows directly from definitions:

Lemma 4.2.
$$(Q^*/Q)/(Q^*_\sigma/Q) \simeq Q^*/Q^*_\sigma \simeq Q_\sigma/Q = (Q^*/Q)^\sigma$$

Definition 4.3. *Let N be the order of σ. Set*

$$C_tT = \{e^{2\pi i f} \in C(S^1,T) | f(\theta + \frac{1}{N}) - \sigma(f(\theta)) \in Q, \theta \in [0,1]\}$$

$$C_NT = \{e^{2\pi i f} \in C(S^1,T) | f(\theta + \frac{1}{N}) - f(\theta) \in Q, \theta \in [0,1]\}$$

$$C_NT_0 = \{e^{2\pi i f} \in C_NT | f(0) - \sigma(f(0)) \in Q\}$$

For each $e^{2\pi i f} \in C_NT_0$ we define a map $\varphi_2 : C_NT_0 \to C_tT$ by $\varphi_2(e^{2\pi i f}) = e^{2\pi i h}$ where h is the unique continuous function which satisfies $h(\theta) = f(\theta)$ for $0 \le \theta \le \frac{1}{N}$ and $h(\theta + \frac{1}{N}) \equiv g(f(\theta))$ modulo Q if $\frac{1}{N} \le \theta \le 1$.

Example 4.4. *(1) Let $e^{2\pi i N\alpha} \in C_tT_0$. Then $\varphi_2(e^{2\pi i N\alpha}) = e^{2\pi i h(\theta)}$ such that $h(\theta) = N\alpha\theta$ for $0 \le \theta \le \frac{1}{N}$ and $h(\theta + \frac{1}{N}) \equiv \sigma(f(\theta))$ modulo Q if $\frac{1}{N} \le \theta \le 1$. There is a unique choice of continuous function $h(\theta)$ with the specified property:*

$$h(\theta) = \alpha + \sigma(\alpha) + \cdots \sigma^{i-1}(\alpha) + (N\theta - i)\sigma^i(\alpha)$$

for $\frac{i}{N} \le \theta \le \frac{i+1}{N}$ and $i = 0, ..., N-1$. From this example one can see that in general φ_2 maps smooth functions to piece-wise smooth functions.

(2) Let $e^{2\pi i h} \in C_NT_0$ be a constant loop. Then $h \equiv g(h)$ modulo Q and $\varphi_2(e^{2\pi i h}) = e^{2\pi i h}$.

As in section 2.4, let $\xi \in S^1$ and identify $\mathbb{R} \simeq S^1 \setminus \{\xi\} \simeq (0,1)$.
Let

$$L_{\mathbb{R}}T = \{e^{2\pi i f} \in LT | \mathrm{supp} e^{2\pi i f} \subset (0,1), f^{(n)}(0) = f^{(n)}(1) = 0, \forall n \ge 1\}.$$

Fix $P = 2NQ \subset Q$ which inherits inner product $\langle \cdot \rangle$ from Q. Note that $\langle \alpha, \beta \rangle \in 4N\mathbb{Z}$ for $\alpha, \beta \in P$. We can choose an equivalence class of 2-cocylces ϵ on Q so that $\epsilon(\alpha,\beta) = 1, \forall \alpha, \beta \in P$ since $\langle \alpha, \beta \rangle \in 4\mathbb{Z}, \forall \alpha, \beta \in P$. Similarly for the central extension $\mathcal{L}T$ associated to $P, \langle , \rangle$ (resp. $\mathcal{L}T$ associated to $P, \frac{1}{N}\langle , \rangle$), we will choose the 2-cocyles as in Definition 3.1 to be trivial since $\langle \alpha, \beta \rangle \in 4N\mathbb{Z}, \forall \alpha, \beta \in P$.

Definition 4.5. *Denote by $\mathcal{L}_{\mathbb{R}}T$ be the subgroup of $\mathcal{L}T$ associated to $P, \langle , \rangle$ (with the trivial 2-cocycle as above) whose projection onto LT is $L_{\mathbb{R}}T$. Denote by $\mathcal{L}_NT_0$, $\mathcal{L}_NT, \mathcal{L}_tT$ the subgroups of $\mathcal{L}T$ associated with $P, \frac{1}{N}\langle , \rangle$ (with the trivial 2-cocycle as above) whose projections onto LT are the smooth loops in C_NT_0, C_NT, C_tT respectively.*

Proposition 4.6. *The homomorphism*

$$\begin{array}{ccccc} L_{\mathbb{R}}T & \overset{\varphi_1}{\to} & L_NT_0 & \overset{\varphi_2}{\to} & L_tT \\ e^{2\pi i f(\theta)} & \overset{\varphi_1}{\mapsto} & e^{2\pi f(N\theta)} & \overset{\varphi_2}{\mapsto} & \varphi_(e^{2\pi i f(\theta)}) := \varphi_2(e^{2\pi i f(N\theta)}) \end{array}$$

can be lifted uniquely to a homomorphism between the central extensions

$$\mathcal{L}_{\mathbb{R}}T \overset{\varphi}{\to} \mathcal{L}_tT.$$

Proof. Note that $\varphi_2\varphi_1$ maps $L_{\mathbb{R}}T$ to smooth loops in LT. The proof is a direct computation by definitions. ■

Let $\mathcal{A}_P$ be the conformal subnet of $\mathcal{A}_Q$ associated to the lattice P and represented on its vacuum Hilbert space H_P, and $\mathcal{A}_{P,\frac{1}{N}}$ the conformal net associated to $(P, \frac{1}{N}\langle\,\cdot\,\rangle)$ on H where the extra subscript $\frac{1}{N}$ indicates that the inner product on P is $\frac{1}{N}\langle\,\cdot\,\rangle$. Using φ_1 we get a covariant representation π of $\mathcal{A}_P$ on H such that

$$\pi(\pi_P(e^{2\pi i f(\theta)})) = \pi(\varphi_1(e^{2\pi i f})) = \pi(e^{2\pi i f(N\theta)})$$

for any localized $e^{2\pi i f}$. Denote by Pr the projection from H to $\overline{\pi(\mathcal{L}_t T)\Omega}$.

Definition 4.7. *Let $e^{2\pi i f} \in \mathcal{L}_{\mathbb{R}}T$, and*

$$\pi(e^{2\pi i f(N\theta)}) = \prod_{i=0}^{N-1} Ad_{R^i}(\pi(e^{2\pi i \hat{f}(\theta)}))$$

with $\hat{f}(\theta) = f(N\theta)$ if $0 \le \theta \le \frac{1}{N}$, and $\hat{f}(\theta) = \Delta_f$ if $\frac{1}{N} \le \theta \le 1$, and R is the rotation by $\frac{2\pi}{N}$. Define

$$\pi_\sigma^{(\xi)}(e^{2\pi i f(\theta)}) = \prod_{i=0}^{N-1} Ad_{\sigma^i R^i}(\pi(e^{2\pi i \hat{f}(\theta)}))Pr.$$

Proposition 4.8. *$\pi_\sigma^{(\xi)}$ extends to a normal representation of $\mathcal{A}_P(\mathbb{R})$ and restricts to a DHR representation of $\mathcal{A}_P^\Gamma$.*

Proof. Fix $I \subset \mathbb{R} \simeq (0,1) \simeq S^1 \setminus \{\xi\}$, and assume that $\mathrm{supp} e^{2\pi i f(\theta)} \subset I$. Then

$$\pi^{(\xi)}(e^{2\pi i f(N\theta)}) = \prod_{i=0}^{N-1} Ad_{R^i}(\pi(e^{2\pi i \hat{f}(\theta)})) \subset \mathcal{A}_{P,\frac{1}{N}}(I_1) \vee \cdots \vee \mathcal{A}_{P,\frac{1}{N}}(I_N)$$

where $I_i^N = I$, $i = 1, ..., N$, as intervals on S^1, and we choose the ordering so that on $(0,1)$, I_i is to the left of I_{i+1}. Since $\mathcal{A}_{P,\frac{1}{N}}$ is split by Prop. 3.11, there is a normal isomorphism

$$\chi_I : \mathcal{A}_{P,\frac{1}{N}}(I_1) \otimes \cdots \otimes \mathcal{A}_{P,\frac{1}{N}}(I_N) \to \mathcal{A}_{P,\frac{1}{N}}(I_1) \vee \cdots \vee \mathcal{A}_{P,\frac{1}{N}}(I_N)$$

so that $\chi_I(x_1 \otimes \cdots \otimes x_N) = x_1 \cdots x_N, x_i \in \mathcal{A}_{P,\frac{1}{N}}(I_i), i = 1, ..., N$. Note that the map $U_\sigma : x_1 \otimes \cdots \otimes x_N \to x_1 \otimes \sigma(x_2) \otimes \cdots \otimes \sigma^{N-1}(x_N)$ is normal. So we have

$$\pi_\sigma^{(\xi)}(e^{2\pi i f(\theta)}) = \chi_I U_\sigma \chi_I^{-1}(\pi(e^{2\pi i f(N\theta)}))$$

if $\mathrm{supp} e^{2\pi i f} \subset I$. Thus $\pi_\sigma^{(\xi)}(x) = \chi_I U_\sigma \chi_I^{-1}(\pi(\varphi_1(x)))$ is a normal representation of $\mathcal{A}_P(I)$.

Note that

$$\prod_{i=0}^{N-1} Ad_{\sigma^i R^i}(Ad_{R^j} x) = \prod_{i=0}^{N-1} Ad_{\sigma^i R^i}(Ad_{\sigma^{-j}} x)$$

for $x \in \mathcal{A}_{P,\frac{1}{N}}(I_1) \vee \cdots \vee \mathcal{A}_{P,\frac{1}{N}}(I_N)$ since σ and R commute.

Also note that $Ad_\sigma \pi(\varphi_1(x)) = \pi(\varphi_1(Ad_\sigma x))$ for $x \in \mathcal{A}_P(I)$. So if $x \in \mathcal{A}_P^{\langle\sigma\rangle}(I)$, then

$$\prod_{i=0}^{N-1} Ad_{\sigma^i R^i}(Ad_{R^j}(\pi(\varphi_1(x)))) = \prod_{i=0}^{N-1} Ad_{\sigma^i R^i}(\pi(\varphi_1(x))).$$

So when we change $\xi \in S^1$ to $\xi^1 \in S^1$ we have $\pi_\sigma^{(\xi)} = \pi_\sigma^{(\xi^1)}$ when restricting to $\mathcal{A}_P^{\langle\sigma\rangle}$. It follows that $\pi_\sigma^{(\xi)}$ restricts to a DHR representation of $\mathcal{A}_P^{\langle\sigma\rangle}$. ■

Remark 4.9. *Note that the definition of soliton above is similar to the soliton given in [19], but without using* $\mathrm{Diff}(S^1)$ *and hence are different. We will write* $\pi_\sigma^{(\xi)}$ *simply as* π_σ *when restricting to* $\mathcal{A}_P^{\langle\sigma\rangle}$.

Proposition 4.10. $\pi_\sigma^{(\xi)}$ *is an irreducible soliton of* $\mathcal{A}_P$.

Proof. First we prove that the representation $\pi(\mathcal{L}_t T)Pr$ on PrH is irreducible. Notice that this is a representation with positive energy, and the identity component $(\mathcal{L}_t T)^o$ is the product of a Heisenberg group and $T_0 := \exp^{2\pi i P_0(\mathbb{R}P)}$. It follows from (1)-(2) of Lemma 3.5 that the representation $\pi(\mathcal{L}_t T)Pr$ of $\mathcal{L}_t T$ is irreducible. To prove the proposition it is enough to show that $\pi_\sigma^{(\xi)}(\mathcal{A}_P(\mathbb{R}))'' = \pi(\mathcal{L}_t T)'' Pr$, and it is sufficient to show that $\pi(e^{2\pi i g})Pr \in \pi_\sigma^{(\xi)}(\mathcal{A}_P(\mathbb{R}))''$ for any $e^{2\pi i g} \in (\mathcal{L}_t T)^o$.

Let $x(\theta)$ be a complex valued function on $[0,1]$ with $x(\theta + \frac{1}{N}) = x(\theta)$, and $x^{(n)}(0) = x^{(n)}(\frac{1}{N}) = 0$ for all n. It follows from Lemma 1.2.2 of [23] that for any $\epsilon > 0$ one can choose x_ϵ such that $||x_\epsilon - 1||_{\frac{1}{2}} < \epsilon$ (cf. §1.2 of [23] for the definition of norm $||.||_{\frac{1}{2}}$), and by Proposition 1.3.2 of [23] we have that $\pi(e^{2\pi i x_\epsilon g}) \to \pi(e^{2\pi i g})$ strongly. Note that $\pi(e^{2\pi i x_\epsilon g}) = \pi(\varphi_2\varphi_1(e^{2\pi i f_\epsilon}))$, where $e^{2\pi i f_\epsilon} \in \mathcal{L}_{\mathbb{R}} T$ with $f_\epsilon(\theta) = x_\epsilon(\frac{\theta}{N}) g(\frac{\theta}{N})$ for $0 \le \theta \le 1$. It follows that $\pi(e^{2\pi i g})Pr \in \pi_\sigma^{(\xi)}(\mathcal{A}_P(\mathbb{R}))''$. ■

Definition 4.11. *Let* $\lambda(\theta) : [0,1] \to \mathbb{C}P$ *be a smooth map with* $\lambda(0) = 0$, $\lambda(1) = \lambda \in P^*$, *and* $\lambda^{(n)}(0) = \lambda^{(n)}(1) = 0$ *for all positive* n. *Define an automorphism* $Ad_{\lambda(\theta)}$ *on* $\mathcal{L}_t T$ *such that*

$$Ad_{\lambda(\theta)}(e^{2\pi i(NP_0(\alpha)\theta + \alpha_*)}, x) = (e^{2\pi i(NP_0(\alpha)\theta+\alpha_*)}, x e^{2\pi i \int \langle \lambda' | P_0(\alpha)\theta + \alpha_* \rangle d\theta}),$$

$$Ad_{\lambda(\theta)}(e^{2\pi i h(\theta)}, y) = (e^{2\pi i h(\theta)}, y e^{2\pi i N \int_0^{\frac{1}{N}} \langle \lambda'(N\theta) | h(\theta) \rangle d\theta})$$

where $h(\theta + \frac{1}{N}) = \sigma h(\theta)$ *for* $0 \le \theta \le 1$, $\int h d\theta = 0$.

Lemma 4.12. *For any* $e^{2\pi i f} \in \mathcal{L}_{\mathbb{R}} T$ *we have*

$$\pi(\varphi_2\varphi_1(Ad_{\lambda(\theta)} e^{2\pi i f})) = \pi(Ad_{\lambda(\theta)}\varphi_2\varphi_1(e^{2\pi i f})).$$

Proof. It is straightforward by definition. ■

Proposition 4.13. $\pi_\sigma^{(\xi)}(Ad_\lambda)$ *is an irreducible soliton of* $\mathcal{A}_P(\mathbb{R})$ *which corresponds to an irreducible representation of* $\mathcal{L}_t T$ *where the central group* $(P^*/P)^\sigma$ *of* $\mathcal{L}_t T$ *acts as* $\mu \in (Q^*/Q)^\sigma \mapsto e^{2\pi i \langle \mu | \lambda \rangle}$.

Proof. $\pi_\sigma^{(\xi)}(Ad_\lambda)$ is irreducible since $\pi_\sigma^{(\xi)}$ is by Proposition 4.10. The second statement follows by lemma 4.12. ■

To make contact with the results in Section 4 of [2], and to motivate the definitions in the next section, let us define (compare to Section 4.3 of [2]).

Definition 4.14. *Let $G_P = S^1 \times \exp(2\pi i P_0(\mathbb{R}P)) \times P$ be the set consisting of elements $ce^h U_\alpha$ ($c \in S^1, h \in 2\pi i P_0(\mathbb{R}P), \alpha \in P$). Define a multiplication in G_P by the formulas*

$$e^h e^{h'} = e^{h+h'}$$

$$e^h U_\alpha e^{-h} = e^{-\langle h|\alpha\rangle} U_\alpha$$

$$U_\alpha U_\beta = U_{\alpha+\beta}.$$

Note that G_P is a group. Recall the Heisenberg group $\tilde{V}_Q$ associated with a lattice Q and inner product $\langle , \rangle$ on Q before Lemma 3.5. We will consider a Heisenberg group $\tilde{V}_{P,\frac{1}{N}}$ associated with the lattice P and inner product $\frac{1}{N}\langle , \rangle$ on P.

Lemma 4.15. *We have the isomorphism*

$$\mathcal{L}_t T \cong G_P \times \tilde{V}_{P,\frac{1}{N}}.$$

Proof. First we note that $\mathcal{L}_t T$ is generated by $e^{2\pi i(NP_0(\alpha)\theta+\alpha_*)}$ and $e^{2\pi i h(\theta)}$ with $h(\theta+\frac{1}{N}) = \sigma h(\theta)$ for $0 \le \theta \le 1$, $\int h d\theta = 0$. For any $\mu \in (P^*/P)^\sigma$, $e^{2\pi i\mu}$ is in the center of $\mathcal{L}_t T$.

Let $\psi : \mathcal{L}_t T \to G_P \times \tilde{V}_{P,\frac{1}{N}}$ be a map such that

$$\varphi(e^{2\pi i h}, x) = (e^{2\pi i h}, x) \in \tilde{V}_{P,\frac{1}{N}}$$

$$\varphi(e^{2\pi i(N\pi_0(\alpha)\theta+\alpha_*)}, y) = yU_\alpha \in G_P$$

$$\varphi(e^{2\pi i h_0}) = e^{2\pi i h_0} \in G_P$$

for $h(\theta)$ with $h(\theta+\frac{1}{N}) = \sigma h(\theta), 0 \le \theta \le 1$, $\int h d\theta = 0$, and $h_0 \in P_0(\mathbb{R}P)$. One checks directly by definition that φ is an isomorphism of groups. ■

We note that a class of irreducible representations of $G_P \times \tilde{V}_{P,\frac{1}{N}}$ is given by Theorem 4.2 of [2], and these irreducible representations are determined uniquely by the action of central subgroup $(Q^*/Q)^\sigma$ of G_P as follows: Given $\mu \in Q^*/Q$, there is an irreducible representation π_μ of $G_P \times \tilde{V}_{P,\frac{1}{N}}$ on a Hilbert space H_μ such that $\lambda \in (Q^*/Q)^\sigma$ acts as $e^{2\pi i\langle\lambda|\mu\rangle}$. By using Proposition 4.13 and Lemma 4.15, it is easy to check that π_μ corresponds to $\pi_\sigma(Ad_{\mu(\theta)})$. Denote by $\pi_{\sigma,\mu} = \pi_\sigma(Ad_{\mu(\theta)})$ and note that using this notation $\pi_\sigma = \pi_{\sigma,0}$.

Proposition 4.16. *Let $\sigma_1 \in \Gamma$. Then*

(1) $\pi_{\sigma,\mu}(Ad\sigma_1) \cong \pi_{\sigma_1^{-1}\sigma\sigma_1,\sigma_1^{-1}\mu}$ as solitons of $\mathcal{A}_P(\mathbb{R})$.

(2) $\pi_{\sigma,\lambda_1} \cong \pi_{\sigma,\lambda_2}$ if and only if $\lambda_1 - \lambda_2 \in P_\sigma^*$.

Proof. (1) First note that $Ad_{\mu(\theta)}Ad_{\sigma_1} = Ad_{\sigma_1}Ad_{\sigma_1^{-1}\mu(\theta)}$ on $\mathcal{L}_{\mathbb{R}}T$, and so it is sufficient to show that $\pi_\sigma(Ad_{\sigma_1}) \cong \pi_{\sigma_1^{-1}\sigma\sigma_1}$. Fix $I \subset \mathbb{R}$, let I_i be intervals on S^1 so that $I_i^N = I$, $i = 1, ..., N$. Then

$$\prod_{i=1}^{N} Ad_{\sigma^i R^i}(Ad_{\sigma_1}x) = Ad_{\sigma_1}(\prod_{i=1}^{N} Ad_{(\sigma_1^{-1}\sigma\sigma_1)^i R^i}(x))$$

for all $x \in A_{P,\frac{1}{N}}(I_1 \vee \cdots \vee I_N)$. It follows that $\pi_\sigma(Ad_{\sigma_1}) \cong \pi_{\sigma_1^{-1}\sigma\sigma_1}$.

(2) By Proposition 4.13, $\pi_{\sigma,\lambda_1} \cong \pi_{\sigma,\lambda_2}$ if and only if $\langle\mu|\lambda_1 - \lambda_2\rangle \in \mathbb{Z}$ for all $\mu \in (P^*/P)^\sigma = P_\sigma/P$. It follows that $\pi_{\sigma,\lambda_1} \cong \pi_{\sigma,\lambda_2}$ if and only if $\lambda_1 - \lambda_2 \in P_\sigma^*$. ∎

Consider $\mathcal{A}_P^{\langle\sigma\rangle} \subset \mathcal{A}_P$, the fixed point subnet of $\mathcal{A}_P$ under the action of σ. Let $V \in \mathcal{A}_P(I)$ be a unitary such that $\sigma(V) = e^{\frac{2\pi i}{N}}V$ (cf. §8 of [19]).

Ad_V induces a DHR representation of $\mathcal{A}_P^{\langle\sigma\rangle}$. By Proposition 4.8 and Prop. 8.2 of [19] π_σ decomposes into a direct sum of N irreducible representations of $\mathcal{A}_P^{\langle\sigma\rangle}$. Let $\pi_{\sigma 0}$ be one of such irreducible representations of $\mathcal{A}_P^{\langle\sigma\rangle}$ obtained from π_σ by projecting onto σ invariant subspace. Set $\pi_\nu = \pi_{\sigma 0}(AdV)$. By Proposition 4.8, π_ν restricts to a DHR representation of $\mathcal{A}_P^{\langle\sigma\rangle}$, and it is a covariant representation of $\mathcal{A}_P^{\langle\sigma\rangle}$. In fact, $\pi(\varphi(\cdot))$ is a covariant representation of $\mathcal{A}_P^{\langle\sigma\rangle}$ as $\mathcal{A}_P^{\langle\sigma\rangle}$ is conformal and $\pi(\varphi(\cdot))$ restricts to a DHR representation of $\mathcal{A}_P^{\langle\sigma\rangle}$ (cf. [1]).

Let

$$U_I = \{g \in \mathbf{G} | gI \cup I \subset S^1 - \{\xi\}\}$$

be a neighborhood of identity in $\mathbf{G}$. For $g \in U_I$ we have

$$\pi_\nu(gxg^*) = \pi_\nu(g)\pi_\nu(x)\pi_\nu(g^*) = \pi_{\sigma 0}(g)\pi_{\sigma 0}(g^*VgV^*)\pi_\nu(x)\pi_{\sigma 0}(Vg^*V^*g)\pi_{\sigma 0}(g)^*$$

for all $x \in \mathcal{A}_P^{\langle\sigma\rangle}$. So we have $\pi_\nu(g)^*\pi_{\sigma 0}(g)\pi_{\sigma 0}(g^*VgV^*)^* \in \mathbb{C}1$ for all $g \in U_I$. It follows that $\pi_\nu(g) = \pi_{\sigma 0}(g)\pi_{\sigma 0}(g^*VgV^*)$ for all $g \in U_I$ as the only one dimensional representation of $\mathbf{G}$ is the trivial representation.

Consider $\theta \to \pi_{\sigma 0}(R_\theta)$ and set

$$F(\theta) = Ad_{\pi(r_{-\frac{\theta}{N}})}(\pi(\varphi(V)))\pi(\varphi(V))^* Pr_{\sigma 0}$$

where $\pi(r_{-\frac{\theta}{N}})$ denotes the unitary operator on H implementing rotation by $-\frac{\theta}{N}$ on $\mathcal{A}_{P,\frac{1}{N}}$, and $Pr_{\sigma 0} = Pr\sum_{1\leq\sigma\leq N}\sigma^i$ is the projection onto the irreducible representation $\pi_{\sigma 0}$ of $\mathcal{A}_P^{\langle\sigma\rangle}$. Note that if $\theta \in U_I$ then $F(\theta) = \pi_\sigma(R_\theta)\pi_{\sigma 0}(R_\theta^* V R_\theta V^*)$. Also $\pi(\varphi(AdR_\theta, y)) = \pi(Ad_{r_{\frac{\theta}{N}}}(\varphi(y)))$ for all $y \in \mathcal{A}_P^{\langle\sigma\rangle}$ by definitions, i.e.,

$$\pi(R_\theta)\pi(\varphi(y))\pi(R_\theta)^* = \pi(r_{\frac{\theta}{N}})\pi(\varphi(y))\pi(r_{\frac{\theta}{N}})^*$$

for $y \in \mathcal{A}_P^{\langle\sigma\rangle}$. Using this one checks easily that

$$F(\theta_1 + \theta_2) = F(\theta_1)F(\theta_2).$$

It follows that $F(\theta) = \pi_\nu(R_\theta)$, for all θ, since both sides are one-parameter group of unitaries which agree on a neighborhood of 0.

On the other hand,

$$
\begin{aligned}
F(2\pi) &= \pi_{\sigma 0}(R_{2\pi} Ad_{\pi(r_{-\frac{2\pi}{N}})}(\pi(\varphi(V)))\pi(\varphi(V^*)) \\
&= \pi_{\sigma 0}(R_{2\pi}\pi(\varphi(\sigma(V)V^*)) \\
&= e^{\frac{2\pi i}{N}}\pi_{\sigma 0}(R_{2\pi}).
\end{aligned}
$$

where we have used $\sigma(V)V^* = e^{\frac{2\pi i}{N}}$. It follows that the univalence of $\pi_{\sigma 0}(AdV)$ is the univalence of $\pi_{\sigma 0}$ multiplied by $e^{\frac{2\pi i}{N}}$. By monodromy equation (cf. [9]), we have proved the following

Proposition 4.17. *$G(\pi_{\sigma 0}, AdV) = e^{\frac{2\pi i}{N}}$, where $G(\cdot,\cdot)$ is defined as in Lemma 8.3 of [19].*

Remark 4.18. *The same argument as in the proof of Proposition 4.17 shows that $K(1) = 1$ in the paragraph after (47) of [19].*

4.1 General case

When $\frac{1}{N}\langle\cdot\rangle$ is not an even integral on Q, we do not have an analogue of Proposition 4.6 and net $\mathcal{A}_{Q,\frac{1}{N}}$. However we have a subnet $\mathcal{A}_{2NQ} \subset \mathcal{A}_Q$ where we can apply the construction of Section 2.1. Also there is no $\mathcal{L}_t T$ in general case, but we have $\tilde{V}_{Q,\frac{1}{N}}$ and an analogue G_Q as defined as follows (cf. Section 4.3 of [2]).

Definition 4.19. *Let $G_Q = S^1 \times \exp(2\pi i P_0(\mathbb{R}Q)) \times Q$ be the set consisting of elements of the form $ce^h U_\alpha$, $(c \in S^1, h \in 2\pi i P_0(\mathbb{R}Q), \alpha \in P)$ with multiplication*

$$e^h e^{h'} = e^{h+h'}$$

$$e^h U_\alpha e^{-h} = e^{-\langle h|\alpha\rangle} U_\alpha$$

$$U_\alpha U_\beta = \epsilon(\alpha,\beta) e^{\pi i \langle\alpha|\beta_*\rangle - \frac{1}{2}\langle\alpha|\beta\rangle + \frac{1}{2}\langle\alpha|P_0(\beta)\rangle} U_{\alpha+\beta}.$$

One checks easily that G_Q is a group.

Remark 4.20. *Our group G_Q is slightly different from G of Section 4.3 of [2], the commutator among U_α, U_β is the complex conjugate of 4.44 of [2]. The reason for defining the multiplication rule for $U_\alpha U_\beta$ comes from the following computations: Let $h(\theta)$ be as in (1) of Examples 4.4, then $h(\theta) = N\pi_0(\alpha)\theta + \alpha_* + h_\alpha(\theta)$. Regarding $e^{2\pi i h_\alpha(\theta)}$ as an element in $\tilde{V}_{Q,\frac{1}{N}}$ we have*

$$e^{2\pi i h_\alpha(\theta)} e^{2\pi i h_\beta(\theta)} = e^{\pi i\langle\alpha|\beta_*\rangle - \frac{1}{2}\langle\alpha|\beta\rangle + \frac{1}{2}\langle\alpha|P_0(\beta)\rangle} e^{2\pi i(h_\alpha(\theta)+h_\beta(\theta))}.$$

Hence if we map $e^{i\alpha\theta}$ to $U_\alpha e^{2\pi i h_\alpha(\theta)}$, to preserve the commutator relations we need $U_\alpha U_\beta = \epsilon(\alpha,\beta) e^{\pi i\langle\alpha|\beta_*\rangle - \frac{1}{2}\langle\alpha|\beta\rangle + \frac{1}{2}\langle\alpha|P_0(\beta)\rangle} U_{\alpha+\beta}$.

We will treat G_P (cf, Definition 4.14) as a subgroup of G_Q under the natural map $G_P \to G_Q$.

According to Lemma 4.15, the analogue of $\mathcal{L}_t T$ is now replaced by the group $G_Q \times \tilde{V}_{Q,\frac{1}{N}}$.

Definition 4.21. *For $\lambda(\theta)$ as in Definition 4.11 with $\lambda \in Q^*$ define an automorphism on $G_Q \times \tilde{V}_{Q,\frac{1}{N}}$ by*

$$Ad_{\lambda(\theta)} U(\alpha) = U(\alpha) e^{2\pi i \int \langle \lambda' | P_0(\alpha)\theta + \alpha_* \rangle}$$

$$Ad_{\lambda(\theta)} e^{ih} = e^{ih} e^{2\pi i N \int_0^{\frac{1}{N}} \langle \lambda'(N\theta) | h(\theta) \rangle d\theta}$$

for h with $h(\theta + \frac{1}{N}) = \sigma(h(\theta))$.

Definition 4.22. *Let $f(\theta) = \Delta_f \theta + f_0 + f_1$. Define $\hat{f}(\theta) = f(N\theta)$, $0 \leq \theta \leq \frac{1}{N}$ and $\hat{f}(\theta)$ is a continuous function with $\hat{f}(\theta + \frac{1}{N}) = \sigma(\hat{f}(\theta))$ modulo Q. It follows that*

$$\hat{f}(\theta) = N P_0(\alpha)\theta + P_0(f_0) + (\Delta_f)_* + \hat{f}_1(\theta)$$

where $\hat{f}_1(\theta + \frac{1}{N}) = \sigma(\hat{f}_1(\theta))$ and $\int_{S^1} \hat{f}_1 d\theta = 0$. Then the map φ is defined as

$$\varphi(e^{if(\theta)}) = U(\Delta_f) e^{2\pi i (P_0(f_0) + \hat{f}_1(\theta))} e^{-\pi i \langle P_0(\Delta f) | f_0 \rangle}.$$

Lemma 4.23. *$\varphi : \mathcal{L}_{\mathbb{R}} T \to G_Q \times \tilde{V}_{Q,\frac{1}{N}}$ is a group homomorphism and $\varphi(Ad_{\lambda(\theta)} x) = Ad_{\lambda(\theta)} \varphi(x)$ for all $x \in \mathcal{L}_{\mathbb{R}} T$.*

Proof. The proof is a direct (but tedious) check by using definitions. ■

Fix $\sigma \in \Gamma$ with $\sigma^N = 1$. Let π_λ be an irreducible representation of $G_Q \times \tilde{V}_{Q,\frac{1}{N}}$ as given by Theorem 4.2 of [2]. This is an irreducible representation of $G_Q \times \tilde{V}_{Q,\frac{1}{N}}$ on a Hilbert space H_λ where the central subgroup $(Q^*/Q)^\sigma$ of G_Q acts as the character $e^{2\pi i \langle \lambda | \mu \rangle}$ for $\mu \in (Q^*/Q)^\sigma$. This representation, when restricting to $G_P \times \tilde{V}_{Q,\frac{1}{N}}$, decomposes into direct sum of finitely many irreducible representations of $G_P \times \tilde{V}_{Q,\frac{1}{N}}$:

$$H_\lambda = \bigoplus_\omega H_{\lambda,\omega} \otimes K_\omega$$

where K_ω is an irreducible representation of $G_P \times \tilde{V}_{Q,\frac{1}{N}}$ as given by Theorem 4.2 of [2] for lattice $P = 2NQ$, and the central subgroup $(P^*/P)^\sigma$ of G_P acts by the character $e^{2\pi i \langle \mu_1 | \omega \rangle}$ for $\mu_1 \in (P^*/P)^\sigma$ and each $H_{\lambda,\omega}$ is of finite dimensional. We note that by Proposition 4.13, K_ω corresponds to representation $\pi_{\sigma,\omega}$ of $\mathcal{A}_P(\mathbb{R})$. Fix an interval $I \subset S^1 - \{\xi\}$, and a set of representatives $\alpha_1, \cdots, \alpha_k$, $k = (\operatorname{rank} Q)^{2N}$ for the finite abelian group Q/P. By abuse of notations, in this section we will use $\mathcal{L}_{\mathbb{R}} T$ to be the central extension of $L_{\mathbb{R}} T$ as in Definition 4.5, but associated with lattice Q. Choose $e^{2\pi i f_{\alpha_i}} \in \mathcal{L}_{\mathbb{R}} T$ so that $\Delta_{f_{\alpha_i}} = \alpha_i$, $i = 1, ..., k$, and $\operatorname{supp} e^{2\pi i f_{\alpha_i}} \subset I$. Note that for each $I \subset J \subset \mathbb{R}^1 - \{\xi\}$, every element $x \in \mathcal{A}_Q(J)$ can be written uniquely as $x = \sum_{s=1}^k x_i \pi(e^{2\pi i f_s})$, where $x_i \in \mathcal{A}_P(J)$.

Definition 4.24. *With notations as above, we define*

$$\pi_{\sigma,\lambda}^{(\xi)}(x) = \sum_{1\leq s\leq k,\omega} Id_{\lambda,\omega} \otimes \pi_{\sigma,\omega}^{(\xi)}(x_i)\pi_\lambda(e^{2\pi i f_s})$$

for $x \in \mathcal{A}_Q(J)$.

Proposition 4.25. $\pi_{\sigma,\lambda}^{(\xi)}$ *as defined in Definition 4.24 is an irreducible soliton representation of* $\mathcal{A}_Q$. $\pi_{\sigma,\lambda}^{(\xi)}(Ad\mu(\theta)) \cong \pi_{\sigma,\lambda+\mu}^{(\xi)}$, $\pi_{\sigma,\lambda}^{(\xi)}(Ad\sigma_1) = \pi^{(\xi)}_{\sigma_1^{-1}\sigma\sigma_1,\sigma_1^{-1}\lambda}$ *for* $\sigma_1 \in \Gamma$, *and* $\pi_{\sigma,\lambda_1}^{(\xi)} \simeq \pi_{\sigma,\lambda_2}^{(\xi)}$ *iff* $\lambda_1 - \lambda_2 \in Q_\sigma^*$. *Moreover* $\pi_{\sigma,\lambda}^{(\xi)}$ *restricts to a DHR representation of* $\mathcal{A}_Q^{\langle\sigma\rangle}$.

Proof. Let J be the interval as defined before Definition 4.24. By Propositions 4.8 and 4.13, the map $x \in \mathcal{A}_Q(J) \to \pi_{\sigma,\lambda}^{(\xi)}(x)$ is normal. To check that $\pi_{\sigma,\lambda}^{(\xi)}$ is a homomorphism, it is enough to check $\pi_{\sigma,\lambda}^{(\xi)}(x_1x_2) = \pi_{\sigma,\lambda}^{(\xi)}(x_1)\pi_{\sigma,\lambda}^{(\xi)}(x_2)$ for x_1, x_2 in a set of generators for $\mathcal{A}_Q(J)$. We can choose this set of generators to be elements of $\mathcal{L}_J T$. It follows from Lemma 4.23 that $\pi_{\sigma,\lambda}$ is a homomorphism. The rest of the proposition follows from Propositions 4.8, 4.16 and definitions. ■

We will use $\pi_{\sigma,\lambda}$ to denote the DHR representation of $\mathcal{A}_Q^{\langle\sigma\rangle}$ from Prop. 4.25 and let $\pi_{\sigma 0}$ be an irreducible subrepresentation of $\pi_{\sigma,0}|_{\mathcal{A}_Q^\sigma}$.

Proposition 4.26. $G(\pi_{\sigma 0}, AdV) = e^{\frac{2\pi i}{N}}$ *where* $G(\cdot,\cdot)$ *is defined as in Lemma 8.3 of [19].*

Proof. Since $\pi_{\sigma 0}(R_\theta) \in \pi_{\sigma 0}(\mathcal{A}_Q^\sigma)''$, it follows by Proposition 4.17 and definition of π_σ that the univaluence of $\pi_{\sigma 0}(\mathcal{A}_Q^\sigma)$ is the univalence of $\pi_{\sigma 0}$ multiplied by $e^{\frac{2\pi i}{N}}$, and this implies the proposition by monodromy equation (cf. [9]). ■

4.2 Irreducible representations of $\mathcal{A}_Q^\Gamma$.

By Prop.4.25, $\pi_{\sigma,\lambda_1}^{(\xi)} \simeq \pi_{\sigma,\lambda_2}^{(\xi)}$ iff $\lambda_1 - \lambda_2 \in Q_\sigma^*$. By using lemma 4.2, we will identify λ with its image in $(Q^*/Q)^\sigma$ under the composition of quotient map $Q^* \to Q^*/Q_\sigma^*$ and $Q^*/Q_\sigma^* \simeq Q_\sigma/Q = (Q^*/Q)^\sigma$ in this section.

Theorem 4.27. *Let* $\sigma_i \in \Gamma$, $\lambda_i \in (Q^*/Q)^\sigma$, *and* $\pi_{\sigma_i,\lambda_i}^{(\xi)}$ *be solitons of* $\mathcal{A}_Q^\Gamma$ *as in proposition 4.25. Then* $\pi_{\sigma_1,\lambda_1}^{(\xi)} \cong \pi_{\sigma_2,\lambda_2}^{(\xi)}$ *as solitons of* $\mathcal{A}_Q$ *if and only if* $\sigma_1 = \sigma_2$ *and* $\lambda_1 = \lambda_2$.

Proof. The proof is similar to that of Theorem 7.1 of [15]. It is sufficient to show that $\pi_{\sigma_1,\lambda_1}^{(\xi)} \cong \pi_{\sigma_2,\lambda_2}^{(\xi)}$ implies $\sigma_1 = \sigma_2$. Note that $\pi_{\sigma_i,\lambda_i}^{(\xi)}$ restrict to DHR representations of $\mathcal{A}_Q^{\langle\sigma_i\rangle}$. It follows that $\pi_{\sigma_1,\lambda_1}^{(\xi)}$ restricts to a DHR representation of $\mathcal{A}_Q^{\langle\sigma_1\rangle} \vee \mathcal{A}_Q^{\langle\sigma_2\rangle}$, and by Proposition 4.26 and the proof (2) of Proposition 7.2 of [15] we must have $\mathcal{A}_Q^{\langle\sigma_1\rangle} \vee \mathcal{A}_Q^{\langle\sigma_2\rangle} = \mathcal{A}_Q^{\langle\sigma_1\rangle}$. By Galois correspondence (cf. [13]) we have $\langle\sigma_2\rangle \subset \langle\sigma_1\rangle$. Exchanging σ_1 and σ_2 we conclude that $\langle\sigma_2\rangle = \langle\sigma_1\rangle$. So $\sigma_2 = \sigma_1^m$ for some integer m

with $(m, order(\sigma_1)) = 1$. By Prop. 4.26, the same proof as in (1) of Proposition 7.2 of [15] shows that $m = 1$. ■

We will use the following simple lemma, and we refer the reader to §2 of [15] for definitions of endomorphisms.

Lemma 4.28. *Let $\rho \in \mathrm{End}(M)$ be an endomorphism and assume that $[\rho\tau_i] = [\rho]$ for $i = 1, ..., k$ where $\tau_i \in \mathrm{Aut}(M)$ and $[\tau_i] \neq [\tau_j]$ if $i \neq j$. Then $d_\rho^2 \geq k$.*

Proof. If $d_\rho = \infty$ there is nothing to prove. Assume that d_ρ is finite. By by Frobenius duality $[\bar{\rho}\rho]$ contains $\bigoplus_{i=1}^k [\tau_i]$ and so $d_\rho^2 \geq k$. ■

Proposition 4.29. *Let $\pi_{\sigma,\lambda}^{(\xi)}$ be as in Proposition 4.25. Then $d(\pi_{\sigma,\lambda}^{(\xi)})^2 \geq \frac{|Q*/Q|}{|(Q^*/Q)^\sigma|}$.*

Proof. By Proposition 4.25,

$$\pi_{\sigma,\lambda}^{(\xi)}(Ad_{\mu(\theta)}) \cong \pi_{\sigma,\lambda+\mu}^{(\xi)}$$

for $\mu \in Q^*/Q$. It follows that $\pi_{\sigma,\lambda}^{(\xi)}(Ad_{\mu(\theta)}) \cong \pi_{\sigma,\lambda}^{(\xi)}$ if and only if $\mu \in Q_\sigma^*/Q$. The inequality now follows from Lemma 4.28 and Lemma 4.2. ■

Let $\pi_{\sigma_i,\lambda_i}^{(\xi)}$ be solitons of $\mathcal{A}_Q$ ($i = 1, 2$). By Proposition 4.4 of [15], $\pi_{\sigma_1,\lambda_1}^{(\xi)} \restriction \mathcal{A}^\Gamma \cong \pi_{\sigma_2,\lambda_2}^{(\xi)} \restriction \mathcal{A}^\Gamma$ if and only if there exists $\sigma \in \Gamma$ such that $\pi_{\sigma_1,\lambda_1}^{(\xi)}(Ad\sigma) = \pi_{\sigma_2,\lambda_2}^{(\xi)}$. By Proposition 4.25 and Theorem 4.27 we have $\sigma^{-1}\sigma_1\sigma = \sigma_2$ and $\sigma^{-1}\lambda_1 = \lambda_2$. This motivates the following consideration. Let $\{[\sigma_1], ..., [\sigma_m]\}$ be a list of conjugacy classes in Γ. Each π_{σ_i,λ_i} restricts to a DHR representation of $\mathcal{A}_Q^\Gamma$. Let $\Gamma_{\sigma_i} := \{\sigma \in \Gamma | \sigma\sigma_i = \sigma_i\sigma\}$ and let $[\lambda_{i,1}], ..., [\lambda_{i,m_i}]$ be the orbits of $(Q^*/Q)^{\sigma_i}$ under the action of Γ_{σ_i}. Let $\Gamma_{\sigma_i,\lambda_{i,s}} := \{\sigma \in \Gamma_{\sigma_i} | \sigma\lambda_{i,s} = \lambda_{i,s}\}$ for $s = 1, ..., m_i$. Then for any $\sigma \in \Gamma_{\sigma_i,\lambda_{i,s}}$, $\pi_{\sigma_i,\lambda_{i,s}}$ and $\pi_{\sigma_i,\lambda_{i,s}}(Ad\sigma)$ are isomorphic as representations of A_Q^Γ and by Theorem 4.8 of [15], there is a projective representation of $\Gamma_{\sigma_i,\lambda_{i,s}}$ on $H_{\sigma_i,\lambda_{i,s}}$ with cocycle $c_{\pi_{\sigma_i,\lambda_{i,s}}}$ such that $H_{\sigma_i,\lambda_{i,s}} = \bigoplus_\delta M_\delta \otimes H_{(\delta,\sigma_i,\lambda_{i,s})}$ where each M_δ is an irreducible representation of $\Gamma_{\sigma_i,\lambda_{i,s}}$ with cocycle $c_{\pi_{\sigma_i,\lambda_{i,s}}}$ (and all irreducible representations of $\Gamma_{\sigma_i,\lambda_{i,s}}$ with the same cocycle $c_{\pi_{\sigma_i,\lambda_{i,s}}}$ appear in the decomposition of $H_{\sigma_i,\lambda_{i,s}}$), and $H_{(\delta,\sigma_i,\lambda_{i,s})}$ supports an irreducible representation of $\mathcal{A}_Q^\Gamma$ with $H_{(\delta,\sigma_i,\lambda_{i,s})} \ncong H_{(\delta',\sigma_i,\lambda_{i,s})}$ if $\delta \neq \delta'$. By Theorem 4.5 of [15] and Prop. 4.29, for fixed $\sigma_i, \lambda_{i,s}$, $s = 1, ..., m_i$,

$$\sum_\delta d(\delta, \sigma_i, \lambda_{i,s})^2 = \frac{|\Gamma|^2}{|\Gamma_{\sigma_i,\lambda_{i,s}}|} d(\pi_{\sigma_i,\lambda_{i,s}})^2 \geq \frac{|\Gamma|^2}{|\Gamma_{\sigma_i,\lambda_{i,s}}|} \frac{|Q^*/Q|}{|(Q^*/Q)^{\sigma_i}|}.$$

By definition we have

$$|(Q^*/Q)^{\sigma_i}| = \sum_{1 \leq s \leq m_i} \frac{|\Gamma_{\sigma_i}|}{|\Gamma_{\sigma_i,\lambda_i,s}|},$$

and so

$$\sum_{\delta, 1 \leq s \leq m_i} d(\delta, \sigma_i, \lambda_{i,s})^2 \geq \frac{|\Gamma|^2}{|\Gamma_{\sigma_i}|} |Q^*/Q|.$$

Now sum over $1 \leq i \leq m$ and note that $|\Gamma| = \sum_{1 \leq i \leq m} \frac{|\Gamma|}{|\Gamma_{\sigma_i}|}$. We have

$$\sum_{\delta, 1 \leq i \leq k, 1 \leq s \leq m_i} d(\delta, \sigma_i, \lambda_{i,s})^2 \geq |\Gamma|^2 |Q^*/Q| = \mu_{\mathcal{A}_Q^\Gamma}$$

where we have used Cor. 3.19 and Th. 2.4. It follows that all " $\geq$" above are " $=$" by Theorem 2.4, and we have proved the following theorem:

Theorem 4.30. *$H_{\delta, \sigma_i, \lambda_{i,s}}$ for $i = 1, ..., m$, $\lambda_i \in (Q^*/Q)^{\sigma_i}$, $1 \leq s \leq m_i$ as above give a complete list of irreducible representations of $\mathcal{A}_Q^\Gamma$, and these representations generate a unitary modular tensor category.*

From the proof and Prop. 4.29 we have also proved:

Corollary 4.31.

$$d(\pi_{\sigma,\lambda}^{(\xi)}) = \sqrt{\frac{|Q^*/Q|}{|(Q^*/Q)^\sigma|}}.$$

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