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Ultradistributions and Time–Frequency Analysis

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Ultradistributions and Time–Frequency Analysis

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Abstract. The aim of the paper is to show the connection between the theory of ultradistributions and time-frequency analysis. This is done through time-frequency representations and modulation spaces. Furthermore, some classes of pseudodifferential operators are observed.

1. Introduction

The aim of the paper is to illustrate the natural connection between time-frequency analysis and the theory of (tempered) ultradistributions. This is done by the use of the so called time-frequency representations, Theorems 3.8 and 3.9. Another result is the definition of certain Gelfand-Shilov type spaces within the framework of modulation spaces, Proposition 4.3.

Pseudodifferential operators serve as yet another example which demonstrates how the combination of the techniques may lead to new results, Theorem 5.5. They are a traditional tool in micro-local analysis, [5], [33], [39], [51]. However, the use of the techniques of time-frequency analysis in the study of pseudodifferential operators lead to a new insight into their properties. Not only some classical results are generalized, e.g. the Calderon - Vaillancourt theorem, but also the scope of the theory has been enlarged. More concretely, operators whose symbols are non smooth, or the ones whose symbols may grow faster than polynomials are studied by the use of time-frequency analysis techniques, [19],[21], [22], [46], [47].

The paper is organized as follows. In Section 2 we introduce the notation. A survey of test function spaces is given in Section 3. Subsection 3.3 contains main results of the first part of the paper. Modulation spaces and their connection to certain test function spaces is given in Section 4. In the last Section we study the action of pseudodifferential operators on modulation spaces.

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2. Basic Notation

By **N**, **Z**, **R** and **C**, we denote sets of positive integers, integers, real numbers and complex numbers respectively, and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. By $\mathbf{R}^d$ ($\mathbf{N}_0^d$), $d \in \mathbf{N}$, we denote set of $d-$dimensional real numbers (nonnegative integers). Euclidean norm of $x \in \mathbf{R}^d$ is given by $|x| = \left(x_1^2 + \cdots + x_d^2\right)^{1/2}$, and $\langle x \rangle = (1 + |x|^2)^{1/2}$. For multiindices $\alpha, \beta \in \mathbf{N}_0^d$, we have $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\alpha! = \alpha_1! \cdots \alpha_d!$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $x \in \mathbf{R}^d$, and, for $\beta \leq \alpha$, i.e. $\beta_j \leq \alpha_j$, $j \in \{1, 2, \ldots, d\}$, $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdot \cdots \cdot \binom{\alpha_d}{\beta_d}$. The letter C denotes a positive constant, not necessarily the same at every appearance. Dual pairing between a test function space $\mathcal{A}$ and its dual $\mathcal{A}'$ is denoted by $\langle \cdot, \cdot \rangle = {}_{\mathcal{A}'}\langle \cdot, \cdot \rangle_{\mathcal{A}}$. By $\check{f}$ we denote the reflection $\check{f}(x) = f(-x)$. The operator of partial differentiation D is given by

$$D^\alpha = D_x^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d} = \left(\frac{1}{2\pi i} \frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{1}{2\pi i} \frac{\partial}{\partial x^d}\right)^{\alpha_d}$$

for all multiindices $\alpha \in \mathbf{N}_0^d$ and all $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$. If f and g are smooth enough, then the Leibnitz formula holds

$$D^\alpha (fg)(x) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^{(\alpha-\beta)} f(x) D^{(\beta)} g(x).$$

By Ω we denote an open subset of $\mathbf{R}^d$. The space of infinitely differentiable functions on Ω is denoted by $C^\infty(\Omega)$. Throughout the paper, the integrals are taken over $\mathbf{R}^d$, or $\mathbf{R}$, unless otherwise indicated. $L^2(\mathbf{R}^d)$ is the Hilbert space of square integrable functions with the inner product $(f, g) = \int f(x)\overline{g(x)}dx = \langle f, \overline{g} \rangle$. By $\|f\|$ we denote norm of $f \in L^2(\mathbf{R}^d)$. The Fourier transform of an integrable function f is defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int e^{-2\pi i x \xi} f(x) dx, \;\; \xi \in \mathbf{R}^d.$$

Translation and modulation operators, T and M are defined by

$$T_x f(\cdot) = f(\cdot - x) \quad \text{and} \quad M_x f(\cdot) = e^{2\pi i x \cdot} f(\cdot), \quad x \in \mathbf{R}^d.$$

The following relations hold

$$M_y T_x = e^{2\pi i x \cdot y} T_x M_y, \;\; (T_x f)\hat{} = M_{-x}\hat{f}, \;\; (M_x f)\hat{} = T_x \hat{f}, \quad x, y \in \mathbf{R}^d, f, g \in L^2(\mathbf{R}^d).$$

3. Test Function Spaces

We suppose that a reader is familiar with the "classical" Schwartz test function spaces $\mathcal{D}(\Omega)$, $\mathcal{E}(\Omega)$, $\mathcal{S}(\mathbf{R}^d)$ and their duals, $\mathcal{D}'(\Omega)$, $\mathcal{E}'(\Omega)$ and $\mathcal{S}'(\mathbf{R}^d)$, [41].

We have $\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}(\Omega) \hookrightarrow \mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$, and

$$\mathcal{D}(\mathbf{R}^d) \hookrightarrow \mathcal{S}(\mathbf{R}^d) \hookrightarrow \mathcal{E}(\mathbf{R}^d) \hookrightarrow \mathcal{E}'(\mathbf{R}^d) \hookrightarrow \mathcal{S}'(\mathbf{R}^d) \hookrightarrow \mathcal{D}'(\mathbf{R}^d),$$

where "$A \hookrightarrow B$" means that A is dense subset of B and that the inclusion mapping is continuous.

$\mathcal{S}(\mathbf{R}^d)$ and $\mathcal{S}'(\mathbf{R}^d)$ play a particularly important role in various applications since the Fourier transform is a topological isomorphism between $\mathcal{S}(\mathbf{R}^d)$ and $\mathcal{S}(\mathbf{R}^d)$ and extends to a continuous linear transform from $\mathcal{S}'(\mathbf{R}^d)$ onto itself.

In order to deal with particular problems in applications various generalizations of the Schwartz type spaces were proposed. We give here only a very brief and incomplete list in order to make our aim more transparent. More precisely, we restrict our exposition to Gevrey classes and certain Gelfand-Shilov type spaces. Their duals are spaces of ultradistributions.

Starting with the work of Beurling [2] and Roumieu [40] several theories of ultradistributions are developed, e.g. Denjoy-Carleman-Komatsu [30], Beurling-Björck [3], Braun-Meise-Taylor [4], Cioranescu-Zsidó [10]. The notion of equivalence of ultradistribution theories allows to transfer results from one theory to another. This includes, for example, topological properties, repesentation and structure theorems, behavior under various integral transforms, hypoellipticity, see [9], [6], [18], [27], [30], [34], [39].

3.1. Gevrey Classes

Gevrey type spaces fill the gap between the space of analytic functions $\mathcal{A}(\Omega)$ and the space of infinitely differentiable functions $C^\infty(\Omega)$. This turns out to be particularly important in the study of operators whose behavior differs in C^∞ and analytic framework (local and microlocal analysis). Recall, the space of analytic functions is defined by

$$\mathcal{A}(\Omega) = \{f \in C^\infty(\Omega) \;\mid\; (\forall K \Subset \Omega)(\exists C > 0)(\exists h > 0) \quad \sup_{x \in K} |\partial^\alpha f(x)| \leq C h^{|\alpha|} |\alpha|!\}.$$

$K \Subset \Omega$ means that K is a compact subset of Ω. For $1 < s < \infty$ we define a Gevrey class by

$$G^s(\Omega) = \{f \in C^\infty(\Omega) \;\mid\; (\forall K \Subset \Omega)(\exists C > 0)(\exists h > 0) \quad \sup_{x \in K} |\partial^\alpha f(x)| \leq C h^{|\alpha|} |\alpha|!^s\}.$$

we denote by $G_0^s(\Omega)$ a subspace of $G^s(\Omega)$ which consists of compactly supported functions. We have $\mathcal{A}(\Omega) \hookrightarrow \cap_{s>1} G^s(\Omega)$ and $\cup_{s\geq 1} G^s(\Omega) \hookrightarrow C^\infty(\Omega)$.

For $1 < s < \infty$ the Gevrey class $\mathcal{E}^{\{s\}}(\Omega)$ (resp. $\mathcal{E}^{(s)}(\Omega)$) consists of all $f \in C^\infty(\Omega)$ such that for any compact set K in Ω there are constants h and C (resp. for any $h > 0$ there is a constant C) such that

$$\sup_{x \in K} |\partial^\alpha f(x)| \leq C h^{|\alpha|} |\alpha|!^s.$$

$\mathcal{D}^{\{s\}}(\Omega)$ (resp. $\mathcal{D}^{\{s\}}(\Omega)$) is the space of all $f \in \mathcal{E}^{\{s\}}(\Omega)$ (resp. $f \in \mathcal{E}^{(s)}(\Omega)$) with compact support under a natural locally convex topology. The corresponding dual spaces $(\mathcal{E}^{(s)}(\Omega))'$ and $(\mathcal{D}^{(s)}(\Omega))'$ are spaces of Beurling ultradistributions, while $(\mathcal{E}^{\{s\}}(\Omega))'$ and $(\mathcal{D}^{\{s\}}(\Omega))'$ are spaces of Roumieu type ultradistributions, see[31]. Common notation for both projective and inductive limit is $*$ in the upper index (instead of (s) or $\{s\}$). The ultradistributions spaces are a generalization of the corresponding Schwartz distribution spaces since

$$\mathcal{E}^*(\Omega) \hookrightarrow \mathcal{E}(\Omega) \quad \text{and} \quad \mathcal{D}^*(\Omega) \hookrightarrow \mathcal{D}(\Omega)$$

imply $\mathcal{D}'(\Omega) \hookrightarrow (\mathcal{D}^*(\Omega))'$ and $\mathcal{E}'(\Omega) \hookrightarrow (\mathcal{E}^*(\Omega))'$. Also, $(\mathcal{E}^*(\Omega))' \hookrightarrow (\mathcal{D}^*(\Omega))'$.

Remark 3.1.

- Global definition, with $\Omega = \mathbf{R}^d$, is obtained by taking the supremum over $\mathbf{R}^d$.
- One may use $|\alpha|^{s|\alpha|}$ or $\Gamma(s|\alpha|+1)$ instead of $|\alpha|!^s$, [39, Proposition 1.4.2].
- Weighted ultradistribution spaces $(\mathcal{D}^*_{L^s}(\mathbf{R}^d))'$, $s \geq 1$, are studied in [6].
- Likewise $\mathcal{D}(\mathbf{R}^d)$ and $\mathcal{E}(\mathbf{R}^d)$, spaces $\mathcal{D}^*(\mathbf{R}^d)$ and $\mathcal{E}^*(\mathbf{R}^d)$ are not invariant under the Fourier transform. Their behavior under the Fourier transform is given by Paley-Wiener type theorems, see e.g. [39, Paragraph 1.6].

The last remark is a motivation for the study of other generalizations. The idea is to obtain Beurling and Roumieu type spaces invariant under the Fourier transform, i.e. to define a natural counterpart of the space of tempered distributions in the framework of ultradistributions.

3.2. Gelfand-Shilov Type Spaces

For the sake of simplicity we skip the definition of Gelfand-Shilov type spaces S_α, S^β, and $W_M^{M^\times}$, and refer the reader to [17], [23], [25].

Definition 3.2. Let there be given $\alpha, \beta \geq 0$, and $A, B > 0$. Gelfand-Shilov type space $S^{\alpha,B}_{\beta,A} = S^{\alpha,B}_{\beta,A}(\mathbf{R}^d)$ is defined by

$$S^{\alpha,B}_{\beta,A} = \{f \in C^\infty(\mathbf{R}^d) \mid (\exists C > 0) \sup_{x \in \mathbf{R}^d} |x^p \partial^q f(x)| \leq C A^p p^{p\alpha} B^q q^{q\beta},\ \ \forall p, q \in \mathbf{N}_0^d\}.$$

We introduce the following notation for projective and inductive limits:

$$\Sigma^\alpha_\beta = S^{\alpha,0}_{\beta,0} = \text{proj} \lim_{A>0,B>0} S^{\alpha,B}_{\beta,A}; \quad S^\alpha_\beta := \text{ind} \lim_{A>0,B>0} S^{\alpha,B}_{\beta,A}.$$

S^α_β is nontrivial if and only if $\alpha+\beta > 1$, or $\alpha+\beta = 1$ and $\alpha\beta > 0$, [17]. Spaces Σ^α_α, $\alpha > 1/2$, were introduced and studied in [34]. Note that $\Sigma^{1/2}_{1/2}$, if defined as above, is trivial. An alternative definition of $\Sigma^{1/2}_{1/2}$ based on the Hermite expansions is given in [34]. We have

$$\Sigma^\alpha_\beta \hookrightarrow S^\alpha_\beta \hookrightarrow \mathcal{S}.$$

For the proof of the following theorem we refer to [17]. See also [9], [27] and [34].

Theorem 3.3. *Let there be given $\alpha, \beta \geq 0$. The Fourier transform is a topological isomorphism between S^β_α and S^α_β ($\mathcal{F}(S^\beta_\alpha) = S^\alpha_\beta$) and extends to a continuous linear transform from $(S^\beta_\alpha)'$ onto $(S^\alpha_\beta)'$. In particular, if $\alpha = \beta$ and $\alpha \geq 1/2$ then $\mathcal{F}(S^\alpha_\alpha) = S^\alpha_\alpha$.*

Similar assertions hold for Σ^α_β.

Therefore we obtain a family of Fourier transform invariant spaces which are contained in $\mathcal{S}$. Corresponding dual spaces are, due to this, called tempered ultradistributions (of Beurling or Roumieu type).

Further on we will use the notation $\mathcal{S}^{(\alpha)} = \Sigma^{\alpha}_{\alpha}$, $\mathcal{S}^{\{\alpha\}} = S^{\alpha}_{\alpha}$, and $\mathcal{S}^*$, $*$ means (α) or $\{\alpha\}$.

The relation between Gevrey type spaces and Gelfand-Shilov type spaces is given by the inclusions

$$\mathcal{D}^*(\mathbf{R}^d) \hookrightarrow \mathcal{S}^*(\mathbf{R}^d) \hookrightarrow \mathcal{E}^*(\mathbf{R}^d).$$

In particular, $G^{\alpha}_0(\mathbf{R}^d) \hookrightarrow S^{\alpha}_{\alpha}(\mathbf{R}^d) \hookrightarrow G^{\alpha}(\mathbf{R}^d)$, $\alpha > 1$.

Apart from being invariant under the Fourier transform, spaces $\mathcal{S}^*$ have another important property. Namely, for $1/2 \leq \alpha < 1$, S^{α}_{α} and Σ^{α}_{α} are subspaces of the space of analytic functions (hence do not contain compactly supported functions except $f \equiv 0$). However, the spaces are "sufficiently rich" in the sense of [17]. On contrary, Gevrey classes can not jump in the quasi-analytic case.

Remark 3.4.

- Quasi-analytic Gelfand-Shilov type spaces are essential for the ultradistributional approach to hyperfunctions, [8]. For example, S^1_1 is known as the Sato test function space for a space of Fourier hyperfunctions.
- For an abstract harmonic analysis approach to Gelfand-Shilov type spaces we refer to [26].
- For a relation between the spaces of type $W_M^{M^\times}$ (studied in [25]) and Gelfand-Shilov type spaces, see [27].

3.3. Time-Frequency Representations

In this subsection we give a characterization of $\mathcal{S}^*$ by means of time-frequency representations. Although Theorems 3.8 and 3.9 are not completely new results we present them not only to make the exposition self contained, but also in order to emphasize the role of time-frequency analysis techniques in the study of ultradistributions.

The following theorem gives a beautiful symmetric characterization of $\mathcal{S}^*$. It has been reinvented several times, [9], [23], [27], [37].

Theorem 3.5. *Let there be given $s \geq 1/2$. The following conditions are equivalent:*

a) $f \in \mathcal{S}^{\{s\}}$ *(resp.* $f \in \mathcal{S}^{(s)}$, $s > 1/2$*)*

b) $\sup_{x\in\mathbf{R}^d} |x^{\alpha} f(x)| \leq C h^{|\alpha|} \alpha!^s$ *and* $\sup_{\xi\in\mathbf{R}^d} |\xi^{\beta} \hat{f}(\xi)| \leq C k^{|\beta|} \beta!^s$ *for some (resp. every)* $h, k > 0$;

c) $\sup_{x\in\mathbf{R}^d} |x^{\alpha} f(x)| \leq C h^{|\alpha|} \alpha!^s$ *and* $\sup_{x\in\mathbf{R}^d} |\partial^{\beta} f(x)| \leq C k^{|\beta|} \beta!^s$ *for some (resp. every)* $h, k > 0$;

d) $\sup_{x\in\mathbf{R}^d} |f(x)| e^{h|x|^{1/s}} < \infty$ *and* $\sup_{\xi\in\mathbf{R}^d} |\hat{f}(\xi)| e^{k|\xi|^{1/s}} < \infty$, *for some (resp. every)* $h, k > 0$.

Proof. For the proof of the inductive limit case see [9]. Projective limit case is observed in [27]. □

The following definition collects some of the most important time-frequency representations.

Definition 3.6. Let $g \in L^2(\mathbf{R}^d) \setminus \{0\}$. The short-time Fourier transform of a signal $f \in L^2(\mathbf{R}^d)$, known also as the Gabor transform, is given by

$$V_g f(x,\xi) = \int e^{-2\pi i t\xi} \overline{g(t-x)} f(t) dt, \quad x,\xi \in \mathbf{R}^d. \tag{3.1}$$

The cross ambiguity function (the Fourier–Wigner transform) of f and g is

$$A(f,g)(x,\xi) = \int e^{2\pi i \xi t} f(t+\frac{x}{2}) \overline{g(t-\frac{x}{2})} dt, \quad x,\xi \in \mathbf{R}^d \tag{3.2}$$

and the cross Wigner distribution of f and g is

$$W(f,g)(x,\xi) = \int e^{-2\pi i \xi t} f(x+\frac{t}{2}) \overline{g(x-\frac{t}{2})} dt, \quad x,\xi \in \mathbf{R}^d. \tag{3.3}$$

The quadratic expressions $Af := A(f,f)$ and $Wf := W(f,f)$ are called the (radar) ambiguity function and the Wigner distribution of f.

For various applications of the time-frequency representations in signal analysis and in harmonic analysis see [16], [19], [20], [22], [24], [25], [27], [49] and the references given there.

For $f, g \in L^2(\mathbf{R}^d)$, following relations hold:

$$A(f,g)(x,\xi) = e^{\pi i x \xi} V_g f(x,\xi),$$
$$W(f,g)(x,\xi) = 2^d e^{4\pi i x\xi} V_{\check{g}} f(2x, 2\xi),$$
$$W(f,g)(x,\xi) = (\mathcal{F}A(f,g))(x,\xi), \quad x,\xi \in \mathbf{R}^d.$$

Thus, for our purpose, it will be enough to choose one of the representations. We consider the cross Wigner distribution. The choice is made due to the role which it plays in the theory of pseudodifferential and localization operators, see the next section.

Proposition 3.7. *We list here some of the properties of the cross Wigner distribution.*

a) $W(f,g)$ *maps* $\mathcal{S}(\mathbf{R}^d) \times \mathcal{S}(\mathbf{R}^d)$ *into* $\mathcal{S}(\mathbf{R}^d \times \mathbf{R}^d)$ *and extends to a map from* $\mathcal{S}'(\mathbf{R}^d) \times \mathcal{S}'(\mathbf{R}^d)$ *into* $\mathcal{S}'(\mathbf{R}^d \times \mathbf{R}^d)$.

b) *For* $f_1, f_2, g_1, g_2 \in L^2(\mathbf{R}^d)$, *the Moyal identity holds:*

$$\langle W(f_1,g_1), W(f_2,g_2)\rangle = \langle f_1, f_2\rangle \overline{\langle g_1, g_2\rangle}.$$

c) $\|W(f,g)\|_\infty \leq 2^d \|f\| \|g\|$.

d) $W(\hat{f},\hat{g})(x,\xi) = W(f,g)(-\xi,x)$.

e) $W(f,g)(x,\xi) = \overline{W(g,f)}(x,\xi)$.

For the proofs we refer to [16], [19], [49].

The space $\Sigma_{1/2}^{1/2} = \mathcal{S}^{(1/2)}$ is not included in the following theorem.

Theorem 3.8. *Let $f,g \in \mathcal{S}^*(\mathbf{R}^d)$. Then $W(f,g)(x,\xi) \in \mathcal{S}^*(\mathbf{R}^d \times \mathbf{R}^d)$. The same is true for the Fourier–Wigner transform and the short-time Fourier transform.*

Proof. The proof for the case $f = g$, i.e. for the Wigner distribution $W(f,f)$ is given in [7] for $\mathcal{S}^{\{s\}}$ and in [27] for $\mathcal{S}^*$. We follow the proof of Proposition 3.7 a) given in [49] and give the proof in two steps.

Step 1. Obviously, $f,g \in \mathcal{S}^*(\mathbf{R}^d)$ implies that $f(x)g(t) \in \mathcal{S}^*(\mathbf{R}^d \times \mathbf{R}^d)$, that is

$$\sup_{x,t\in\mathbf{R}^d} |x^\alpha t^\beta \partial_x^\gamma \partial_t^\delta f(x)g(t)| \leq Ch^{|\alpha|+|\beta|+|\gamma|+|\delta|}|\alpha|^{s|\alpha|}\cdot \beta^{s|\beta|}\cdot |\gamma|^{s|\gamma|}\cdot \delta^{s|\delta|}, \tag{3.4}$$

for some (resp. all) $h > 0$. Here we use $|\cdot|^{s|\cdot|}$ instead of $|\cdot|!^s$ for the convenience, see Remark 3.1. Recall, $\mathcal{S}^*$ denotes $\mathcal{S}^{(s)}$ or $\mathcal{S}^{\{s\}}$, $s \geq 1/2$. Let us show that, under the same assumptions $\varphi(x,t) := f(x+\frac{t}{2})g(x-\frac{t}{2}) \in \mathcal{S}^*(\mathbf{R}^d \times \mathbf{R}^d)$. By Theorem 3.5 c) we need to show

$$\sup_{x,t\in\mathbf{R}^d} |x^\alpha t^\beta \varphi(x,t)| \leq Ch^{|\alpha|+|\beta|}|\alpha|^{s|\alpha|}\cdot \beta^{s|\beta|},$$

and

$$\sup_{x,t\in\mathbf{R}^d} |\partial_x^\alpha \partial_t^\beta \varphi(x,t)| \leq Ck^{|\alpha|+|\beta|}|\alpha|^{s|\alpha|}\cdot \beta^{s|\beta|}$$

for some (resp. every) $h,k > 0$. The first inequality easily follows from assumptions on f and g and a change of variables:

$$\begin{aligned}\sup_{x,t\in\mathbf{R}^d} |x^\alpha t^\beta f(x+\frac{t}{2})g(x-\frac{t}{2})| &= \sup_{y,t\in\mathbf{R}^d} |(y-t/2)^\alpha t^\beta f(y)g(y-t)|\\ &\leq 2^{-|\alpha|}\sup_{y,t\in\mathbf{R}^d} |(y-(t-y))^\alpha (-1)^{|\beta|}((y-t)-y)^\beta f(y)g(y-t)|\\ = 2^{-|\alpha|}\sup_{y,z\in\mathbf{R}^d} |(y-z)^\alpha (z-y)^\beta f(y)g(z)| &\leq C_{\alpha,\beta}\sup_{y,z\in\mathbf{R}^d} |(z-y)^{\alpha+\beta} f(y)g(z)|.\end{aligned}$$

Now, (3.4) implies $\sup_{x,t\in\mathbf{R}^d} |x^\alpha t^\beta \varphi(x,t)| \leq Ch^{|\alpha|+|\beta|}|\alpha|^{s|\alpha|}\cdot \beta^{s|\beta|}$ for some (resp. every) $h > 0$.

In order to prove the second inequality recall that in the definition of $\mathcal{S}^*$ we may take $|\alpha|^{s|\alpha|}$ instead of $\alpha!^s$. The Leibniz formula gives

$$\begin{aligned}|\partial_x^\alpha \partial_t^\beta \varphi(x,t)| &= |\sum_{\delta\leq\alpha}\sum_{\gamma\leq\beta}\binom{\alpha}{\delta}\binom{\beta}{\gamma}\frac{1}{2^{|\alpha|+|\beta|}}\partial_x^\delta \partial_t^\gamma f(x+t/2)\partial_x^{\alpha-\delta}\partial_t^{\beta-\gamma}g(x-t/2)|\\ &\leq C_{\alpha,\beta}\sup_{x,t\in\mathbf{R}^d} |\partial_x^\delta \partial_t^\gamma f(x+t/2)\partial_x^{\alpha-\delta}\partial_t^{\beta-\gamma}g(x-t/2)|.\end{aligned}$$

Now, $m^{sm}n^{sn} \leq (m+n)^{s(m+n)}$ and (3.4) imply

$$\sup_{x,t\in\mathbf{R}^d} |\partial_x^\alpha \partial_t^\beta \varphi(x,t)| \leq Ck^{|\alpha|+|\beta|}|\alpha|^{s|\alpha|}\cdot \beta^{s|\beta|}$$

for some (resp. all) $k > 0$.

Therefore, $\varphi(x,t) \in \mathcal{S}^*(\mathbf{R}^d \times \mathbf{R}^d)$.

Step 2. Now, we show that $\Phi(x,\xi) = \int e^{-2\pi i t\xi}\varphi(x,t)dt \in \mathcal{S}^*(\mathbf{R}^d \times \mathbf{R}^d)$ if $\varphi \in \mathcal{S}^*(\mathbf{R}^d \times \mathbf{R}^d)$.

Again, we use Theorem 3.5 c) and show that

$$\sup_{x,\xi\in\mathbf{R}^d} |x^\alpha \xi^\beta \Phi(x,\xi)| \leq C h^{|\alpha|+|\beta|}\alpha!^s\beta!^s, \quad \text{and} \quad \sup_{x,\xi\in\mathbf{R}^d} |\partial_x^\alpha \partial_\xi^\beta \Phi(x,\xi)| \leq C k^{|\beta|}\alpha!^s\beta!^s$$

for some (resp. every) $h, k > 0$. Integration by parts gives

$$|x^\alpha\xi^\beta \int e^{-2\pi i t\xi}\varphi(x,t)dt| = |\int (-1)^\beta D_t^\beta e^{-2\pi i t\xi} x^\alpha \varphi(x,t)dt|$$

$$= |\int e^{-2\pi i t\xi} x^\alpha D_t^\beta \varphi(x,t)dt| \leq C\int (1+t^2)^{-d}(1+t^2)^d x^\alpha \partial_t^\beta \varphi(x,t)dt|.$$

Since $\varphi \in \mathcal{S}^*(\mathbf{R}^d \times \mathbf{R}^d)$ we have $|(1+t^2)^d x^\alpha \partial_t^\beta \varphi(x,t)| \leq C\tilde{h}^{2d+\alpha+\beta}(2d)^s\alpha!^s\beta!^s$, for some (any) $\tilde{h} > 0$. Then, an easy case studies examination shows that for some (resp. for all) $h > 0$, $\sup_{x,\xi\in\mathbf{R}^d} |x^\alpha\xi^\beta\Phi(x,\xi)| \leq C h^{|\alpha|+|\beta|}\alpha!^s\beta!^s$.

For the second equality, we have

$$|\partial_x^\alpha \partial_\xi^\beta \int e^{-2\pi i t\xi}\varphi(x,t)dt| = |\int (-2\pi i t)^\beta e^{-2\pi i t\xi}\partial_x^\alpha \varphi(x,t)dt|$$

$$= |\int (2\pi i)^{|\beta|} e^{-2\pi i t\xi} t^\beta \partial_x^\alpha \varphi(x,t)dt| \leq |2\pi i|^{|\beta|}|\int e^{-2\pi i t\xi}(1+t^2)^{-d} t^\beta \partial_x^\alpha \psi(x,t)dt|,$$

where $\psi = (1+t^2)^d\varphi$. Now $\varphi \in \mathcal{S}^*$ implies that $\psi \in \mathcal{S}^*$ hence

$$|t^\beta \partial_x^\alpha \psi(x,t)| \leq C h^{|\alpha|+|\beta|}\alpha!^s\beta!^s \Rightarrow \sup_{x,\xi\in\mathbf{R}^d} |\partial_x^\alpha \partial_\xi^\beta \Phi(x,\xi)| \leq C k^{|\beta|}\alpha!^s\beta!^s.$$

Theorem 3.8 is therefore proved in a general case for transforms of the type $\int e^{-2\pi i t\xi}\varphi(x,t)dt$ with $\varphi \in \mathcal{S}^*$. In particular, the assertion holds for the cross Wigner ditribution as claimed. Proofs for the Fourier Wigner transform and the short- ime Fourier transform are analogous.

Note that Theorem 3.8 for the inductive limit case and the short-time Fourier transform is proved in [23], see also [19, Section 11.2]. The main difference is that we use Theorem 3.5 c) which allows to estimate actions of multiplication and differentiation separately. □

Theorem 3.9. *Let $g \in \mathcal{S}^*(\mathbf{R}^d)$ and let $f \in \mathcal{S}'^*(\mathbf{R}^d)$. If the cross Wigner distribution $W(f,g)(x,\xi)$ (resp. the cross ambiguity function or the short-time Fourier transform) belongs to $\mathcal{S}^*(\mathbf{R}^d \times \mathbf{R}^d)$ then $f \in \mathcal{S}^*(\mathbf{R}^d)$.*

Proof. We combine the ideas of the proof of similar assertions given in [23] and [7]. Firstly, we show the inversion formula for the cross Wigner distribuiton in the spirit of [19, Section 3.2]. First, note that, for given $g_1, g_2 \in L^2(\mathbf{R}^d)$ and $\langle g_1, g_2\rangle \neq 0$,

$$l(h) = \frac{1}{\langle g_1, g_2\rangle} 2^d \int\int e^{4\pi i x\xi} W(f,g_1)(x,\xi)\langle h, M_{2\xi}T_{2x}\check{g}_2\rangle dx d\xi, \quad h \in L^2(\mathbf{R}^d),$$

is a bounded functional on $L^2(\mathbf{R}^d)$ and therefore it defines a unique function $\tilde{f} \in L^2(\mathbf{R}^d)$,

$$\tilde{f}(t) = 2^d \int\int e^{4\pi i x\xi} W(f,g)(x,\xi) M_{2\xi} T_{2x} \check{g}(t) dx d\xi \tag{3.5}$$

such that $l(h) = \langle \tilde{f}, h\rangle$, for all $h \in L^2(\mathbf{R}^d)$.

Let us show that for all $f \in L^2(\mathbf{R}^d)$ the following inversion formula holds

$$f = \frac{1}{\langle g_1, g_2\rangle} 2^d \int\int e^{4\pi i x\xi} W(f,g_1)(x,\xi) M_{2\xi} T_{2x} \check{g}_2 dx d\xi.$$

Actually, (3.5), the Moyal formula and

$$W(f,g)(x,\xi) = \langle f, 2^d e^{4\pi i x\xi} M_{2\xi} T_{2x} \check{g}\rangle$$

imply

$$\langle \tilde{f}, h\rangle = \frac{1}{\langle g_1, g_2\rangle} \int\int W(f,g_1)(x,\xi) W(h,g_2) dx d\xi = \langle f, h\rangle.$$

Thus $\tilde{f} = f$ in $L^2(\mathbf{R}^d)$. Moreover, it can be shown that $f \in \mathcal{S}(\mathbf{R}^d)$, [19, Proposition 11.2.4].

We are now in a position to show that $f \in \mathcal{S}^*(\mathbf{R}^d)$. We will again use Theorem 3.5. Now it is convenient to use part d) of the Theorem. Therefore, we need to show that $\sup_{x\in\mathbf{R}} |f(x)| e^{h|x|^{1/s}} < \infty$ and $\sup_{\omega\in\mathbf{R}} |\hat{f}(\omega)| e^{k|\omega|^{1/s}} < \infty$, for some (resp. for every) $h, k \geq 0$.

Note that $e^{h|t|^{1/s}} M_{2\xi} T_{2x} \check{g}(t) = M_{2\xi} T_{2x} e^{h|t+2x|^{1/s}} \check{g}(t)$. The inversion formula implies

$$\sup_{t\in\mathbf{R}^d} e^{h|t|^{1/s}} \frac{1}{\langle g_1, g_2\rangle} |2^d \int\int e^{4\pi i x\xi} W(f,g_1)(x,\xi) M_{2\xi} T_{2x} \check{g}_2 dx d\xi|$$

$$\leq \sup_{t\in\mathbf{R}^d} |2^d \int\int e^{4\pi i x\xi} W(f,g_1)(x,\xi) e^{h|2x|^{1/s}} M_{2\xi} T_{2x} e^{h|t|^{1/s}} \check{g}(t) dx d\xi$$

$$\leq C \int\int e^{-\tilde{h}|x|^{1/s}} e^{-k|2x|^{1/s}} e^{h|2x|^{1/s}} dx d\xi < \infty,$$

since $W(f,g_1) \in \mathcal{S}^*(\mathbf{R}^d \times \mathbf{R}^d)$ and $g \in \mathcal{S}^*$. In order to prove

$$\sup_{\omega\in\mathbf{R}} |\hat{f}(\omega)| e^{k|\omega|^{1/s}} < \infty$$

we use $(M_{2\xi} T_{2x} \check{g}(t))\hat{}(\omega) = T_{2\xi} M_{-2x} \hat{\check{g}}(\omega)$ and the same arguments as above. □

Remark 3.10. Theorems 3.8 and 3.9 are proved in [23] for the inductive limit case and the short-time Fourier transform. The proof for the Wigner distribution $W(f,f)$ is given in [7].

3.4. Tempered Ultradistributions

This subsection is given for the sake of completeness, having in mind that Gevrey and (non-quasianalytic) Gelfand-Shilov type spaces might be considered as special cases of the following construction [30].

Let $(M_p) = (M_p)_{p\in\mathbf{N}_0}$ be a sequence of positive numbers which satisfies some of the following conditions:

$(M.1)$ (logarithmic convexity) $M_p^2 \le M_{p-1}M_{p+1}, \quad p \in \mathbf{N}$;

$(M.2)'$ (stability under the action of differential operators) There exist positive constants A, H such that $M_{p+1} \le AH^p M_p$;

$(M.2)$ (stability under the action of ultradifferential operators) There exist positive constants A, H such that

$$M_{p+1} \le AH^p \min{}_{0\le q\le p} M_{p-q}M_q, \quad p, q \in \mathbf{N}_0;$$

$(M.3)'$ (non-quasi-analicity) $\sum_{p\in\mathbf{N}} \frac{M_{p-1}}{M_p} < \infty$;

$(M.3)$ (strong non-quasi-analicity) There exists $A > 0$, such that

$$\sum_{p=q+1}^{\infty} \frac{M_{p-1}}{M_p} < Aq\frac{M_{q+1}}{M_q}, \quad q \in \mathbf{N}.$$

We assume $M_0 = 1$. Obviously $(M.2) \Rightarrow (M.2)'$, $(M.3) \Rightarrow (M.3)'$.

The so-called *associated function* for a sequence (M_p) is defined by

$$M(\rho) = \sup_{p\in\mathbf{N}_0} \ln \frac{\rho^p M_0}{M_p}, \quad 0 < \rho < \infty.$$

Properties $(M.1)-(M.3)$ can be expressed via the associated function. It plays an important role in the proofs of properties of ultradistributions, see [30, page 29].

We give the definition in one dimension for the sake of simplicity only.

Definition 3.11. Let there be given $h \ge 0$, and a sequence (M_p) such that $(M.1)-(M.3)'$ holds. The space $\mathcal{S}_h^{(M_p)} = \mathcal{S}_h^{(M_p)}(\mathbf{R})$ is the space of smooth functions f on $\mathbf{R}$ such that

$$\sup_{x\in\mathbf{R}} \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{h^{\alpha+\beta}|x^\alpha f^{(\beta)}(x)|}{M_\alpha M_\beta} < \infty.$$

It is a Banach space with the norm

$$\sigma_{m,\infty}(f) = \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{h^{\alpha+\beta}}{M_\alpha M_\beta}\|x^\alpha f^{(\beta)}(x)\|_\infty, \quad f \in \mathcal{S}_h^{(M_p)}.$$

Let $\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$ be a projective (when $h \to \infty$) and an inductive (when $h \to 0$) limit of $\mathcal{S}_h^{(M_p)}$, and $\mathcal{S}_h^{\{M_p\}}$, respectively. Dual spaces of $\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$, denoted by $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$, are then spaces of tempered ultradistributions of Beurling and Roumieu type respectively. They are studied in [25], [9], [27], [30], [31], [34], where one can find expansion of the elements of $\mathcal{S}^{(M_p)}$, $\mathcal{S}^{\{M_p\}}$ and their duals via Hermite functions, as well as their characterization through the Fourier transform, the Wigner distribution and the Bargman transform. Structural theorems, continuity of ultradifferential operators, the Hilbert transform and the convolution are studied there as well.

The following inclusions hold

$$\mathcal{D}^{(M_p)}(\mathbf{R}) \hookrightarrow \mathcal{S}^{(M_p)}(\mathbf{R}) \hookrightarrow \mathcal{S}(\mathbf{R}) \hookrightarrow L^2(\mathbf{R}) \hookrightarrow \mathcal{S}'(\mathbf{R}) \hookrightarrow \mathcal{S}^{'(M_p)}(\mathbf{R}) \hookrightarrow \mathcal{D}^{'(M_p)}(\mathbf{R}).$$

Remark 3.12.

- If $s > 1$ examples of (M_p) sequences which satisfy some of the above conditions are $M_p = p!^s$, $M_p = p^{sp}$, $M_p = \Gamma(sp+1)$, where Γ denotes the Gamma function. Therefore spaces $\mathcal{S}^*$ of the preceding paragraph are examples of spaces obtained by the above construction.
- Spaces $\mathcal{D}^*$ and $\mathcal{E}^*$ and corresponding weighted versions are introduced and studied within the above approach in [6], [30], [31], [35], [36].

4. Modulation Spaces

Modulation spaces consist of functions or distributions whose short-time Fourier transform satisfies some prescribed decay at infinity as well as some integrability conditions. They are recognized as the most important spaces in time-frequency analysis, [19], [13], [1]. The decay of the short-time Fourier transform is controlled by a weight function, i.e. a non negative locally integrable function on $\mathbf{R}^{2d}$. It is often sufficient to suppose that weights have (at most) polynomial growth. However, in order to deal with ultradistribution, almost exponential growth should be allowed.

Definition 4.1. Let $\gamma \in [0,1)$. A strictly positive and continuous function w_γ on $\mathbf{R}^d \times \mathbf{R}^d$ is called an exp–type weight if there exist $h \geq 0$ and $C > 0$ such that

$$w_\gamma(x+y, \xi+\eta) \leq C e^{h(|x|^\gamma + |\xi|^\gamma)} w_\gamma(y,\eta), \quad x, y, \xi, \eta \in \mathbf{R}^d$$

and $w_\gamma(x, \epsilon_1 \cdot \xi_1, \dots, \epsilon_d \cdot \xi_d) = w_\gamma(x,\xi)$, $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in \{-1,1\}^d$.

A typical example of an exp–type weight is

$$w_\gamma(x,\xi) = e^{h_1|x|^\gamma + h_2|\xi|^\gamma}, \quad x, \xi \in \mathbf{R}^d,\ h_1, h_2 \geq 0. \tag{4.1}$$

More generally, $(1+|x|+|\xi|)^a e^{b|x|^{\gamma_1} + c|\xi|^{\gamma_2}}$ is an exp-type weight for any $a, b, c \geq 0$ and $\gamma_1, \gamma_2 \in [0,1)$.

Definition 4.2. Let there be given $\gamma \in [0,1)$, an exp-type weight w_γ, $t \in \mathbf{R}$, and $1 \leq p, q < \infty$. Let $0 \not\equiv g \in \mathcal{S}^{(1/\gamma)}$ (if $\gamma = 0$, then $0 \not\equiv g \in \mathcal{S}$). Then

$$M_{p,q}^{w_\gamma,t} = \left\{ f \in \mathcal{S}^{'(1/\gamma)} : \|f\|_{M_{p,q}^{w_\gamma,t}} < \infty \right\}, \tag{4.2}$$

$$\|f\|_{M_{p,q}^{w_\gamma,t}} = \left[\int \left(\int |V_g f(x,\xi)|^p\, w_\gamma^p(x,\xi) e^{t(|x|^\gamma + |\xi|^\gamma)} dx \right)^{q/p} d\xi \right]^{1/q}$$

is called the modulation space.

$M^{w_\gamma,t}_{p,q}$ is a Banach space [19, Theorem 11.3.5.]. If $t = 0$, we write $M^{w_\gamma}_{p,q} = M^{w_\gamma,0}_{p,q}$, for short. The above definition is independent of the choice of g, $0 \not\equiv g \in \mathcal{S}^{(1/\gamma)}$, in the sense that different functions define the same modulation space and equivalent norms [13].

Modulation spaces $M^w_{p,q}$, where $1 \le p, q < \infty$ and $w(x + y, \xi + \eta) \le C(1 + |x| + |\xi|)^s w(y, \eta)$, for some $C > 0$ and all $x, y, \xi, \eta \in \mathbf{R}^d$, are studied in [1], [15], [19] and [43]. Obviously, every weight defined in such a way is an exp–type weight. In particular, for $w_s(x, \xi) = (1 + |x| + |\xi|)^s$, $x, \xi \in \mathbf{R}^d$, $s \ge 0$ properties of pseudodifferential operators whose symbols belong to the corresponding modulation spaces are given in [22], see also [15], [19], [29], [46], [47] for some generalizations.

Inclusion relations between modulation spaces and other spaces of functions or distributions (L^p, Besov and Sobolev spaces) are given in [13], [19], [28], [46]. Note that for $w_s(x, y) = (1 + |x| + |y|)^s$, $x, y \in \mathbf{R}^d$,

$$\mathcal{S} = \operatorname{proj} \lim_{s\to\infty} M^{w_s}_{p,q}, \quad 1 \le p, q < \infty,$$

[14], [47]. Definition 4.2 and Theorems 3.8 and 3.9 imply that an analogous statement should hold for Gelfand-Shilov type spaces if polynomial weights are replaced by weights of almost exponential growth. Indeed,

Proposition 4.3. *Let* $1 \le p < \infty$ *and fix* $\gamma \in (0, 1)$. *Put* $w_{\gamma,s}(x, \xi) = e^{s(|x|^\gamma + |\xi|^\gamma)}$, $s \in \mathbf{R}$. *We have*

$$\mathcal{S}^{(1/\gamma)} = \operatorname{proj} \lim_{s\to\infty} M^{w_{\gamma,s}}_{2,2}, \quad \mathcal{S}^{\{1/\gamma\}} = \operatorname{ind} \lim_{s\to\infty} M^{w_{\gamma,s}}_{p,p}.$$

For the proof of projective limit case we refer to [37]. The inductive limit case is proved in [23].

5. Pseudodifferential Operators

Pseudodifferential operators in Gevrey classes are studied in [5], [18], [33], [39], [51]. For operators which map $G^s_0(\Omega)$ to $G^s(\Omega)$, $s > 1$, almost exponential (subexponential) growth of corresponding symbols is allowed. This gives rise to operators of "infinite order" in contrast to "finite order" symbols which are slowly increasing at infinity. Depending on the problem at hand various classes of symbols are introduced. Just to give an example, a class of symbols $S^{\infty,\theta}_{\varrho,\delta}(\mathbf{R}^{2d})$ consists of all functions $\sigma \in C^\infty(\mathbf{R}^{2d})$ for which there exist constants $C > 0$ and $B \ge 0$ such that for every $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ such that

$$\sup_{x\in\mathbf{R}^d} \left| D^\alpha_x D^\beta_\xi \sigma(x, \xi) \right| \le c_\varepsilon C^{|\alpha+\beta|} \alpha!^{\theta(\varrho-\delta)} \beta! (1 + |\xi|)^{\delta|\alpha| - \varrho|\beta|} e^{(\varepsilon|\xi|)^{1/\theta}} \tag{5.1}$$

for every $\alpha, \beta \in \mathbf{N}^d_0$ and every $|\xi| \ge B|\beta|^\theta$, [51].

In order to deal with Gelfand-Shilov type spaces, we use their definition via modulation spaces. In the context of modulation spaces with polynomial weights smooth symbols with at most polynomial growth are studied in [12], [43], [44]. The following class, which includes functions of almost exponential growth, is studied in [38].

Definition 5.1. Let $\gamma \in (0,1)$. Let there be given $L_1, L_2 \geq 0$, and $\lambda, \tau \in \mathbf{R}$. We define the symbol class $S^{\lambda,\tau,\gamma}_{L_1,L_2}(\mathbf{R}^{2d}) = S^{\lambda,\tau,\gamma}_{L_1,L_2}$ as the set of $\sigma \in C^\infty(\mathbf{R}^{2d})$ satisfying

$$|\frac{L_1^{|\alpha|}}{\alpha!^{1/\gamma}}\frac{L_2^{|\beta|}}{\beta!^{1/\gamma}} D^\alpha_x D^\beta_\xi \sigma(x,\xi)| \leq C e^{\lambda|x|^\gamma + \tau|\xi|^\gamma}, \quad \alpha,\beta \in \mathbf{N}^d_0, x,\xi \in \mathbf{R}^d$$

for some positive constant $C = C_\sigma$ depending on L_1, L_2, λ, τ and γ. The infimum of such constants C_σ will be denoted by $||\sigma||^{\lambda,\tau,\gamma}_{L_1,L_2}$.

$S^{\lambda,\tau,\gamma}_{L_1,L_2}$ contains polynomial symbols as well as some ultrapolynomial symbols. For example, if $|a_n| \leq C\frac{k^n}{n!^{1/\gamma}}$, for some positive constants C and k, and all $n \in \mathbf{N}_0$ then

$$\sigma(x,\xi) := \sum_{n=0}^{\infty} a_n \left(1 + |x|^2 + |\xi|^2\right)^{n/2}, \in S^{\lambda,\tau,\gamma}_{L_1,L_2}$$

for all $L_1, L_2 > 0$ and

$$\lambda, \tau \geq (k^\gamma 2^{\gamma/2}(1 + d(L_1^2 + L_2^2))^{\gamma/2})/\gamma.$$

For a relation between $S^{\lambda,\tau,\gamma}_{L_1,L_2}$ and classes of symbol introduced in [5], [43] and [51], as well as for the continuity properties of the corresponding operators we refer to [38]. As an illustration we give the following result from [38]. The notion of Weyl symbol is given in Definition 5.3.

Proposition 5.2. *Let $\gamma \in (0,1)$. Let there be given $L_1, L_2 \geq 0$, and $\lambda, \tau \in \mathbf{R}$, and $L_1 \geq \frac{-\tau}{2^{\gamma+1/\gamma}}$ if $\tau < 0$.*

Observe a pseudodifferential equation $\sigma(x,D)u = f$, where the Weyl symbol of $\sigma(x,D)$ belongs to the class $S^{\lambda,\tau,\gamma}_{A,B}$, with

$$A > 2\tilde{\gamma}L_1, \quad B > 2\tilde{\gamma}L_2, \quad \textit{where} \quad \tilde{\gamma} = (\frac{4}{1-\gamma^2})^{1/\gamma}.$$

If $u \in \mathcal{S}^{(1/\gamma)}$, then $f \in \mathcal{S}^{(1/\gamma)}$ as well. Moreover, the mapping $\sigma(x,D) : \mathcal{S}^{(1/\gamma)} \mapsto \mathcal{S}^{1/(\gamma)}$ is continuous.

Another consequence is the boundedness of the operator $\sum_{|\alpha|\leq n} a_\alpha D^\alpha$, on $\mathcal{S}^{(1/\gamma)}$. It follows from the fact that its Weyl symbol belongs to the class $S^{\lambda,\tau,\gamma}_{A,B}$ for any choice of $A, B > 0$.

Definition 5.3. Let $\mathcal{A}$ be a class of symbols. We say that $\sigma \in \mathcal{A}$ is the Weyl symbol of the operator $\sigma(x,D)$ if and only if

$$(5.2) \quad \sigma(x,D)f(x) = \int\int \sigma\left(\frac{x+y}{2},\xi\right) e^{2\pi i(x-y)\xi} f(y)dyd\xi, \quad f \in \mathcal{S}^{(1/\gamma)}(\mathbf{R}^d).$$

We also say that $\sigma(x,D)$ is the Weyl transforms of the symbol $\sigma(x,\xi)$ and refer to (5.2) as the Weyl correspondence between an operator and a symbol.

The Weyl correspondence can be defined by the means of the cross Wigner distribution [49]. Namely, for $f, g \in L^2(\mathbf{R}^d)$ and $\sigma \in L^2(\mathbf{R}^{2d})$

$$(\sigma(x,D)f,g) = (\sigma, W(g,f)) = \langle \sigma, W(\bar{g},f)\rangle = \int\int \sigma(x,\xi) W(f,g)(x,\xi) dx d\xi.$$

Remark 5.4.

- The cross Wigner distribution and the short-time Fourier transform play important role in the study of localization operators, [1], [11], [50]. For given "windows" $\varphi_1, \varphi_2 \in L^2(\mathbf{R}^d)$ and a suitable symbol a, the time-frequency localization operator $A_a^{\varphi_1,\varphi_2}$ is defined by

$$A_a^{\varphi_1,\varphi_2} f = \int\int a(x,\xi) V_{\varphi_1} f(x,\xi) M_\xi T_x \varphi_2 dx d\xi.$$

  For various choices of windows and symbols we refer to [11]. We are interested here in a representation of localization operator through the cross Wigner distribution. It can be shown that $A_a^{\varphi_1,\varphi_2}$ is operator whose Weyl symbol σ is given by $\sigma = a * W(\varphi_1, \varphi_2)$.
- The so called anti-Wick operators arise as a special case of localization operators, [1, Chapters 8,9]. Recall, an anti-Wick symbol σ determines Weyl symbol $\sigma * (2^{2d} e^{-|\cdot|^2})$. Let $\sigma(x,\xi) \in \mathcal{S}^{'(\gamma)}$. Then there exists $s > 0$ such that $\sigma * \left(2^{2d} e^{-|\cdot|^2}\right) \in S^{\lambda,\tau,\gamma}_{L_1,L_2}$ for any $\lambda, \tau \geq s$, and $0 \leq L_1, L_2 \leq 2^{-1/2}$, [38].
- Another time-frequency representation, Rihaczek distribution is closely related to the Kohn-Nirenberg correspondence between symbols and operators, see [19], [21].

Let us now consider a class of symbols $\sigma \in C^\infty(\mathbf{R}^d \times \mathbf{R}^d)$ satisfying:
(S1) $\sigma(z) \geq 1$, $z = (x,\xi) \in \mathbf{R}^d \times \mathbf{R}^d$.
(S2) $(\exists C > 0)$ $(\exists \eta > 0)$ such that

$$\sigma(z+w) \leq C e^{\eta |z|^\gamma} \sigma(w), \quad z, w \in \mathbf{R}^{2d}.$$

(S3) $(\forall h \geq 0)$ $(\exists C > 0)$ $(\exists \tilde{s} \geq 0)$ such that

$$\sup_{\alpha \in \mathbf{N}_0^{2d}, |\alpha| \geq 1} \left| \frac{h^{|\alpha|}}{\alpha!^{1/\gamma}} D^\alpha \sigma(z) \right| \leq C \frac{\sigma(z)}{(1+|z|)^{\tilde{s}}}, \quad z \in \mathbf{R}^{2d}.$$

(S4) $\sigma(x,\xi) \leq \sigma(x,\xi')$ for all $\xi, \xi' \in \mathbf{R}^d$ such that $|\xi| \leq |\xi'|$.

Note, by (S2), (S3) and $|z| \leq |x|^\gamma + |\xi|^\gamma$, we have $\sigma(x,\xi) \in S^{\eta,\eta}_{L_1,L_2}$ for every $L_1, L_2 \geq 0$. Also, condition (S2) implies that σ is an exp-type weight and $\sigma(z) \geq \frac{\sigma(0)}{C} e^{-\eta |z|^\gamma}$ for all $z \in \mathbf{R}^{2d}$.

Example. The following functions satisfy conditions (S1)-(S4).

a) $\sigma(z) = \sum_{k=0}^n a_k \langle z \rangle^{2k}$, $z = (x,\xi) \in \mathbf{R}^d \times \mathbf{R}^d$, where $a_0 \geq 1$, $a_k > 0$;
b) $\sigma(x,\xi) = (1 + |x|^2 + |\xi|^2)^{s/2}$, $s \geq 0$, $x, \xi \in \mathbf{R}^d$. In particular,

$$\sigma(\xi) = (1+|\xi|^2)^{s/2}, \quad s \geq 0, \ \xi \in \mathbf{R}^d;$$

c) $\sigma(x,\xi) = |\xi|^2 + V(x)$, $x, \xi \in \mathbf{R}^d$, where
(V1) $V \in C^\infty(\mathbf{R}^d)$, $V \geq 1$, $V(x) \to \infty$ when $|x| \to \infty$.

(V2) $(\exists C > 0)$ $(\exists \eta > 0)$ such that $V(x+y) \leq Ce^{\eta |x|^\gamma} V(y)$, $x, y \in \mathbf{R}^d$.

(V3) $(\forall h \geq 0)$ $(\exists C > 0)$ such that

$$\sup_{\alpha \in \mathbf{N}_0^d, |\alpha| \geq 1} \left| \frac{h^{|\alpha|}}{\alpha!^{1/\gamma}} D^\alpha V(x) \right| \leq CV(x), \quad x \in \mathbf{R}^d;$$

d) $\sigma(x,\xi) = e^{(1+|x|^2+|\xi|^2)^{\gamma/2}}$, $x, \xi \in \mathbf{R}^d$;

Example c) implies that we may observe the Schrödinger operator with potential $V(x)$ which may have sub-exponential growth.

Theorem 5.5. *Assume that $\sigma(x,\xi)$ satisfies (S1)-(S4). Then*

a) *Let $1 \leq p, q < \infty$ and $s \geq 0$. For every $f \in M^{\sigma,s}_{p,q}$ there exist positive constants C_1, C_2 and C_3 such that*

$$C_1 \|f\|_{M^{\sigma,s}_{p,q}} \leq \|\sigma(x,D)f\|_{M^{1,s}_{p,q}} + C_2 \|f\|_{M^{1,0}_{p,q}} \leq C_3 \|f\|_{M^{\sigma,s}_{p,q}}. \tag{5.3}$$

b) *If, additionally, $\sigma(z) \geq Ce^{\mu|z|^\gamma}$ for $|z| \geq K$, for some positive constants C, μ and K, and if $\sigma(x,D)f \in M^{1,s}_{p,q}$ for $f \in M^{1,0}_{p,q}$, then f belongs to $M^{1,s+\mu}_{p,q}$.*

c) *Let*

$$\sigma(x,\xi) \sim e^{\mu(|x|^\gamma + |\xi|^\gamma)} \quad \text{for some} \quad \mu > 0 \quad \text{when} \quad |x| + |\xi| \to \infty. \tag{5.4}$$

Then $\sigma(x,D)|_{\mathcal{S}^{(1/\gamma)}}$ is essentially self-adjoint in L^2 and the domain of its unique self adjoint extension L is $M^{\sigma,0}_{2,2}$. Furthermore, the self adjoint operator L is compact operator and its eigenfunctions belong to $\mathcal{S}^{(1/\gamma)}$.

Proof. Part a) is proved in [38], see also [44]. Part b) is a corollary of the result. For simplicity we prove b) for $p = q = 2$. Let $B(0,K) = \{z \in \mathbf{R}^{2d} \mid |z| \leq K\}$.

$$\|f\|^2_{M^{1,s+\mu}_{2,2}} = \int\int |V_g f(x,y)|^2 e^{2(s+\mu)(|x|^\gamma + |y|^\gamma)} dxdy$$

$$\leq \iint_{B(0,K)} |V_g f(x,y)|^2 e^{2(s+\mu)(|x|^\gamma+|y|^\gamma)} dxdy + \iint_{\mathbf{R}^d \setminus B(0,K)} |V_g f(x,y)|^2 e^{2(s+\mu)(|x|^\gamma+|y|^\gamma)} dxdy$$

$$\leq C + \Big| \iint_{\mathbf{R}^d \setminus B(0,K)} |V_g f(x,y)|^2 \sigma^2(x,y) e^{2s(|x|^\gamma+|y|^\gamma)} dxdy \Big| \leq C + \|f\|_{M^{\sigma,s}_{2,2}}$$

$$\leq C + \|\sigma(x,D)f\|_{M^{1,s}_{2,2}} + C_2 \|f\|_{M^{1,0}_{2,2}} < \infty,$$

where we used a) together with the fact that $f \in M^{1,0}_{2,2}$ implies $V_g f \in M^{1,0}_{2,2}$.

c) We prove that $\sigma(x,D)|_{\mathcal{S}^{(1/\gamma)}}$ is essentially self-adjoint by using [42, Theorem 26.1]. To that end we need to show the $\mathcal{S}^{(1/\gamma)}$ hypoellipticity. This follows from b) since $\mathcal{S}^{(1/\gamma)} = \bigcap_{s\geq 0} M^{1,s}_{2,2}$. It remains to show that L is compact.

Now we need the following fact from time-frequency analysis. Recall, a set of functions $\{g_{k,n},\ k,n\in\mathbf{Z}\}$, is called a *Gabor system* if

$$g_{k,n}(x) = e^{2\pi bnx} g(x-ak) = M_{bn}T_{ak}g, \quad k,n\in\mathbf{Z},$$

for a fixed function g and time-frequency shift parameters $a,b>0$. A Gabor system is a *Gabor frame* in $L^2(\mathbf{R})$ if there exists $0<A\leq B<\infty$ such that

$$A\|f\|^2 \leq \sum_{k,n\in\mathbf{Z}} |\langle g_{k,n}, f\rangle|^2 \leq B\|f\|^2.$$

If $A=B$ the frame is tight. For a tight Gabor frame $\{g_{k,n},\ k,n\in\mathbf{Z}\}$, we have

$$\|f\|^2 = \sum_{k\in\mathbf{N}^d, n\in\mathbf{Z}^d} |\langle g_{k,n}, f\rangle|^2. \tag{5.5}$$

To show that L is compact it is sufficient to prove that the set

$$S = \{f\in L^2 \ :\ \sum_{k\in\mathbf{N}^d, n\in\mathbf{Z}^d} |\langle g_{k,n}, f\rangle|^2 \sigma_{k,n} \leq 1\}$$

where $\sigma_{k,n} = \sigma(n+\frac{\kappa}{2}, l)$ for $k = 2l+\kappa\in\mathbf{N}^d$, $\kappa\in\{0,1\}^d$, is compact in $L^2(\mathbf{R}^d)$.

Let $(f_n)_{n\in\mathbf{N}}$ be a sequence in S. We need to show that it contains a convergent subsequence. Since $(f_n)_{n\in\mathbf{N}}$ is bounded in $L^2(\mathbf{R}^d)$ it contains a weakly convergent subsequence $(f_{l_n})_{l_n\in\mathbf{N}}$, i.e. there exists $f\in S$ such that

$$\lim_{n\to\infty} (f_{l_n} - f, \phi) = 0, \quad \forall\phi\in L^2(\mathbf{R}^d). \tag{5.6}$$

Let us show that this convergence is strong, i.e. $\lim_{n\to\infty}\|f_{l_n} - f\| = 0$, or, equivalently

$$\lim_{n\to\infty} \sum_{k\in\mathbf{N}^d, m\in\mathbf{Z}^d} |\langle f_{l_n} - f, g_{k,m}\rangle|^2 = 0. \tag{5.7}$$

Since $\lim_{|k|+|m|\to\infty}\sigma_{k,m} = \infty$, for arbitrary $\varepsilon/4>0$ there is a finite number of indices (k,m) such that $\dfrac{1}{\sigma_{k,m}} \leq \dfrac{\varepsilon}{4}$. Denote the set of indices by I_1. Then we have

$$\sum_{k,m)\notin I_1} |\langle f_{l_n} - f, g_{k,m}\rangle|^2 \frac{\sigma_{k,m}}{\sigma_{k,m}}$$

$$< 2\frac{\varepsilon}{4}\Big(\sum_{(k,m)\notin I_1} |\langle f_{l_n}, g_{k,m}\rangle|^2\sigma_{k,m} + \sum_{(k,m)\notin I_1} |\langle f, g_{k,m}\rangle|^2\sigma_{k,m}\Big) < \varepsilon.$$

Furthermore, since (5.6) implies that

$$\lim_{n\to\infty} \sum_{(k,m)\in I_1} |\langle f_{l_n} - f, g_{k,m}\rangle|^2 = 0,$$

by taking e.g. $\varepsilon = 2^{-l_n}$ we obtain (5.7), hence S is compact in $L^2(\mathbf{R}^d)$.

The fact that eigenfunctions of L belong to $\mathcal{S}^{(1/\gamma)}$ is derived by part b) and $\cap_{s\geq 0} M^{1,s}_{2,2} = \mathcal{S}^{(1/\gamma)}$. □

An immediate consequence of Theorem 5.5 is the continuity of the mapping $\sigma : M^{\sigma,s}_{p,q} \mapsto M^{1,s}_{p,q}$. Moreover, $\sigma(M^{\sigma,s}_{p,q})$, the image of $M^{\sigma,s}_{p,q}$ under σ, is a Banach subspace of $M^{1,s}_{p,q}$.

For spectral asymptotics we refer to [12],[38], [44].

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