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Notes on Sobolev Spaces on Compact Classical Groups and Stein–Sahi Representations

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Vienna, Preprint ESI 1553 (2004)

December 6, 2004

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at

Notes on
Sobolev spaces on compact classical groups and Stein–Sahi representations

Yurii A. Neretin

We discuss kernels on complact classical groups $G = \mathrm{U}(n)$, $\mathrm{O}(2n)$, $\mathrm{Sp}(n)$ defined by the formula $K(z,u) = |\det(1-zu^*)|^s$. We obtain the explicit Plancherel formula for these kernels and the interval of positive-definiteness. We also obtain explicit models for Sahi's 'unipotent' representations.

In [15], the author proposed a class of natural kernels on pseudo-Riemannian symmetric spaces G/H and conjectured that these kernels admit explicit Plancherel formula. The purpose of these notes is to verify that at least in the case then G/H is a compact group $G\times G/G \sim \mathrm{U}(n)$, $\mathrm{O}(2n)$, $\mathrm{Sp}(n)$ this Plancherel formula can be written.[1]

0.1. A preliminary example. Recall the construction of unitary representations of the complementary series of the group $\mathrm{SL}(2,\mathbb{R})$ (Bargmann [1], 1947, see also [6]). Consider the unit circle S^1: $z = e^{i\varphi}$ and the following Hermitian form in the space of smooth functions on S^1

$$\langle f,g\rangle_s = \int_0^{2\pi}\int_0^{2\pi} \bigl|\sin(\varphi/2)\bigr|^{1-s} f(\varphi_1)\,\overline{g(\varphi_2)}\,d\varphi_1\,d\varphi_2.$$

This Hermitian form is invariant with respect to the Möbius transformations of the circle in the following sence. Consider the group $\mathrm{SL}(2,\mathbb{R}) \simeq \mathrm{SU}(1,1)$ consisting of all complex 2×2-matrices $g = \begin{pmatrix} a & b\\ \overline{b} & \overline{a}\end{pmatrix}$, satisfying $|a|^2 - |b|^2 = 1$. Then the operators

$$\rho_s\begin{pmatrix} a & b\\ \overline{b} & \overline{a}\end{pmatrix} f(z) = f\bigl((\overline{a}+\overline{b}z)^{-1}(a+bz)\bigr)\,|\overline{a}+\overline{b}z|^{-1+s}$$

preserve the form $\langle f,g\rangle_s$, and moreover this property uniquely determines our form.

For $-1 < s < 1$ our Hermitian form is positive definite. This fact follows from the identity

$$\langle \sum a_n e^{in\varphi}, b_n e^{in\varphi}\rangle = \mathrm{const}\sum \frac{\Gamma(n+(1-s)/2)}{\Gamma(n+(1+s)/2)} a_n \overline{b}_n.$$

We observe, that for $-1 < s < 1$, all the gamma-coefficients are positive, and hence $\langle f,f\rangle_s$ is positive for all f.

[1] Of course, the compact groups are the most simple objects among symmetric spaces. Nevertheless, our calculation (see also [16]) is an important heurictic argument for a computability of the Plancherel measure in the general case. Indeed, Oshima [18] recentely observed that difference equations for c-function are the same for different real forms of one complex symmetric space $G_{\mathbb{C}}/H_{\mathbb{C}}$ (and hence c-functions of different real forms differs by trigonometric factors; actual evaluation of this trigonometric factor is a highly nontrivial result of Oshima). It is natural to believe that this phenomenon survives in our situation.

It is natural to consider the completion H_s of the space $C^\infty(S^1)$ with respect to this inner product. Obviously,

$$f(\varphi) = \sum a_n e^{in\varphi} \in H_s \quad \Longleftrightarrow \quad \sum \frac{\Gamma(n+(1-s)/2)}{\Gamma(n+(1+s)/2)} |a_n|^2 < \infty \quad \Longleftrightarrow$$
$$\Longleftrightarrow \quad \sum (1+|n|)^{-s} |a_n|^2 < \infty.$$

The last condition shows that H_s is a Sobolev space.

This construction is the simplest example of a representation that is unitary by some nontrivial reason. In fact, Hilbert spaces with various inner product having the form

$$\langle f, g\rangle = \iint_{X\times X} K(x,y) f(x)\,\overline{g}(y)\,dx\,dy,$$

where $K(x,y)$ is a distribution, are usual in the representation theory.

For instance (see Vilenkin, [26], X.2), conformally invariant inner products in spaces of functions on a spere $|x_1|^2 + \cdots + |x_n|^2 = 1$ are given by the formula

$$\langle f, g\rangle = \iint_{S^{n-1}\times S^{n-1}} \|x-y\|^{-\lambda} f(x)\overline{g(y)}\,dx\,dy \tag{0.1}$$

and this inner product is positive definite iff $0 < \lambda < n-1$.

The Hilbert spaces defined by the inner products (0.1) are Sobolev spaces. In the representation theory also there are few cases (related to the groups $\mathrm{U}(1,n)$, $\mathrm{Sp}(1,n)$) when some anisotropic Sobolev spaces arise in a natural way, but usually the situation is more complicated (some discussion of functional-theoretical phenomena is contained in [17]).

0.2. Stein kernels. Consider the space Mat_n consisting of $n\times n$ complex matrices. Consider the following Hermitian form on $C^\infty(\mathrm{Mat}_n)$

$$\langle f, g\rangle = \iint_{\mathrm{Mat}_n\times \mathrm{Mat}_n} |\det(x-y)|^{-2n+\sigma} f(x)\overline{g(y)}\,dx\,dy \tag{0.2}$$

(for $\sigma > 2n$ the integral is well defined, further we can consider the meromorphic continuation in σ). The Hermitian form (0.2) is invariant with respect to the transformations

$$\rho_\sigma \begin{pmatrix} a & b \\ c & d\end{pmatrix} f(x) = f((a+xc)^{-1}(b+xd))|\det(a+xc)|^{-2n-\sigma},$$

where $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{GL}(2n,\mathbb{C})$ is an arbitrary invertible $(n+n)\times(n+n)$ matrix.

Stein's theorem. *The inner product (0.2) is positive definite iff* $-1 < \sigma < 1$.

Vogan [27], 1986, extended Stein's construction to the groups $\mathrm{GL}(n)$ over real numbers $\mathbb{R}$ and quaternions $\mathbb{H}$, see also [20]. Sahi [21], [22] extended the construction to other series of classical groups, precisely to the groups $\mathrm{O}(2n, 2n)$, $\mathrm{U}(n, n)$, $\mathrm{Sp}(n, n)$, $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SO}^*(4n)$, $\mathrm{Sp}(4n, \mathbb{C})$, and $\mathrm{O}(2n, \mathbb{C})$,

The inner products (0.1) and (0.2) seem similar, but the kernel $\|x-y\|^{-\lambda}$ has a singularity on the diagonal $x = y$; on the contrary $|\det(x-y)|^{-2n-\sigma}$ has a singularity on a complicated surface in $\mathrm{Mat}_n \times \mathrm{Mat}_n$ containing the diagonal. In particular, the Hilbert space defined by the Stein inner product (0.2) is not a Sobolev space in the standard sence.

Similarily, various natural integral operators that appear in the representation theory usually are not pseudodifferential operators in the standard sence.

0.3. Sobolev kernels on the orthogonal groups $\mathrm{O}(2n)$. Consider the orthogonal group $\mathrm{O}(2n)$, i.e., the group of real $2n \times 2n$-matrices h satisfying $h^t h = 1$. We consider the Hermitian form on $C^\infty(\mathrm{O}(2n))$ given by

$$\langle F_1, F_2 \rangle_\lambda = \iint_{\mathrm{O}(2n)\times \mathrm{O}(2n)} |\det(u-v)|^\lambda F_1(u) \overline{F_2(v)}\, d\mu(u)\, d\mu(v), \qquad (0.3)$$

where μ is the Haar measure on $\mathrm{O}(2n)$. This form is well defined for $\lambda > 0$, for $\lambda < 0$ we consider its meromorphic continuation.

In this paper, we obtain the explicit expression of this form in characters (Theorem 3.2) and also show that for $-n < \lambda < -n+1$ our form is positive definite

We consider the group $\mathrm{O}(2n, 2n)$ consisting of $(2n+2n) \times (2n+2n)$-matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where the symbol ^{t} denotes the transposition of a matrix.

For an orthogonal matrix $h \in \mathrm{O}(2n)$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{O}(2n, 2n)$, we have

$$(a + hc)^{-1}(b + hd) \in \mathrm{O}(2n).$$

It is easy to verify that the Hermitian form (0.3) is invariant with respect to the operators $C^\infty(\mathrm{O}(2n)) \to C^\infty(\mathrm{O}(2n))$ given by

$$\rho_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} F(h) = F\big((a+hc)^{-1}(b+hd)\big) \det(a+hc)^{-2n+1-\lambda}.$$

Thus our Theorem 3.2 gives another proof of Sahi's theorem about existence of Stein-type series for the groups $\mathrm{O}(2n, 2n)$. Also, we obtain new models of the "unipotent representations" of the groups $\mathrm{O}(2n, 2n)$ (see Sahi, [23], Dvorsky and Sahi, [4], [5]).

0.4. Structure of the paper. We consider the groups $\mathrm{U}(n)$, $\mathrm{O}(2n)$, $\mathrm{Sp}(n)$ in Sections 2, 3, 4 respectively. We also obtain models of Stein–Sahi representations for the groups $\mathrm{U}(n,n)$, $\mathrm{O}(2n,2n)$, $\mathrm{Sp}(n,n)$ In particular, we obtain an independent proof of the corresponding Sahi's results.

In all the cases, we reduce the problem to evaluation of some determinants, necessary determinant calculations are collected in preliminary Section 1 (our calculations are more uniform than it seems at the first glance, see [16].

In Section 5, we discuss models of 'unipotent representations' from [23], [4], [5].

0.5. Notation. We denote the Haar measure on $\mathrm{U}(n)$, $\mathrm{SO}(n)$, $\mathrm{Sp}(n)$ by μ. We assume that the measure of the whole group is 1.

The Pochhammer symbol is given by

$$(a)_n = a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{0.4}$$

The second expression also has sence for negative n.

Acknowledgements. I am grateful to A.Dvorsky for his explanations on 'unipotent representations'. This work was done during my visit to the Schrödinger Institute, Vienna, winter 2001–2002. I thank the administrators of the Institute for their hospitality.

1. Some determinant identities

Let $A = \{a_{kl}\}$ be a square $n \times n$ matrix, $k, l = 1, 2, \dots, n$, We denote its determinant by $\det_{k,l} a_{kl}$.

1.1. A determinant of Cauchy type. Recall that the *Cauchy determinant* (see, for instance, [11]) is given by

$$\det_{kl} \frac{1}{x_k + y_l} = \frac{\prod_{1\leqslant k<l\leqslant n}(x_k - x_l) \cdot \prod_{1\leqslant k<l\leqslant n}(y_k - y_l)}{\prod_{1\leqslant k,l\leqslant n} (x_k + y_l)}. \tag{1.1}$$

The following variant of the Cauchy determinant, is also well known.

LEMMA 1.1.

$$\det \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \frac{1}{x_1+b_1} & \frac{1}{x_2+b_1} & \frac{1}{x_3+b_1} & \dots & \frac{1}{x_n+b_1} \\ \frac{1}{x_1+b_2} & \frac{1}{x_2+b_2} & \frac{1}{x_3+b_2} & \dots & \frac{1}{x_n+b_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_1+b_{n-1}} & \frac{1}{x_2+b_{n-1}} & \frac{1}{x_3+b_{n-1}} & \dots & \frac{1}{x_n+b_{n-1}} \end{pmatrix} =$$

$$= \frac{\prod_{1\leqslant k<l\leqslant n}(x_k - x_l)\prod_{1\leqslant \alpha<\beta\leqslant n-1}(b_\alpha - b_\beta)}{\prod_{\substack{1\leqslant k\leqslant n\\ 1\leqslant\alpha\leqslant n-1}} (x_k + b_\alpha)}. \tag{1.2}$$

Proof. Let Δ be the Cauchy determinant (1.1). Then

$$y_1\Delta = \begin{pmatrix} \frac{y_1}{x_1+y_1} & \frac{y_1}{x_2+y_1} & \dots & \frac{y_1}{x_1+y_1} \\ \frac{1}{x_1+y_2} & \frac{1}{x_2+y_2} & \dots & \frac{1}{x_n+y_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_1+y_n} & \frac{1}{x_2+y_n} & \dots & \frac{1}{x_n+y_n} \end{pmatrix}.$$

We consider $\lim_{y_1\to\infty} y_1\Delta$ and substitute $y_{\alpha+1} = b_\alpha$.

1.2. One standard determinant. The following determinant is equivalent to Lemma 3 from Krattenthaller, [11].

Lemma 1.2.

$$\det\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \frac{x_1+b_1}{x_1+a_1} & \frac{x_2+a_1}{x_2+b_1} & \frac{x_3+a_1}{x_3+b_1} & \dots & \frac{x_n+a_1}{x_n+b_1} \\ \frac{(x_1+a_1)(x_1+a_2)}{(x_1+b_1)(x_1+b_2)} & \frac{(x_2+a_1)(x_2+a_2)}{(x_2+b_1)(x_2+b_2)} & \frac{(x_3+a_1)(x_3+a_2)}{(x_3+b_1)(x_3+b_2)} & \dots & \frac{(x_n+a_1)(x_n+a_2)}{(x_n+b_1)(x_n+b_2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\prod\limits_{1\leqslant m\leqslant n-1}(x_1+a_m)}{\prod\limits_{1\leqslant m\leqslant n-1}(x_1+b_m)} & \frac{\prod\limits_{1\leqslant m\leqslant n-1}(x_2+a_m)}{\prod\limits_{1\leqslant m\leqslant n-1}(x_2+b_m)} & \frac{\prod\limits_{1\leqslant m\leqslant n-1}(x_3+a_m)}{\prod\limits_{1\leqslant m\leqslant n-1}(x_3+b_m)} & \dots & \frac{\prod\limits_{m:\,1\leqslant m\leqslant n-1}(x_n+a_m)}{\prod\limits_{m:\,1\leqslant m\leqslant n-1}(x_n+b_m)} \end{pmatrix} =$$

$$= \frac{\prod\limits_{1\leqslant k<l\leqslant n}(x_k-x_l)\prod\limits_{1\leqslant\alpha\leqslant\beta\leqslant n-1}(a_\alpha-b_\beta)}{\prod\limits_{1\leqslant k\leqslant n,1\leqslant\beta\leqslant n-1}(x_k+b_\beta)}. \quad (1.3)$$

Proof. Decomposing a matrix element into the sum of partial fractions, we obtain

$$\frac{(x_k+a_1)\dots(x_k+a_\alpha)}{(x_k+b_1)\dots(x_k+b_\alpha)} = 1+\sum_{1\leqslant\beta\leqslant\alpha}\frac{(a_1-b_\beta)(a_2-b_\beta)\dots(a_\alpha-b_\beta)}{(b_1-b_\beta)\dots(b_{\beta-1}-b_\beta)(b_{\beta-1}-b_\beta)}\cdot\frac{1}{x_k+b_\beta}$$

We observe that the $(\alpha+1)$-th row is a linear combination of the rows

$$\begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix},$$
$$\begin{pmatrix} \frac{1}{x_1+b_1} & \frac{1}{x_2+b_1} & \dots & \frac{1}{x_n+b_1} \end{pmatrix},$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$\begin{pmatrix} \frac{1}{x_1+b_\alpha} & \frac{1}{x_2+b_\alpha} & \dots & \frac{1}{x_n+b_\alpha} \end{pmatrix}.$$

Thus our determinant is

$$\prod_{\alpha=1}^{l-1}\frac{\prod_{j=1}^{\alpha}(a_j-b_\alpha)}{\prod_{j=1}^{\alpha-1}(b_j-b_\alpha)}\cdot\det\begin{pmatrix} 1 & 1 & \dots & 1 \\ \frac{1}{x_1+b_1} & \frac{1}{x_2+b_1} & \dots & \frac{1}{x_n+b_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_1+b_\alpha} & \frac{1}{x_2+b_\alpha} & \dots & \frac{1}{x_n+b_\alpha} \end{pmatrix}.$$

and we reduce the evaluation of our determinant to Lemma 1.1.

1.3. Two determinants.

LEMMA 1.3. *Concider the $n\times n$ matrix Q having elements*

$$q_{lk}=\frac{(x_k+a_1)(x_k+a_2)\dots(x_k+a_l)}{(x_k+b_1)(x_k+b_2)\dots(x_k+b_l)}-\frac{(x_k-a_1)(x_k-a_2)\dots(x_k-a_l)}{(x_k-b_1)(x_k-b_2)\dots(x_k-b_l)}.$$

Then

$$\det Q=\frac{2^n\prod_{1\leqslant k\leqslant n}x_k\cdot\prod_{1\leqslant k<l\leqslant n}(x_k^2-x_l^2)\cdot\prod_{1\leqslant\alpha<\beta\leqslant n}(b_\alpha+b_\beta)\cdot\prod_{1\leqslant\alpha\leqslant\beta\leqslant n}(a_\alpha-b_\beta)}{\prod_{1\leqslant k\leqslant n,1\leqslant\alpha\leqslant n-1}(x_k^2-b_\alpha^2)}.$$

PROOF. We expand a matrix element in a sum of partial fractions.

$$q_{kl}=\sum_{\alpha=1}^{l}\frac{\prod_{j=1}^{l}(a_j-b_\alpha)}{\prod_{j=1}^{\alpha-1}(b_j-b_\alpha)}\Big\{\frac{1}{x_k+b_\alpha}+\frac{1}{x_k-b_\alpha}\Big\}.$$

We write

$$\frac{1}{x_k+b_\alpha}+\frac{1}{x_k-b_\alpha}=\frac{2x_k}{x_k^2-b_\alpha^2}$$

and observe that an l-th row is a linear combination of vector-rows

$$\begin{matrix}\begin{pmatrix}\frac{x_1}{x_1^2-b_1^2} & \dots & \frac{x_k}{x_n^2-b_1^2}\end{pmatrix},\\ \dots\quad\dots\quad\dots\quad\dots\quad\dots\\ \begin{pmatrix}\frac{x_1}{x_1^2-b_l^2} & \dots & \frac{x_k}{x_n^2-b_l^2}\end{pmatrix}.\end{matrix}$$

Thus our determinant is

$$\prod_{l=1}^{n}\frac{\prod_{j=1}^{l}(a_j-b_\alpha)}{\prod_{\alpha=1}^{\alpha-1}(b_j-b_\alpha)}\cdot\prod_{j=1}^{n}(2x_j)\cdot\det_{1\leqslant j,l\leqslant n}\frac{1}{x_j^2-b_l^2}.$$

and the last factor is reduced to the classical Cauchy determinant (1.1).

LEMMA 1.4.

$$\det_{k,l}\left\{\frac{(x_k+a_1)(x_k+a_2)\dots(x_k+a_l)}{(x_k+b_1)(x_k+b_2)\dots(x_k+b_l)}+\frac{(x_k-a_1)(x_k-a_2)\dots(x_k-a_l)}{(x_k-b_1)(x_k-b_2)\dots(x_k-b_l)}\right\}=$$
$$=\frac{2\prod_{1\leqslant k<l\leqslant n}(x_k^2-x_l^2)\cdot\prod_{1\leqslant\alpha\leqslant\beta\leqslant n-1}(b_\alpha+b_\beta)\cdot\prod_{1\leqslant\alpha\leqslant\beta\leqslant n-1}(a_\alpha-b_\beta)}{\prod_{1\leqslant k\leqslant n,1\leqslant\alpha\leqslant n-1}(x_k^2-b_\alpha^2)}.\quad(1.4)$$

REMARK. In particular the first row of our matrix is $\begin{pmatrix}2 & \dots & 2\end{pmatrix}$.

PROOF. Decomposing a matrix element into a sum of prime fractions, we obtain

$$q_{kl}=2+\sum_{\alpha=1}^{l}\frac{\prod_{j=1}^{\alpha}(a_j-b_\alpha)}{\prod_{j=1}^{\alpha-1}(b_j-b_\alpha)}\Big(\frac{1}{x_k+b_j}-\frac{1}{x_k-b_j}\Big)$$

Since,

$$\frac{1}{x_k+b_l}-\frac{1}{x_k-b_l}=\frac{-2b_k}{x_k^2-b_l^2},$$

the l-th row is a linear combination of the vectors-rows

$$\begin{matrix} (& 2 & \dots & 2 &) \\ (& \frac{b_1}{x_1^2-b_1^2} & \dots & \frac{b_1}{x_n^2-b_1^2} &) \\ \dots & \dots & \dots & \dots & \dots \\ (& \frac{b_{l-1}}{x_1^2-b_{n-1}^2} & \dots & \frac{b_n}{x_n^2-b_{n-1}^2} &) \end{matrix}$$

and we obtain

$$2\prod_{1\leqslant j\leqslant n-1} 2b_j \prod_{1\leqslant l\leqslant n-1}\frac{(a_1-b_l)(a_2-b_l)\dots(a_l-b_l)}{(b_1-b_l)(b_2-b_l)\dots(b_{l-1}-b_l)}\times \\ \times\det_{k,l}\begin{pmatrix} 1 & 1 & \dots & 1 \\ \frac{1}{x_1^2-b_1^2} & \frac{1}{x_2^2-b_1^2} & \dots & \frac{1}{x_n^2-b_1^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_1^2-b_{n-1}^2} & \frac{1}{x_2^2-b_{n-1}^2} & \dots & \frac{1}{x_n^2-b_{n-1}^2}\end{pmatrix}.$$

It remains to apply Lemma 1.1.

2. Sobolev kernel on the unitary group.

2.1. Definition of kernel. Let z be an $n\times n$ matrix with norm <1. For $\sigma\in\mathbb{C}$, we define the function $\det(1-z)^\sigma$ by

$$\det(1-z)^\sigma:=\det\Big[1-\sigma z+\frac{\sigma(\sigma-1)}{2!}g^2-\frac{\sigma(\sigma-1)(\sigma-2)}{3!}g^3+\dots\Big].$$

We also define this function for z satisfying $\|z\|=1$, $\det(1-z)\neq 0$ being

$$\det(1-z)^\sigma:=\lim_{u\to z,\,|u\|<1}\det(1-u)^\sigma.$$

The expression $\det(1-z)^\sigma$ is continuous on the domain $\|z\|\leqslant 1$ except the surface $\det(1-z)=0$.

We denote by $\det(1-z)^{\{\sigma|\tau\}}$ the function

$$\det(1-z)^{\{\sigma|\tau\}}:=\det(1-z)^\sigma\det(1-\overline{z})^\tau.$$

We define the function $\ell_{\sigma,\tau}(g)$ on the unitary group $\mathrm{U}(n)$ by

$$\ell_{\sigma,\tau}(g):=2^{-(\sigma+\tau)n}\det(1-z)^{\{\sigma|\tau\}}.$$

Obviously,

$$\ell_{\sigma,\tau}(h^{-1}gh)=\ell_{\sigma,\tau}(g)\qquad\text{for } g,\ h\in\mathrm{U}(n). \tag{2.1}$$

LEMMA 2.1. *Let $e^{i\psi_1}$, ..., $e^{i\psi_n}$ be the eigenvalues of $g \in \mathrm{U}(n)$; we assume $0 \leqslant \psi_k < 2\pi$. Then*

$$\ell(g) = \exp\big\{(\sigma - \tau)\sum_k(\psi_k - \pi)/2\big\}\prod_{k=1}^{n}\sin^{\sigma+\tau}\frac{\psi_k}{2}.$$

PROOF. It is sufficient to verify this statement for diagonal matrices, or equivalenly we can check the identity

$$(1 - e^{i\psi})^{\{\sigma|\tau\}} = \exp\big\{(\sigma - \tau)(\psi - \pi)/2\big\}\sin^{\sigma+\tau}\frac{\psi}{2}.$$

We have

$$\frac{1}{2}(1 - e^{i\psi}) = \exp\{i(\psi - \pi)/2\}\sin\frac{\psi}{2}.$$

Further, in the equality

$$2^{-\sigma}(1 - e^{i\psi})^{\sigma} = \exp\{i\sigma(\psi - \pi)/2\}\sin^{\sigma}\frac{\psi}{2},$$

the both sides are real-analytic on $(0, 2\pi)$ and the substitution $\psi = \pi$ gives 1 in the both sides. □

We also define the kernel $L_{\sigma,\tau}(g, h)$ on $\mathrm{U}(n)$ by

$$L_{\sigma,\tau}(g, h) = \ell_{\sigma,\tau}(gh^{-1}). \tag{2.2}$$

Obviously, this kernel is invariant with respect to left and right shifts on $\mathrm{U}(n)$, i.e.,

$$L_{\sigma,\tau}(r_1gr_2, r_1hr_2) = L_{\sigma,\tau}(g, h) \qquad \text{for } g,\ h,\ r_1,\ r_2 \in \mathrm{U}(n).$$

2.2. Characters, see Weyl book [28]. The set of finite dimensional representations of $\mathrm{U}(n)$ is parametrized by collections of integers (signatures)

$$\mathbf{m}: \quad m_1 > m_2 > \cdots > m_n.$$

The character $\chi_{\mathbf{m}}$ of representation $\pi_{\mathbf{m}}$ (a Schur function) corresponding to a signature $\mathbf{m}$ is given by

$$\chi_{\mathbf{m}}(g) = \frac{\det_{k,j=1,2,\dots,n}\left\{e^{im_j\psi_k}\right\}}{\det_{k,j=1,2,\dots,n}\left\{e^{i(j-1)\psi_k}\right\}}, \tag{2.3}$$

where $e^{i\psi_k}$ is the eigenvalues of g. Recall that the denominator admits decomposition

$$\prod_{l<k}(e^{i\psi_l} - e^{i\psi_k}). \tag{2.4}$$

The dimension of π_m is

$$\dim\pi_{\mathbf{m}} = \chi_{\mathbf{m}}(1) = \frac{\prod_{0\leqslant\alpha<\beta\leqslant n}(m_\alpha - m_\beta)}{\prod_{j=1}^{n} j!}. \tag{2.5}$$

A function $F(g)$ on $\mathrm{U}(n)$ is *central* if it satisfies the identity $F(h^{-1}gh) = F(g)$.

Consider the Haar measure μ on $\mathrm{U}(n)$ normalized by the condition: the measure of the whole group is 1. For a central function on $\mathrm{U}(n)$, the following *Weyl integration formula* holds

$$\int_{\mathrm{U}(n)} F(g)\, d\mu(g) =$$

$$= \frac{1}{(2\pi)^n n!} \int_{0<\psi_1<2\pi} \dots \int_{0<\psi_n<2\pi} F\big(\mathrm{diag}(e^{i\psi_1}, \dots, e^{i\psi_n})\big) \Big| \prod_{m<k} (e^{i\psi_m} - e^{i\psi_k}) \Big|^2 \prod_{k=1}^{n} d\varphi_k, \tag{2.6}$$

where $\mathrm{diag}(\cdot)$ is a diagonal matrix with given entries.

A central function $F \in L^2(\mathrm{U}(n))$ admits an expansion in characters,

$$F(g) = \sum_{\mathbf{m}} c_{\mathbf{m}} \chi_m.$$

where the summation is given over all the signatures $\mathbf{m}$ and the coefficients $c_{\mathbf{m}}$ are the L^2-inner products

$$c_{\mathbf{m}} = \int_{\mathrm{U}(n)} F(g) \overline{\chi_{\mathbf{m}}(g)}\, d\mu(g).$$

Applying formula (2.6), explicit expression (2.3) for characters, and formula (2.4) for the denominator, we obtain

$$c_{\mathbf{m}} = \frac{1}{(2\pi)^n n!} \int_{0<\psi_1<2\pi} \dots \int_{0<\psi_n<2\pi} F\Big(\mathrm{diag}\{e^{i\psi_1}, \dots, e^{i\psi_n}\}\Big) \times$$

$$\times \det_{k,j=1,2,\dots,n} \big\{e^{i(j-1)\psi_k}\big\} \det_{k,j=1,2,\dots,n} \big\{e^{-im_j\psi_k}\big\} \prod_{k=1}^{n} d\varphi_k. \tag{2.7}$$

For calculation of such expressions, we will use the following evident Lemma (see, for instance, [15],)

LEMMA 2.2. *Let X be a set,*

$$\int_{X^n} \prod_{k=1}^{n} f(x_k) \det_{k,l=1,\dots n} \{u_l(x_k)\} \det_{k,l=1,\dots n} \{v_l(x_k)\} \prod_{j=1}^{n} dx_j =$$

$$= n! \det_{l,m=1,\dots,n} \Big\{ \int_X f(x) u_l(x) v_m(x)\, dx \Big\}$$

2.3. Lobachevsky beta-integrals. We will use two following integrals, see [7], 3.631,1, 3.631,8,

$$\int_0^\pi \sin^{\mu-1}(\varphi)\, e^{ibx}\, dx = \frac{2^{1-\mu}\pi\Gamma(\mu)e^{ib\pi/2}}{\Gamma\big((\mu+b+1)/2\big)\Gamma\big((\mu-b+1)/2\big)} \tag{2.8}$$

$$\int_0^\pi \sin^{\mu-1}(\varphi)\, \cos(bx)\, dx = \frac{2^{1-\mu}\pi\Gamma(\mu)\cos(b\pi/2)}{\Gamma\big((\mu+b+1)/2\big)\Gamma\big((\mu-b+1)/2\big)} \tag{2.9}$$

In some sence, our integral evaluations below, 2.4, 3.3, 4.3, are multivariate analogs of these integrals.

2.4. Expansion of the function $\ell_{\sigma,\tau}$.

THEOREM 2.3. *Let* $\mathrm{Re}(\sigma+\tau)<1$. *Then*

$$\ell_{\sigma,\tau}(g) = \\ = \frac{(-1)^{n(n-1)/2}\sin^n(\pi\sigma)2^{-(\sigma+\tau)n}}{\pi^n}\prod_{j=1}^n \Gamma(\sigma+\tau+j)\times \\ \times\sum_{\mathbf{m}}\Bigg\{\prod_{1\leqslant\alpha<\beta\leqslant n}(m_\alpha-m_\beta)\prod_{j=1}^n\frac{\Gamma(-\sigma+m_j-n+1)}{\Gamma(\tau+m_j+1)}\chi_{\mathbf{m}}(g)\Bigg\} = \tag{2.10}$$

$$= (-1)^{n(n-1)/2}2^{-(\sigma+\tau)n}\prod_{j=1}^n\Gamma(\sigma+\tau+j)\times \\ \times\sum_{\mathbf{m}}\Bigg\{\frac{(-1)^{\sum m_j}\prod\limits_{1\leqslant\alpha<\beta\leqslant n}(m_\alpha-m_\beta)}{\prod\limits_{j=1}^n\Gamma(\sigma-m_j+n)\Gamma(\tau+m_j+1)}\chi_{\mathbf{m}}(g)\Bigg\}. \tag{2.11}$$

PROOF. We must evaluate the inner product

$$\int_{\mathrm{U}(n)}\ell_{\sigma,\tau}(g)\,\overline{\chi_{\mathbf{m}}(g)}\,d\mu(g)$$

Applying (2.7), we obtain

$$\frac{1}{(2\pi)^n\, n!}\int\limits_{0<\psi_k<2\pi}\prod_{j=1}^n\Big[\sin^{\sigma+\tau}(\psi_j/2)\cdot\exp\big\{i(\sigma-\tau)(\psi_j-\pi)/2\big\}\Big]\times \\ \times\det_{1\leqslant k,l\leqslant n}\big\{e^{-im_k\psi_l}\big\}\cdot\det_{1\leqslant k,l\leqslant n}\big\{e^{i(k-1)\varphi_l}\big\}\prod_{l=1}^n d\varphi_l.$$

By Lemma 2.2, we reduce this integral to

$$\frac{1}{(2\pi)^n}\det_{1\leqslant k,j\leqslant n} I(k,j),$$

where

$$I(k,j) = e^{-i(\sigma-\tau)\pi/2} \int_0^{2\pi} \sin^{\sigma+\tau}(\psi/2) \cdot \exp\{i((\sigma+\tau)/2+k-1-m_j)\}\, d\varphi.$$

We apply (2.8) and obtain

$$I(k,j) = \frac{2^{1-\sigma-\tau}\pi\Gamma(\sigma+\tau+1)\,(-1)^{k-1-m_j}}{\Gamma(\sigma+k-m_j)\Gamma(\tau-k+m_j+2)}$$

Applying standard formulae for Γ-function, we obtain

$$I(k,j) = 2^{1-\sigma-\tau}\Gamma(\sigma+\tau+1)\,\sin(-\sigma\pi)\cdot\frac{\Gamma(-\sigma+m_j-k+1)}{\Gamma(\tau+m_j-k+2)} =$$
$$= 2^{1-\sigma-\tau}\Gamma(\sigma+\tau+1)\,\sin(-\sigma\pi)\cdot\frac{\Gamma(-\sigma+m_j-n+1)}{\Gamma(\tau+m_j-n+2)}\cdot\boxed{\frac{(-\sigma+m_j-n+1)_{n-k}}{(\tau+m_j-n+2)_{n-k}}}$$

The factors outside the box do not depend on on k. Thus, we must evaluate the determinant

$$\det_{1\leqslant k,j\leqslant n} \frac{(-\sigma+m_j-n+1)_{n-k}}{(\tau+m_j-n+2)_{n-k}}$$

Up to a permutation of rows, it is a determinant of the form described in Lemma 1.2 with

$$x_j = m_j, \qquad a_j = -\sigma-n+j, \qquad \tau = -n+j+1.$$

After a simple rearrangement of the factors, we obtain the required result □

2.5. Hermitians forms defined by kernels. First, recall some standard facts on characters of compact groups, for details see, for instance, [9], 9.2, 11.1.

Let K be a compact Lie group equipped with the Haar measure μ, we assume that the measure of the whole group is 1. Let π_1, π_2, ...be the complete collection of pairwise distinct irreducible representations. Let χ_1, χ_2, ...be their characters. Recall the orthogonality relations

$$\langle\chi_k,\chi_l\rangle_{L^2} := \int_K \chi_k(h)\overline{\chi_l(h)}\, d\mu(h) = \delta_{k,l} \tag{2.12}$$

and

$$\chi_k * \chi_l = \begin{cases} \frac{1}{\dim \pi_k}\chi_k, & \text{if } k=1 \\ 0, & \text{if } k \neq l \end{cases} \tag{2.13}$$

where $*$ denotes the convolution on the group, $u*v(g) = \int u(gh^{-1})v(h)d\mu(h)$.

We consider the action of the group $K\times K$ on K by the left and right shifts $(k_1,k_2):\ g\mapsto k_1gk_2$. The representation $K\times K$ in $L^2(K)$ is a multiplicity free direct sum of irreducible represntations having the form $\pi_k\otimes\pi_k^*$, where π_k^* is the dual representation,

$$L^2(K) \simeq \bigoplus_k \rho_k\otimes\rho_k^*. \tag{2.14}$$

Each distribution f on K is a sum of "elementary harmonics"

$$f = \sum_k f^{(k)}, \qquad f_k \in V_k.$$

The summands of this sum correspond to the decomposition (2.14).

The projector to a subspace $\rho_k \otimes \rho_k^*$ of k-th elementary harmonics is the convolution with the corresponding character,

$$f^{\{k\}} = \frac{1}{\dim \pi_k} f * \chi_k \tag{2.15}$$

(in particular, $f^{(k)}$ is smooth).

For a central distribution Ξ on K, consider the Hermitian form

$$\langle u, v\rangle = \iint_{K\times K} \Xi(h,g)\, u(h)\overline{v(g)}\, d\mu(u)\, d\mu(g).$$

Consider the expansion of Ξ in characters

$$\Xi = \sum_k c_k \chi_k.$$

LEMMA 2.4.

$$\langle u, v\rangle = \sum_k \frac{c_k}{\dim \pi_k} \int_{\mathrm{U}(n)} u^{\{k\}}(h)\overline{v^{\{k\}}(h)}\, d\mu(h). \tag{2.16}$$

PROOF. We can assume $u = \chi_k$, $v = \chi_l$. We evaluate

$$\iint_{K\times K} \sum_k c_k \Xi(gh^{-1})\, \chi_k(h)\overline{\chi}_l(g)\, d\mu(g)\, d\mu(h).$$

using (2.12) and (2.13). □

2.6. Positivity. Let $\mathrm{Re}(\sigma+\tau) < 1$. Consider the sesquilinear form on $C^\infty(\mathrm{U}(n))$ given by

$$\langle q, r\rangle_{\sigma,\tau} = \iint_{\mathrm{U}(n)\times\mathrm{U}(n)} L_{\sigma,\tau}(g,h) q(g)\overline{r(h)}\, dg\, dh, \tag{2.17}$$

where the kernel $L_{\sigma,tau}$ is the same as above. Obviously, for fixed q, r, this expression admits a meromorphic continuation in σ, τ to the whole $\mathbb{C}^2$. Moreover, Theorem 2.3 allows to write an explicit expression for this continuation. Expanding q and r in elementary harmonics

$$q(h) = \sum_{\mathbf{m}} \varkappa_{\mathbf{m}}(h), \qquad r(h) = \sum_{\mathbf{m}} \theta_{\mathbf{m}}(h),$$

we obtain (see Lemma 2.4)

$$\langle q, r\rangle_{\sigma,\tau} = \sum_{\mathbf{m}} \frac{c_{\mathbf{m}}}{\dim \pi_{\mathbf{m}}} \int_{\mathrm{U}(n)} \varkappa_{\mathbf{m}}(h)\overline{\theta_{\mathbf{m}}(h)}\, d\mu(h),$$

where the meromorphic expressions for $c_{\mathbf{m}}$ were obtained in Theorem 2.3.

If σ, $\tau \in \mathbb{R}$, then our kernel $L_{\sigma,\tau}$ is Hermitian, i.e., $L_{\sigma,\tau}(h,g) = \overline{L_{\sigma,\tau}(g,h)}$, or eguivalently

$$\langle q, r\rangle_{\sigma,\tau} = \overline{\langle r, q\rangle}_{\sigma,\tau}$$

COROLLARY 2.5. *For $\sigma, \tau \in \mathbb{R} \setminus \mathbb{Z}$, the inner product (2.17) is positive definite (up to a sign), iff fractional parts of $-\sigma - n$ and τ are equal.*

The domain of positivity is the union of the dotted squares on Figure 1.

For σ, τ satisfying this corollary, denote by $H_{\sigma,\tau}$ the completion of $C^\infty(\mathrm{U}(n))$ with respect to our inner product.

2.7. Action of $\mathrm{U}(n,n)$ on the space $\mathrm{U}(n)$. Consider the linear space $\mathbb{C}^n \oplus \mathbb{C}^n$ equipped with the indefinite Hermitian form

$$\{v \oplus w, v' \oplus w'\} = \langle v, v'\rangle_{\mathbb{C}^n \oplus 0} - \langle w, w'\rangle_{0 \oplus \mathbb{C}^n}$$

where $\langle \cdot, \cdot\rangle$ is the standard inner product in $\mathbb{C}^n$. Denote by $\mathrm{U}(n,n)$ the group of linear operators in $\mathbb{C}^n \oplus \mathbb{C}^n$ preserving the form $\{\cdot,\cdot\}$. We write elements of this group as block $(n+n)\times(n+n)$ matrices $g := \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$. By definition, these matrices satisfy the condition

$$g\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} g^* = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2.18}$$

For $h \in \mathrm{U}(n)$, consider its graph $\mathrm{graph}(h)$ in $\mathbb{C}^n \oplus \mathbb{C}^n$. It is an n-dimensional linear subspace, consisting of all vectors $z \oplus zh$, where a vector-row z ranges in $\mathbb{C}^n$. Since $h \in \mathrm{U}(n)$, the subspace $\mathrm{graph}(h)$ is isotropic[2] with respect to our Hermitian form $\{\cdot,\cdot\}$. Conversely, any n-dimensional isotropic subspace in $\mathbb{C}^n \oplus \mathbb{C}^n$ is a graph of a unitary operator $h \in \mathrm{U}(n)$.

Thus we have one-to-one correspondence between the group $\mathrm{U}(n)$ and the Grassmannian of n-dimensional isotropic subspaces in $\mathbb{C}^n \oplus \mathbb{C}^n$.

The group $\mathrm{U}(n,n)$ acts on the Grassmannian in an obvious way, and hence $\mathrm{U}(n,n)$ acts on the space $\mathrm{U}(n)$. An explicit formula for the latter action can be easily written:

$$g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}: \quad h \mapsto h^{[g]} := (a + zc)^{-1}(b + zd), \qquad h \in \mathrm{U}(n),\ g \in \mathrm{U}(n,n) \tag{2.19}$$

LEMMA 2.6. a) *For the Haar measure $\mu(h)$ on $\mathrm{U}(n)$, we have*

$$\mu(h^{[g]}) = |\det{}^{-2n}(\alpha + z\gamma)| \cdot \mu(h)$$

b) *The kernel $L_{\sigma,\tau}$ satisfies the identity*

$$L_{\sigma,\tau}(u^{[g]}, v^{[g]}) = L_{\sigma,\tau}(u,v)\det(\alpha + u\gamma)^{\{\sigma|\tau\}}\det(\alpha + v\gamma)^{\{\tau|\sigma\}}. \tag{2.20}$$

[2] A subspace V in a linear space is *isotropic* with respect to an Hermitian (or bilinear) form Q if Q equals 0 on V.

PROOF. a) The differential of the map $u \mapsto u^{[g]}$ is given by

$$du \mapsto (a + u\gamma)^{-1}\, du\, (-\gamma u^{[g]} + \delta), \qquad (2.21)$$

see, for instance, [13], Lemma 1.1.

The rational map $u \mapsto u^{[g]}$ is defined on the space of all complex $n \times n$ matrices. By (2.21), its complex Jacobian is

$$J(g, u) := \det{}^{-n}(a + zc)^{-n} \det{}^{n}(-\gamma u^{[g]} + \delta).$$

Hence the real Jacobian on the space of all matrices is $|J(g, u)|^2$, and the Jacobian of the map $\mathrm{U}(n) \to \mathrm{U}(n)$ is $|J(g, u)|$. It can easily be checked (see [13], Lemma 1.2) that

$$\det(-\gamma u^{[g]} + \delta) = \det(\alpha + z\gamma)^{-1} \det(g).$$

By (2.18), $|\det g| = 1$, and this finishes proof.

b) A direct calculation.

2.8. Unitary representations of $\mathrm{U}(n, n)$**.** Let $\mathrm{U}(n, n)$ acts in the space of C^∞-functions on $\mathrm{U}(n)$ by the operators

$$\rho_{\sigma,\tau}(g)F(h) = F(h^{[g]}) \det{}^{\{-n-\sigma|-n-\tau\}}(\alpha + h\gamma). \qquad (2.22)$$

REMARK. Let us explain the sence of the complex power in this formula. It can easily be checked with (2.18), that $\|\gamma\alpha^{-1}\| < 1$. Hence, for all matrices h satisfying $\|h\| \leqslant 1$, the matrix $\alpha + h\gamma$ is invertible. Hence the function

$$\ln\det(\alpha + h\gamma)$$

has a countable family of continuous branches on the set $\|h\| \leqslant 1$ and in particular on $\mathrm{U}(n)$. We define

$$\det{}^{\{-n-\sigma|-n-\tau\}}(\alpha + h\gamma) := \exp\{-(n+\sigma)\ln\det(\alpha + h\gamma) - \\ - (n+\tau)\overline{\ln\det(\alpha + h\gamma)} - 2\pi i k(\sigma - \tau)\}. \qquad (2.23)$$

Thus, we can think that for each $g \in \mathrm{U}(n, n)$ formula (2.22) defines a countable family of operators $\rho_{\sigma,\tau}(g)$, they differs one from another by constant factors $\exp\{2\pi i k(\sigma - \tau)\}$. These operators define a projective representation (see [9],14) of the group $\mathrm{U}(n, n)$

$$\rho_{\sigma,\tau}(g)\rho_{\sigma,\tau}(g') = \lambda(g, g')\rho_{\sigma,\tau}(gg'), \qquad \lambda(g, g') \in \mathbb{C}.$$

Equivalently, we can consider the multi-valued operator-valued function $\rho_{\sigma,\tau}(g)$ on $\mathrm{U}(n, n)$ as a single-valued function on the universal covering group $\mathrm{U}(n, n)^\sim$ of $\mathrm{U}(n, n)$. Then $\rho_{\sigma,\tau}(g)$ became a linear representation of $\mathrm{U}(n, n)^\sim$.

If $(\sigma - \tau) \in \mathbb{Z}$, then $\exp\{2\pi i k(\sigma - \tau)\}$ and (2.23) is a well defined single-valued expression. In this case $\rho_{\sigma,\tau}$ is a linear representation of $\mathrm{U}(n, n)$.

Proposition 2.7. *The operators $\rho_{\sigma,\tau}(g)$ preserve the form $\langle\cdot,\cdot\rangle_{\sigma,\tau}$.*

Proof. First, let $\mathrm{Re}(\sigma+\tau)<1$. Substitute $h_1=u_1^{[g]}$, $h_2=u_2^{[g]}$ to the integral

$$\iint_{\mathrm{U}(n)\times\mathrm{U}(n)} L_{\sigma,\tau}(h_1,h_2)\,q(h_1)\overline{r(h_2)}\,d\mu(h_1)\,d\mu(h_2).$$

By Lemma 2.5, we obtain

$$\iint_{\mathrm{U}(n)\times\mathrm{U}(n)} L_{\sigma,\tau}(u_1,u_2)\det(\alpha+u_1\gamma)^{\{-\sigma|-\tau\}}|\det(\alpha+u_2\gamma)|^{-\tau|-\sigma\}}\times$$
$$\times\ q(h_1)\overline{r(h_2)}|\det(\alpha+u_1\gamma)|^{-2n}|\det(\alpha+u_2\gamma)|^{-2n}\,d\mu(u_1)\,d\mu(u_2).$$

Thus, our operators preserve the form $\langle\cdot,\cdot\rangle_{\sigma,\tau}$.

For general σ, τ, we consider the analytic continuation.

Corollary 2.8. *For σ, τ satisfying the positivity conditions of Corollary 2.5, the representation $\rho_{\sigma,\tau}$ is unitary.*

2.9. Some remarks. a) *The case $n=1$, $\sigma=\tau=s-1$* gives precisely the complementary series of representations of $\mathrm{SL}(2,\mathbb{R})\sim\mathrm{SU}(1,1)$ described above in 0.1.

b) The representations $\rho_{\sigma,\tau}$ are very degenerated in the followng sense. For any irreducible unitary representation of a semisimple group G, its restriction to the maximal compact subgroup K has a spectrum with finite multiplicities, but usually these multiplicities are not bounded.

In our case, i.e., $G=\mathrm{U}(n,n)$, $K=\mathrm{U}(n)\times\mathrm{U}(n)$, the restriction of $\rho_{\sigma,\tau}$ to K is the multiplicity free sum $\rho_{\mathbf{m}}\otimes\rho^*_{\mathbf{m}}$ (thus, only few representations of K are present in the spectrum).

c) *Shifts of parameters.* For integer k, the *projective* representations $\rho_{\sigma+k,\tau-k}$ and $\rho_{\sigma,\tau}$ are equivalent. The intertwining operator is the multiplication by the determinant

$$F(h)\mapsto F(h)\det(h)^k.$$

This operator also defines an isometry of forms $L_{\sigma+k,\tau-k}$ and $L_{\sigma,\tau}$

d) *Symmetry.* Representations $\rho_{-n/2-p,-n/2-q}$ and $\rho_{-n/2+p,-n/2+q}$ are dual. The invariant pairing is given by the formula

$$(F_1,F_2)\mapsto\int_{\mathrm{U}(n)}F_1(h)F_2(h)\,d\mu(h). \tag{2.24}$$

For verification of this statement, we substitute $h\mapsto h^{[g]}$ and apply the formula for the Jacobian.

In particular, the point $(\sigma,\tau)=(n/2,n/2)$ corresponds to a representation of a unitary principle series of $\mathrm{U}(n,n)$ (it is unitary in the space $L^2(\mathrm{U}(n))$).

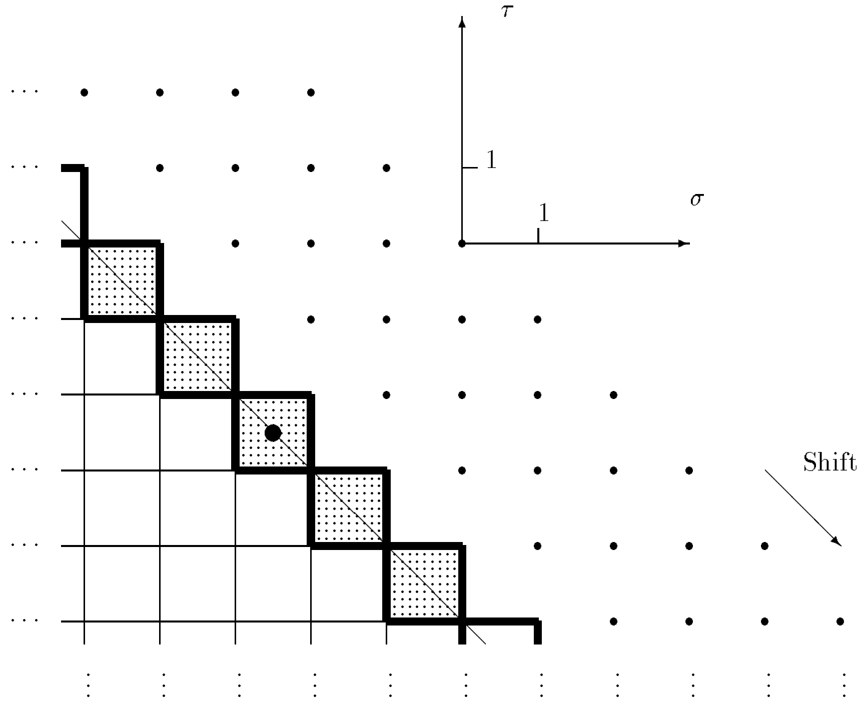

FIGURE. Groups $\mathrm{U}(n)$. The case $n = 5$.

1. The dotted squares correspond to unitary representations $\rho_{\sigma,\tau}$.

2. Vertical and horizontal rays in the south-west of Figure correspond to nondegenerated highest weight and lowest weight representations. Fat points correspond to degenerated highest and lowest weight, representations, and also to unipotent representations. The point $(\sigma, \tau) = (0, 0)$ corresponds to the trivial one-dimensional representation.

3. In points of the thick segments, we have some exotic unitary sub-factors.

4. The shift $(\sigma, \tau) \mapsto (\sigma + 1, \tau - 1)$ transform a projective representation $\rho_{\sigma,\tau}$ to an equivalent representation.

5. The permutation of the axes $(\tau, \sigma) \mapsto (\sigma, \tau)$ gives a complex conjugate representation.

6. The symmetry with respect to the point $(-n/2, -n/2)$ (black circle) gives a dual representation (for odd n this point is a center of a dotted square; for even n this point is a common vertex of two dotted squares).

7. For $\sigma + \tau = n$ (the skew diagonal line) our Hermitian form is the standard L^2-product. The represenations $\rho_{\sigma,\tau}$ in this case are in the untary principal series.

8. Linear (non-projective) reprsentations of $\mathrm{U}(n, n)$ correspond to the family of parallel lines $\sigma - \tau \in \mathbb{Z}$.

e) *Another symmetry.* The representation $\rho_{\tau,\sigma}$ is complex cojugate to $\rho_{\sigma,\tau}$

f) *Problem of unitarisability of subquotients.* Corollary 2.8 gives a classification of unitary representations among $\rho_{\sigma,\tau}$. But for integer σ or integer τ, the representation $\rho_{\sigma,\tau}$ can contain a unitary subrepresentation, a unitary factor-representation, or a unitary sub-factor.

g) *Unitary highest weight representations and Sahi's unipotent representations.* The kernels

$$L_{\sigma,0}(u,v) = \det{}^{\sigma}(1 - uv^{-1})$$

are well-known, and they define highest weight representations. By a well-known theorem (Berezin, Gindikin, Rossi–Vergne, Wallach), the form $L_{\sigma,0}$ is a nonnegative Hermitian form iff

$$\sigma < -n+1 \qquad \text{and for } \sigma = -n+1, -n+2, \ldots, 0. \tag{2.25}$$

First, let $\sigma < -(n-1)$, If m_n is negative, then the factor $\Gamma(0+m_j+1)$ in the denominator of (2.10) is infinity, and hence (2.10) is 0. If $m_n \geqslant 0$, then all other coeffitients $c_{\mathbf{m}}$ have the same sign (all signs are positive or all are negative). It is easy to observe this from (2.10) for noninteger σ. For integer σ, this follows from the limit considerations: numerator and denominator in (2.19) have poles of the same order, the limit as $\sigma \to -l$ have to be nonnegative as a limit of a nonnegative function.

Second, let σ be in the discrete part of the set (2.25), $\sigma = -n + \theta$. Then the factor $\prod \Gamma(\alpha + j)$ has a pole of order $n - j$. The factor $\prod \Gamma(\sigma + n - m_j)$ has a pole of order $\geqslant n - j$ (since $m_n \geqslant 0$, $m_{n-1} \geqslant 1, \ldots, m_1 \geqslant n-1$). Hence the ratio is finite, it is nonvanishing if the order of a pole of denominator is precisely $n - j$. This happens iff $m_n = 0$, $m_1 = 1, \ldots, m_{n-j+1} = j-1$. After this, it remains to follow signs in (2.11).

Remark. This consideration almost coincides with the original Berezin's proof [2]. Hua Loo Keng [8] have obtained an expansion of the kernel $L_{\sigma,0}$ in a series of characters (it is a partial case] of our Theorem 2.3), and Berezin checked signs in this expansion.

In integer points lying in the strip $0 \geqslant \sigma + \tau \geqslant -n+1$ also there are located unipotent representations [23], see below Section 5.

h) *Some other unitary subquotients.* Consider a representation $\rho_{\sigma,\tau}$ lying on the boundary of the dotted domain on Fig.1. For definiteness, assume $\tau = 0$. Obviously, each term of its Jordan–Holder series is unitarizable (since a limit of positive inner products is positive). Denote by Y_j the set of all the signatures $\mathbf{m}$ satisfying the condition

$$m_{n-j} \geqslant 0, \qquad m_{n-j+1} < 0$$

(i.e., precisely j terms of the signature are negarive). For $\mathbf{m} \in Y_j$, the expression $c_{\mathbf{m}}$ (2.19) has a zero of order j.

Denote by W_j, the subspace in $C^\infty(\mathrm{U}(n))$ spanned by all harmonics with signatures lying in $\cup_{k \geqslant j} Y_j$. We obtain an invariant filtration

$$C^\infty(\mathrm{U}(n)) = W_0 \supset W_1 \supset \cdots \supset W_n$$

The representation of $\mathrm{U}(n,n)$ in each sub-factor W_j/W_{j+1} is unitaty.

i) *Matrix Sobolev spaces of an arbitrary order.* Denote

$$s = -\sigma - \tau + n.$$

Let F be a distribution on $\mathrm{U}(n)$, let $F = \sum F_{\mathbf{m}}$ be its expansion in a series of elementary harmonics. We have

$$F \in H_{\sigma,\tau} \qquad \Longleftrightarrow \qquad \sum_{\mathbf{m}} \frac{c_{\mathbf{m}}}{\dim \pi_{\mathbf{m}}} \|F_{\mathbf{m}}\|^2_{L^2} < \infty \qquad \Longleftrightarrow$$
$$\Longleftrightarrow \qquad \sum_{\mathbf{m}} \Big\{ \|F_{\mathbf{m}}\|^2_{L^2} \prod_{j=1}^{n} (1+|m_j|)^s \Big\} < \infty, \quad (2.26)$$

where $\|F_{\mathbf{m}}\|_{L^2}$ denotes

$$\|F_{\mathbf{m}}\|_{L^2} := \Big(\int_{\mathrm{U}(n)} |F_{\mathbf{m}}(h)|^2 \, d\mu(h) \Big)^{1/2}$$

Our Hermitian form defines a norm only in the case $|s|<1$, but (2.26) has sence for arbitrary real s, and thus *we have a possibility to define a Sobolev space* $\mathcal{H}_s$ *on* $\mathrm{U}(n)$ *of an arbitrary order.*

Author do not know applications of this remark, but it seems that it can be useful in two following situations.

First, a reasonable harmonic analysis related to semisimple Lie groups is the analysis of unitary represenations. But near 1980 Molchanov observed that there are many identities with special function that admits interpretations on "physical level of rigor" as formulae of nonunitary harmonic analysis. Up to now, there are no reasonable interpretations of this phenomenon (see, for instance [3], see also [14], Section 1-32 and formula (2.6)–(2.15)). In particular, we do not know reasonable functional spaces that can be place of action of this analysis. It seems that our spaces H_s can be possible candidates.

Second, natural integral operators in the noncomutative harmonic analysis seem similar to pseudo-differential operators, but they are not pseudo-differential operators in the usual sence. In particular, they are not well compatible with the standard scales of functional spaces. It can happened that our spaces $\mathcal{H}_s$ can be more reasonable in this situation.

j) *An identity for formal series.* We have (see notation (0.4))

$$\big|\sin(\varphi/2)\big|^{\sigma+\tau} e^{(\sigma-\tau)\varphi} = \frac{2^{\sigma+\tau}\Gamma(\sigma+\tau+1)}{\Gamma(\sigma)\Gamma(\tau)} \sum_{m=-\infty}^{\infty} \frac{(-1)^m e^{im\varphi}}{(\tau+1)_m(\sigma+1)_{-m}}.$$

Introduce new variables $x_p = e^{i\varphi_p}$. We can rewrite Theorem 2.3 as the following identity for formal series

$$\prod_{p=1}^{n}\Big\{\sum_{m=-\infty}^{\infty}\frac{(-1)^m x_p^l}{(\tau+1)_m(\sigma+1)_{-m}}\Big\}=$$
$$=\frac{\prod_{k=1}^{n-1}(\sigma+\tau+1)_k}{(\sigma)_n^n}\cdot\sum_{\mathbf{l}}\Big[\prod_{j=1}^{n}\frac{(-1)^{j_j}}{(\tau+1)_{l_j}(\sigma+n)_{-l_j}}\Big]\frac{\{\det_{1\leqslant j,p\leqslant n}x_p^{l_j}\}}{\{\det_{1\leqslant j,p\leqslant n}x_p^{j-1}\}}$$

3. Orthogonal groups

3.1. Definition of the kernel. We consider the (disconnected) group $\mathrm{O}(2n)$ as a basic object[3]. Each element of this group can be reduced by a conjugation $g\mapsto hgh^{-1}$ to the block diagonal form with 2×2-blocks

$$\begin{pmatrix} A(\varphi_1) & 0 & \dots & 0\\ 0 & A(\varphi_2) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & A(\varphi_n)\end{pmatrix}, \text{ where } A(\varphi_j)=\begin{pmatrix}\cos\varphi_j & \sin\varphi_j\\ -\sin\varphi_j & \cos\varphi_j\end{pmatrix}. \tag{3.1}$$

The collection $(\varphi_1,\dots,\varphi_n)$ is uniquely determined by g modulo permutations and arbitrary transformations $\varphi_j\mapsto-\varphi_j$. The numbers $e^{\pm i\varphi_j}$ coincide with the eigenvalues of g.

We define the function $\ell_\lambda(g)$ on $\mathrm{O}(2n)$ by

$$\ell_\lambda(g)=\begin{cases}\det\big((1-g)/2\big)^\lambda, & \text{, if } g\in\mathrm{SO}(2n);\\ 0, & \text{if } g\notin\mathrm{SO}(2n).\end{cases}$$

the derminant is nonegative and hence its complex powers are well-defined. In the terms of the eigenvalues $e^{\pm i\varphi_j}$, the function $\ell_\lambda(g)$ coincides with

$$\ell_\lambda(g)=\prod_{j=1}^{n}\big|\sin(\psi_j/2)\big|^{2\lambda}.$$

We define the kernel $L_\lambda(\cdot,\cdot)$ on $\mathrm{O}(2n)$ as

$$L_\lambda(g,h)=\ell_\lambda(gh^{-1}).$$

3.2. Characters, see [28]. Irreducible representations π_λ of $\mathrm{SO}(2n)$ are parametrized by collections of numbers

$$(\mathbf{l},\varepsilon):\quad l_1>l_2>\dots>l_n>0,\qquad \varepsilon=\pm1,$$

[3]It is also possible to concider its connectd subgroup $\mathrm{SO}(2n)$; in this case we must replace the group $\mathrm{O}(2n,2n)$ below by it connected component $\mathrm{SO}_0(2n,2n)$, consider a connected component of the Grassmannian and also do some obvious minor changes.

or

$$\mathbf{l}: \quad l_1 > l_2 > \dots > l_n = 0.$$

Formulae for the characters are slightly different in these two cases.

If $l_n = 0$, we have

$$\chi_{\mathbf{l}}(g) = \frac{\det_{1\leqslant k,m\leqslant n}\{\cos(l_k\varphi_m)\}}{\det_{1\leqslant k,m\leqslant n}\{\cos(k-1)\varphi_m\}}.$$

For $l_n \neq 0$, we have

$$\chi_{\mathbf{l}}^{\varepsilon}(g) = \frac{\det_{1\leqslant k,m\leqslant n}\{\cos(l_k\varphi_m)\} + \varepsilon\det_{1\leqslant k,m\leqslant n}\{\sin(l_k\varphi_m)\}}{2\det_{1\leqslant k,m\leqslant n}\{\cos(k-1)\varphi_m\}}.$$

Denote by J the diagonal matrix $\in \mathrm{O}(2n)$ having $(2n-1)$ entries $(+1)$ and one (-1). The map $h \mapsto JhJ^{-1}$ is an interior automorphism of $\mathrm{O}(2n)$ and an exterior automorphism of $\mathrm{SO}(2n)$. The representations $\rho_{\mathbf{l}}^{\pm}$ of $\mathrm{SO}(2n)$ corresponding to the characters $\chi_{\mathbf{l}}^{\pm}$ are twins in the following sence

$$\pi_{\mathbf{l}}^{\pm}(JhJ^{-1}) = \pi_{\mathbf{l}}^{\mp}(h).$$

Also the substitution

$$\varphi_1 \mapsto \varphi_1, \qquad \dots \qquad \varphi_{n-1} \mapsto \varphi_{n-1}, \qquad \varphi_n \mapsto -\varphi_n$$

changes $\chi_{\mathbf{l}}^{+}$ and $\chi_{\mathbf{l}}^{-}$.

For $l_n = 0$, the representation $\pi_{\mathbf{l}}$ is stable with respect to the exterior automorphism $h \mapsto JhJ^{-1}$.

This digression about two types of characters is almost non-essential for us, since really we will consider arbitrary

$$\mathbf{l}: \quad l_1 > l_2 > \dots > l_n \geqslant 0,$$

and the functions $\chi_{\mathbf{l}}$ given by

$$\chi_{\mathbf{l}}(g) := \frac{\det_{1\leqslant k,m\leqslant n}\{\cos(l_k\varphi_m)\}}{\det_{1\leqslant k,m\leqslant n}\{\cos(k-1)\varphi_m\}} = \begin{cases} \chi_{\mathbf{l}}(g), & \text{if } l_n = 0, \\ \chi_{\mathbf{l}}^{+}(g) + \chi_{\mathbf{l}}^{-}(g), & \text{if } l_n > 0. \end{cases} \tag{3.2}$$

REMARK. The functions (3.2) are precisely restrictions of the characters of the disconnected group $\mathrm{O}(2n)$ to the connected group $\mathrm{SO}(2n)$.

Let F be a central functions on $\mathrm{SO}(2n)$. Let f be its restriction to the maximal torus, i.e., to the set of matrices having the form (3.1). The *Weyl integration formula* for central functions on $\mathrm{SO}(2n)$ has the form

$$\int_{\mathrm{SO}(2n)} F(g)d\mu(g) =$$

$$= \frac{1}{\pi^n n!} \int_{0<\psi_1<2\pi} \dots \int_{0<\psi_n<2\pi} f(\varphi_1,\dots,\varphi_n)\Big(\det_{1\leqslant k,m\leqslant n}\{\cos(k-1)\varphi_m\}\Big)^2 d\varphi_1\dots d\varphi_n.$$

3.3. Expansion of ℓ_λ in characters.

THEOREM 3.2. *Let $\lambda < 1/2$. For $g \in \mathrm{SO}(2n)$,*

$$\ell_\lambda(g) = \sum_{\mathbf{l}:\, l_n = 0} c_{\mathbf{l}} \chi_{\mathbf{l}} + \frac{1}{2} \sum_{\mathbf{l}:\, l_n > 0} c_{\mathbf{l}} (\chi_{\mathbf{l}}^+ + \chi_{\mathbf{l}}^-),$$

where

$$c_{\mathbf{l}} = (-1)^{n(n-1)/2} 2^{2n\lambda+1} \pi^{-n} \sin^n \pi\lambda \prod_{k=1}^{n} \Gamma(2\lambda + 2k - 1) \times \\ \times \prod_{1 \leqslant \alpha < \beta \leqslant n} (l_\alpha^2 - l_\beta^2) \cdot \prod_{j=1}^{n} \frac{\Gamma(l_j - \lambda - n + 1)}{\Gamma(l_j + \lambda + n)} = \quad (3.3)$$

$$= (-1)^{n(n-1)/2} 2^{2n\lambda+1} \prod_{k=1}^{n} \Gamma(2\lambda + 2k - 1) \times \\ \times \frac{(-1)^{\sum l_j} \prod_{1 \leqslant \alpha < \beta \leqslant n} (l_\alpha^2 - l_\beta^2)}{\prod_{j=1}^{n} \Gamma(-l_j + \lambda + n)\Gamma(l_j + \lambda + n))}. \quad (3.4)$$

PROOF. First, $\ell_\lambda(JhJ^{-1}) = \ell_\lambda(h)$. Hence

$$\langle \ell_\lambda, \chi_{\mathbf{l}}^+ \rangle_{L^2} = \langle \ell_\lambda, \chi_{\mathbf{l}}^- \rangle_{L^2} = \frac{1}{2} \langle \ell_\lambda, \chi_{\mathbf{l}} \rangle_{L^2}$$

Evaluating the last expression, we obtain

$$\frac{1}{\pi^n n!} \int\limits_{0<\varphi_1<2\pi} \dots \int\limits_{0<\varphi_n<2\pi} \prod_{k=1}^{n} \left|\sin(\varphi_k/2)\right|^{2\lambda} \times \\ \times \det_{1 \leqslant j,k \leqslant n} \left\{\cos l_j \varphi_k\right\} \det_{1 \leqslant j,k \leqslant n} \left\{\cos(j-1)\varphi_k\right\} \prod_{k=1}^{n} d\varphi_k.$$

Applying Lemma 2.2, we transform this integral to

$$\pi^{-n} \det I(j, m),$$

where

$$I(j, m) = \int_0^{2\pi} \left|\sin(\varphi/2)\right|^{2\lambda} \cos(l_j \varphi) \cos[(m-1)\varphi]\, d\varphi.$$

Expanding the product of cosines, we obtain a sum of two integrals of the form (2.9)

$$\int_0^{\pi} \left|\sin\varphi\right|^{2\lambda} \Big[\cos 2(l_j + m - 1)\varphi + \cos 2(l_j - m + 1)\varphi\Big]\, d\varphi =$$

$$= \pi 2^{-2\lambda}\Gamma(2\lambda+1)\Big[\frac{(-1)^{l_j+m-1}}{\Gamma(\lambda+l_j+m)\Gamma(\lambda-l_j-m+2)}+ \\ + \frac{(-1)^{l_j+m-1}}{\Gamma(\lambda+l_j-m+2)\Gamma(\lambda-l_j+m)}\Big].$$

After simple transformations, we get

$$2^{-2\lambda}\Gamma(2\lambda+1)\sin\lambda\pi\Big[\frac{\Gamma(-1-\lambda+l_j+m)}{\Gamma(\lambda+l_j+m)}+\frac{\Gamma(1-\lambda+l_j-m)}{\Gamma(2+\lambda+l_j-m)}\Big]=$$

$$=\frac{\Gamma(2\lambda+1)\sin(\lambda\pi)}{2^{2\lambda}}\cdot\frac{\Gamma(l_j-\lambda)}{\Gamma(l_j+\lambda+1)}\boxed{\Big(\frac{(l_j-\lambda)_{m-1}}{(l_j+\lambda+1)_{m-1}}+\frac{(l_j+\lambda-m+2)_{m-1}}{(l_j-\lambda-m+1)_{m-1}}\Big)}$$

The factors outside the boxed equation do not depend on m. Thus, we must evaluate

$$\det_{1\leqslant j,m\leqslant n}\Big[\frac{(l_j-\lambda)_{m-1}}{(l_j+\lambda+1)_{m-1}}+\frac{(l_j+\lambda-m+2)_{m-1}}{(l_j-\lambda-m+1)_{m-1}}\Big].$$

A matrix element can be represented in the from

$$h_{jm}=\frac{(l_j-\lambda)(l_j-\lambda+1)\dots(l_j-\lambda+m-2)}{(l_j+\lambda+1)(l_j+\lambda+2)\dots(l_j+\lambda+m-1)}+ \\ +\frac{(l_j+\lambda)(l_j+\lambda-1)\dots(l_j+\lambda-m+2)}{(l_j-\lambda-1)(l_j-\lambda-2)\dots(l_j-\lambda-m+1)}.$$

(in each numerator and denominator we have $(m-1)$ factor; in particular for $m=1$ we have $h_{j1}=2$). Thus we obtain a determinat evaluated in Lemma 1.4 with

$$x_j=l_j,\qquad a_j=-\lambda+j-1,\qquad b_j=\lambda_j+j.$$

After some rearangement of the factors we obtain the required result.

3.4. Representations. Now we can repreate the considerations of 2.6-2.8. Consider the linear space $\mathbb{R}^{2n}\oplus\mathbb{R}^{2n}$ equipped with the indefinite bilinear form

$$\{v\oplus w, v'\oplus w'\}=\langle v,v'\rangle_{\mathbb{R}^{2n}\oplus 0}-\langle w,w'\rangle_{0\oplus\mathbb{R}^{2n}},$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^{2n}$. Denote by $\mathrm{O}(2n,2n)$ the group of linear operators in $\mathbb{R}^{2n}\oplus\mathbb{R}^{2n}$ preserving this form.

For $h\in\mathrm{O}(2n)$ consider its graph graph(h) in $\mathbb{R}^{2n}\oplus\mathbb{R}^{2n}$. Since $h\in\mathrm{O}(2n)$, the subspace graph(h) is isotropic with respect to our bilinear form $\{\cdot,\cdot\}$.

Thus we have one-to-one correspondence between the group $\mathrm{O}(2n)$ and the Grassmannian of $2n$-dimensional isotropic subspaces in $\mathbb{R}^{2n}\oplus\mathbb{R}^{2n}$.

The group $\mathrm{O}(2n,2n)$ acts on the Grassmannian and hence $\mathrm{O}(2n,2n)$ acts on the space $\mathrm{O}(2n)$. The explicit formula for the latter action is the same as above, see (2.19).

We consider the inner product in $C^\infty(\mathrm{O}(2n))$ given by

$$\langle F_1,F_2\rangle_\lambda=\iint_{\mathrm{O}(2n)\times\mathrm{O}(2n)}L_\lambda(h_1,h_2)\,F(h_1)\,\overline{F(h_2)}\,d\mu(h_1)\,d\mu(h_2). \qquad (3.5)$$

This integral is convergent if $\lambda > -1/2$, for general λ we consider the analytic continuation.

PROPOSITION 3.2. a) *The inner product (3.5) is invariant with respect to the transformations*

$$\rho_\lambda(g)F(h) = F(h^{[g]})\det(\alpha + h\gamma)^{-2n+1-\lambda},$$

where $g \in \mathrm{O}(2n, 2n)$, $h \in \mathrm{O}(2n)$.

b) *If* $-(n-1) > \lambda > -n$, *then the inner product* $\langle \cdot, \cdot \rangle_\lambda$ *is positive definite. In other words, the representation* ρ_λ *in this case is unitary.*

PROOF. Statement b) follows from Theorem 3.1. For evaluation of the Jacobian, see, for instance, [13], 1.2–1.3.

4. Symplectic groups

4.1. Quaternionic matrices. We denote by $\mathbb{H}$ the quaternionic field. Operators in the quaternionic coordinate space $\mathbb{H}^n$ can written in the form

$$v \mapsto vQ,$$

where Q is an $n \times n$-matrix with quaternionic elements, and $v \in \mathbb{H}^n$ is a vector-row. More formaly, these transformations are endomorphisms of a left n-dimensional module over the field $\mathbb{H}$.

Let Q be a quaternionic matrix. It defines the operator $\mathbb{H}^n \to \mathbb{H}^n$, identifying $\mathbb{H}^n$ with $\mathbb{R}^{4n}$, Hence we obtain an operator $Q_\mathbb{R} : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$. We define the determinant of Q by

$$\det Q = \sqrt[4]{\det Q_\mathbb{R}}.$$

($\det Q_\mathbb{R}$ is a nonnegative real number, hence $\det Q$ also is a positive real number).

A quaterninic matrix Q is unitary, if it preserves the Hermitian form

$$\{v, w\} = \sum v_j \overline{w}_j.$$

In other words

$$QQ^* = 1,$$

where Q^* is a matrix obtained from Q by transposition and element-wise quaternionic conjugation; if q_{kl} are the matrix elements of Q, then the matrix elements of Q^* are $\overline{q}_{lk}$.

4.2. Symplectic group, see [28]. The group of all $n \times n$ quaterninic unitary matrices is called "symplectic group" or "compact symplectic group", the standard notation is $\mathrm{Sp}(n)$.

Using a group conjugation $h \mapsto g^{-1}hg$, each element of $\mathrm{Sp}(n)$ can be reduced to a diagonal matrix with the entries $e^{ik\varphi}$. The collection ψ_j is defined uniquely modulo permutations and change of signs: $\psi_j \mapsto (-1)^{\sigma_j}\psi_j$.

Let F be a central function on $\mathrm{Sp}(n)$, let f be its restriction to the set of diagonal matrices. The following *Weyl integration formula* holds

$$\int_{\mathrm{Sp}(n)} F(g)\, d\mu(g) = \frac{1}{\pi^n n!} \int\limits_{0<\psi_1<2\pi} \dots \int\limits_{0<\psi_n<2\pi} f(\varphi_1, \dots, \varphi_n) \times \\ \times \Big| \det_{1\leqslant j,k\leqslant n} \big\{ \sin(k\psi_j) \big\} \Big|^2 \prod_{j=1}^{n} d\varphi_j .$$

Finite-dimensional irreducible representations of $\mathrm{Sp}(2n)$ are enumerated by collections $\mathbf{l}$ of integers having the form

$$\mathbf{l}: \quad l_1 > l_2 > \dots > l_n > 0.$$

The corresponding character is

$$\chi_l(g) = \frac{\det_{1\leqslant j,k\leqslant n} \big\{ \sin(l_k \psi_j) \big\}}{\det_{1\leqslant j,k\leqslant n} \big\{ \sin(k\psi_j) \big\}}.$$

We define the function ℓ_λ on $\mathrm{Sp}(n)$ by

$$\ell_\lambda(g) = \det\big((1-g)/2\big)^{2\lambda} = \prod_{j=1}^{n} |\sin(\psi_j/2)|^{2\lambda}.$$

4.3. Expansion of l_λ.

THEOREM 4.1. *For* $\mathrm{Re}\,\lambda < 1/2$,

$$\ell_\lambda(g) = \sum_{\mathbf{l}} c_{\mathbf{l}} \chi_{\mathbf{l}},$$

where

$$c_{\mathbf{l}} = \pi^{-n} 2^{-2n\lambda} \sin^n \lambda\pi \prod_{k=1}^{n} \Gamma(2\lambda + 2k) \times \\ \times \prod_{\alpha=1}^{n} (2 l_\alpha) \prod_{0\leqslant\alpha<\beta\leqslant n} (l_\alpha^2 - l_\beta^2) \prod_{j=1}^{n} \frac{\Gamma(l_j - \lambda - k)}{\Gamma(l_j + \lambda + 1 + k)}. \qquad (4.1)$$

PROOF is very similar to the orthogonal case. We must evaluate the determinat whose matrix elements are

$$I(k,j) := \int_0^{2\pi} \big|\sin(\varphi/2)\big|^{2\lambda} \sin(l_j\varphi)\, \sin(k\varphi)\, d\varphi.$$

Expanding the product of sines, we obtain a sum of two integrals of the form (2.9)

$$\int_0^\pi \left|\sin(\varphi/2)\right|^{2\lambda}\left[\cos 2(l_j-k)\varphi-\cos 2(l_j+k)\varphi\right]d\varphi =$$
$$=\pi 2^{-2\lambda}\Gamma(2\lambda+1)\times$$
$$\times\Bigl[\frac{(-1)^{l_j-k}}{\Gamma(l_j+\lambda-k+1)\Gamma(-l_j+\lambda+k+1)}-\frac{(-1)^{l_j+k}}{\Gamma(l_j+\lambda+k+1)\Gamma(-l_j+\lambda-k+1)}\Bigr]=$$
$$=\pi 2^{-2\lambda}\Gamma(2\lambda+1)\Bigl[\frac{\Gamma(l_j-\lambda-k)}{\Gamma(l_j+\lambda-k+1)}+\frac{\Gamma(l_j-\lambda+k)}{\Gamma(l_j+\lambda+k+1)}\Bigr].$$

We transform the expression in the brackets to

$$\frac{\Gamma(l_j-\lambda)}{\Gamma(l_j+\lambda)}\cdot\Bigl\{\frac{(l_j+\lambda-k+1)_k}{(l_j-\lambda-k)_k}-\frac{(l_j-\lambda)_k}{(l_j+\lambda+1)_k}\Bigr\}.$$

The factor in the front of brackets does not depend on k and hence it is suffitient to evaluate the determinant of the matrix

$$u_{kj}=\frac{(l_j+\lambda-k+1)_k}{(l_j-\lambda-k)_k}-\frac{(l_j-\lambda)_k}{(l_j+\lambda)_k}.$$

This matrix has the form described in Lemma 1.3, with

$$x_j=l_j,\qquad a_k=\lambda+1-k,\qquad b_k=-\lambda-k.$$

4.4. Unitary representations of $Sp(n,n)$**.** The pseudounitary quaternionic group $\mathrm{Sp}(n,n)$ is the group of quaternionic $(n+n)\times(n+n)$-matrices preserving the Hermitian form

$$\{v,w\}:=\sum_{j=1}^{n} v_j\overline{w}_j-\sum_{j=n+1}^{2n} v_j\overline{w}_j.$$

We consider its action on $\mathrm{Sp}(n)$ by linear fractional transformations (2.19) as above. We also consider the representation $\rho_\lambda(g)$ of $\mathrm{Sp}(n,n)$ in the space $C^\infty(\mathrm{Sp}(n))$ given by

$$\rho_\lambda(g)F(h)=F(h^{[g]})\det(\alpha+h\gamma)^{-2n-1-\lambda}$$

PROPOSITION 4.2. a) *The operators* $\rho_\lambda(g)$ *preserve the Hermitian form*

$$\langle F_1,F_2\rangle_\lambda=\iint_{\mathrm{Sp}(n)\times\mathrm{Sp}(n)}\ell_\lambda(gh^{-1})F_1(g)\overline{F(h)}\,d\mu(g)\,d\mu(h).$$

b) *If* $-n>\lambda>-n-1$*, then the Hermitian form* $\langle\cdot,\cdot\rangle_\lambda$ *is positive definite and thus the representation* ρ_λ *is unitary.*

The nontrivial part of the statement is positivity of the Hermitian form. This follows from Theorem 4.1.

5. Unipotent representations.

Here we discuss models of "unipotent" representations of Sahi [23] and Dvorsky–Sahi, [4]–[5].

5.1. The case $\mathrm{O}(2n, 2n)$**.** In notation of Section 3, we suppose that

$$\lambda = -n + \alpha \qquad \alpha = 1, 2, \ldots, n. \tag{5.1}$$

The first row in (3.3) has zero of order α at our λ. The second row in (3.3) has a pole of order $\leqslant \alpha$. Hence the total expression is nonzero iff the order of the pole is precisely α, i.e.,

$$l_n = 0, \quad , l_{n-1} = 1, \quad \ldots l_{n-\alpha+1} = \alpha - 1$$

Under this condition, all the coefficients $c_{\mathbf{l}}$ are positive.

Thus, for λ having the form (5.1), the inner product (3.5) is non-negative definite (and degenerated) and the operators (3.6) are unitary with respect to this inner product.

Remark. For $\alpha = 0$, our representation is the one-dimensional representation.

For $\alpha = 1$, the representation obtained in this way is an element of the Molchanov degenerated discrete series, see [12], now there exists a wide literature devoted to this representation, see, for instance [10].

5.2. Groups $\mathrm{U}(n, n)$**.** Now we assume

$$\tau = 0, \qquad \sigma = -n + \alpha, \qquad \text{where } \alpha = 1, \ldots, n-1. \tag{5.2}$$

The coefficient $c_{\mathbf{m}}$ given by (2.10) is non-zero, if $\mathbf{m}$ is contained in the union of the following disjoint sets Z_j, $j = 0, 1, \ldots, n - \alpha$,

$$Z_\theta : \mathbf{m} \quad \text{has form} \quad (m_1, \ldots, m_{n-\alpha-\theta}, \alpha - 1, \alpha - 2, \ldots, 0, m_{n-\theta+1}, \ldots m_n)$$

Denote by $V_{\mathbf{m}}$ be the $\mathrm{U}(n) \times \mathrm{U}(n)$-submodule in $C^\infty(\mathrm{U}(n))$ corresponding a signature $\mathbf{m}$, see 2.5.

Proposition 5.1. *The subspace*

$$W_{tail} := \oplus V_\mu \subset C^\infty(\mathrm{U}(n)), \quad \textit{where } \mu \notin \cup Z_j,$$

is a $\mathrm{U}(n, n)$*-invariant subspace.*

b) *The quotient* $C^\infty(\mathrm{U}(n))/W_{tail}$ *is a sum* $n - \alpha + 1$ *submodules*

$$W_j = \oplus_{\mu \in Z_j} V_\mu$$

The representation of $\mathrm{U}(n, n)$ *in each* V_j *is unitary.*

A proof is given below.

5.3. Blow-up. The distribution $\ell_{\sigma,\tau}$ depends meromorphically in the two complex variables σ, τ. Its poles and zeros are located at $\sigma \in \mathbb{Z}$ and in $\tau \in \mathbb{Z}$

and hence values of this distributions at points $(\sigma,\tau)\in\mathbb{Z}^2$ generally are not uniquely defined. Passing to this point from different directions, we can obtain different limits.

It is convenient to formulate this more formally.

We fix a point $(\sigma,\tau)=(-n+\alpha,0)$ and introduce the new local coordinates near this point[4] by

$$\sigma=-n+\alpha+s\varepsilon,\qquad \tau=t\varepsilon \tag{5.3}$$

The new coordinates are defined up to the equivalence

$$(s,t,\varepsilon)\sim(su,tu,\varepsilon/u) \tag{5.4}$$

We also think that

$$|s|,\quad |t|,\quad |\varepsilon| \text{ are suffitiently small, and } (s,t)\neq(0,0). \tag{5.5}$$

If we replace $\varepsilon=1/R$, then the collections (s,t,R) are defined up to the equivalence

$$(s,t,\varepsilon)\sim(su,tu,Ru)$$

i.e., s,t,u is a point of the projective plane. Thus the set (5.5) can be consired as a subset in projective plane. In the new coordinates, the point $(-n+\alpha,0)\in\mathbb{C}^2$ corresponds to the whole complex projective line $(s,t,0)$.

Thus replacing a neibourhood of $(-n+\alpha,0)\in\mathbb{C}^2$ by the set (5.5), we obtain a new complex manifold, denote it by $\widetilde{\mathbb{C}}^2$.

PROPOSITION 5.2. *The distribution*

$$\ell^{s,t,\varepsilon}(z):=\ell_{\sigma,\tau}(z)$$

is a meromorphic distribution-valued function on $\widetilde{\mathbb{C}}^2$ in the domain (5.5). The unique pole (or order n) near $(s,t)=(0,0)$ is the line $s+t=0$.

PROOF. Each Fourier coefficient $c_{\mathbf{m}}$ in the formula (2.10) has the form

$$\varepsilon^k\frac{t^j s^{n-\alpha-j}}{(s+t)^{n-\alpha}}R_{\mathbf{m}}(s,t,\varepsilon) \tag{5.6}$$

where the last factor is holomorphic and nonvanishing near the line $(s,t,0)$, and the powers k, j, $n-\alpha-j$ are nonnegative.

Thus, the Fourier coefficients $c_{\mathbf{m}}$ of the distribution $\ell^{s,t,\varepsilon}$ are meromorphic in the our region.

We must verify that the derivatives $\partial\ell^{s,t,\varepsilon}/\partial t$, $\partial\ell^{s,t,\varepsilon}/\partial s$, $\partial\ell^{s,t,\varepsilon}/\partial\varepsilon$ are well-defined distributions.

A central function on $\mathrm{U}(n)$ is a distribution, if its Forier coefficient have at most polynomial growth in $\mathbf{m}$. For (5.6), this can be verified by a direct tracing.

[4]This construction is the *blow-up* of the plane $\mathbb{C}^2$ at the point $(-n+\alpha,0)$.

5.4. The family of invariant Hermitian forms. Proof of Proposition 5.1. Thus, for $(\sigma,\tau)=(-n+\alpha,0)$, we have the following family of Hermitian forms invariant with respect to the operators (2.22)

$$\Lambda_{s,t}(f,g)=\iint_{\mathrm{U}(n)\times\mathrm{U}(n)}\ell^{s,t,0}(zu^*)f(z)\overline{f(u)}\,d\mu(z)\,d\mu(u)$$

Now we emphasis some additional properties of the formula (5.6).

First, the exponent k in (5.6) is zero iff $\mathbf{m}\in\cup Z_j$. Hence W_{tail} is the common kernel of all the forms $\Lambda_{s,t}$. This implies the statement a) of Proposition 5.1.

By the invariance, we subspaces W_j are pairwise orthogonal with respect to all the forms $\Lambda^{s,t,0}$.

Secondly, $R_{\mathbf{m}}(s,t,0)$ is a constant which depend only on $\mathbf{m}$ (since nonconstant summands in the linear factors of $R_{\mathbf{m}}$ have the form $s\varepsilon$), $t\varepsilon$).

Thirdly, if $\mathbf{m}\in Z_j$, then the exponent j in (5.6) is our j. This means, that the restriction of $\Lambda^{s,t,0}$ to W_j has the the form

$$\frac{t^j s^{n-\alpha-j}}{(s+t)^{n-\alpha}}\Xi_j(f,g) \tag{5.7}$$

where $\Xi(f,g)$ is independent of t, s.

The Fourier coefficients of Ξ_j are the factors $R_{\mathbf{m}}(s,t,0)$ from (5.6), and they can easily be written. It is easy to observe that all these coefficients have the same sign.

This finishes proof of Proposition 5.1.b).

5.5. Expression for the kernel. We can concider the form Ξ_j as a form on $C^\infty(\mathrm{U}(n))$ extending it to other harmonics as zero.

PROPOSITION 5.3. *The form Ξ_j is defined by the distribution*

$$\frac{1}{n!}\frac{\partial^j}{\partial t^j}(1+t)^{n-\alpha}\ell^{1,t,0}(zu^*)\Big|_{t=0}=$$
$$=\frac{1}{n!}\lim_{\varepsilon\to 0}\frac{\partial^j}{\partial t^j}(1+t)^{n-\alpha}\det(1-zu^*)^{\{n+\alpha+\varepsilon|t\varepsilon\}}\Big|_{t=0,\varepsilon=0}$$

PROOF. The first expression is obtained by differention of the block decomposition (5.7) of the Hermitian forms $\Lambda_{s,t}$; the right-hand side is a result of changing of the limit passing and differentiation.

REMARK. The sum (5.8) contains the well-defined term $\det(1-zu^*)^{-n+\alpha}:=\ell^{1,0,0}(zu^*)$ (that looks like a function) and sum of distributions supported by the submanifold $\mathrm{rk}(1-zu^*)=n-1,\ n-2,\ \dots,\ n-\alpha$ (the analysis of the determinantal distributions was done by Ricci, Stein [19]).

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