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Decay of Correlations for the Rauzy–Veech–Zorich Induction Map on the Space of Exchanges of Four Intervals

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1 Introduction

The aim of this paper is to prove a stretched-exponential bound for the decay of correlations for the Rauzy-Veech-Zorich induction map on the space of exchanges of four intervals (Theorem 4).

This is done by approximating the map by a Markov chain satisfying the Doeblin condition, the method of Sinai [13] and Bunimovich–Sinai [14]. The main "loss of memory" estimate is Lemma 4.

1.1 Interval exchange transformations.

Let m be a positive integer. Let π be a permutation on m symbols. The permutation π will always be assumed *irreducible*, which means that $\pi\{1,\ldots,k\} = \{1,\ldots,k\}$ only if $k = m$.

Let λ be a vector in $\mathbb{R}_+^m$, $\lambda = (\lambda_1,\ldots,\lambda_m)$, $\lambda_i > 0$ for all i. Denote

$$|\lambda| = \sum_{i=1}^{m} \lambda_i.$$

Consider the half-open interval $[0, |\lambda|)$. Consider the points $\beta_i = \sum_{j<i} \lambda_j$, $\beta_i^\pi = \sum_{j<i} \lambda_{\pi^{-1}j}$.

Denote $I_i = [\beta_i, \beta_{i+1})$, $I_i^\pi = [\beta_i^\pi, \beta_{i+1}^\pi)$. The length of I_i is λ_i, whereas the length of I_i^π is $\lambda_{\pi^{-1}i}$.

Set

$$T_{(\lambda,\pi)}(x) = x + \beta_i^\pi - \beta_i \text{ for } x \in I_i.$$

The map $T_{(\lambda,\pi)}$ is called an interval exchange transformation corresponding to (λ,π).

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The map $T_{(\lambda,\pi)}$ is an order-preserving isometry from I_i onto $I^{\pi}_{\pi(i)}$.

We say that λ is *irrational* if there are no rational relations between $|\lambda|$, λ_1,λ_2, $\dots \lambda_{m-1}$.

Theorem 1 (Oseledets([5]), Keane([9])) *Let π be irreducible and λ irrational. Then for any $x \in [0, \sum_{i=1}^m \lambda_i)$, the set $\{T^n_{(\lambda,\pi)}x, n \geq 0\}$ is dense in $[0, \sum_{i=1}^m \lambda_i)$.*

1.2 Rauzy operations a and b.

Let (λ, π) be an interval exchange. Assume that π is irreducible and λ is irrational.

Following Rauzy [6], consider the induced map of (λ,π) on the interval $[0, |\lambda| - min(\lambda_m, \lambda_{\pi^{-1}(m)}))$. The induced map is again an interval exchange of m intervals. For $i,j = 1,\dots,m$, denote by E_{ij} an $m\times m$ matrix of which the i,j-th element is equal to 1, all others to 0. Let E be the $m \times m$-identity matrix.

1.2.1 Case a: $\lambda_{\pi^{-1}m} > \lambda_m$.

Define

$$A(\pi, a) = \sum_{i=1}^{\pi^{-1}(m)} E_{ii} + E_{m,\pi^{-1}m+1} + \sum_{i=\pi^{-1}m+1}^{m} E_{i,i+1}$$

$$a\pi(j) = \begin{cases} \pi j, & \text{if } j \leq \pi^{-1}m; \\ \pi m, & \text{if } j = \pi^{-1}m+1; \\ \pi(j-1), & \text{other } j. \end{cases}$$

If $\lambda_{\pi^{-1}m} > \lambda_m$, then the induced interval exchange of $T_{(\lambda,\pi)}$ on the interval $[0, \sum_{i\neq m}\lambda_i)$ is $T_{(\lambda',\pi')}$, where $\lambda' = A(a,\pi)^{-1}\lambda$ and $\pi' = a\pi$.

1.2.2 Case b: $\lambda_m > \lambda_{\pi^{-1}m}$.

Define

$$A(\pi, b) = E + E_{m,\pi^{-1}m}$$

$$b\pi(j) = \begin{cases} \pi j, & \text{if } \pi j \leq \pi m; \\ \pi j + 1, & \text{if } \pi m < \pi j < m; \\ \pi m + 1, & \text{if } \pi j = m. \end{cases}$$

If $\lambda_m > \lambda_{\pi^{-1}m}$, then the induced interval exchange on the interval $[0, \sum_{i\neq\pi^{-1}m}\lambda_i)$ has the form (λ',π'), where $\lambda' = A(b,\pi)^{-1}\lambda$ and $\pi' = b\pi$.

Note that operations a and b are invertible, namely, we have:

$$a^{-1}\pi(j) = \begin{cases} \pi(j), & \text{if } j \le \pi^{-1}(m); \\ \pi(j+1), & \text{if } \pi^{-1}(m)+1 < j < m; \\ \pi(\pi^{-1}(\pi(m)+1), & \text{if } j = m. \end{cases}$$

$$b^{-1}\pi(j) = \begin{cases} \pi(j), & \text{if } \pi(j) \le \pi(m) \\ m, & \text{if } j = \pi^{-1}(\pi(m)+1); \\ \pi(j)-1, & \text{if } \pi(j) > \pi(m)+1. \end{cases}$$

For $(\lambda, \pi) \in \Delta(\mathcal{R})$, denote

$$T_{a^{-1}}(\lambda, \pi) = (A(a^{-1}\pi, a)\lambda, a^{-1}\pi), T_{b^{-1}}(\lambda, \pi) = (A(b^{-1}\pi, b)\lambda, b^{-1}\pi). \quad (1)$$

The interval exchange $T_{a^{-1}}(\lambda, \pi)$ is the preimage of (λ, π) under the operation a, and the intervals exchange $T_{b^{-1}}(\lambda, \pi)$ is the preimage of (λ, π) under the operation b.

Normalize (dividing by $|\lambda| = \lambda_1 + \cdots + \lambda_m$) and set:

$$t_{a^{-1}}(\lambda, \pi) = (\frac{A(a^{-1}\pi, a)\lambda}{|A(a^{-1}\pi, a)\lambda|}, a^{-1}\pi), t_{b^{-1}}(\lambda, \pi) = (\frac{A(b^{-1}\pi, b)\lambda}{|A(b^{-1}\pi, b)\lambda|}, b^{-1}\pi). \quad (2)$$

1.3 Rauzy class and Rauzy graph.

If π is an irreducible permutation, then its *Rauzy class* is the set of all permutations that can be obtained from π by applying repeatedly the operations a and b; the Rauzy class of the permutation π is denoted $\mathcal{R}(\pi)$. Rauzy class has a natural structure of an oriented labelled graph: namely, the permutations of the Rauzy class are the vertices of the graph, and if $\pi = a\pi'$ then we draw an edge from π to π' and label it by a, and if $\pi = b\pi'$ then we draw an edge from π to π' and label it by b. This labelled graph will be called the *Rauzy graph* of the permutation π.

For a permutation π, consider the set $\{a^n\pi, n \ge 0\}$. This set forms a cycle in the Rauzy graph which will be called the *a-cycle* of π. Similarly, the set $\{b^n\pi, n \ge 0\}$ will be called the *b-cycle* of π.

1.4 The Rauzy-Veech-Zorich induction.

Denote

$$\Delta_{m-1} = \{\lambda \in \mathbb{R}^m_+ : |\lambda| = 1\},$$

$$\Delta^+_\pi = \{\lambda \in \Delta_{m-1}, \lambda_{\pi^{-1}m} > \lambda_m\}, \Delta^-_\pi = \{\lambda \in \Delta_{m-1}, \lambda_m > \lambda_{\pi^{-1}m}\},$$

$$\Delta(\mathcal{R}) = \Delta_{m-1} \times \mathcal{R}(\pi).$$

Define a map

$$\mathcal{T} : \Delta(\mathcal{R}) \to \Delta(\mathcal{R})$$

by

$$\mathcal{T}(\lambda,\pi)=\begin{cases}(\frac{A(\pi,a)^{-1}\lambda}{|A(\pi,a)^{-1}\lambda|},a\pi), & \text{if } \lambda\in\Delta_\pi^+;\\ (\frac{A(\pi,b)^{-1}\lambda}{|A(\pi,b)^{-1}\lambda|},b\pi), & \text{if } \lambda\in\Delta_\pi^-.\end{cases}$$

Each $(\lambda,\pi)\in\Delta(\mathcal{R})$ has exactly two preimages under the map $\mathcal{T}$, namely, $t_{a^{-1}}(\lambda,\pi)$ and $t_{b^{-1}}(\lambda,\pi)$ (2).

The set $\Delta(\mathcal{R})$ is a finite union of simplices. Let $\mathbf{m}$ be the Lebesgue measure on $\Delta(\mathcal{R})$ normalized in such a way that $\mathbf{m}(\Delta(\mathcal{R}))=1$.

Theorem 2 (Veech[1]) *The map $\mathcal{T}$ has an infinite conservative ergodic invariant measure, absolutely continuous with respect to Lebesgue measure on $\Delta(\mathcal{R})$.*

From this result Veech [1] derives that almost all (with respect to $\mathbf{m}$) interval exchange transformations are uniquely ergodic.

Denote

$$\Delta^+=\cup_{\pi'\in\mathcal{R}(\pi)}\Delta_{\pi'}^+,\ \Delta^-=\cup_{\pi'\in\mathcal{R}(\pi)}\Delta_{\pi'}^-.$$

Following Zorich [4], we define the function $n(\lambda,\pi)$ in the following way.

$$n(\lambda,\pi)=\begin{cases}\inf\{k>0:\mathcal{T}^k(\lambda,\pi)\in\Delta^-\}, & \text{if } \lambda\in\Delta_\pi^+;\\ \inf\{k>0:\mathcal{T}^k(\lambda,\pi)\in\Delta^+\}, & \text{if } \lambda\in\Delta_\pi^-.\end{cases}$$

Define

$$\mathcal{G}(\lambda,\pi)=\mathcal{T}^{n(\lambda,\pi)}(\lambda,\pi).$$

The map $\mathcal{G}$ will be referred to as *the Rauzy-Veech-Zorich induction map* [6, 1, 4].

For $(\lambda,\pi)\in\Delta(\mathcal{R})$, denote

$$t_{a^{-n}}(\lambda,\pi)=t^n_{a^{-1}}(\lambda,\pi),t_{b^{-n}}(\lambda,\pi)=t^n_{b^{-1}}(\lambda,\pi),T_{a^{-n}}(\lambda,\pi)=T^n_{a^{-1}}(\lambda,\pi),T_{b^{-n}}(\lambda,\pi)=T^n_{b^{-1}}(\lambda,\pi).$$

Under the map $\mathcal{G}$, each interval exchange (λ,π) has countably many preimages:

$$\mathcal{G}^{-1}(\lambda,\pi)=\begin{cases}\{t_{a^{-n}}(\lambda,\pi),n\in\mathbb{N}\}, & \text{if } (\lambda,\pi)\in\Delta^+;\\ \{t_{b^{-n}}(\lambda,\pi),n\in\mathbb{N}\}, & \text{if } (\lambda,\pi)\in\Delta^-.\end{cases}$$

Theorem 3 (Zorich[4]) *The map $\mathcal{G}$ has an ergodic invariant probability measure, absolutely continuous with respect to Lebesgue on $\Delta(\mathcal{R})$.*

Denote this invariant measure by ν; the probability with respect to ν will be denoted by $\mathbb{P}$.

Let $\rho(\lambda,\pi)$ be the density of ν with respect to the Lebesgue measure $\mathbf{m}$. Zorich [4] showed that for any $\pi\in\mathcal{R}$ there exist two rational homogeneous of degree $-m$ functions ρ_π^+, ρ_π^- such that

$$\rho(\lambda,\pi) = \begin{cases} \rho^+_\pi(\lambda), & \text{if } \lambda \in \Delta^+_\pi; \\ \rho^-_\pi(\lambda), & \text{if } \lambda \in \Delta^-_\pi. \end{cases}$$

The map $\mathcal{G}$ is not mixing: indeed, from the definition of $\mathcal{G}$, we have

$$\mathcal{G}(\Delta^+) = \Delta^-,\ \mathcal{G}(\Delta^-) = \Delta^+.$$

Let $\mathcal{B}$ be the Borel σ-algebra on $\Delta(\mathcal{R})$, and let $\mathcal{F}_n = \mathcal{G}^{-n}\mathcal{B}$. We have $\mathcal{F}_{n+2} \subset \mathcal{F}_n$. Recall [23] that *exactness* of the map $\mathcal{G}^2$ means, by definition, that the σ-algebra $\cap_{n=1}^{\infty}\mathcal{F}_{2n}$ is trivial [23] (in other words, that Kolmogorov's $0-1$ law holds for the map $\mathcal{G}^2$.)

Proposition 1 *The map $\mathcal{G}^2 : \Delta^+ \to \Delta^+$ is exact with respect to $\nu|_{\Delta^+}$.*

This Proposition is proven in Section 4; it implies mixing of the map $\mathcal{G}^2$.

1.5 The Birkhoff metric on $\Delta(\mathcal{R})$

Introduce a metric on Δ_{m-1} by setting

$$d(\lambda,\lambda') = \log \frac{\max_i \frac{\lambda_i}{\lambda'_i}}{\min_i \frac{\lambda_i}{\lambda'_i}}. \tag{3}$$

Now introduce a metric on $\Delta(\mathcal{R})$ by setting

$$d((\lambda,\pi),(\lambda',\pi')) = \begin{cases} 2, & \text{if } \pi \neq \pi'; \\ d(\lambda,\lambda'), & \text{if } \pi = \pi'. \end{cases}$$

For $\alpha > 0$, let $H(\alpha)$ be the space of functions $\phi : \Delta(\mathcal{R}) \to \mathbb{R}$ such that if $d((\lambda,\pi),(\lambda',\pi') \leq 1$, then $|\phi(\lambda,\pi) - \phi(\lambda',\pi')| \leq Cd((\lambda,\pi),(\lambda',\pi'))^\alpha$ for some constant C.

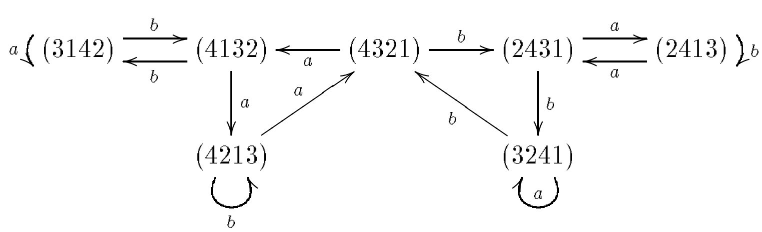
Define

$$C_{H(\alpha)}(\phi) = \max_{d((\lambda,\pi),(\lambda',\pi'))\leq 1} \frac{|\phi(\lambda,\pi) - \phi(\lambda',\pi')|}{d((\lambda,\pi),(\lambda',\pi'))^\alpha},$$

1.6 Four intervals and the main result.

Consider the case of interval exchanges of four intervals with the permutation (4321).

The Rauzy graph of this permutation looks as follows:



The main result of this paper is

Theorem 4 *Let $\mathcal{G} : \Delta(4321) \to \Delta(4321)$ be the Rauzy-Veech-Zorich induction map corresponding to the permutation (4321) and let ν be the corresponding invariant measure.*

Then, for any $\alpha > 0$, there exist positive constants C, δ such that for any $\phi \in H(\alpha) \cap L_4(\Delta^+(4321), \nu)$ and $\psi \in L_2(\Delta^+(4321), \nu)$ we have

$$|\int \phi \ \times \psi \circ \mathcal{G}^{2n} d\nu - \int \phi d\nu \int \psi d\nu| \leq C \exp(-\delta n^{1/6})(C_{H(\alpha)}(\phi) + |\phi|_{L_4})(|\psi|_{L_2}).$$

Denote by $\mathcal{N}(0, \sigma)$ the Gaussian distribution with mean 0 and variance σ. By [7, 8, 17], we have

Corollary 1 *Let $\phi \in H(\alpha) \cap L_4(\Delta(4321)^+, \nu)$, $\int \phi d\nu = 0$. Assume that there does not exist $\psi \in L_2(\Delta(4321)^+, \nu)$ such that $\phi = \psi \circ \mathcal{G}^2 - \psi$. Then there exists $\sigma > 0$ such that*

$$\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \phi \circ \mathcal{G}^{2n} \xrightarrow{d} \mathcal{N}(0, \sigma) \text{ as } N \to \infty.$$

An advantage of the method of Markov approximations of Sinai [13], Bunimovich-Sinai [14] is that, along with the result for $\mathcal{G}^2$, one automatically obtains the same rate for the decay of correlations for its natural extension (in the sense of Rokhlin [24]). In our case, the natural extension of the map $\mathcal{G}$ can be identified with the induction map on Veech's space of zippered rectangles [1]. For this invertible induction map one therefore immediately obtains the same rate for the decay of correlations and, consequently, the Central Limit Theorem.

Theorem 4 has an application to the Teichmueller flow on the moduli space of abelian differentials. To every Rauzy class there corresponds a connected component of a stratum in the moduli space of abelian differentials on compact surfaces [26]. Just as the geodesic flow on a certain finite cover of the modular surface is a special flow over the natural extension of the Gauss map on the unit interval [18, 19], the Teichmueller flow on a certain finite cover of the moduli space corresponding to a Rauzy class is a special flow over the natural extension of $\mathcal{G}$ [1]. The roof function is Hoelder, unbounded, with logarithmic growth at infinity. This allows to apply the theorem of Melbourne and Torok [15] and derive the Central Limit Theorem for the Teichmueller flow. These applications will be discussed in greater detail in a sequel to this paper.

The stratum corresponding to the Rauzy class of (4321) consists of differentials on a surface of genus two having one singularity point of order two. Let $\mathcal{M}(2)$ be this stratum, let g_t be the Teichmueller flow on $\mathcal{M}(2)$, let X_t be the vector field generating the flow g_t, and, finally, let μ_2 be the smooth invariant probability measure [21]. Denote by $H(\alpha)(\mathcal{M}(2))$ the space of functions satisfying the α-Hoelder condition with respect to the Teichmueller metric.

Corollary 2 *Let $\phi \in H(\alpha)(\mathcal{M}(2))$ have compact support and assume $\int \phi d\mu_2 = 0$. Assume that there does not exist $\psi \in L_2(\mathcal{M}(2), \mu_2)$ such that $\phi = X_t \psi$. Then there exists $\sigma > 0$ such that*

$$\frac{1}{\sqrt{T}} \int_0^T \phi \circ g_t \xrightarrow{d} \mathcal{N}(0, \sigma) \text{ as } T \to \infty.$$

1.7 Outline of the Proof of Theorem 4.

First, one takes a subset of the space $\Delta(\mathcal{R})$ such that the induced map of $\mathcal{G}$ is uniformly expanding (namely, the set of all interval exchanges such that the renormalization matrix for them is a fixed matrix all whose elements are positive, see Proposition 4; note that the return map on such a subset is an essential element in Veech's proof of unique ergodicity [1]). Then one estimates the statistics of return times in this subset, in the spirit of Lai-Sang Young [11]. After that, the method of Markov approximations, due to Sinai [13], Bunimovich and Sinai [14], is used to complete the proof.

The paper is organized as follows. In Section 3, we state auxiliary propositions about unimodular matrices. In Section 4, following Veech [1] and Zorich [4], we construct symbolic dynamics for the Rauzy-Veech-Zorich induction map $\mathcal{G}$ and compute its transition probabilities. In Section 5, we establish the exactness of $\mathcal{G}^2$. Note that so far all results hold for exchanges of any number of intervals. In Section 6, we consider the Rauzy class of the permutation (4321) and state Lemma 4, the main step in the proof of Theorem 4. Lemma 4 is proven in Sections 7 – 10. In the remainder of the paper we apply the Markov approximation method of Sinai [13], Bunimovich and Sinai [14], in order to derive the decay of correlations from Lemma 4. Note that the arguments here are again general, i.e., valid for exchanges of any number of intervals. If a statement similar to Lemma 4 were obtained in the general case, then the decay of correlations would also follow automatically. The argument of Sections 7 – 10, however, uses rather heavily the particular properties of the Rauzy class $\mathcal{R}(4321)$.

2 Matrices

Let A be an $m \times m$-matrix with positive entries.

Denote

$$|A| = \sum_{i,j=1}^{m} A_{ij}$$

$$col(A) = \max_{i,j,k} \frac{A_{ij}}{A_{kj}},$$

$$row(A) = \max_{i,j,k} \frac{A_{ij}}{A_{ik}}$$

Proposition 2 *Let Q be a matrix with positive entries, A a matrix with non-negative entries without zero columns or rows.*

Then all entries of the matrices AQ and QA are positive, and, moreover, we have

$$row(AQ) \leq row(Q), col(QA) \leq col(Q)$$

Corollary 3 *Let Q be a matrix with positive entries, A a matrix with nonnegative entries without zero columns or rows.*

$$row(QAQ) \leq row(Q), col(QAQ) \leq col(Q)$$

Let A be an $m \times m$ matrix with nonnegative entries and determinant 1. Consider the map $J_A : \Delta_{m-1} \to \Delta_{m-1}$ given by

$$J_A(\lambda) = \frac{A\lambda}{|A\lambda|}.$$

Then

$$detDJ_A(\lambda) = \frac{1}{|A\lambda|^m}. \tag{4}$$

Suppose all entries of A are positive; then, for any $\lambda, \lambda' \in \Delta_{m-1}$, we have

$$row(A)^{-m} \leq \frac{detDJ_A(\lambda)}{detDJ_A(\lambda')} \leq row(A)^m, \tag{5}$$

whence we have the following

Proposition 3 *Let $C \subset \Delta_{m-1}$ and let A be a matrix with positive entries and determinant 1. Then*

$$row(A)^{-m} \leq \frac{\mathbf{m}(J_A(C))}{\mathbf{m}(C)} \leq row(A)^m.$$

We also note the following well-known Lemma (see, for example, [17]):

Lemma 1 *Suppose all entries of the matrix A are positive. Then the map J_A is uniformly contracting with respect to the Birkhoff metric.*

3 Markov Partition for $\mathcal{G}$.

First, using the results of Veech [1] and Zorich [4], we construct a symbolic dynamics for the map $\mathcal{G}^2$, and then we give a formula for transition probabilities in the sense of Sinai [25].

3.1 The alphabet

Let $\pi \in \mathcal{R}$, and let n be a positive integer.

Set

$$\Lambda(n, a, \pi) = \{\lambda : \text{there exists } (\lambda', \pi') \text{ such that } \lambda' \in \Delta^+_{\pi'} \text{ and } (\lambda, \pi) = t_{a^{-n}}(\lambda', \pi')\}$$

$$\Delta(n, a, \pi) = \{(\lambda, \pi), \lambda \in \Lambda(a, n, \pi)\}$$

In other words, $\Delta(n, a, \pi)$ is the set of interval exchange transformations such that the application of the Zorich induction results in the application of the a-operation n times.

The sets $\Delta(n, a, \pi)$ and $\Delta(n', a, \pi')$ are disjoint unless $n = n'$, $\pi = \pi'$, and

$$\Delta^-_\pi = \cup_{n=1}^\infty \Delta(n, a, \pi)$$

up to a set of measure zero (namely, a union of countably many hyperplanes on which Zorich induction is not defined).

If $\pi' = a^n \pi$, then we have

$$\mathcal{G}\Delta(n, a, \pi) = \Delta^+_{\pi'}.$$

Similarly, for $\pi \in \mathcal{R}$, and n a positive integer, set

$$\Lambda(n, b, \pi) = \{\lambda : \text{there exists } (\lambda', \pi') \text{ such that } \lambda' \in \Delta^-_{\pi'} \text{ and } (\lambda, \pi) = t_{b^{-n}}(\lambda', \pi')\}.$$

$$\Delta(n, b, \pi) = \{(\lambda, \pi), \lambda \in \Lambda(n, b, \pi)\}.$$

In other words, $\Delta(n, b, \pi)$ is the set of interval exchange transformations such that the application of the Zorich induction results in the application of the b-operation n times.

The sets $\Delta(n, b, \pi)$ and $\Delta(n', b, \pi')$ are disjoint unless $n = n'$, $\pi = \pi'$, and

$$\Delta^+_\pi = \cup_{n=1}^\infty \Delta(n, b, \pi)$$

up to a set of measure zero (namely, a union of countably many hyperplanes on which the Zorich induction is not defined).

If $\pi' = b^n \pi$, then, clearly,

$$\mathcal{G}(\Delta(n, b, \pi)) = \Delta^+_{\pi'}.$$

Note that the sets $\Delta(n, a, \pi)$ and $\Delta(n', b, \pi')$ are always disjoint, since we have $\Delta(n, a, \pi) \subset \Delta^-_\pi$, $\Delta(n', b, \pi') \subset \Delta^+_{\pi'}$.

The sets $\Delta(n, a, \pi)$, $\Delta(n, b, \pi)$, for all $n > 0$ and all $\pi \in \mathcal{R}$, form a Markov partition for $\mathcal{G}$.

3.2 Words

Consider the alphabet

$$\mathcal{A} = \{(c, n, \pi), c = a \text{ or } b\}$$

For $w_1 \in \mathcal{A}$, $w_1 = (c_1, n_1, \pi_1)$, we write $c_1 = c(w_1), \pi_1 = \pi(w_1), n_1 = n(w_1)$.

For $w_1, w_2 \in \mathcal{A}$, $w_1 = (c_1, n_1, \pi_1)$, $w_2 = (c_2, n_2, \pi_2)$, define the function $B(w_1, w_2)$ in the following way: $B(w_1, w_2) = 1$ if $c_1^{n_1}\pi_1 = \pi_2$ and $c_1 \neq c_2$ and $B(w_1, w_2) = 0$ otherwise.

Let

$$W_{\mathcal{A},B} = \{w = w_1 \dots w_n, w_i \in \mathcal{A}, B(w_i, w_{i+1}) = 1 \text{ for all } i = 1, \dots, n\}.$$

For $w_1 \in \mathcal{A}$, $w_1 = (c_1, n_1, \pi_1)$, set

$$A(w) = A(c_1, c_1^{-n_1}\pi_1) \dots A(c_1, c_1^{-1}\pi_1)A(c_1, \pi_1),$$

and for $w \in W_{\mathcal{A},B}$, $w = w_1 \dots w_n$, set

$$A(w) = A(w_1) \dots A(w_n).$$

Also, for $w_1 \in \mathcal{A}$, $\pi \in \mathcal{R}$, set $w_1^{-1}\pi = c_1^{-n_1}\pi$, and for $w \in W_{\mathcal{A},B}$, $w = w_1 \dots w_n$, set

$$w^{-1}\pi = w_1^{-1} \dots w_n^{-1}\pi.$$

For $w \in W_{\mathcal{A},B}$, define a map $t_w : \Delta(\mathcal{R}) \to \Delta(\mathcal{R})$ by

$$t_w(\lambda, \pi) = (\frac{A(w)\lambda}{|A(w)\lambda|}, w^{-1}\pi)$$

Consider also the map

$$T_w(\lambda, \pi) = (A(w)\lambda, w^{-1}\pi)$$

For $w_1 \in \mathcal{A}$, $w_1 = (c_1, n_1, \pi_1)$, we write $\Delta(w_1) = \Delta(c_1, n_1\pi_1)$.

For $w \in W_{\mathcal{A},B}$, $w = w_1 \dots w_n$, denote

$$\Delta(w) = t_w(\Delta(\mathcal{R})).$$

Then, by definition,

$$\Delta(w) = \{(\lambda, \pi) : (\lambda, \pi) \in \Delta(w_1), \mathcal{G}(\lambda, \pi) \in \Delta(w_2), \dots, \mathcal{G}^{n-1}(\lambda, \pi) \in \Delta(w_n)\}.$$

Say that $w_1 \in \mathcal{A}$ is compatible with $(\lambda, \pi) \in \Delta(\mathcal{R})$ if

1. either $\lambda \in \Delta_\pi^+$, $c_1 = a$, and $a^{n_1}\pi_1 = \pi$
2. or $\lambda \in \Delta_\pi^-$, $c_1 = b$, and $b^{n_1}\pi_1 = \pi$.

Say that a word $w \in W_{\mathcal{A},B}$, $w = w_1 \dots w_n$ is compatible with (λ, π) if w_n is compatible with (λ, π).

We can write

$$\mathcal{G}^{-n}(\lambda, \pi) = \{t_w(\lambda, \pi) : |w| = n \text{ and } w \text{ is compatible with } (\lambda, \pi)\}.$$

Suppose that a word $w \in W_{\mathcal{A},B}$ is compatible with both (λ, π) and (λ', π). Then

$$d(t_w(\lambda, \pi), t_w(\lambda', \pi)) \leq d((\lambda, \pi), (\lambda', \pi')).$$

If, moreover, all entries of the the matrix $A(w)$ are positive, then, by Lemma 1, there exists $\alpha(w)$, $0 < \alpha(w) < 1$, such that

$$d(t_w(\lambda, \pi), t_w(\lambda', \pi)) \leq \alpha(w) d((\lambda, \pi), (\lambda', \pi')).$$

We therefore have

Proposition 4 *Let $w \in W_{\mathcal{A},B}$ be such that all entries of the matrix $A(w)$ are positive. Then the return map of $\mathcal{G}$ on $\Delta(w)$ is uniformly expanding with respect to the Birkhoff metric.*

3.3 Sequences

Now let

$$\Omega_{\mathcal{A},B} = \{\omega = \omega_1 \dots \omega_n \dots, \ \omega_n \in \mathcal{A}, B(\omega_n, \omega_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\}$$

and

$$\Omega^{\mathbb{Z}}_{\mathcal{A},B} = \{\omega = \dots \omega_{-n} \dots \omega_1 \dots \omega_n \dots, \ \omega_n \in \mathcal{A}, B(\omega_n, \omega_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}\}$$

Denote by σ the shift on both these spaces.

There is a natural map $\Phi : \Delta \to \Omega_{\mathcal{A},B}$ given by the formula

$$\Phi(\lambda, \pi) = \omega_1 \dots \omega_n \dots$$

if

$$\mathcal{G}^n(\lambda, \pi) \in \Delta(\omega_n)$$

The measure ν projects under Φ to a σ-invariant measure on $\Omega_{\mathcal{A},B}$; probability with respect to that measure will be denoted by $\mathbb{P}$.

For $w \in W_{\mathcal{A},B}$, $w = w_1 \dots w_n$, let

$$C(w) = \{\omega \in \Omega_{\mathcal{A},B} : \omega_1 = w_1, \dots, \omega_n = w_n\}.$$

We have then

$$\Delta(w) = \Phi^{-1}(C(w)).$$

W. Veech [1] has proved the following

Proposition 5 *The map Φ is ν-almost surely bijective.*

We thus obtain a symbolic dynamics for the map $\mathcal{G}$.

3.4 The natural extension.

Consider the natural extension for the map $\mathcal{G}$.

The phase space is the space of sequences of interval exchanges; it will be convenient to number them by negative integers. We set:

$$\overline{\Delta}(\mathcal{R}) =$$

$$\{\mathbf{x} = \dots(\lambda(-n), \pi(-n)), \dots, (\lambda(0), \pi(0)) | \, \mathcal{G}(\lambda(-n), \pi(-n)) = (\lambda(1-n), \pi(1-n)), n = 1, \dots\}$$

The map $\mathcal{G}$ and the invariant measure ν are extended to $\overline{\Delta}$ in the natural way. We shall still denote the probability with respect to the extended measure by $\mathbb{P}$.

We extend the map Φ to a map

$$\overline{\Phi} : \overline{\Delta} \to \Omega^{\mathbb{Z}}_{\mathcal{A},B},$$

$$\overline{\Phi}(\lambda) = \dots\omega_{-n} \dots \omega_0 \dots \omega_n \dots,$$

if $(\lambda(-n), \pi(-n) \in \Delta(\omega_{-n})$, and $\mathcal{G}^n(\lambda(0), \pi(0)) \in \Delta(\omega_n)$.

The space $\overline{\Delta}(\mathcal{R})$ can be naturally identified with Veech's space of zippered rectangles [1].

3.5 Transition probabilities.

Take a sequence $c_1 \dots c_n \dots \in \Omega_{\mathcal{A},B}$. Following Sinai [25], consider the *transition probability*

$$\mathbb{P}(\omega_1 = c_1 | \omega_2 = c_2, \dots, \omega_n = c_n, \dots) = \lim_{n\to\infty} \frac{\mathbb{P}(c_1 c_2 \dots c_n)}{\mathbb{P}(c_2 \dots c_n)}.$$

In this subsection, we give a formula for this probability in terms of $(\lambda, \pi) = \Phi^{-1}(c_2 \dots c_n \dots)$.

Assume $w_1 \in \mathcal{A}$ is compatible with (λ, π).

Denote

$$\mathbb{P}(w_1 | (\lambda, \pi)) = \mathbb{P}(((\lambda(-1), \pi(-1)) = t_{w_1}(\lambda(0), \pi(0)) | (\lambda(0), \pi(0)) = (\lambda, \pi)).$$

If $w_1 \in \mathcal{A}$ is compatible with (λ, π), from the definition of $\mathcal{G}$ and from (4) we have

$$\mathbb{P}(w_1 | (\lambda, \pi)) = \frac{\rho(t_w(\lambda, \pi))}{\rho(\lambda, \pi) |A(w_1)\lambda|^m} \tag{6}$$

Let $w = w_1 \dots w_n$ be compatible with (λ, π).

Denote

$$\mathbb{P}(w|(\lambda,\pi)) = \mathbb{P}((\lambda(-k),\pi(-k)) = t_{w_{n-k}}(\lambda(1-k),\pi(1-k)), k=1,\dots,n|(\lambda(0),\pi(0)) = (\lambda,\pi)).$$

From (6), by induction, we have

$$\mathbb{P}(w|(\lambda,\pi)) = \frac{\rho(t_w(\lambda,\pi))}{\rho(\lambda,\pi)|A(w)\lambda|^m} \tag{7}$$

By homogeneity of the density, we can write

$$\mathbb{P}(w|(\lambda,\pi)) = \frac{\rho(T_w(\lambda,\pi))}{\rho(\lambda,\pi)} \tag{8}$$

Corollary 4 *There exists $C > 0$ such that the following is true. Suppose $w \in W_{\mathcal{A},B}$ is compatible with (λ,π). Then*

$$\mathbb{P}(w|(\lambda,\pi)) \geq \frac{C}{\rho(\lambda,\pi)|A(w)|^m}$$

Proof: the invariant density is bounded from below: there exists $C > 0$ such that $\rho(\lambda,\pi) > C$ for all $(\lambda,\pi) \in \Delta(\mathcal{R})$. In particular, $\rho(t_w(\lambda,\pi)) > C$. Substituting into (7), we obtain the result.

For $\epsilon : 0 < \epsilon < 1$, let

$$\Delta_\epsilon = \{(\lambda,\pi) \in \Delta(\mathcal{R}), \min|\lambda_i| \geq \epsilon\}.$$

For any $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that for any $(\lambda,\pi) \in \Delta_\epsilon$ we have $\rho(\lambda,\pi) < C(\epsilon)$.

Corollary 5 *For any $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that if $(\lambda,\pi) \in \Delta_\epsilon$, then*

$$\mathbb{P}(w|(\lambda,\pi)) \geq \frac{C(\epsilon)}{|A(w)|^m}.$$

3.6 An estimate on the number of Rauzy operations.

Let $(\lambda,\pi) \in \Delta(\mathcal{R})$. Define the permutations $\pi^{(n)}$ and the letters $w_1(n) \in \mathcal{A}$ in the following way. If $\lambda \in \Delta_\pi^+$, then set $\pi^{(n)} = a^{-n}\pi$, $w_1(n) = (a,n,\pi^{(n)})$; if $\lambda \in \Delta_\pi^-$, then set $\pi^{(n)} = b^{-n}\pi$, $w_1(n) = (b,n,\pi^{(n)})$.

Define $\lambda^{(n)}, \Lambda^{(n)}$ by relations

$$(\lambda^{(n)},\pi^{(n)}) = t_{w_1(n)}(\lambda,\pi), (\Lambda^{(n)},\pi^{(n)}) = T_{w_1(n)}(\lambda,\pi)$$

Clearly,

$$\mathcal{G}^{-1}(\lambda,\pi) = \{(\lambda^{(n)},\pi^{(n)}), n = 1,\dots\}.$$

Proposition 6 *Let*

$$l = \begin{cases} \text{the length of the } a-cycle\ of\ \pi, & if\ \lambda \in \Delta_\pi^+; \\ \text{the length of the } b-cycle\ of\ \pi, & if\ \lambda \in \Delta_\pi^-. \end{cases}$$

Then

$$\sum_{n=kl+1}^{\infty} \mathbb{P}(w_1(n)|(\lambda,\pi)) = \frac{\rho(\Lambda^{(kl)},\pi)}{\rho(\lambda,\pi)}.$$

Proof:
Using homogeneity of the invariant density, write

$$\mathbb{P}(w_1(n)|(\lambda,\pi)) = \frac{\rho(\lambda^{(n)},\pi^{(n)})}{|A(w_1)\lambda|^m \rho(\lambda,\pi)} = \frac{\rho(\Lambda^{(n)},\pi^{(n)})}{\rho(\lambda,\pi)}.$$

The $\mathcal{G}$-invariance of the measure ν gives

$$\rho(\lambda,\pi) = \sum_{n=1}^{\infty} \rho(\Lambda^{(n)},\pi^{(n)}) \tag{9}$$

Note that $\pi^{(kl)} = \pi$ and

$$T_{w_1(n)}(\Lambda^{(kl)},\pi) = (\Lambda^{(n+kl)},\pi^{(n)}).$$

Substituting into (9), we obtain

$$\rho(\Lambda(kl),\pi) = \sum_{n=kl+1}^{\infty} \rho(\Lambda^{(n)},\pi^{(n)}) = \rho(\lambda,\pi)(\sum_{n=kl+1}^{\infty} \mathbb{P}(w_1(n)|(\lambda,\pi))),$$

whence

$$\sum_{n=kl+1}^{\infty} \mathbb{P}(w_1(n)|(\lambda,\pi)) = \sum_{n=kl+1}^{\infty} \frac{\rho(\Lambda^{(n)},\pi^{(n)})}{\rho(\lambda,\pi)} = \frac{\rho(\Lambda^{(kl)},\pi^{(kl)})}{\rho(\lambda,\pi)}.$$

4 Proof of the Exactness

First, one notes that the discrete parameter π does not give rise to any period, and then the proof follows the standard pattern [27, 17]: since almost every point of any measurable subset is a density point, bounded distortion estimates of Proposition 3 imply that if the measure of a tail event is positive, then it must be arbitrarily close to 1.

In more detail, observe that there exists an integer M such that for any $n > M$ and for any $\pi, \pi' \in \mathcal{R}$ there exist $k_1, \ldots, k_{2n}$ such that $a^{k_1}b^{k_2}\ldots a^{k_{2n-1}}b^{k_{2n}}\pi = \pi'$. This follows from conmnectedness of the Rauzy graph and the fact that for any $\pi \in \mathcal{R}$ there exist n_1, n_2 such that $a^{n_1}\pi = b^{n_2}\pi = \pi$.

Let α_0 be the partition of Δ^+ into Δ_π^+, $\pi \in \mathcal{R}$, and let α_n be the partition into the cylinders $\Delta(w)$, where $w \in \mathcal{W}_{\mathcal{A},B}$, $|w| = 2n$.

Lemma 2 *There exists $k > 0$ such that the following is true. Suppose $C \subset \Delta^+$, and there exists $\pi \in \mathcal{R}$ such that $\Delta^+_\pi \subset C$. Then $\mathcal{G}^{2k} C = \Delta^+(\mathcal{R})$.*

This implies

Lemma 3 *There exists $k > 0$ such that the following holds. For any $\varepsilon > 0$ there is $\delta > 0$ such that for any $C \subset \Delta^+(\mathcal{R})$ satisfying $\mathbf{m}(C \triangle \Delta^+_\pi) < \delta$, we have $\mathbf{m}(\mathcal{G}^{2k} C \triangle \Delta^+) < \varepsilon$.*

Now suppose $C \subset \Delta^+$ is a $\mathcal{G}^2$-tail event, i.e., for any $n > 0$ there exists B_n such that $C = \mathcal{G}^{-2n} B_n$ and $0 < \nu(C) < 1$. Then $\nu(B_n) = \nu(C)$ and, by Lemma 3, we can assume that there exists $\varepsilon > 0$ such that for any $\pi \in \mathcal{R}$, we have

$$\mathbf{m}((\Delta^+ \setminus C) \cap \Delta^+_\pi) \geq \varepsilon \tag{10}$$

Let $\mathbf{q} = q_1 \dots q_l$ be a word such that the matrix $A(\mathbf{q})$ is positive.

For almost any $(\lambda, \pi) \in C$ we have

$$\lim_{n \to \infty} \frac{\mathbf{m}(\alpha_n(\lambda, \pi) \cap C)}{\mathbf{m}(\alpha_n(\lambda, \pi))} = 1 \tag{11}$$

Now let n be such that $\mathcal{G}^{2n}(\lambda, \pi) \in \Delta(\mathbf{q})$. Denote $(\lambda', \pi') = \mathcal{G}^{2n}(\lambda, \pi)$. Let A be the corresponding renormalization matrix, that is, $\lambda = J_A \lambda'$. Then $A = A_1 A(\mathbf{q})$ for some (unimodular nonnegative integer) matrix A_1. We have $\alpha_n(\lambda, \pi) = J_A(\Delta^+_{\pi'})$. By Proposition 3, from (10), we deduce that there exists ε', not depending on n such that

$$\frac{\mathbf{m}(\alpha_n(\lambda, \pi) \cap (\Delta^+ \setminus C))}{\mathbf{m}(\alpha_n(\lambda, \pi))} \geq \varepsilon'.$$

Since, by ergodicity, for almost any (λ, π) we can find infinitely many n such that $\mathcal{G}^{2n}(\lambda, \pi) \in \Delta(\mathbf{q})$, we arrive at a contradiction with (11), which gives the exactness of $\mathcal{G}^2$.

5 Four Intervals

Starting in this Section and until Section 10 we only consider the Rauzy class $\mathcal{R}(4321)$.

5.1 The Main Lemma

For $\epsilon : 0 < \epsilon < 1$, define, in the same way as above,

$$\Delta_\epsilon = \{(\lambda, \pi) \in \Delta(4321), \min |\lambda_i| \geq \epsilon\}.$$

Lemma 4 *There exist positive constants γ, K, p such that the following is true for any $\epsilon > 0$. Suppose $(\lambda, \pi) \in \Delta_\epsilon$. Then*

$$\mathbb{P}\{\exists n \leq K |\log \epsilon|, (\lambda(-n), \pi(-n)) \in \Delta_\gamma | (\lambda(1), \pi(1)) = (\lambda, \pi))\} \geq p.$$

From Corollary 5, we obtain

Corollary 6 *Let* $\mathbf{q} \in W_{\mathcal{A},B}$, $\mathbf{q} = q_1 \ldots q_l$ *be such that all entries of the matrix* $A(\mathbf{q})$ *are positive. Then there exist positive constants* $K(\mathbf{q}), p(\mathbf{q})$ *such that the following is true for any* $\epsilon > 0$. *Suppose* $(\lambda, \pi) \in \Delta_\epsilon$. *Then*

$$\mathbb{P}\{\exists n \leq K(\mathbf{q})|\log \epsilon|, (\lambda(-n), \pi(-n)) \in \Delta(\mathbf{q})|(\lambda(1), \pi(1)) = (\lambda, \pi))\} \geq p(\mathbf{q}).$$

Informally, the proof of Lemma 4 proceeds by getting rid of small intervals. If there is only one small interval, then there is nothing to prove, because the invariant density in this case is bounded (Corollary 7). If there are two small intervals, then one can in finitely many steps reduce to the case of (λ, π) such that $\pi = a\pi$ and λ_3, λ_4 are small or (λ, π) such that $b\pi = \pi$ and $\lambda_{\pi^{-1}3}$, $\lambda_{\pi^{-1}4}$ are small (Proposition 8, proven in Section 8). In either of these cases, a direct computation shows that, with positive probability, there will remain at most one small interval after one inverse Zorich step (Proposition 7 proven in Section 7).

If there are three small intervals, then the proof proceeds as follows. If the large interval gets added to a small one, then we already have two large ones. Assume, therefore, that the small intervals are added between themselves. It is proven that their total length grows exponentially with the number of Zorich steps: therefore, if ϵ is the length of the shortest subinterval, then in $\log \epsilon$ steps there must appear a second large interval. This argument is made precise in Section 9.

We now proceed to precise formulations. Set

$$\Delta_{1,\gamma} = \{(\lambda, \pi) \in \Delta : \exists i_1 \in \{1, 2, 3, 4\} \text{ such that } \lambda_{i_1} > \gamma\},$$

$$\Delta_{2,\gamma} = \{(\lambda, \pi) \in \Delta : \exists i_1, i_2 \in \{1, 2, 3, 4\} \text{ such that } \lambda_{i_1} > \gamma, \lambda_{i_2} > \gamma\},$$

$$\Delta_{3,\gamma} = \{(\lambda, \pi) \in \Delta : \exists i_1, i_2, i_3 \in \{1, 2, 3, 4\} \text{ such that } \lambda_{i_1} > \gamma, \lambda_{i_2} > \gamma, \lambda_{i_3} > \gamma\}.$$

Lemma 5 *There exist positive constants* N, q, α *such that the following is true for any* $\gamma > 0$. *Suppose* $(\lambda, \pi) \in \Delta_{2,\gamma}$. *Then*

$$\mathbb{P}\{(\lambda(-N), \pi(-N)) \in \Delta_{\alpha\gamma}|(\lambda(1), \pi(1)) = (\lambda, \pi))\} \geq q.$$

Lemma 6 *There exist positive constants* γ, K, p *such that the following is true for any* $\epsilon > 0$. *Suppose* $\lambda \in \Delta_\epsilon$. *Then*

$$\mathbb{P}\{\exists n \leq K|\log \epsilon|, (\lambda(-n), \pi(-n)) \in \Delta_{2,\gamma}|(\lambda(1), \pi(1)) = (\lambda, \pi))\} \geq p.$$

Lemmas 5, 6 yield Lemma 4.

5.2 Invariant densities

The invariant densities for the Rauzy-Veech-Zorich map $\mathcal{G}$ on the space $\Delta = \Delta_3 \times \mathcal{R}(4321)$ are given by the following formulas, found by Zorich [4]:

For (4321)

$$\rho^+_{(4321)}(\lambda) = \frac{1}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_3+\lambda_4)(\lambda_2+\lambda_3+\lambda_4)} \tag{12}$$

$$\rho^-_{(4321)}(\lambda) = \frac{1}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_3+\lambda_4)(\lambda_1+\lambda_2+\lambda_3)} \tag{13}$$

For (3142)

$$\rho^+_{(3142)}(\lambda) = \frac{1}{\lambda_4(\lambda_1+\lambda_2+\lambda_4)(\lambda_1+\lambda_3+\lambda_4)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)} \tag{14}$$

$$\rho^-_{(3142)}(\lambda) = \frac{1}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_2+\lambda_4)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}\left(\frac{1}{\lambda_1+\lambda_2+\lambda_3}+\frac{1}{\lambda_1+\lambda_3+\lambda_4}\right) \tag{15}$$

For (2413)

$$\rho^+_{(2413)}(\lambda) = \frac{1}{(\lambda_3+\lambda_4)(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}\left(\frac{1}{\lambda_1+\lambda_3+\lambda_4}+\frac{1}{\lambda_2+\lambda_3+\lambda_4}\right) \tag{16}$$

$$\rho^-_{(2413)}(\lambda) = \frac{1}{\lambda_2(\lambda_1+\lambda_2+\lambda_3)(\lambda_2+\lambda_3+\lambda_4)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)} \tag{17}$$

For (4213)

$$\rho^+_{(4213)}(\lambda) = \frac{1}{(\lambda_2+\lambda_3)(\lambda_2+\lambda_4)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}\left(\frac{1}{\lambda_1+\lambda_2+\lambda_4}+\frac{1}{\lambda_2+\lambda_3+\lambda_4}\right) \tag{18}$$

$$\rho^-_{(4213)}(\lambda) = \frac{1}{\lambda_1(\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_4)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)} \tag{19}$$

For (3241)

$$\rho^+_{(3241)}(\lambda) = \frac{1}{\lambda_4(\lambda_1+\lambda_2)(\lambda_2+\lambda_3+\lambda_4)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)} \tag{20}$$

$$\rho^-_{(3241)}(\lambda) = \frac{1}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}\left(\frac{1}{\lambda_2+\lambda_3+\lambda_4}+\frac{1}{\lambda_1+\lambda_2+\lambda_3}\right) \tag{21}$$

For (4132)

$$\rho^{+}_{(4132)}(\lambda) = \frac{1}{(\lambda_2+\lambda_4)(\lambda_3+\lambda_4)(\lambda_1+\lambda_2+\lambda_3)}\left(\frac{1}{(\lambda_2+\lambda_3+\lambda_4)}+\frac{1}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}\right) \quad (22)$$

$$\rho^{-}_{(4132)}(\lambda) = \frac{1}{(\lambda_3+\lambda_4)(\lambda_1+\lambda_3)(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)} \quad (23)$$

For (2431)

$$\rho^{+}_{(2431)}(\lambda) = \frac{1}{(\lambda_2+\lambda_3)(\lambda_3+\lambda_4)(\lambda_1+\lambda_3+\lambda_4)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)} \quad (24)$$

$$\rho^{-}_{(2431)}(\lambda) = \frac{1}{(\lambda_2+\lambda_3)(\lambda_1+\lambda_3+\lambda_4)(\lambda_1+\lambda_2)}\left(\frac{1}{(\lambda_1+\lambda_2+\lambda_3)}+\frac{1}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}\right) \quad (25)$$

Note the following important relation:

$$\rho^{+}_{\pi}(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \rho^{-}_{\pi^{-1}}(\lambda_{\pi(1)},\lambda_{\pi(2)},\lambda_{\pi(3)},\lambda_{\pi(4)}) \quad (26)$$

Remark: This relation holds in general, for any number of intervals: for any π, we have

$$\rho^{+}_{\pi}(\lambda_1,\ldots\lambda_m) = \rho^{-}_{\pi^{-1}}(\lambda_{\pi(1)},\ldots\lambda_{\pi(m)})$$

This follows from Zorich's construction of the invariant densities [4].

The form of the invariant densities implies

1. $\rho(\lambda,\pi) \geq \frac{1}{100}$.
2. If $(\lambda,\pi)\in\Delta_{3,\gamma}$, then $\rho(\lambda,\pi)\leq\frac{1}{\gamma^4}$.

Corollary 7 *Let $(\lambda,\pi)\in\Delta_{3,\gamma}$. Let w be a word compatible with (λ,π). Then*

$$\mathbb{P}(w|(\lambda,\pi)) \geq \frac{\gamma^4}{100|A(w)|^4}.$$

6 Two Small Intervals and the Proof of Lemma 5.

Lemma 7 *There exists $\alpha>0$, $p>0$ such that for any $\gamma>0$ the following is true.*

Let $(\lambda,\pi)\in\Delta_{2,\gamma}$. Then

$$\mathbb{P}((\lambda(-10),\pi(-10))\in\Delta_{3,\alpha\gamma}|(\lambda(0),\pi(0))=(\lambda,\pi))\geq p$$

This Lemma will be deduced from two propositions that follow:

Proposition 7 *Suppose $\pi \in \mathcal{R}(4321)$ is such that either $a\pi = \pi$ or $b\pi = \pi$. There exists $\alpha > 0$, $p > 0$ such that for any $\gamma > 0$ the following is true. Assume $(\lambda, \pi) \in \Delta_{2,\gamma}$. Then*

$$\mathbb{P}((\lambda(-1), \pi(-1) \in \Delta_{3,\alpha\gamma} | (\lambda(0), \pi(0)) = (\lambda, \pi)) \geq p$$

It suffices to prove this proposition for a-invariant permutations, as the other case follows from the relation (26) and the observation that $a\pi = \pi$ if and only if $b\pi^{-1} = \pi^{-1}$.

Let $i_1, i_2 \in \{1, 2, 3, 4\}$, $\pi \in \mathcal{R}(4321)$. The pair i_1, i_2 is called $(\pi, +)$-*-critical* if

1. The set $\{\lambda_{i_1} = 0, \lambda_{i_2} = 0\}$ belongs to the closure of Δ_π^+.
2. If $\lambda_{i_1} = \lambda_{i_2} = 0$ then $\rho_\pi^+(\lambda) = \infty$.

Similarly, the pair i_1, i_2 is called $(\pi, -)$-*-critical* if

1. The set $\{\lambda_{i_1} = 0, \lambda_{i_2} = 0\}$ belongs to the closure of Δ_π^-.
2. If $\lambda_{i_1} = \lambda_{i_2} = 0$ then $\rho_\pi^-(\lambda) = \infty$.

Proposition 8 *Let (i_1, i_2) be a $(\pi, +)$-critical pair or a $(\pi, -)$-critical pair, and let j_1, j_2 be the two remaining indices, that is, $\{i_1, i_2, j_1, j_2\} = \{1, 2, 3, 4\}$. Let $\gamma > 0$. Suppose $\lambda_{j_1} > \gamma$, $\lambda_{j_2} > \gamma$. Then there exists a word w, $|w| < 10$ satisfying*

$$\mathbb{P}(w | (\lambda, \pi))) > \frac{\gamma}{100}$$

and such that for $(\lambda', \pi') = t_w(\lambda, \pi)$ either

1. *$\lambda'_1 > \frac{\gamma}{10}, \lambda'_2 > \frac{\gamma}{10}$ and $a\pi' = \pi'$; or*
2. *$\lambda'_{\pi^{-1}1} > \frac{\gamma}{10}, \lambda'_{\pi^{-1}2} > \frac{\gamma}{10}$ and $b\pi' = \pi'$.*

It suffices to prove this proposition for $(\pi, +)$-critical pairs, as the other case follows from the relation (26) and the observation that there is a bijection between $(\pi, +)$ and $(\pi^{-1}, -)$-critical pairs.

We prove Propositions 7, 8 by consideration of cases.

7 Permutations Invariant Under a and the Proof of Proposition 7.

7.1 The case of (3241).

Consider the permutation (3241). Note that $a(3241) = (3241)$.

Let $\lambda \in \Delta^{+}_{(3241)}$, in other words, $\lambda_3 > \lambda_4$. Consider the interval exchange $(\lambda, (3241))$.

Similarly to Subsection 3.6, denote $w_1(n) = (a, n, (3241))$, and define vectors $\lambda^{(n)}$ by the formula

$$(\lambda^{(n)}, (3241)) = t_{w_1(n)}(\lambda, (3241)).$$

We have then

$$\lambda_1^{(n)} = \frac{\lambda_1}{1+n\lambda_4}, \lambda_2^{(n)} = \frac{\lambda_2}{1+n\lambda_4}, \lambda_3^{(n)} = \frac{\lambda_3 + n\lambda_4}{1+n\lambda_4}, \lambda_4^{(n)} = \frac{\lambda_4}{1+n\lambda_4},$$

and

$$\mathcal{G}^{-1}(\lambda, (3241)) = \{(\lambda^{(n)}, (3241)), n \in \mathbb{N}\}$$

Here the parameter n is equal to the number of a-operations needed to pass from $(\lambda^{(n)}, (3241))$ to $(\lambda, (3241)$.

By (8), we have

$$\mathbb{P}(w_1(n)|(\lambda, (3241)) = \frac{\rho^{-}_{(3241)}(\lambda_1, \lambda_2, \lambda_3 + n\lambda_4, \lambda_4)}{\rho^{+}_{(3241)}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}$$

From the invariance of the measure we have

$$\rho^{+}_{(3241)}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \sum_{s=0}^{\infty} \rho^{-}_{(3241)}(\lambda_1, \lambda_2, \lambda_3 + s\lambda_4, \lambda_4)$$

This implies

$$\rho^{+}_{(3241)}(\lambda_1, \lambda_2, \lambda_3 + n\lambda_4, \lambda_4) = \sum_{s=n+1}^{\infty} \rho^{-}_{(3241)}(\lambda_1, \lambda_2, \lambda_3 + s\lambda_4, \lambda_4)$$

and, consequently, for any positive integer n we have

$$\sum_{s \geq n} \mathbb{P}(w_1(s)|(\lambda, (3241)) = \frac{\rho^{+}_{(3241)}(\lambda_1, \lambda_2, \lambda_3 + n\lambda_4, \lambda_4)}{\rho^{+}_{(3241)}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}$$

(This is a particular case of Proposition 6).

From the formula for $\rho^{+}_{(3241)}$, we obtain

$$\sum_{s \geq n} \mathbb{P}(w_1(s)|(\lambda, (3241)) = \frac{\lambda_2 + \lambda_3 + \lambda_4}{(\lambda_2 + \lambda_3 + (n+1)\lambda_4)(1+n\lambda_4)}$$

Pick $\alpha > 0$, $\beta > 0$, and set $n_\alpha = \alpha(\frac{\lambda_2+\lambda_3+\lambda_4}{\lambda_4})$, $n_\beta = \beta(\frac{\lambda_2+\lambda_3+\lambda_4}{\lambda_4})$. Then

$$\frac{1}{1+\alpha} \geq \sum_{n \geq n_\alpha} \mathbb{P}(w_1(n)|(\lambda, (3241)) \geq \frac{1}{(1+\alpha)^2}$$

whence

$$\sum_{n_\beta \geq n \geq n_\alpha} \mathbb{P}(w_1(n)|(\lambda,(3241)) \geq \frac{1}{(1+\alpha)^2} - \frac{1}{1+\beta}.$$

Lemma 8 *Let $0 < \gamma < 1$ and assume $\lambda_1 > \gamma, \lambda_2 > \gamma$. Then for any $\alpha, \beta > 0$, we have*

$$\mathbb{P}(\lambda_1^{(n)} \geq \frac{\gamma}{1+\beta}, \lambda_2^{(n)} \geq \frac{\gamma}{1+\beta}, \lambda_3^{(n)} \geq \frac{\alpha\gamma}{1+\beta}) \geq \frac{1}{(1+\alpha)^2} - \frac{1}{1+\beta}.$$

Proof. Indeed, since

$$\lambda_1^{(n)} = \frac{\lambda_1}{1+n\lambda_4}, \lambda_2^{(n)} = \frac{\lambda_2}{1+n\lambda_4}, \lambda_3^{(n)} = \frac{\lambda_3+n\lambda_4}{1+n\lambda_4}, \lambda_4^{(n)} = \frac{\lambda_4}{1+n\lambda_4},$$

if

$$\beta(\frac{\lambda_2+\lambda_3+\lambda_4}{\lambda_4}) \geq n \geq \alpha(\frac{\lambda_2+\lambda_3+\lambda_4}{\lambda_4}) \geq \frac{1}{(1+\alpha)^2} - \frac{1}{1+\beta},$$

then

$$\lambda_1^{(n)} \geq \frac{\gamma}{1+\beta}, \lambda_2^{(n)} \geq \frac{\gamma}{1+\beta}, \lambda_3^{(n)} \geq \frac{\alpha\gamma}{1+\beta}.$$

7.2 The case of (3142).

Consider the permutation (3142). Note that $a(3142) = (3142)$.

Let $\lambda \in \Delta^+_{(3142)}$, in other words, $\lambda_3 > \lambda_4$. Consider the interval exchange $(\lambda, (3142))$.

Similarly to the previous subsection, denote $w_1(n) = (a, n, (3142))$, and define vectors $\lambda^{(n)}$ by the formula

$$(\lambda^{(n)}, (3142)) = t_{w_1(n)}(\lambda, (3142)).$$

We have then

$$\lambda_1^{(n)} = \frac{\lambda_1}{1+n\lambda_4}, \lambda_2^{(n)} = \frac{\lambda_2}{1+n\lambda_4}, \lambda_3^{(n)} = \frac{\lambda_3+n\lambda_4}{1+n\lambda_4}, \lambda_4^{(n)} = \frac{\lambda_4}{1+n\lambda_4},$$

and

$$\mathcal{G}^{-1}(\lambda, (3241)) = \{(\lambda^{(n)}, (3241)), n \in \mathbb{N}\}$$

Here the parameter n is equal to the number of a-operations needed to pass from $(\lambda^{(n)}, (3142))$ to $(\lambda, (3142)$.

By (8), we have

$$\mathbb{P}(w_1(n)|(\lambda,(3142)) = \frac{\rho^-_{(3142)}(\lambda_1, \lambda_2, \lambda_3 + n\lambda_4, \lambda_4)}{\rho^+_{(3142)}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}$$

From the invariance of the measure we have

$$\rho^+_{(3142)}(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \sum_{s=0}^{\infty} \rho^-_{(3142)}(\lambda_1,\lambda_2,\lambda_3+s\lambda_4,\lambda_4)$$

This implies

$$\rho^+_{(3142)}(\lambda_1,\lambda_2,\lambda_3+n\lambda_4,\lambda_4) = \sum_{s=n+1}^{\infty} \rho^-_{(3142)}(\lambda_1,\lambda_2,\lambda_3+s\lambda_4,\lambda_4)$$

and, consequently, for any positive integer n we have

$$\sum_{s\geq n} \mathbb{P}(w_1(s)|(\lambda,(3142)) = \frac{\rho^+_{(3142)}(\lambda_1,\lambda_2,\lambda_3+n\lambda_4,\lambda_4)}{\rho^+_{(3142)}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}$$

(This is a particular case of Proposition 6).
From the formula for $\rho^+_{(3142)}$, we obtain

$$\sum_{s\geq n} \mathbb{P}(w_1(s)|(\lambda,(3142)) = \frac{\lambda_1+\lambda_3+\lambda_4}{(\lambda_1+\lambda_3+(n+1)\lambda_4)(1+n\lambda_4)}$$

Pick $\alpha>0$, $\beta>0$, and set $n_\alpha = \alpha(\frac{\lambda_1+\lambda_3+\lambda_4}{\lambda_4})$, $n_\beta = \beta(\frac{\lambda_1+\lambda_3+\lambda_4}{\lambda_4})$. Then

$$\frac{1}{1+\alpha} \geq \sum_{n\geq n_\alpha} \mathbb{P}(w_1(n)|(\lambda,(3142)) \geq \frac{1}{(1+\alpha)^2}$$

whence

$$\sum_{n_\beta\geq n\geq n_\alpha} \mathbb{P}(w_1(n)|(\lambda,(3142)) \geq \frac{1}{(1+\alpha)^2} - \frac{1}{1+\beta}.$$

Lemma 9 *Let $0<\gamma<1$ and assume $\lambda_1>\gamma, \lambda_2>\gamma$. Then for any $\alpha,\beta>0$, we have*

$$\mathbb{P}(\lambda_1^{(n)} \geq \frac{\gamma}{1+\beta}, \lambda_2^{(n)} \geq \frac{\gamma}{1+\beta}, \lambda_3^{(n)} \geq \frac{\alpha\gamma}{1+\beta}) \geq \frac{1}{(1+\alpha)^2} - \frac{1}{1+\beta}.$$

Proof. Indeed, since

$$\lambda_1^{(n)} = \frac{\lambda_1}{1+n\lambda_4}, \lambda_2^{(n)} = \frac{\lambda_2}{1+n\lambda_4}, \lambda_3^{(n)} = \frac{\lambda_3+n\lambda_4}{1+n\lambda_4}, \lambda_4^{(n)} = \frac{\lambda_4}{1+n\lambda_4},$$

if

$$\beta(\frac{\lambda_1+\lambda_3+\lambda_4}{\lambda_4}) \geq n \geq \alpha(\frac{\lambda_1+\lambda_3+\lambda_4}{\lambda_4}) \geq \frac{1}{(1+\alpha)^2} - \frac{1}{1+\beta},$$

then

$$\lambda_1^{(n)} \geq \frac{\gamma}{1+\beta}, \lambda_2^{(n)} \geq \frac{\gamma}{1+\beta}, \lambda_3^{(n)} \geq \frac{\alpha\gamma}{1+\beta}.$$

8 Critical Pairs and the Proof of Proposition 8.

First we list the +-critical pairs for all permutations in $\mathcal{R}(4321)$:

For (4321) — $(1,2),(2,3)$. Note that $(3,4)$ is not a +-critical pair for (4321), since $\lambda_4 > \lambda_1$ on $\Delta^+_{(4321)}$.

For (4132) — none. Note that neither $(3,4)$ nor $(2,4)$ is a critical pair since $\lambda_4 > \lambda_1$ on $\Delta^+_{(4132)}$.

For (2431) — $(2,3)$ only (and, again, $(3,4)$ is not, for same reasons as above).

For (3142) — $(3,4)$.

For (2413) — none.

For (4213) — $(2,3)$ only.

For (3241) — $(3,4)$ and $(1,2)$.

Note that for the pairs $((3241),(3,4)),(3142,(3,4))$ there is nothing to prove.

For each remaining critical pair, we explicitly construct a path to an a or b-invariant permutation and compute its probability.

8.1 (4321) and $(1,2)$.

$$w = (a, 1, (4213))$$

$$(\Lambda,\Pi) = T_w(\lambda,\pi) = ((\lambda_1+\lambda_2,\lambda_3,\lambda_4,\lambda_2),(4213))$$

$$\mathbb{P}(w|\lambda,\pi) = \frac{\rho^-_{((4213)}(\Lambda)}{\rho^+_{(4321)}(\lambda)} = \frac{(\lambda_2+\lambda_3)(\lambda_2+\lambda_3+\lambda_4)}{(\lambda_1+2\lambda_2+\lambda_3)(1+\lambda_2)}$$

8.2 (4321) and $(2,3)$.

$$w = ((b,1,(3142))(a,2,(4132))$$

$$(\Lambda,\Pi) = T_w(\lambda,\pi) = ((\lambda_1+\lambda_2+\lambda_3,\lambda_4,\lambda_2,\lambda_2+\lambda_3),(3142).$$

and

$$\mathbb{P}(w|\lambda,\pi) = \frac{\rho^+_{((3142)}(\Lambda)}{\rho^+_{(4321)}(\lambda)} = \frac{(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)(\lambda_2+\lambda_3+\lambda_4)}{(1+\lambda_2+\lambda_3)(\lambda_1+3\lambda_2+2\lambda_3)(1+2\lambda_2+\lambda_3)}$$

8.3 (4213) and $(2,3)$.

$$w = (b, 1, (3241))(a, 2, (4321))$$

$$(\Lambda,\Pi) = T_w(\lambda,\pi) = ((\lambda_1+\lambda_2+\lambda_3,\lambda_4,\lambda_2,\lambda_2+\lambda_3),(3241)),$$

$$\mathbb{P}(w|\lambda,\pi) = \frac{\rho^+_{(3241)}(\Lambda)}{\rho^+_{(4213)}(\lambda)} = \frac{(\lambda_2+\lambda_4)(\lambda_1+\lambda_2+\lambda_4)(\lambda_2+\lambda_3+\lambda_4)}{(2\lambda_2+\lambda_3+\lambda_4)(1+2\lambda_2+\lambda_3)(1+\lambda_2+\lambda_4)}$$

8.4 (2431) and $(2,3)$.

$$(w = (a, 1, (2413))$$

$$(\Lambda, \Pi) = T_w(\lambda, \pi) = (\lambda_1, \lambda_2 + \lambda_3, \lambda_4, \lambda_3), (2413),$$

$$\mathbb{P}(w|\lambda, \pi) = \frac{\rho^-_{((2413)}(\Lambda)}{\rho^+_{(2431)}(\lambda)} = \frac{(\lambda_3 + \lambda_4)(\lambda_1 + \lambda_3 + \lambda_4)}{(\lambda_2 + 2\lambda_3 + \lambda_4)(1 + \lambda_3)}$$

8.5 (3241) and $(1,2)$.

$$w = (a, 1, (4213))(b, 2, (4321))(a, 1, (3241))$$

$$(\Lambda, \Pi) = T_w(\lambda, \pi) = (\lambda_1 + \lambda_2, \lambda_3 + \lambda_4, \lambda_1 + \lambda_2 + \lambda_4, \lambda_2), (4213))$$

$$\mathbb{P}(w|\lambda, \pi) = \frac{\rho^-_{((2413)}(\Lambda)}{\rho^+_{(2431)}(\lambda)} = \frac{(\lambda_4)(\lambda_2 + \lambda_3 + \lambda_4)}{(1 + \lambda_4)(1 + \lambda_2)(1 + \lambda_1 + 2\lambda_2 + \lambda_4)}.$$

9 Estimates on the Length Growth and the Proof of Lemma 6.

9.1 Bounded Growth.

Let $\mathbf{x} \in \overline{\Delta}$, that is, $\mathbf{x} = (\ldots, (\lambda(-n), \pi(-n), \ldots, (\lambda, \pi))$, where, as usual, $\mathcal{G}(\lambda(-n), \pi(-n)) = (\lambda(1-n), \pi(1-n))$. Define the words $w(n)$ by the relation $(\lambda(-n), \pi(-n)) = t_{w(n)}(\lambda, \pi)$. Set $(\Lambda(-n), \pi(-n)) = T_{w(n)}(\lambda, \pi)$.

Lemma 10

$$\mathbb{P}(\Lambda(-1) > K) < \frac{2}{K-1}.$$

For definiteness, assume $\lambda \in \Delta^+_\pi$. (the proof is completely identical in the other case). Denote $w_1(n) = (a, n, a^{-n}\pi)$. Then $\mathcal{G}$-preimages of (λ, π) are $(\lambda^{(n)}, \pi^{(n)}) = t_{a^{-n}}(\lambda, \pi) = t_{w_1(n)}(\lambda, \pi)$, $n = 1, 2, \ldots$.

Let l be the length of the a-cycle of π, that is, the smallest such number that $a^l\pi = \pi$.

There are two possibilities: either $\rho^+_\pi(\lambda) = \frac{1}{l_1(\lambda)l_2(\lambda)l_3(\lambda)l_4(\lambda)}$, where $l_i(\lambda) = \sum_{j=1}^m a_{ij}\lambda_j$ and $a_{ij} = 0$ or 1, or $\rho^+_\pi(\lambda) = \frac{1}{l_1(\lambda)\ldots l_4(\lambda)} + \frac{1}{m_1(\lambda)m_2(\lambda)m_3(\lambda)m_4(\lambda)}$, where l_i are as above and $m_i = \sum_{j=1}^m b_{ij}\lambda_j$, $b_{ij} = 0$ or 1.

From Proposition 6 we have

$$\sum_{n=kl+1}^{\infty} \mathbb{P}(w_1(n)|(\lambda, \pi)) = \frac{\rho^+_\pi(\lambda)}{\rho^+_\pi(\Lambda(kl))}.$$

Renumbering, if necessary, the linear forms l_i, we may assume that $a_{1,\pi^{-1}(4)} = 1$ (and, in the second case, that $b_{1,\pi^{-1}(4)} = 1$ also).

Since under the application of the a-operation the only length that changes is that of $I_{\pi^{-1}m}$, we have

$$|\Lambda^{(n)}| - 1 = \Lambda^{(n)}_{\pi^{-1}(4)} - \lambda_{\pi^{-1}(4)},$$

whence

$$\Lambda^{(n)}_{\pi^{-1}(4)} = \lambda_{\pi^{-1}(4)} + |\Lambda^{(n)}| - 1.$$

From here we have

$$\sum_{n=kl+1}^{\infty} \mathbb{P}(w_1(n)|(\lambda,\pi)) \leq \frac{1+\lambda_{\pi^{-1}(4)}}{\lambda_{\pi^{-1}(4)} + |\Lambda^{(n)}| - 1}.$$

Now let k be the smallest integer such that $|\Lambda^{(n)}| > K$. Let $s = [k/l]$. Then $\Lambda^{(sl)} > K - 1$, and

$$\mathbb{P}(|\Lambda^{(n)}| > K) \leq \sum_{n=sl+1}^{\infty} \mathbb{P}(w_1(n)|(\lambda,\pi)) \leq \frac{1+\lambda_{\pi^{-1}(4)}}{\lambda_{\pi^{-1}(4)} + |\Lambda^{(n)}| - 1} \leq \frac{2}{K-1}.$$

The lemma is proved.

Remark. This lemma is true in complete generality, i.e., for interval exchanges of any number of intervals. The proof is identical.

9.2 Exponential growth.

Lemma 11 *There exists N such that the following is true.*

For any point $\mathbf{x} \in \overline{\Delta}(4321)$, there exist $i_1, i_2 \in \{1,2,3,4\}$ such that

$$\Lambda(-N)_{i_1} + \Lambda(-N)_{i_2} \geq 2(\lambda(0)_{i_1} + \lambda(0)_{i_2})$$

One directly observes the following properties of the Rauzy graph of (4321).

Proposition 9 *Any simple cycle in the Rauzy graph of* (4321) *is either an a-cycle or a b-cycle of a permutation.*

To a word $w \in \mathcal{W}_{\mathcal{A}}$ naturally corresponds a path in the Rauzy graph. Denote this path by $\mathbf{p}(w)$. To a path $\mathbf{p}$ in the Rauzy graph, there naturally corresponds a renormalization matrix $A(\mathbf{p})$.

Proposition 10 *Let $\mathbf{p}$ be the a-cycle or the b-cycle of some permutation in $\mathcal{R}(4321)$. Then all diagonal entries of the matrix $A(\mathbf{p})$ are positive.*

Edges marked with a will be called a-edges, edges marked with b will be called b-edges.

Lemma 12 *Suppose $\mathbf{p}$ is a path in $\mathcal{R}(4321)$ that starts and ends at the same permutation. Then all diagonal entries of the matrix $A(\mathbf{p})$ are positive.*

Proof: For a-cycles and b-cycles this can be checked directly, and since they represent all simple loops in our Rauzy graph, the general statement follows.

A 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

will be called a hyperbolic block if $a, b, c, d > 0$, $a + b > 2$.

An $m \times m$ matrix will be said to have a diagonal hyperbolic block if one of its diagonal 2×2 minors is hyperbolic.

Lemma 13 *Let $\pi \in \mathcal{R}(4321)$, let A_1 be the matrix corresponding to the a-cycle of π, A_2 the matrix corresponding to the b-cycle of π. Then the minor of $A_1 A_2$ in positions $(4, \pi^{-1}(4))$ is hyperbolic.*

This is checked directly by considerations of cases.

Lemma 14 *Suppose $\mathbf{p}$ is a path in $\mathcal{R}(4321)$ starting at π and ending at π, and such that the edge $\pi \xrightarrow{a} a\pi$ appears in the path. Then all permutations of the a-cycle of π appear among the vertices of the path $\mathbf{p}$.*

Suppose $\mathbf{p}$ is a path in $\mathcal{R}(4321)$ starting at π and ending at π, and such that the edge $\pi \xrightarrow{b} b\pi$ appears in the path. Then all permutations of the b-cycle of π appear among the vertices of the path $\mathbf{p}$.

This is directly seen from the form of Rauzy graph of $\mathcal{R}(4321)$.

Lemma 15 *Suppose $\mathbf{p}$ is a path starting at π and ending at π, and such that both edges $\pi \xrightarrow{a} a\pi$ and $\pi \xrightarrow{b} b\pi$ appear in the path. Then the matrix $A(\mathbf{p})$ has a hyperbolic block.*

Now, in order to prove Lemma 11, it suffices to establish the following

Lemma 16 *Suppose $w \in \mathcal{W}_{\mathcal{A}}$, $|w| \geq 1000$. Then $A(w)$ has a hyperbolic diagonal block.*

Proof. Consider the path $\pi_1 \ldots \pi_2 \ldots\ldots \pi_n$, $n \geq 1000$. corresponding to w. In this path, there are at least 1000 switches from a-edges to b-edges.

We can find a permutation π and four numbers $k_1 < k_2 < k_3 < k_4$ such that $\pi_{k_1} = \pi_{k_2} = \pi_{k_3} = \pi_{k_4} = \pi$, that the edges of the path starting at π_{k_i} are all marked with a and, finally, that there are both a-edges and b-edges between π_{k_i} and $\pi_{k_{i+1}}$.

The first assumption implies, in particular, that all vertices and all edges of the a-cycle of π appear between π_{k_i} and $\pi_{k_{i+1}}$.

Now pick the first b-edge between π_{k_2} and π_{k_3}. Let this be the edge $\pi_{n_1} \xrightarrow{b} \pi_{n_1+1}$. Then π_{n_1} belongs to the a-cycle of π. But then, there is an appearance of the edge $\pi_{n_1} \xrightarrow{a} a\pi_{n_1}$ between $\pi_{k_1} = \pi$ and $\pi_{k_2} = \pi$— indeed, this is so because all vertices and all edges of the a-cycle of π must appear between π_{k_1} and π_{k_2}.

Similarly, the edge $\pi_{n_1} \xrightarrow{a} a\pi_{n_1}$ must appear between $\pi_{k_3} = \pi$ and $\pi_{k_4} = \pi$.
We have therefore found a subpath of our original path of the form

$$\pi' \xrightarrow{a} \ldots\ldots \pi' \xrightarrow{b} \ldots \pi'.$$

By the previous lemma, the matrix corresponding to such path has a hyperbolic block.

9.3 Proof of Lemma 6

The constant γ will be chosen to be smaller than $\frac{1}{1000}$.

If (λ, π) has two intervals larger than $\frac{1}{1000}$, then there is nothing to prove.

Suppose therefore that we have three intervals smaller than $\frac{1}{1000}$. These will be called "small intervals". The remaining interval is larger than $\frac{999}{1000}$. It will be referred to as the "large interval".

In the remainder of the proof, it will be convenient to adopt the Kerckhoff [20] convention of numbering the subintervals of our interval exchange.

Namely, as we apply the inverse Rauzy induction, if the operation b^{-1} is performed, then the numeration remains the same, from left to right, whereas if the operation a^{-1} is performed then the order of the subintervals in the preimage is cyclically shifted after the $\pi^{-1}4$-th interval. This way of numbering is convenient to relate the subintervals at step n and at step $n+1$ of the (inverse) Rauzy induction.

For definiteness, let I_1 be the large interval, I_2, I_3, I_4 be the small intervals.

We shall say that a subinterval is *in critical position* at step n if it is the one whose length changes at step n.

Proposition 11 *If I_j is in critical position at step n then it was added to some interval at step $n-1$.*

This follows directly from the definition of the Rauzy induction.

Now consider two cases:

1. I_1 is in critical position at step 0.
2. A small interval is at critical position at step 0.

Start with the first case.

By Lemma 10, there exists numbers L, q that $\Lambda(1) < L$ with probability at least q. Now let n be the first moment of time at which $\Lambda(n) \geq 2L$. By Lemma 10, we can assume $\Lambda(n) < 2L^2$ (this occurs with probability at least q) . By the Lemma 11, $n < C(\log L + |\log \epsilon|)$.

Consider two cases:

1. There exists l, $1 < l \leq n$, such that I_1 is in critical position at time l
2. I_1 is never in critical position at times $1, \ldots, n$.

In the first case, if I_1 is in critical position at time l, then it was added to some interval at time $l-1$, say, to I_2. But then, since $|I_1(1)| < |\Lambda(l)| < 2L$ and $|I_1| > \frac{1}{2}$, we have

$$\frac{|I_1(l)|}{\Lambda(l)} > \frac{1}{4L}, \frac{|I_2(l)|}{\Lambda(l)} > \frac{1}{4L},$$

and the Lemma is proved in this case.

In the second case, $|I_1(n)| = |I_1(1)| < L$. Then $|I_2(n)| + |I_3(n)| + |I_4(n)| \geq L$, and therefore, $\max |I_2(n)|, |I_3(n)|, |I_4(n)| \geq L/3$. Assume the maximum is achieved at I_2.

Then, since $|I_1(n)| = |I_1(1)| > 1/2$, we have again

$$\frac{|I_1(n)|}{\Lambda(n)} > \frac{1}{4L^2}, \frac{|I_2(n)|}{\Lambda(n)} > \frac{1}{4L^2},$$

and the Lemma is proved in this case also.

Now consider the second case, namely, when I_1 is not in critical position at step 0, so $|I_1(1)| = |I_1(0)|$.

Let n be the first moment at which $\Lambda(n)| > 2$. Just as in the previous case, by Lemma 10, we can assume that $|\Lambda(n)| < 2L$.

By Lemma 11, $n < C|\log \epsilon|$.

Consider two cases:

1. There exists l, $1 < l \leq n$, such that I_1 is in critical position at time l
2. I_1 is never in critical position at times $1, \ldots, n$.

In the first case, if I_1 is for the first time in critical position at time l, then it was added to some interval at time $l-1$, say, to I_2. But then, since $|I_1(l-1)| = |I_1(0)|$ and $|I_1| > \frac{1}{2}$, we have

$$\frac{|I_1(l)|}{\Lambda(l)} > \frac{1}{4}, \frac{|I_2(l)|}{\Lambda(l)} > \frac{1}{4},$$

and the Lemma is proved in this case.

In the second case, $|I_1(n)| = |I_1(1)|$ and therefore $1 > |I_1(n)| > \frac{1}{2}$. Then $|I_2(n)| + |I_3(n)| + |I_4(n)| \geq 1$, and therefore, $\max |I_2(n)|, |I_3(n)|, |I_4(n)| \geq 1/3$. Assume the maximum is achieved at I_2. We have then

$$\frac{|I_1(n)|}{\Lambda(n)} > \frac{1}{12L}, \frac{|I_2(n)|}{\Lambda(n)} > \frac{1}{12L},$$

and the Lemma is proved in this case also.

9.4 Estimate of the measure.

Lemma 17 *There exists a constant C such that*

$$\nu(\Delta \setminus \Delta_\epsilon) < C\epsilon$$

The proof repeats that of Proposition 13.2 in Veech [1].

Lemma 4 and Corollary 6 therefore imply the following

Corollary 8 *Let* $\mathbf{q} \in W_{\mathcal{A},B}$*,* $\mathbf{q} = q_1 \dots q_l$ *be such that all entries of the matrix* $A(\mathbf{q})$ *are positive.*

There exist $C > 0, \alpha > 0$ *such that the following is true for any* n*.*

$$\mathbb{P}((\lambda,\pi) : \mathcal{G}^{2k}(\lambda,\pi) \notin \Delta(\mathbf{q}) \text{ for all } \ k, 1 \le k \le n) \le C\exp(-\alpha\sqrt{n}).$$

Proof: Let $n = r^2$ and denote

$$X(n,\mathbf{q}) = \{(\lambda,\pi) : \mathcal{G}^{2k}(\lambda,\pi) \notin \Delta(\mathbf{q}) \text{ for all } \ k, 1 \le k \le n)\}.$$

Take

$$B(n) = \{(\lambda,\pi) : \mathcal{G}^{2k}(\lambda,\pi) \notin \Delta_{\exp(-r)} \text{ for some } \ k, 1 \le k \le n)\}$$

Then, by the previous Lemma, $\nu(B(n)) \le Cr^2\exp(-r)$, whereas, by Corollary 6,

$$\nu(X(n,\mathbf{q}) \setminus B(n)) \le p(\mathbf{q})^r,$$

and Corollary 8 is proven.

Remark. This result allows to use the methods of L.-S. Young [11] and of V.Maume-Deschmaps [12] to obtain better decay rates for bounded Lipschitz and Hoelder functions.

10 Inequalities

Let

$$W^+_{\mathcal{A},B} = \{w \in W_{\mathcal{A},B} : \ |w| \text{ is even }, \ \Delta(w) \subset \Delta^+\}.$$

Lemma 18 *For any* $C_1, C_2 > 0$ *there exists* $C_3 > 0$ *such that the following is true.*

Suppose $row(A) < C_1$ *and* $\lambda \in \Delta_{C_2}$*.*

Then

$$\frac{1}{C_3} \le \frac{|A\lambda|^m}{\Pi_{j=1}^m \sum_{i=1}^m A_{ij}} \le C_3$$

Proof:

Denote $A_j = \sum_{i=1}^m A_{ij}$, so that $|A| = \sum_{j=1}^m A_j$.

Then

$$\frac{A_j}{A_k} \le row(A),$$

whence

$$\frac{A_j}{|A|} \ge \frac{1}{m \ row(A)}.$$

Finally, if $\lambda \in \Delta_{C_2}$, then

$$|A\lambda| \geq C_2|A|,$$

which completes the proof.

Corollary 9 *For any $C_4 > 0$, $C_5 > 0$ there exists $C_6 > 0$ such that the following is true. Suppose $(\lambda, \pi) \in \Delta_{C_4}$. Suppose $w \in \mathcal{W}_{\mathcal{A},B}$ is compatible with (λ, π) and such that $row(A(w)) < C_5$. Then*

$$\frac{1}{C_6} \leq \frac{\mathbf{m}(C(w))}{\mathbb{P}(w|(\lambda, \pi))} \leq C_6$$

Corollary 10 *For any $C_7 > 0$, $C_8 > 0$ $C_9 > 0$, there exists $C_{10} > 0$ such that the following is true.*

Suppose $(\lambda, \pi) \in \Delta_{C_7}$.

Suppose $w \in \mathcal{W}_{\mathcal{A},B}$ is compatible with (λ, π) and furthermore satisfies

$$row(A(w)) < C_8,\ \ \Delta(w) \subset \Delta_{C_9}$$

Then

$$\frac{1}{C_{10}} \leq \frac{\mathbb{P}(C(w))}{\mathbb{P}(w|(\lambda, \pi))} \leq C_{10}$$

Corollary 11 *Let M be such that for any $n > M$ any two vertices in the Rauzy graph can be joined in n steps.*

Then for any $C_{17} > 0$, $C_{18} > 0$ $C_{19} > 0$, there exists $C_{20} > 0$ such that the following is true.

Suppose $(\lambda, \pi) \in \Delta^+ \cap \Delta_{C_{17}}$.

Suppose $w \in W^+_{\mathcal{A},B}$ satisfies

$$row(A(w)) < C_{18},\ \ \Delta(w) \subset \Delta^+ \cap \Delta_{C_{19}}$$

Then for any $n \geq M$, we have

$$\frac{1}{C_{20}} \leq \frac{\mathbb{P}(C(w))}{\mathbb{P}^{(2n)}(w|(\lambda, \pi))} \leq C_{20}$$

From the definition (3) of the Birkhoff metric it easily follows that for any $\lambda, \lambda' \in \Delta_{m-1}$ we have

$$e^{-d(\lambda,\lambda')}\lambda'_i \leq \lambda_i \leq e^{d(\lambda,\lambda')}\lambda'_i. \qquad (27)$$

Proposition 12 *Assume $\lambda, \lambda' \in \Delta^+_\pi$. Then*

$$\exp(-md(\lambda, \lambda')) \leq \frac{\rho(\lambda, \pi)}{\rho(\lambda', \pi)} \leq \exp(md(\lambda, \lambda'))$$

Proof. Indeed, there exist linear forms

$$l_i^{(j)}(\lambda) = \sum_{k=1}^{m} a_{ik}^{(j)} \lambda_k,$$

where $a_{ik}^{(j)}$ are nonnegative integers (in fact, either 0 or 1, but we do not need this here),

such that

$$\rho(\lambda, \pi) = \sum_{j=1}^{s} \frac{1}{l_1^{(j)}(\lambda) l_2^{(j)}(\lambda) \dots l_m^{(j)}(\lambda)}.$$

Clearly, if for all $i = 1, \dots, m$ and some $\alpha > 0$, we have $\alpha^{-1}\lambda_i \le \lambda_i' \le \alpha\lambda_i$, then

$$\alpha^{-m} \le \frac{\rho(\lambda, \pi)}{\rho(\lambda', \pi)} \le \alpha^m,$$

and the Proposition is proved.

For similar reasons we have

Proposition 13 *Assume $\lambda, \lambda' \in \Delta_\pi^+$ and let A be an arbitrary matrix with nonnegative integer entries. Then*

$$\exp(-md(\lambda, \lambda')) \le \frac{\rho(A\lambda, \pi)}{\rho(A\lambda', \pi)} \le \exp(md(\lambda, \lambda'))$$

From these propositions and the formula 7 we obtain

Corollary 12 *Let $c \in \mathcal{A}$ be compatible with π. Then for any $\lambda, \lambda' \in \Delta_\pi^+$ we have*

$$\exp(-2md(\lambda, \lambda')) \le \frac{\mathbb{P}(c|(\lambda, \pi))}{\mathbb{P}(c|(\lambda', \pi))} \le \exp(2md(\lambda, \lambda'))$$

This Corollary implies the following

Lemma 19 *Let $w \in W_{\mathcal{A},B}^+$ be such that the cylinder $C(w)$ has finite Birkhoff diameter.*

Then for any c compatible with w and any $(\lambda_0, \pi) \in C(w)$ we have

$$\exp(-2m \; diamC(w)) \le \frac{\mathbb{P}(c|(\lambda_0, \pi))}{\mathbb{P}(\omega_0 = c|\omega|_{[1,|w|]} = w)} \le \exp(2m \; diamC(w))$$

Proof: We have

$$\nu(C(cw)) = \int_{C(w)} \mathbb{P}(c|(\lambda, \pi)) d\nu(\lambda, \pi)$$

Let $d = diamC(w)$. For any $(\lambda,\pi), (\lambda',\pi) \in C(w)$, we have, by Corollary 12,

$$\exp(-2md) \leq \frac{\mathbb{P}(c|(\lambda,\pi))}{\mathbb{P}(c|(\lambda',\pi))} \leq \exp(2md).$$

Fix an arbitrary $(\lambda_0,\pi) \in \Delta_w$.

Then, from the above,

$$\nu(C(w))P(c|(\lambda_0,\pi))\exp(-2md) \leq \int_{C(w)} P(c|(\lambda,\pi))d\nu(\lambda,\pi) \leq$$
$$\leq \nu(C(w))P(c|(\lambda_0,\pi))\exp(2md),$$

and, since, by definition, we have

$$\mathbb{P}(\omega_0 = c|\omega|_{[1,|w|]} = w) = \frac{\mathbb{P}(cw)}{\mathbb{P}(w)},$$

the Lemma is proved.

11 Good Cylinders

Let M be a number such that for any $N \geq M$ any two vertices of the Rauzy graph can be connected in N steps.

Lemma 20 *For any $\gamma > 0$, $N \geq M$ there exists a constant C_0 depending only on γ and N such that for any word $w \in \mathcal{W}^+_{\mathcal{A},B}$ and any $(\lambda,\pi) \in \Delta_\gamma$*

$$\mathbb{P}^{(2N)}(w|(\lambda,\pi)) \geq \frac{C_0}{|A(w)\lambda|^m}$$

Proof:

Let $w = w_1 \dots w_{2n}$, and let $w_{2n} = (a, m_1, \pi_1)$.

Let $\pi'_1\pi'_2 \dots \pi'_{2N}$ a path of length $2N$ between π and π_1 (here $\pi'_1 = \pi$, $\pi'_{2n} = \pi_1$, $\pi_{2k+1} = a\pi_{2k}$, $\pi_{2k+2} = b\pi_{2k+1}$.

Denote $w_{n+2i+1} = (a, 1, \pi_{2i+1})$, $w_{n+2i} = (b, 1, \pi_{2i})$. In other words, the word $= w_{2n+1} \dots w_{2n+2N} \in \mathcal{W}_{\mathcal{A},B}$ is the word correspoding to the path $\pi'_1\pi'_2 \dots \pi'_{2N}$ in the Rauzy graph. Then $w' = w_1 \dots w_{2n+2N}$ is a word compatible with (λ,π). Besides,

$$|A(w_{2n+1}c_{n+2} \dots w_{2n+2N})| < (2N)^{(2N)}.$$

We have

$$P^{(2n)}(w|(\lambda,\pi)) \geq P(w'|(\lambda,\pi)) = \frac{\rho(T_{w'}(\lambda), w'\pi)}{|A(w')\lambda|^m \rho(\lambda,\pi)},$$

There exists a universal constant C_1 such that $\rho(\lambda',\pi') > C_1$ for any $(\lambda',\pi') \in \Delta^+$ (the density of the invariant measure is bounded from below).

Then, $|A(w')\lambda|^m \leq |A(w')|^m \leq (2N)^{2mN}|A(w)|^m$.

Finally, there exists a C_2 depending on c only such that if $\lambda_i > c$ for all i then $\rho(\lambda,\pi) > C_2$.

Combining all of the above, we obtain the result of the Lemma.

11.1 Definition of good cylinders

Let $\mathbf{q} = q_1 \dots q_l$ be a word such that all entries of the matrix $A(\mathbf{q})$ are positive. Fix $\epsilon > 0$ and let et k_0 be such that

$$\mathbb{P}(C(\mathbf{q}) \cap \mathcal{G}^{-2n} C(\mathbf{q})) \geq \epsilon \text{ for } n > k_0. \tag{28}$$

Take $k \geq k_0$. Let $r = 2(K+1)k + 2M$, where K is the constant from the Lemma 4.

Let θ, $0 < \theta < 1$ be arbitrary.

A word $w = w_1 \dots w_k$ is called *good* if

1. $C(w) \subset \Delta_{\exp(-k)}$.
2. the word $\mathbf{q}$ appears at least $\frac{k^\theta}{l}$ times in w (we only count disjoint appearances).

A word $w_1 \dots w_r$ is called good if $w_1 \dots w_k$ is good, a word $w_1 \dots w_{Nr}$ is called good if all words $w_1 \dots w_r$, $w_{r+1} \dots w_{2r}$, $\dots w_{(N-1)r+1} \dots w_{Nr}$ are good, and a word $w_1 \dots w_{Nr+L}$, $L < r$, is good if $w_1 \dots w_{Nr}$ is good and either $L < k$ or $w_{Nr+1} \dots w_{Nr+k}$ is good.

We denote by $\mathbf{G}(N)$ the set of all good words of length N.

Let

$$\Delta(G(N)) = \cup_{w \in \mathbf{G}(N)} C(w),$$

and

$$\Delta(B(N)) = \Delta^+ \setminus \Delta(G(N))$$

By Corollary 8, there exist constants C_{31}, C_{32} such that for all r we have

$$\mathbb{P}(\Delta(B(N)) \leq C_{31} N \exp(-C_{32} r^{(1-\theta)/2}). \tag{29}$$

From Corollary 12 we deduce that there exists a constant C_{33} such that for any $(\lambda, \pi), (\lambda', \pi) \in \Delta_{\mathbf{q}}$, and any word w compatible with $\mathbf{q}$, we have

$$\frac{1}{C_{33}} \leq \frac{\mathbb{P}(w|(\lambda,\pi))}{\mathbb{P}(w|(\lambda',\pi))} \leq C_{33}.$$

Finally, by Lemma 20, there exists a constant C_{34} such that for any $w \in \mathcal{W}_{\mathcal{A},B}$ and for any $N > M$ we have

$$\frac{1}{C_{34}} \leq \frac{\mathbb{P}^{(2N)}(w|(\lambda,\pi))}{\mathbb{P}^{(2N)}(w|(\lambda',\pi))} \leq C_{34}.$$

Take an arbitrary point $(\lambda, \pi) \in \Delta_{\mathbf{q}}$. Define a new measure φ on Δ^+. Namely, for a set $A \subset \Delta^+$ put

$$\varphi(A) = \mathbb{P}(\lambda(-2M), \pi(-2M)) \in A | \lambda(0), \pi(0) = (\lambda, \pi)) \tag{30}$$

Lemma 21 *There exists a constant $\alpha > 0$ such that the following is true for any r. Let $\mathcal{C}_1, \mathcal{C}_2 \in \mathbf{G}(r)$.*

Then

$$\mathbb{P}(\omega|_{[1,r]} = \mathcal{C}_1, \omega|_{[r+1,2r]} \in \mathbf{G}(r)\ |\omega|_{[2r+1,3r]} = \mathcal{C}_2) \geq \alpha\varphi(\mathcal{C}_1)$$

Indeed, we have the following propositions:

Proposition 14 *There exist a constant p_1 such that the following is true for all r and all $n \geq r$.*

Let $C_2 \in \mathbf{G}(r)$, $(\lambda, \pi) \in \mathcal{C}_2$. Then

$$\mathbb{P}((\lambda(-2n), \pi(-2n)) \in \Delta_{\mathbf{q}}|(\lambda(0), \pi(0)) = (\lambda, \pi)) \geq p_1.$$

Proposition 15 *There exists a constant p_2 such that the following is true for all k.*

$$\mathbb{P}(\omega|_{[1,r]} \in \mathbf{G}(r), \omega|_{[2M+1,l+2M+1]} = \mathbf{q}\ |\omega|_{[r+1,r+l+1]} = \mathbf{q}) \geq p_2$$

Proposition 16 *There exists a constant p_3 such that the following is true for all r. Let $c_1 \ldots c_n \dots \in C(\mathbf{q})$.*

$$\mathbb{P}(\omega|_{[1,r]} = \mathcal{C}_1|\omega_{r+2M+1} = c_1, \omega_{r+2M+2} = c_2, \ldots) \geq p_3\varphi(C_1)$$

These Propositions imply the Lemma.

12 Markov Approximation and the Doeblin Condition

We define a new measure $\mathbf{p}_{r,\theta}$ on the set $\mathbf{G}(r^2)$ of good cylinders of length r^2.

Let $\mathcal{C} = c_1 \ldots c_{r^2}$ be a (r, θ)-good cylinder. Set $\mathcal{C}_i = c_{ir+1} \ldots c_{(i+1)r}$.

Define

$$\mathbf{p}_{r,\theta}(\mathbf{C}) = \mathbb{P}(\omega|_{[1,r]} = \mathcal{C}_1|\omega|_{[r+1,2r]} = \mathcal{C}_2)\mathbb{P}(\omega|_{[r+1,2r]} = \mathcal{C}_2|\omega|_{[2r+1,3r]} = \mathcal{C}_3) \ldots \mathbb{P}(\omega|_{[r^2-r+1,r^2]} = \mathcal{C}_r).$$

If D is not a good cylinder, then $\mathbf{p}_{r,\theta}(D) = 0$.

Normalize to get a probability measure:

$$\mathbf{P}_{r,\theta}(\mathcal{C}) = \frac{\mathbf{p}_{r,\theta}(\mathcal{C})}{\sum_{\mathcal{D} \in \mathbf{G}(r^2)} \mathbf{p}_{r,\theta}(\mathcal{D})}.$$

$\mathbf{P}_{r,\theta}$ is a Markov measure of memory r (in general, non-homogeneous), as is shown by the following well-known Lemma [14].

Lemma 22 *For any k, $0 < k < r$, we have*

$$\mathbf{P}_{r,\theta}(\omega|_{[kr+1,(k+1)r]} = \mathcal{C}_k|\omega|_{[(k+1)r+1,r^2]}) = \mathcal{C}_{k+1} \ldots \mathcal{C}_r) =$$

$$\mathbf{P}_{r,\theta}(\omega|_{[kr+1,(k+1)r]} = \mathcal{C}_k|\omega|_{[(k+1)r+1,(k+2)r]}) = \mathcal{C}_{k+1}).$$

13 The Doeblin Condition

In this section, we establish the Doeblin Condition for the measure $\mathbf{P}_{r,\theta}$.

Proposition 17 *There exist constants C_{41}, C_{42} such that the following is true for any r.*

Let $c_1 \dots c_n \dots \in \Omega_{\mathcal{A},B}$ and assume $c_{n+1} \dots c_{n+r} \in \mathbf{G}(r)$. Then

$$\exp(-C_{41}\exp(-C_{42}k^\theta)) \leq$$
$$\leq \frac{P(\omega_1 = c_1, \dots, \omega_n = c_n | \omega_{n+1} = c_{n+1}, \dots, \omega_{n+r} = c_{n+r})}{\mathbb{P}(\omega_1 = c_1, \dots, \omega_n = c_n | \omega_{n+1} = c_{n+1}, \dots, \omega_{n+i} = c_{n+i}, \dots)} \leq$$
$$\leq \exp(C_{41}\exp(-C_{42}k^\theta))$$

Corollary 13 *There exist constants C_{43}, C_{44} such that the following is true for any r. Let $A \in \mathcal{F}_n$, let $c_{n+1} \dots c_{n+i} \dots \in \Omega_{\mathcal{A}}$, and assume $c_{n+1} \dots c_{n+r} \in \mathbf{G}(r)$. Then*

$$\exp(-C_{43}\exp(-C_{44}k^\theta)) \leq \frac{\mathbb{P}(A|\omega_{n+1} = c_{n+1}, \dots, \omega_{n+r} = c_{n+r})}{\mathbb{P}(A|\omega_{n+1} = c_{n+1}, \dots, \omega_{n+i} = c_{n+i}, \dots} \leq \exp(C_{43}\exp(-C_{44}k^\theta))$$

Lemma 23 *There exist constants $C_{45}, C_{46}, C_{47}, C_{48}$ such that the following is true for any r. Let $c_1 \dots c_{r^2} \in \mathbf{G}(r^2)$. Then for any l, $1 \leq l \leq r$, we have*

$$\exp(-C_{45}l\exp(-C_{46}k^\theta)) \leq$$
$$\leq \frac{\mathbb{P}(\omega_1 = c_1, \dots, \omega_{lr} = c_{lr} | \omega_{lr+1} = c_{lr+1}, \dots, \omega_{r^2} = c_{r^2})}{\mathbf{p}_{r,\theta}(\omega_1 = c_1, \dots, \omega_{lr} = c_{lr} | \omega_{lr+1} = c_{lr+1}, \dots, \omega_{r^2} = c_{r^2})} \leq$$
$$\leq \exp(C_{45}l\exp(-C_{46}k^\theta))$$

and

$$\exp(-C_{47}l\exp(-C_{48}k^\theta)) \leq \frac{\mathbb{P}(\omega_1 = c_1, \dots, \omega_{lr} = c_{lr})}{\mathbf{p}_{r,\theta}(\omega_1 = c_1, \dots, \omega_{lr} = c_{lr})} \leq \exp(C_{47}l\exp(-C_{48}k^\theta))$$

Corollary 14 *There exist constants C_{49}, C_{50} such that the following is true for any r. Let $c_1 \dots c_{r^2} \in \mathbf{G}(r^2)$. Then for any l, $1 \leq l \leq r$, and any $A \in \mathcal{F}_{lr}$, we have*

$$\exp(-C_{49}l\exp(-C_{50}k^\theta)) \leq \frac{\mathbb{P}(A \cap \mathbf{G}(lr)|\omega_{lr+1} = c_{lr+1}, \dots, \omega_{r^2} = c_{r^2})}{\mathbf{p}_{r,\theta}(A|\omega_{lr+1} = c_{lr+1}, \dots, \omega_{r^2} = c_{r^2})} \leq \exp(C_{49}l\exp(-C_{50}k^\theta))$$

and

$$\exp(-C_{49}l\exp(-C_{50}k^\theta)) \leq \frac{\mathbb{P}(A \cap \mathbf{G}(lr))}{\mathbf{p}_{r,\theta}(A)} \leq \exp(C_{49}l\exp(-C_{50}k^\theta))$$

Using (29), we obtain

Corollary 15 *There exist constants C_{51}, C_{52} such that the following is true for any r.*

$$\mathbf{p}_{r,\theta}(\mathbf{G}(r^2)) \geq \exp(-C_{51} r \exp(-C_{52} k^{(1-\theta)/2}))$$

Corollary 16 *There exist constants $C_{53}, C_{54}, C_{55}, C_{56}$ such that the following is true for any r. Let $c_1 \dots c_{r^2} \in \mathbf{G}(r^2)$. Then for any l, $1 \leq l \leq r$, and any $A \in \mathcal{F}_{lr}$, we have*

$$\exp(-C_{53} l \exp(-C_{54} k^\theta) - C_{55} r \exp(-C_{56} k^{(1-\theta)/2})) \leq$$
$$\leq \frac{\mathbb{P}(A \cap \mathbf{G}(lr) | \omega_{lr+1} = c_{lr+1}, \dots, \omega_{r^2} = c_{r^2})}{\mathbf{P}_{r,\theta}(A | \omega_{lr+1} = c_{lr+1}, \dots, \omega_{r^2} = c_{r^2})} \leq$$
$$\leq \exp(C_{53} l \exp(-C_{54} k^\theta) + C_{55} r \exp(-C_{56} k^{(1-\theta)/2})$$

and

$$\exp(-C_{53} l \exp(-C_{54} k^\theta) - C_{55} r \exp(-C_{56} k^{(1-\theta)/2})) \leq$$
$$\leq \frac{\mathbb{P}(A \cap \mathbf{G}(lr))}{\mathbf{P}_{r,\theta}(A)} \leq$$
$$\leq \exp(C_{53} l \exp(-C_{54} k^\theta) + C_{55} r \exp(-C_{56} k^{(1-\theta)/2}).$$

Corollary 17 *There exist constants $C_{57}, C_{58}, C_{59}, C_{60}$ such that the following is true for any r. Let $c_1 \dots c_{r^2} \in \mathbf{G}(r^2)$. Then for any l, $1 \leq l \leq r$, we have*

$$\mathbb{P}((\omega_1 \dots \omega_{lr}) \in \mathbf{G}(lr) | \omega_{lr+1} = c_{lr+1}, \dots, \omega_{r^2} = c_{r^2}) \geq \exp(-C_{57} l \exp(-C_{58} k^\theta) - C_{59} r \exp(-C_{60} k^{(1-\theta)/2}))$$

Proof: Indeed,

$$\mathbf{P}_{r,\theta}((\omega_1 \dots \omega_{lr}) \in \mathbf{G}(lr) | \omega_{lr+1} = c_{lr+1}, \dots, \omega_{r^2} = c_{r^2}) = 1.$$

Now let $c_1 \dots c_{r^2} \in \mathbf{G}(r^2)$. Denote $\mathcal{C}_i = c_{ir+1} \dots c_{(i+1)r}$.
Lemma 21 implies the following

Corollary 18 *There exist constants C_{61}, C_{62} such that the following is true For any l, $1 \leq l \leq r$, we have*

$$\mathbb{P}(\omega|_{[1,lr]} \in \mathbf{G}(lr), \omega|_{[lr+1,(l+1)r]} = \mathcal{C}_l, \omega|_{[(l+1)r+1,(l+2)r]} \in \mathbf{G}(r) | \omega_{(l+2)r+1,(l+3)r])} = \mathcal{C}_3) \geq C_{61} \times \varphi(\mathcal{C}_l)$$

and

$$\mathbf{P}_{r,\theta}(\omega|_{[1,lr]} \in \mathbf{G}(lr), \omega|_{[lr+1,(l+1)r]} = \mathcal{C}_l, \omega|_{[(l+1)r+1,(l+2)r]} \in \mathbf{G}(r) | \omega_{(l+2)r+1,(l+3)r])} = \mathcal{C}_3) \geq C_{62} \times \varphi(\mathcal{C}_l)$$

This is the Doeblin Condition for the measure $\mathbf{P}_{r,\theta}$ (see [13], [14], [22]). The Doeblin Condition implies that there exist constants C_{63}, C_{64} such that for any $\mathcal{C}_1, \mathcal{C}_2 \in \mathbf{G}(r)$, we have

$$\exp(-C_{63}\exp(-C_{64}r)) \le \frac{\mathbf{P}_{r,\theta}(\omega|_{[1,r]} = \mathcal{C}_1 | \omega|_{[r^2,r^2+r]} = \mathcal{C}_2)}{\mathbf{P}_{r,\theta}(\mathcal{C}_1)} \le \exp(C_{63}\exp(-C_{64}r)),$$

whence

Proposition 18 *There exist constants $C_{71}, C_{72}, C_{73}, C_{74}$ such that the following is true for any r.*

$$\exp(-C_{71}\exp(-C_{72}r - C_{73}r^{\theta} - C_{74}r^{(1-\theta)/2})) \le$$

$$\le \frac{\mathbb{P}(\omega|_{[1,r]} = \mathcal{C}_1 | \omega|_{[r+1,r^2]} \in \mathbf{G}(r^2 - r), \omega|_{[r^2,r^2+r]} = \mathcal{C}_2)}{\mathbb{P}(\mathcal{C}_1)} \le$$

$$\le \exp(C_{71}\exp(-C_{72}r - C_{73}r^{\theta} - C_{74}r^{(1-\theta)/2})).$$

Moreover, in view of Proposition 14, the same estimate, upto a constant, takes place for any $n \ge r^2$.

Proposition 19 *There exist constants $C_{75}, C_{76}, C_{77}, C_{78}$ such that the following is true for all r and all $n \ge r^2$.*

$$\exp(-C_{75}\exp(-C_{76}r - C_{77}r^{\theta} - C_{78}r^{(1-\theta)/2})) \le$$

$$\le \frac{\mathbb{P}(\omega|_{[1,r]} = \mathcal{C}_1 | \omega|_{[r+1,n]} \in \mathbf{G}(n - r), \omega|_{[n,n+r]} = \mathcal{C}_2)}{\mathbb{P}(\mathcal{C}_1)} \le$$

$$\le \exp(C_{75}\exp(-C_{76}r - C_{77}r^{\theta} - C_{78}r^{(1-\theta)/2})).$$

14 Approximation of Hoelder Functions

Proposition 20 *Let $\phi \in H(\alpha)$, $\phi \ge 1$. Then*

$$\frac{\phi(x)}{\phi(y)} \le 1 + Cd(x,y)^{\alpha}$$

Proof: Follows from

$$\frac{\phi(x)}{\phi(y)} \le 1 + |\phi(x) - \phi(y)|$$

Recall that $\mathcal{F}_n$ is the σ-algebra of sets of the form $\mathcal{G}^{-2n}(A)$, $A \subset \Delta$.

To prove the Theorem 4, we need to estimate the L_2-norm of $E(\phi|\mathcal{F}_n)$ for $\phi \in H(\alpha)$.

Proposition 21 *Let $\theta \in \mathbb{R}$, $0 < \theta < 1$. There exist constants C_{81}, C_{82} such that the following is true for any r and any $n \geq r^2$.*

Let $\phi \in H(\alpha) \cap L_4(\Delta(4321)^+, \nu)$ satisfy $\phi \geq 1$.

Then $\phi = \phi_1 + \phi_2$ where

1. *$\phi_1 \geq 1$ where $\phi_1 \neq 0$.*
2. *$|\frac{E(\phi_1|\mathcal{F}_n)}{E(\phi_1)} - 1| \leq \exp(-C_{81}(r^{(1-\theta)/2} + r^{\theta})$.*
3. *$||\phi||_{L_2} \leq \exp(-C_{82} r^{(1-\theta)/2})$.*

Proof: For any good word $w = w_1 \dots w_r$, choose a point $x_{w_1 \dots w_r} \in C(w)$.

Now consider a function

$$\tilde{\phi} = \sum_{w \in G(n)} \phi(x_{w_1 \dots w_r}) \chi_{C(w)}.$$

(here $\chi_{C(w)}$ stands for the characteristic function of $C(w)$).

Note that if $\omega|_{[1,r]} \in \mathbf{G}(r)$, then there exist constants C_{91}, C_{92} such that

$$|\frac{\tilde{\phi}(\omega)}{\phi(\omega)} - 1| < C_{91} \exp(-C_{92} r)$$

Let $\omega \in \Omega_{\mathcal{A},B}$ be such that $\omega|_{[n-r,n]} \in \mathbf{G}(r)$.

Then, from Lemma 18, and because of the Hoelder property of ϕ, we clearly have

$$|\frac{E(\tilde{\phi}|\mathcal{F}_n)}{E(\tilde{\phi})} - 1| < C_{93} |\phi|_{H(\alpha)} \exp(-C_{94}(r^{\theta} + r^{(1-\theta)/2}))$$

Finally, to estimate the input of bad cylinders, observe that

$$E(|\phi \chi_{\Delta(B(n))}|^2) \leq |\phi|_{L_4} \mathbb{P}(\Delta(B(n)),$$

and because of the estimate (29), the Proposition is proved.

Proposition 21 with $\theta = 1/3$ yields Theorem 4.

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