ESI The Erwin Schrödinger International Institute for Mathematical Physics

Boltzmanngasse 9
A-1090 Wien, Austria

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K. Khanin
D. Khmelёv
A. Sobolevskiĭ

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A blow-up Phenomenon in the Hamilton–Jacobi Equation in an Unbounded Domain

K. Khanin, D. Khmelëv, and A. Sobolevskiĭ

Abstract. We construct an example of blow-up in a flow of min-plus linear operators arising as solution operators for a Hamilton–Jacobi equation $\partial S/\partial t + |\nabla S|^\alpha/\alpha + U(x,t) = 0$, where $\alpha > 1$ and the potential $U(x,t)$ is uniformly bounded together with its gradient. The construction is based on the fact that the absolute value of velocity for a Lagrangian minimizer on a time interval of length T is bounded by $O\left((\log T)^{2-2/\alpha}\right)$, and that this estimate is asymptotically sharp. Implications of this example for existence of global generalized solutions to randomly forced Hamilton–Jacobi or Burgers equations is discussed.

1. Introduction

In this paper we present an example of blow-up in a flow of min-plus linear integral operators arising as solution operators for a class of Hamilton–Jacobi equations. As we shall see, existence of such blow-up has interesting consequences for the application of idempotent functional analysis to stochastic partial differential equations.

1.1. Consider the inviscid Burgers equation in the d-dimensional space

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla U(x,t), \tag{1.1}$$

where $u(x,t) = (u_1(x,t), u_2(x,t), \dots, u_d(x,t))$ is a potential velocity field, so that $u(x,t) = \nabla S(x,t)$. The potential $S(x,t)$ must satisfy the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left|\nabla S\right|^2 + U(x,t) = 0. \tag{1.2}$$

Here and below, ∇ denotes the vector of derivatives with respect to components of the vector $x \in \mathbf{R}^d$.

It is well-known that the Cauchy problems for nonlinear equations (1.1) and (1.2) fail to have global in time classical solutions: they develop infinite velocity gradients in finite time. There exist several ways to extend solutions beyond formation of such singularities in a suitable generalized sense, allowing for discontinuities of velocities [**Hop50, Lio82, CL83, Sub95, KM97**]. Under an additional stability

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hypothesis (the entropy condition), all of them become essentially equivalent (see, e.g., the paper [**Rou**] in the present volume), and the corresponding solutions admit an explicit representation in terms of the Lax–Oleĭnik variational principle.

Namely, a generalized solution to a Cauchy problem for the Hamilton–Jacobi equation (1.2) with the initial condition $S(x,0) = S_0(x)$ has the form

$$S(x,t) = \inf_{\gamma(t)=x} \big(A_{0,t}[\gamma] + S_0(\gamma(0))\big), \tag{1.3}$$

where the action functional $A_{\cdot,\cdot}[\cdot]$ is given by

$$A_{t_1,t_2}[\gamma] \equiv \int_{t_1}^{t_2} L(\dot\gamma(s), \gamma(s), s)\, ds \tag{1.4}$$

for any t_1 and t_2 with $t_1 < t_2$, the Lagrangian has the form $L(v,x,t) = |v|^2/2 - U(x,t)$, and the infimum is taken over all absolutely continuous trajectories $\gamma(\cdot)$ defined over $[0,t]$ and satisfying $\gamma(t) = x$. Define further

$$A_{t_1,t_2}(y,x) = \inf_{\gamma(t_1)=y,\ \gamma(t_2)=x} A_{t_1,t_2}[\gamma]. \tag{1.5}$$

Under mild conditions on the Lagrangian, this infimum, as well as the infimum in (1.3), is attained at a trajectory $\gamma^{y,x}_{t_1,t_2} \colon [t_1,t_2] \to \mathbf{R}^d$ (see, e.g., [**Fat01**]); below we call such trajectories Lagrangian minimizers. The solution to the Cauchy problem for the Burgers equation (1.1) on the time interval $[0,t]$ with the initial condition $u(x,0) = \nabla S_0(x)$ is then given by $u(x,t) = \dot\gamma^x_{0,t}(t)$, where $\gamma^x_{0,t}$ is a Lagrangian minimizer corresponding to the minimum of the right-hand side in

$$S(x,t) = T_{0,t} S_0(x) = \min_y \big(A_{0,t}(y,x) + S_0(y)\big). \tag{1.6}$$

For the purposes of the present paper, the Lax–Oleĭnik formula (1.3) or (1.6) constitutes a sufficient replacement for definitions of generalized solutions. Note that in its form (1.6), the Lax–Oleĭnik formula becomes a min-plus integral operator representation of a solution. The solution operators $T_{\cdot,cdot}$ form a flow, i.e., they satisfy $T_{t_2,t_3} T_{t_1,t_2} = T_{t_1,t_3}$ for any $t_1 < t_2 < t_3$; however, this flow fails to be a semigroup unless $U(x,t)$ does not depend on time.

We note that the duality between representations of solutions in terms of the value function $S(x,t)$ or minimizers $\gamma^x_{0,t}$ is more than a heuristic relation; when one relaxes the action minimization problem in the spirit of Kantorovich, allowing measure-valued solutions instead of classic minimizing curves, the function $S(x,t)$ becomes the dual variable in a correponding infinite-dimensional linear program (see, e.g., [**Mat89, EG02**]).

1.2. Our interest in solution operators of the form (1.6) is motivated by the theory of global (time-stationary) viscosity solutions in the case of randomly forced inviscid Burgers and Hamilton–Jacobi equations, which was developed recently in [**EKMS00**], [**IK03**] and [**GIKP03**]. The crucial role in the construction of this global solution is played by Lagrangian minimizers γ^x_t defined over a semi-infinite time interval $(-\infty, t]$: namely, a global solution to the random forced inviscid Burgers is given by $u(x,t) = \dot\gamma^x_t(t)$. To prove that such semi-infinite minimizers exist, one has to take a limit as $T \to \infty$ for minimizers $\gamma^x_{t-T,t}$ defined on finite time intervals of the form $[t-T,t]$. Existence of this limit follows from a uniform bound on the absolute value of a velocity $|\dot\gamma^x_{t-T,t}(t)|$, which thus becomes the central problem for the theory.

Observe first that the velocity of a minimizer is uniformly bounded if the state space of the Lagrangian system is a compact manifold M. Indeed, in this case the displacement of a minimizer for any time interval is bounded by the diameter of the manifold, so action minimizing trajectories cannot have large velocities. The simplest example is given by the d-dimensional torus $\mathbf{R}^d/\mathbf{Z}^d$. Hence, the uniform bound on velocities holds in the case of $\mathbf{Z}^d$-periodic potential $U(x,t)$, satisfying $U(x+k,t) = U(x,t)$ for all $k \in \mathbf{Z}^d$. It turns out that, for the randomly forced Burgers equation on a compact manifold, a unique global solution $u(x,t)$ exists with probability 1. In fact the whole theory is developed at the moment only in the case of compact manifolds, where the bound on velocities can be easily proved. At present almost nothing is known about global solutions in the case of $\mathbf{R}^d$ (however see [**HK03**] for some results and discussions).

In the case of non-periodic potentials one can imagine a situation where a minimizer spends almost all its time in a very favourable part of $\mathbf{R}^d$ which may lie far away from its prescribed endpoint x, and then goes very quickly to x. Such scenario will lead to a large terminal velocity at point x which might depend on the time interval where minimization is performed. There are two cases, however, when such behaviour is impossible. The first one corresponds to the autonomous bounded potential: $U(x,t) = U(x)$, for which the energy

$$H(p,x,t) = \max_{v \in \mathbf{R}^d} \big(p \cdot v - L(v,x,t)\big) = \frac{|p|^2}{2} + U(x,t) \tag{1.7}$$

is conserved and the velocity of any Lagrangian trajectory is uniformly bounded if this trajectory is at rest at the initial moment of time. Since all minimizers are Lagrangian trajectories, the bound on their velocities follows immediately.

The second case corresponds to a potential $U(x,t)$ that depends on time periodically. Here the situation is more delicate. It is not true anymore that the velocities of Lagrangian trajectories are bounded. Moreover, it was shown recently by J. Mather that Lagrangian trajectories can be accelerated by a periodic potential to an arbitrary large velocity. However, A.Fathi was able to show with methods developed in [**Fat01**] that the velocities of minimizers are still bounded; his elegant unpublished proof is recalled in Appendix B below.

The examples constructed in this paper show that for special potentials $U(x,t)$ the velocity of a minimizer may be arbitrarily large; in fact, one can construct a potential $U(x,t)$ defined for all $t < 0$ that accelerates minimizers to infinite velocitites. Because of this blow-up in velocity, for such potentials even generalized global solutions do not exist. The simple remarks we just made demonstrate that for this blow-up effect it is crucial that the system be defined on an unbounded manifold (say $\mathbf{R}^d$) and the potential $U(x,t)$ depend on time non-periodically. Implications of our examples to the existence of global solutions in the randomly forced case is discussed in the conclusion to this paper.

1.3. We pass now to precise formulation of our results. Below we consider not (1.2) but a more general Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H(\nabla S, x, t) = 0, \tag{1.8}$$

where the Hamiltonian has the form

$$H(p,x,t) = \frac{1}{\alpha}|p|^\alpha + U(x,t). \tag{1.9}$$

The corresponding Lagrangian system has the Lagrangian

$$L(v,x,t) = \frac{1}{\beta}|v|^\beta - U(x,t), \tag{1.10}$$

where $\alpha^{-1}+\beta^{-1}=1$. Suppose that $\alpha,\beta>1$ and the potential $U(\cdot,t)$ is a C^1 function of x for any t, uniformly bounded together with its spatial derivative:

$$0 \le U(x,t) \le C,\ |\nabla U(x,t)| \le C, \quad x\in\mathbf{R}^d, t\in\mathbf{R}. \tag{1.11}$$

Let the trajectory $\gamma^x_{t_1,t_2}\colon [t_1,t_2]\to\mathbf{R}^d$ be a (not necessarily unique) Lagrangian minimizer for the action A_{t_1,t_2}, satisfying the conditions $\dot\gamma^x_{t_1,t_2}(t_1)=0$, $\gamma^x_{t_1,t_2}(t_2)=x$. Note for future references that, under the above conditions on Lagrangian, $\gamma^x_{t_1,t_2}$ is a classical solution of the Euler–Lagrange equation

$$\frac{d}{dt}(\dot\gamma(t)|\dot\gamma(t)|^{\beta-2}) = -\nabla U \tag{1.12}$$

(see, e.g., [**Fat01**]), where the dot notation stands for the ordinary derivative with respect to time variable.

Theorem 1. *There exists $K=K(C,\beta)>0$ such that for any $[t_1,t_2]$ with large enough $T\equiv t_2-t_1$ and any $x\in\mathbf{R}$*

$$|\dot\gamma^x_{t_1,t_2}(t_2)| \le K(\log T)^{2/\beta}. \tag{1.13}$$

Theorem 2. *There exists $K=K(C,\beta)>0$ such that for any $[t_1,t_2]$ with large enough $T\equiv t_2-t_1$ and any $y\in\mathbf{R}^d$ there is a potential $U(\cdot,t)$, defined on the time interval $[t_1,t_2]$ and satisfying* (1.11), *such that*

$$|\dot\gamma^x_{t_1,t_2}(t_2)| \ge \frac{K(\log T)^{2/\beta}}{2^{\beta/(\beta-1)}} \tag{1.14}$$

for any x with $|x-y|\le R_T\equiv\frac{K}{2}(\log T)^{2/\beta}$.

Constants in Theorems 1 and 2 are not necessarily the same. In what follows they will be given explicit expression in terms of the parameters C and β.

Theorem 3. *There exists a potential $U(x,t)$ defined for all $t<0$ and satisfying* (1.11) *such that for all $x\in\mathbf{R}^d$*

$$\limsup_{t\to-\infty}|\dot\gamma^x_{t,0}(0)| = \infty. \tag{1.15}$$

The paper is organized as follows. Theorem 1 is proved in Section 2. Theorems 2 and 3 are proved in Section 3. In Section 4, we make concluding remarks and indicate several directions in which one can generalize the results of the present paper. In Appendix A we give the technical proof of Lemma 4, deferred from the main text. Appendix B, included for completeness, contains A. Fathi's argument that rules out blow-up if the potential $U(x,t)$ is periodic in time.

To simplify notation we denote below the minimizer $\gamma^x_{t_1,t_2}$ by γ^x and assume that all constants may have implicit dependence on the parameters C and β. For convenience we introduce a positive variable $s=t_2-t$ for $t\in[t_1,t_2]$ and denote by $w(s)$ the absolute value of the average velocity over $[0,s]$:

$$w(s) \equiv \frac{|\gamma^x(t_2)-\gamma^x(t_2-s)|}{s}. \tag{1.16}$$

2. Proof of the upper bound on velocity

Before giving the proof of Theorem 1 in full generality, we observe that it becomes particularly simple in the case of $\beta = 2$. Fix a time interval $[t_1, t_2]$ and a minimizer γ^x with final position $\gamma^x(t_2) = x$. Take s_1 and s_2 with $0 \le s_1 \le s_2 \le T$, where $T \equiv t_2 - t_1$, and suppose that the absolute value of the average velocity of the minimizer increases from $w_2 \equiv w(s_2)$ to $w_1 \equiv w(s_1)$ over the time interval $[t_2 - s_2, t_2 - s_1]$.

Observe that minimization of the action allows to control the increase in the average velocity:

$$1 + \frac{(w_1 - w_2)^2}{2C} \le \frac{s_2}{s_1}. \tag{2.1}$$

To see this, note that

$$\begin{aligned} A_{t_2-s_2,t_2}[\gamma^x] &= A_{t_2-s_2,t_2-s_1}[\gamma^x] + A_{t_2-s_1,t_2}[\gamma^x] \\ &\ge \frac{1}{2}\big(s_1 w_1^2 + \frac{1}{s_2 - s_1}(s_1 w_1 - s_2 w_2)^2\big) - C s_2, \end{aligned} \tag{2.2}$$

where to estimate the action we use (1.11) and Jensen's inequality, taken in the form

$$\int_{t'}^{t''} |\dot\gamma(t)|^\beta \, dt \ge (t'' - t')^{1-\beta} |\gamma(t'') - \gamma(t')|^\beta \tag{2.3}$$

for $\beta > 1$ and an arbitrary C^1 curve $\gamma(t)\colon [t', t''] \to \mathbf{R}^d$. On the other hand, consider a trajectory $\gamma(t)$, $t \in [t_2 - s_2, t_2]$, that has the same endpoints as γ^x but keeps constant velocity, which is equal to w_2. By action minimization and (1.11),

$$A_{t_2-s_2,t_2}[\gamma^x] \le A_{t_2-s_2,t_2}[\gamma] \le \frac{1}{2} s_2 w_2^2. \tag{2.4}$$

Combining (2.2) and (2.4), after some simple algebra we arrive at (2.1).

The meaning of inequality (2.1) is that increasing the absolute value of the average velocity in *arithmetic* progression requires a *geometric* progression in time steps. Therefore the largest possible increase over a time interval of length T is proportional to $\log T$. The desired bound (1.13) on the terminal velocity $\dot\gamma^x(t_2)$ may now be inferred from (i) the observation that the smaller is the time interval, the closer are the absolute values of average and terminal velocity, and (ii) the boundedness of the average velocity $w(T)$ at the earliest time moment $t_1 = t_2 - T$, which we prove in a separate lemma for future reference.

LEMMA 1. $w(T) \le (C\beta)^{1/\beta}$.

PROOF. Using (2.3) and (1.11), it is easy to see that

$$A_{t_1,t_2}[\gamma^x] \ge (T/\beta)(w(T))^\beta - CT. \tag{2.5}$$

On the other hand, the action of the curve $\gamma(t) = x$ for all $t \in [t_1, t_2]$, satisfies the estimate $A_{t_1,t_2}[\gamma] \le 0$. Since $A_{t_1,t_2}[\gamma^x] \le A_{t_1,t_2}[\gamma]$, we have $w(T) \le (C\beta)^{1/\beta}$. □

Turning now to the proof of Theorem 1, we start with two auxiliary results. The first lemma extends inequality (2.1) to the case of general $\beta > 1$.

LEMMA 2. *For* $0 \le s_1 \le s_2 \le T$ *denote*

$$w_1 \equiv w(s_1), \quad w_2 \equiv w(s_2), \quad \Delta \equiv w_1 - w_2 = \xi w_1^{(2-\beta)/2} \tag{2.6}$$

and assume $0 < \Delta < w_1$*. There exists* $W = W(\xi) > 0$ *such that if* $w_1 > W$*, then*

$$1 + \frac{\xi^2(\beta - 1)}{3C} \leq \frac{s_2}{s_1}. \tag{2.7}$$

PROOF. Using (2.3) and (1.11), we get

$$\begin{aligned} A_{t_2-s_2,t_2}[\gamma^x] &= A_{t_2-s_1,t_2}[\gamma^x] + A_{t_2-s_2,t_2-s_1}[\gamma^x] \\ &\geq \frac{s_1 w_1^\beta}{\beta} + \frac{(s_2 - s_1)^{1-\beta}}{\beta}|\gamma^x(t_2 - s_1) - \gamma^x(t_2 - s_2)|^\beta - Cs_2 \\ &\geq \frac{1}{\beta}\left(s_1 w_1^\beta + (s_2 - s_1)^{1-\beta}|s_2 w_2 - s_1 w_1|^\beta\right) - Cs_2. \end{aligned} \tag{2.8}$$

Denote by $\gamma(t)$, $t \in [t_2 - s_2, t_2]$, the trajectory of a point which moves with constant velocity from $(\gamma^x(t_2 - s_2), t_2 - s_2)$ to $(\gamma^x(t_2), t_2)$. Since

$$A_{t_2-s_2,t_2}[\gamma] \leq \frac{s_2 w_2^\beta}{\beta} = \frac{s_2(w_1 - \Delta)^\beta}{\beta} \tag{2.9}$$

and $A_{t_2-s_2,t_2}[\gamma^x] \leq A_{t_2-s_2,t_2}[\gamma]$, inequalities (2.8) and (2.9) imply

$$s_1 w_1^\beta + (s_2 - s_1)\left|w_1 - \frac{s_2}{s_2 - s_1}\Delta\right|^\beta - C\beta s_2 \leq s_2(w_1 - \Delta)^\beta. \tag{2.10}$$

With the notation $\sigma \equiv s_2/(s_2 - s_1)$, this inequality is equivalent to

$$\left|1 - (\sigma\Delta/w_1)\right|^\beta \leq 1 + \sigma\Big(\big(1 - (\Delta/w_1)\big)^\beta - 1 + C\beta w_1^{-\beta}\Big). \tag{2.11}$$

Using in the right-hand side of this inequality the Taylor expansion $(1 - z)^\beta = 1 - \beta z + \frac{\beta(\beta-1)}{2}z^2(1 - \theta z)^{\beta-2}$ with $\theta = \theta(z) \in [0, 1]$, we get:

$$\left|1 - \frac{\sigma\Delta}{w_1}\right|^\beta \leq 1 - \frac{\beta\sigma\Delta}{w_1}\left(1 - \frac{(\beta - 1)}{2}\frac{\Delta}{w_1}\left(1 - \vartheta\frac{\Delta}{w_1}\right)^{\beta-2} - \frac{C}{\Delta w_1^{\beta-1}}\right), \tag{2.12}$$

where $\vartheta = \theta(\Delta/w_1)$. Since $\Delta/w_1 = 1/(\Delta w_1^{\beta-1}) = O(w_1^{-\beta/2})$ for fixed ξ, the value of the largest parenthesis in the right-hand side of (2.12) lies between $2(1 + \beta)^{-1}$ and 1 if $w_1 > W$ with a suitably large $W = W(\xi)$. Since the left-hand side of (2.12) is nonnegative, this implies

$$\frac{\sigma\Delta}{w_1} \leq \frac{\beta + 1}{2\beta} < 1 \tag{2.13}$$

and enables us to use the same expansion in the left-hand side of (2.12). After some cancellations this leads to the inequalities

$$\begin{aligned} \sigma &\leq \frac{1}{(1 - \vartheta^*\sigma\Delta/w_1)^{\beta-2}}\left((1 - \vartheta\Delta/w_1)^{\beta-2} + \frac{2C}{\xi^2(\beta - 1)}\right) \\ &\leq \max\left\{1, \left(\frac{2\beta}{\beta - 1}\right)^{\beta-2}\right\}\left(1 + \frac{2C}{\xi^2(\beta - 1)}\right), \end{aligned} \tag{2.14}$$

where $\vartheta^* = \theta(\sigma\Delta/w_1)$ and the last line follows from (2.13) if $w_1 > W$. Using this upper estimate on σ and enlarging W if necessary, we can ensure that for $w_1 > W$ the parentheses containing ϑ, ϑ^* in (2.14) are arbitrarily close to unity and therefore

$$\sigma \leq 1 + \frac{3C}{\xi^2(\beta - 1)}, \tag{2.15}$$

which implies (2.7). □

Note that in (2.15), as well as in (2.7), the constant 3 may be replaced by any number greater than 2.

Using inequality (2.7), one can replace the arithmetic progression in the w variable, suggested by bound (2.1), by a more general sequence that still leads to a power-law estimate in $\log T$ for the average velocity. The following lemma, employed several times throughout this paper, shows that such estimate allows to control the terminal velocity $\dot\gamma^x(t_2)$.

LEMMA 3. *If* $w(s) \le (2Cs)^{1/(\beta-1)}$ *then* $|\dot\gamma^x(t_2)| \le (3Cs)^{1/(\beta-1)}$. *If* $w(s) > (2Cs)^{1/(\beta-1)}$ *then*

$$(1/2)^{1/(\beta-1)} w(s) \le |\dot\gamma^x(t_2)| \le (3/2)^{1/(\beta-1)} w(s). \tag{2.16}$$

PROOF. The minimizer $\gamma^x(t)$ satisfies the Euler–Lagrange equation (1.12). This together with (1.11) implies

$$\big| |\dot\gamma^x(t')|^{\beta-1} - |\dot\gamma^x(t'')|^{\beta-1} \big| \le Cs \tag{2.17}$$

for all $t', t'' \in [t_2 - s, t_2]$. Since the Lagrangian (1.10) is strictly convex, $\gamma^x(t)$ is a C^1 curve, and there exists $t^* \in [t_2 - s, t_2]$ such that $|\dot\gamma^x(t^*)| = w(s)$. It follows from (2.17) that

$$|\dot\gamma^x(t^*)|^{\beta-1} - Cs \le |\dot\gamma^x(t_2)|^{\beta-1} \le |\dot\gamma^x(t^*)|^{\beta-1} + Cs, \tag{2.18}$$

which implies the statement. □

PROOF OF THEOREM 1. Somewhat departing from notation of Lemma 2, denote $w_1 \equiv |\gamma(t_2-1) - \gamma(t_2)|$, $\Delta \equiv w_1^{(2-\beta)/2}$, $\bar W \equiv \sup\{\, W(\xi) \mid 2^{(2-\beta)/2} \le \xi \le 1 \,\}$. Suppose that $w_1 > \max\{2(C\beta)^{1/\beta}, 2\bar W, (2C)^{1/(\beta-1)}\}$. Since $w(T) \le (C\beta)^{1/\beta}$ by Lemma 1 and $w(s)$ is a continuous function, there exists an increasing sequence $1 \le s_i \le T$, $0 \le i \le n = [\frac{w_1}{2\Delta}]$, where $[\cdot]$ stands for the integer part, such that $s_0 = 1$ and $w(s_i) = w_1 - i\Delta$ for $i > 0$. Denote $\xi_i \equiv \Delta w(s_i)^{(\beta-2)/2}$. Since $w_1/2 \le w(s_i) \le w_1$ for $0 \le i \le n$, all ξ_i satisfy the inequalities $2^{(2-\beta)/2} \le \xi_i \le 1$ and therefore all $w(s_i)$ satisfy the condition of Lemma 2: $w(s_i) > \bar W \ge W(\xi_i)$. Hence

$$T \ge s_n = \prod_{i=1}^{n} \frac{s_i}{s_{i-1}} \ge \left(1 + \frac{2^{2-\beta}(\beta-1)}{3C}\right)^n. \tag{2.19}$$

It follows that $n = [w_1^{\beta/2}/2] \le \widetilde K \log T$, where $\widetilde K = \big(\log(1+\frac{2^{2-\beta}(\beta-1)}{3C})\big)^{-1}$. Therefore $w_1 \le \max\{2(C\beta)^{1/\beta}, 2\bar W, (2C)^{1/(\beta-1)}, (2\widetilde K \log T + 2)^{2/\beta}\}$ for any $T > 0$. The statement now follows from Lemma 3. □

3. Construction of accelerating potentials

Recall that $[t_1, t_2]$ is a fixed time interval with $t_2 - t_1 = T$. To prove Theorems 2 and 3, it is enough to construct in this time interval an example of a potential that depends only on one spatial coordinate. Hence, without loss of generality, we may assume $d = 1$, $x \in \mathbf{R}$.

Observe that setting s_0 equal to s instead of 1 in the proof of Theorem 1 gives for the average velocity of a minimizer at time $t_2 - s$ the bound $O\big((\log(T/s))^{2/\beta}\big)$, which can be turned into a similar bound on $\dot\gamma(t_2 - s)$ by an argument analogous to that of Lemma 3. For $s \in [0, T]$ and any $K > 0$, define

$$g_T(s) \equiv K \int_0^s (\log(T/u))^{2/\beta}\, du. \tag{3.1}$$

The trajectory $-g_T(t_2 - t)$ has therefore the largest velocity possible for a minimizer at all times $t \in [t_1, t_2]$, up to the constant factor K; accelerating potentials constructed below confine minimizers to lie as close to this trajectory as possible.

Before starting the proofs of Theorems 2 and 3, we collect here some properties of the function $g_T(\cdot)$ for future references.

LEMMA 4. *Let $0 \le s \le T$. Then*

$$\int_0^s \frac{1}{\beta}(\dot g_T(u))^\beta\, du = \frac{K^\beta}{\beta} s\big((\log(T/s))^2 + 2\log(T/s) + 2\big), \tag{3.2}$$

$$g_T(s) = Ks\big(\log(T/s)\big)^{2/\beta}\left(1 + \frac{2}{\beta\log(T/s)} + \frac{2(2-\beta)}{\beta^2} r\big(\log(T/s)\big)\right), \tag{3.3}$$

where $0 \le r(z) \le z^{-2}$ for $z > 0$, and

$$\int_0^s \frac{1}{\beta}(\dot g_T(u))^\beta\, du - \frac{s^{1-\beta}}{\beta}\big(g_T(s)\big)^\beta < \frac{4K^\beta s}{\beta}. \tag{3.4}$$

If $T > T_0$ for a suitable $T_0 = T_0(\beta)$ and $3 < s \le T$, then there exists $\bar M = \bar M(\beta)$ such that

$$\frac{(g_T(s) - g_T(1))^\beta}{(s-1)^{\beta-1}} - \frac{(g_T(s))^\beta}{s^{\beta-1}} \le \bar M K^\beta (\log T)^2. \tag{3.5}$$

The proof is postponed to Appendix A.

3.1. Proof of Theorem 2. For any $y \in \mathbf{R}$ define on the time interval $[t_1, t_2]$ a potential

$$U(x,t) \equiv U_C(x - y + g_T(t_2 - t)), \tag{3.6}$$

where $U_C(\cdot)$ is a C^1 function that satisfies the conditions $0 \le U_C(x) \le C$ for all $x \in \mathbf{R}$, $U_C(x) = C$ for $x \le -2$, $U_C(x) = 0$ for $x \ge 0$, and $-C \le U_C'(x) \le 0$ for $x \in [-2, 0]$. Note that the potential $U(x,t)$ satisfies (1.11).

Let $\gamma^x(t)$, $t \in [t_1, t_2]$, be a minimizer with

$$|\gamma^x(t_2) - y| = |x - y| \le R_T \equiv K(\log T)^{2/\beta}/2. \tag{3.7}$$

Without loss of generality suppose that $y = 0$. To establish Theorem 2, we consider three possible cases: (i) $\gamma^x(t_2 - 1) \le -g_T(1)$, (ii) $\gamma^x(t_2 - 1) > -g_T(1)$ and $x \ge 0$, and (iii) $\gamma^x(t_2 - 1) > -g_T(1)$ and $x < 0$. Lemmas 5–7 cover each of these cases and together complete the proof.

LEMMA 5 (case (i)). *If $\gamma^x(t_2 - 1) \le -g_T(1)$, then for any $K > 0$ there holds $\dot\gamma^x(t_2) \ge K(\log T)^{2/\beta}/2^{\beta/(\beta-1)}$ for T large enough.*

PROOF. For the average velocity of γ^x at the instant $t_2 - 1$ we have

$$w(1) = |x - \gamma^x(t_2 - 1)| \ge g_T(1) - R_T \ge K(\log T)^{2/\beta}/2, \tag{3.8}$$

where we use inequalities (3.3) and (3.7). Thus the hypothesis of Lemma 3 is satisfied if $K(\log T)^{2/\beta} \ge 2(2C)^{1/(\beta-1)}$, which by the first of inequalities (2.16) then implies that $\dot\gamma^x(t_2) \ge 2^{-1/(\beta-1)} w(1)$ and, together with estimate (3.8) for $w(1)$, gives the statement of the lemma. □

LEMMA 6 (case (ii)). *Let* $\gamma^x(t_2-1) > -g_T(1)$, $x \geq 0$ *and* $K = (C\beta/5)^{1/\beta}$. *Then* $\dot\gamma^x(t_2) \geq K(\log T)^{2/\beta}/2$ *for* T *large enough.*

PROOF. We first note that the minimizer γ^x cannot stay in the domain where $U = 0$ for all $t \in [t_1, t_2]$. More formally, define

$$\bar s \equiv \inf\{\, s \in (1, T) \mid \gamma^x(t_2 - s) \leq -g_T(s)\,\}; \tag{3.9}$$

then $\bar s < T$ and $\gamma^x(t_2-\bar s) = -g_T(\bar s)$. Indeed, otherwise the velocity of the minimizer γ^x would vanish for all t and we would have $A_{t_1,t_2}[\gamma^x] = 0$. Consider a continuous trajectory $\bar\gamma$ defined on $[t_1, t_2]$ by

$$\bar\gamma(t_2 - s) \equiv \begin{cases} x - (x + g_T(1) + 2)s, & s \in [0,1),\\ -g_T(s) - 2, & s \in [1, T].\end{cases} \tag{3.10}$$

Using (1.11) and (3.2), we obtain the following estimate for the action $A_{t_1,t_2}[\bar\gamma]$:

$$\begin{aligned} A_{t_1,t_2}[\bar\gamma] &= A_{t_2-1,t_2}[\bar\gamma] + A_{t_1,t_2-1}[\bar\gamma]\\ &\leq \frac{(x+g_T(1)+2)^\beta}{\beta} + \frac{1}{\beta}\int_0^T (\dot g_T(s))^\beta\,ds - C(T-1)\\ &= \tfrac{(x+g_T(1)+2)^\beta}{\beta} + \tfrac{2K^\beta T}{\beta} - C(T-1). \end{aligned} \tag{3.11}$$

Observing that $A_{t_1,t_2}[\bar\gamma] \geq A_{t_1,t_2}[\gamma^x] = 0$ and using the fact that $K^\beta = C\beta/5$, we derive

$$\frac{3}{5}T \leq 1 + \frac{(x+g_T(1)+2)^\beta}{\beta C}. \tag{3.12}$$

Since, for T large enough, $x \leq K(\log T)^{2/\beta}/2$ and $g_T(1) \leq 2K(\log T)^{2/\beta}$ by (3.7) and (3.3), we see that the hypothesis $\bar s = T$ leads to a contradiction.

If $\bar s \leq 3$, then the statement of this lemma is established by the same argument as in Lemma 5. Therefore assume that $\gamma^x(t_2 - \bar s) = -g_T(\bar s)$ with $3 < \bar s < T$ and consider the continuous trajectory γ defined for $t \in [t_2 - \bar s, t_2]$ by

$$\gamma(t_2 - s) \equiv \begin{cases} x - (x+g_T(1))s, & s \in [0,1),\\ -g_T(s) - 2(s-1), & s \in [1,2),\\ -g_T(s) - 2, & s \in [2, \bar s - 1),\\ -g_T(s) - 2(\bar s - s), & s \in [\bar s - 1, \bar s].\end{cases} \tag{3.13}$$

For the action $A_{t_2-\bar s,t_2}[\gamma]$ we get using (1.11) that

$$A_{t_2-\bar s,t_2}[\gamma] \leq \frac{(x+g_T(1))^\beta}{\beta} + \frac{1}{\beta}\int_1^{\bar s} (\dot g_T(s))^\beta\,ds + \frac{K^\beta}{\beta}(I_1 + I_2) - C(\bar s - 3), \tag{3.14}$$

where I_1 and I_2 are defined by

$$\begin{aligned} I_1 &\equiv \frac{1}{K^\beta}\int_1^2 \left((\dot g_T(s)+2)^\beta - (\dot g_T(s))^\beta\right) ds,\\ I_2 &\equiv \frac{1}{K^\beta}\int_{\bar s-1}^{\bar s} \left(|\dot g_T(s) - 2|^\beta - (\dot g_T(s))^\beta\right) ds. \end{aligned} \tag{3.15}$$

Note also that by (3.4)

$$\begin{aligned}\frac{1}{\beta}\int_1^{\bar s}(\dot g_T(s))^\beta\,ds &< \frac{4K^\beta\bar s}{\beta}+\frac{(g_T(\bar s))^\beta}{\beta s^{\beta-1}}-\int_0^1(\dot g_T(s))^\beta\,ds\\ &\le \frac{4K^\beta\bar s}{\beta}+\frac{(g_T(\bar s)-g_T(1))^\beta}{\beta(\bar s-1)^{\beta-1}},\end{aligned}\tag{3.16}$$

where the last line follows from Jensen's inequality. On the other hand, since for $t\in[t_2-\bar s,t_2]$ the minimizer γ^x stays in the domain where $U=0$, is velocity remains constant and we have

$$A_{t_2-\bar s,t_2}[\gamma^x]=\frac{(x+g_T(\bar s))^\beta}{\beta\bar s^{\beta-1}}\tag{3.17}$$

Plugging (3.14), (3.16) and (3.17) into the inequality $A_{t_2-\bar s,t_2}[\gamma]-A_{t_2-\bar s,t_2}[\gamma^x]\ge 0$ gives

$$\bar s<15+I_1+I_2+\frac{(x+g_T(1))^\beta}{K^\beta}+\frac{(g_T(\bar s)-g_T(1))^\beta}{K^\beta(\bar s-1)^{\beta-1}}-\frac{(x+g_T(\bar s))^\beta}{K^\beta\bar s^{\beta-1}}.\tag{3.18}$$

where we took into account that $C=5K^\beta\beta^{-1}$.

We now estimate terms in the right-hand side of (3.18). Note first that for T large enough

$$\begin{aligned}I_1 &= \frac{1}{K^\beta}\int_1^2\Big((\dot g_T(s)+2)^\beta-(\dot g_T(s))^\beta\Big)\,ds\\ &< (\log T)^2\int_1^2\left(\left(1-\frac{\log s}{\log T}\right)^{2/\beta}+\frac{2}{K(\log T)^{2/\beta}}\right)ds<2(\log T)^2\end{aligned}\tag{3.19}$$

and similarly $I_2<2(\log T)^2$. Second, note that if T is so large that the right-hand side of (3.3) is less than $2K(\log T)^{2/\beta}$ for $s=1$, then by (3.3) and (3.7)

$$\frac{(x+g_T(1))^\beta}{K^\beta}\le(5/2)^\beta(\log T)^2.\tag{3.20}$$

Third, note that since $x\ge 0$ we can use (3.5) to get

$$\frac{(g_T(\bar s)-g_T(1))^\beta}{K^\beta(\bar s-1)^{\beta-1}}-\frac{(x+g_T(\bar s))^\beta}{K^\beta\bar s^{\beta-1}}<\bar M(\log T)^2.\tag{3.21}$$

Taking the estimates for I_1 and I_2 (see (3.19)), (3.20) and (3.21) into account in (3.18), we get $\bar s\le M(\log T)^2$ for T large enough with a suitable constant $M=M(\beta)$.

Now, using again the fact that the velocity of the minimizer γ^x stays constant for $t\in[t_2-\bar s,t_2]$, we get from (3.3) for large enough T that

$$\dot\gamma^x(t_2)=\frac{x+g_T(\bar s)}{\bar s}\ge\frac{g_T(\bar s)}{\bar s}>K(\log T)^{2/\beta}\left(1-\frac{\log\bar s}{\log T}\right)^{2/\beta}>\frac{K}{2}(\log T)^{2/\beta},\tag{3.22}$$

which establishes the statement of Lemma 6. □

Lemma 7 (case (iii)). *Let* $\gamma^x(t_2-1)>-g_T(1)$, $x<0$ *and* $K=(C\beta/5)^{1/\beta}$. *Then* $\dot\gamma^x(t_2)\ge K(\log T)^{2/\beta}/2^{\beta/(\beta-1)}$ *for* T *large enough.*

PROOF. Take $(t_2-\bar{s}, t_2-\bar{\bar{s}})$ to be the largest neighbourhood of the instant t_2-1 in which the minimizer γ^x stays in the domain where $U=0$. More formally, define

$$\begin{aligned} \bar{s} &\equiv \inf\{\, s\in(1,T) \mid \gamma^x(t_2-s) \le -g_T(s)\,\}, \\ \bar{\bar{s}} &\equiv \sup\{\, s\in(0,1) \mid \gamma^x(t_2-s) \le -g_T(s)\,\}. \end{aligned} \tag{3.23}$$

Since $x<0$, the minimizer γ^x must intersect the curve $-g_T(t_2-t)$ for $t>t_2-1$, so $\gamma^x(t_2-\bar{\bar{s}}) = -g_T(\bar{\bar{s}})$. Moreover, observe that $\bar{s}<T$ and $\gamma^x(t_2-\bar{s}) = -g_T(\bar{s})$. Indeed, otherwise the minimizer γ^x would necessarily stay in the domain where $U=0$ for all $t\in[t_1, t_2-\bar{\bar{s}}]$, so its velocity would have to vanish and we would have $\gamma^x(t)=\gamma^x(t_2-\bar{\bar{s}})$ and $A_{t_1,t_2-\bar{\bar{s}}}[\gamma^x]=0$. Assuming, without loss of generality, that $T\equiv t_2-t_1>\bar{\bar{s}}+1$, consider a continuous trajectory $\bar{\gamma}$ defined on $[t_1,t_2]$ by

$$\bar{\gamma}(t_2-s) \equiv \begin{cases} -g_T(s) - 2(s-\bar{\bar{s}}), & s\in[\bar{\bar{s}}, \bar{\bar{s}}+1), \\ -g_T(s) - 2, & s\in[\bar{\bar{s}}+1, T]. \end{cases} \tag{3.24}$$

Assuming T so large that $\dot{g}_T(s) = K\big(\log(T/s)\big)^{2/\beta} > 2$ for $s\in[0,2]$, and using (1.11), the inequalities $0\le\bar{\bar{s}}<1$, and (3.2), we obtain the following estimate for the action $A_{t_1,t_2-\bar{\bar{s}}}[\bar{\gamma}]$:

$$\begin{aligned} A_{t_1,t_2-\bar{\bar{s}}}[\bar{\gamma}] &= \int_{\bar{\bar{s}}}^{\bar{\bar{s}}+1} \left(\frac{1}{\beta}(\dot{g}_T(s)+2)^\beta - U\big(\bar{\gamma}(t_2-s), t_2-s\big) \right) ds \\ &\quad + \frac{1}{\beta}\int_{\bar{\bar{s}}+1}^{T} (\dot{g}_T(s))^\beta\, ds - C(T-\bar{\bar{s}}-1) \\ &\le \frac{1}{\beta}\int_0^2 (\dot{g}_T(s)+2)^\beta\, ds + \frac{1}{\beta}\int_0^T (\dot{g}_T(s))^\beta\, ds - C(T-2) \\ &< \tfrac{2(2K)^\beta}{\beta}\big((\log\tfrac{T}{2}+1)^2+1\big) + 2C - \left(C - \tfrac{2K^\beta}{\beta}\right) T. \end{aligned} \tag{3.25}$$

Now note that γ^x is a minimizer, so we must have $A_{t_1,t_2-\bar{\bar{s}}}[\bar{\gamma}] \ge A_{t_1,t_2-\bar{\bar{s}}}[\gamma^x]=0$. Thus the hypothesis $\bar{s}=T$ leads to contradiction, since for $K=(C\beta/5)^{1/\beta}$ the right-hand side of the last inequality becomes negative for large T.

If $\bar{s}\le 3$, then the statement of this lemma is established by the same argument as in Lemma 5. Assuming that $\gamma^x(t_2-\bar{s}) = -g_T(\bar{s})$ and $\gamma^x(t_2-\bar{\bar{s}}) = -g_T(\bar{\bar{s}})$ with $0\le\bar{\bar{s}}<1$ and $3<\bar{s}<T$, consider the continuous trajectory γ defined for $t\in[t_2-\bar{s}, t_2-\bar{\bar{s}}]$ by

$$\gamma(t_2-s) \equiv \begin{cases} -g_T(s) - 2(s-\bar{\bar{s}}), & s\in[\bar{\bar{s}}, \bar{\bar{s}}+1), \\ -g_T(s) - 2, & s\in[\bar{\bar{s}}+1, \bar{s}-1), \\ -g_T(s) - 2(\bar{s}-s), & s\in[\bar{s}-1, \bar{s}]. \end{cases} \tag{3.26}$$

Using (1.11), we estimate the action $A_{t_2-\bar{s},t_2-\bar{\bar{s}}}[\gamma]$ by

$$A_{t_2-\bar{s},t_2}[\gamma] \le C(\bar{\bar{s}}+2-\bar{s}) + \frac{1}{\beta}\left(\int_{\bar{\bar{s}}}^{\bar{s}} (\dot{g}_T(s))^\beta\, ds\right) + \frac{K^\beta}{\beta}(I_1+I_2), \tag{3.27}$$

where I_1 and I_2 are defined by formulas (3.15), except that I_1 invovles integration from $\bar{\bar{s}}$ to $\bar{\bar{s}}+1$. Note also that, similarly to (3.16),

$$\frac{1}{\beta}\int_{\bar{\bar{s}}}^{\bar{s}} (\dot{g}_T(s))^\beta\, ds < \frac{4K^\beta \bar{s}}{\beta}. + \frac{(g_T(\bar{s}) - g_T(\bar{\bar{s}}))^\beta}{\beta(\bar{s}-\bar{\bar{s}})^{\beta-1}} \tag{3.28}$$

On the other hand, since for $t \in [t_2 - \bar{s}, t_2 - \bar{\bar{s}}]$ the minimizer γ^x stays in the domain where $U \equiv 0$, its velocity remains constant and we have

$$A_{t_2-\bar{s},t_2-\bar{\bar{s}}}[\gamma^x] = \frac{(g_T(\bar{s}) - g_T(\bar{\bar{s}}))^\beta}{\beta(\bar{s} - \bar{\bar{s}})^{\beta-1}}. \tag{3.29}$$

Plugging (3.27), (3.28) and (3.29) into the inequality $A_{t_2-\bar{s},t_2-\bar{\bar{s}}}[\gamma] - A_{t_2-\bar{s},t_2-\bar{\bar{s}}}[\gamma^x] \geq 0$ and taking into account that $C = 5K^\beta \beta^{-1}$, we get a simpler form of inequality (3.18):

$$\bar{s} < 5(\bar{\bar{s}} + 2) + I_1 + I_2. \tag{3.30}$$

However, this time we need a more accurate estimate of the sum $I_1 + I_2$ than (3.19) can give. Indeed, in the present case, unlike case (ii), we have only indirect control over $\dot{\gamma}^x(t_2)$, namely that provided by Lemma 3; this requires a more stringent constraint on $\bar{s}$.

Recall that (3.19) tells that I_1 and I_2, and therefore $\bar{s}$, are not larger than $O\big((\log T)^2\big)$. Thus for suitably large T we can expand integrands in I_1 and I_2:

$$\begin{aligned} I_1 &= \frac{1}{K^\beta} \int_{\bar{\bar{s}}}^{\bar{\bar{s}}+1} (\dot{g}_T(s))^\beta \left(\left(1 + \frac{2}{\dot{g}_T(s)}\right)^\beta - 1 \right) ds \\ &\leq \int_{\bar{\bar{s}}}^{\bar{\bar{s}}+1} \left(\frac{2\beta}{K} \big(\log(T/s)\big)^{2-2/\beta} + M_1(K) \big(\log(T/s)\big)^{2-4/\beta} \right) ds, \\ I_2 &\leq \int_{\bar{s}-1}^{\bar{s}} \left(-\frac{2\beta}{K} \big(\log(T/s)\big)^{2-2/\beta} + M_1(K) \big(\log(T/s)\big)^{2-4/\beta} \right) ds, \end{aligned} \tag{3.31}$$

where $M_1(K)$ does not depend on T.

It is easy to check that for s such that $0 < s \leq O\big(\log T)^2\big)$

$$\int_s^{s+1} \big(\log(T/u)\big)^{2-2/\beta} du = (\log T)^{2-2/\beta} + O\big(((\log T)^{1-2/\beta} \max\big(1, (\log(s+1))^3\big)\big). \tag{3.32}$$

It follows from (3.32) that

$$\int_{\bar{\bar{s}}}^{\bar{\bar{s}}+1} \big(\log(T/s)\big)^{2-2/\beta} ds - \int_{\bar{s}-1}^{\bar{s}} \big(\log(T/s)\big)^{2-2/\beta} ds \leq M_2 (\log T)^{1-2/\beta} (\log \bar{s})^3. \tag{3.33}$$

Suppose $1 < \beta \leq 2$; then $1 - 2/\beta \leq 0$ and we have $I_1 + I_2 \leq M_2 (\log \bar{s})^3 + 2M_1$. If $\beta > 2$, then $2 - 4/\beta > 1 - 2/\beta > 0$, so that

$$\int_{\bar{\bar{s}}}^{\bar{\bar{s}}+1} \big(\log(T/s)\big)^{2-4/\beta} ds < \int_0^1 \big(\log(T/s)\big)^{2-4/\beta} ds = M_3 (\log T)^{2-4/\beta} \tag{3.34}$$

and the rightmost part of (3.34) grows with T faster than $M_2(\log T)^{1-2/\beta} \log \bar{s} + M_1$; thus

$$\bar{s} \leq \begin{cases} M_2 \log \bar{s} + 2M_1 + 5(\bar{\bar{s}} + 2), & 1 < \beta \leq 2, \\ M_3 (\log T)^{2-4/\beta} + 5(\bar{\bar{s}} + 2), & \beta > 2, \end{cases} \tag{3.35}$$

or $\bar{s} \leq M_4 (\log T)^{\max\{0, 2-4/\beta\}}$ with a suitable constant $M_4 = M_4(K)$, for large enough T. Note that for such $\bar{s}$ by (3.3) and (3.7) we have

$$w(\bar{s}) = \frac{|x - g_T(\bar{s})|}{\bar{s}} \geq \frac{K}{2} (\log T)^{2/\beta}. \tag{3.36}$$

Since $2/\beta > \max\{0, 2-4/\beta\}/(\beta-1)$ for $\beta > 1$, the condition of Lemma 3 is satisfied for large enough T, so that $|\dot\gamma^x(t_2)| \geq K(\log T)^{2/\beta}/2^{\beta/(\beta-1)}$. This establishes the statement of Lemma 7 and concludes the proof of Theorem 2. □

3.2. Proof of Theorem 3. In the proof of Theorem 2 we constructed an accelerating potential $U(x,t)$ corresponding to any long enough time interval $[t_1, t_2]$, $t_2 - t_1 \equiv T$, and any ball $|x| \leq R_T$ of terminal positions x at time t_2. We now glue together a sequence of such potentials to define for all $t < 0$ a potential $U_\infty(x,t)$ that accelerates minimizers indefinitely.

Fix $K = (C\beta/5)^{1/\beta}$. Define increasing sequences T_n and S_n for $n \geq 1$:

$$T_1 \equiv S_1 \equiv \max(1, \bar T), \quad T_n \equiv \exp\left(S_{n-1}^{1/\epsilon}\right), \quad S_n \equiv S_{n-1} + T_n, \quad n \geq 2, \tag{3.37}$$

where $\bar T$ is large enough so that Theorem 2 holds for $T > \bar T$, and ϵ is any positive number satisfying $\epsilon < 2(\beta-1)/\beta^2$. Define also

$$X_0 \equiv 0, \quad X_n \equiv \sum_{i=1}^{n} g_{T_i}(T_i), \quad n \geq 1. \tag{3.38}$$

Note that $g_T(T) = KT\int_0^1 |\log x|^{2/\beta}\,dx$ and therefore $X_n = K_1 S_n$, where $K_1 = K\int_0^1 |\log x|^{2/\beta}\,dx$. Finally, define

$$U_\infty(x,t) \equiv U_C(x - X_{n-1} + g_{T_n}(-t - S_{n-1})) \tag{3.39}$$

for $t \in (-S_n, -S_{n-1}]$, $n \geq 1$, where $S_0 \equiv 0$.

Consider a terminal position x and take n large enough so that $|x| \leq \frac{1}{2}R_{T_n} = \frac{K}{4}(\log T_n)^{2/\beta}$. Denote by $\gamma_n^x(t)$ a minimizer on the time interval $t \in [-S_n, 0]$ such that $\gamma_n^x(0) = x$ and $\dot\gamma_n^x(-S_n) = 0$. To establish Theorem 3, we now show that for all n large enough

$$|\dot\gamma_n^x(0)| \geq \frac{K_0}{2}(\log T_n)^{\frac{2}{\beta}-\epsilon}, \tag{3.40}$$

where $K_0 = K/2^{2+1/(\beta-1)}$; since $2/\beta - \epsilon > 0$, this implies the statement of the theorem.

To prove (3.40), we consider two cases. First assume that $|\gamma_n^x(-S_{n-1}) - X_{n-1}| \leq R_{T_n} = \frac{K}{2}(\log T_n)^{2/\beta}$. Since $\gamma_n^x(t)$ is a minimizer on the time interval $[-S_n, -S_{n-1}]$ with $\dot\gamma_n^x(-S_n) = 0$, it follows from Theorem 2 (with $y = X_{n-1}$) that $|\dot\gamma_n^x(-S_{n-1})| \geq 2K_0(\log T_n)^{2/\beta}$. Using (2.17) in an argument similar to that of Lemma 3, we obtain

$$|\dot\gamma_n^x(0)| \geq \left(\left(2K_0(\log T_n)^{2/\beta}\right)^{\beta-1} - CS_{n-1}\right)^{\frac{1}{\beta-1}}. \tag{3.41}$$

Observing that $S_{n-1} = (\log T_n)^\epsilon$ and increasing n if necessary, we get $|\dot\gamma_n^x(0)| \geq K_0(\log T_n)^{2/\beta}$, which is even stronger than (3.40).

In the second case, when $|\gamma_n^x(-S_{n-1}) - X_{n-1}| > R_{T_n}$, observe that the average velocity $w(S_{n-1})$ on the interval $[-S_{n-1}, 0]$ satisfies the inequality $w(S_{n-1}) \geq \left(\frac{1}{2}R_{T_n} - X_{n-1}\right)/S_{n-1}$. Taking into account that $X_{n-1} = K_1 S_{n-1}$, we obtain for large enough n that

$$w(S_{n-1}) \geq \frac{R_{T_n}}{4S_{n-1}} = \frac{K}{8}(\log T_n)^{\frac{2}{\beta}-\alpha}. \tag{3.42}$$

Thus for all n large enough we can ensure that $w(S_{n-1}) > (2CS_{n-1})^{1/(\beta-1)}$, which by Lemma 3 implies (3.40). □

4. Conclusion

The results of this paper can be generalized in several directions. One can consider Lagrangian systems with discrete time. In this situation one has to find a minimizing sequence $\{ x_i \in \mathbf{R}^d \colon N_1 \le i \le N_2 \}$ for the action

$$A_{N_1,N_2}[\{x_i\}] = \sum_{i=N_1}^{N_2-1} \left[\frac{1}{\beta} |x_{i+1} - x_i|^\beta - U_i(x_i) \right], \tag{4.1}$$

subject to the condition $x_{N_2} = x$. In physics literature such systems are called non-stationary Frenkel–Kontorova type models. Notice that the discrete-time case corresponds to "kicked forcing" in the continuous-time setting, i.e., to a forcing of the form $U(x,t) = \sum_i U_i(x)\delta(t-i)$ (see, e.g., [**BFK00**]). The results in the discrete situation are the same as in the continuous-time setting.

It is also possible to consider more general natural Lagrangian systems where a Lagrangian has the following form $L(x,v,t) = L_0(v) - U(x,t)$. This and other generalizations will be discussed in a forthcoming publication.

It is interesting to study whether in Theorem 3 it is possible to replace the one-sided (upper) limit by the two-sided limit. We believe that the answer to this question is affirmative.

Notice that for the potentials constructed in this paper the partial derivative $\partial U/\partial t$ is unbounded. It is natural to ask whether velocity can grow with T in the case when

$$\left| \frac{\partial U(x,t)}{\partial t} \right| \le C, \quad x \in \mathbf{R}^d, t \in \mathbf{R}. \tag{4.2}$$

It is important to mention that all the "accelerating" potentials constructed in this paper have a very specific form. We expect that for generic bounded time-dependent potentials the velocity of minimizers is bounded. Below we formulate this statement as a conjecture in the case of random potentials.

Conjecture 1. *Let*

$$U(x,t) = \sum_{j=1}^{N} U_j(x) a_j^\omega(t), \quad x \in \mathbf{R}^d, t \in \mathbf{R}, \tag{4.3}$$

where $U_j(x)$ are fixed non-random potentials of class C^1 satisfying condition (1.11) *and $(a_j^\omega(t), 1 \le j \le N)$ is a realization of a stationary vector-valued random process with exponentially decaying correlation, where ω is a point of the corresponding probability space and $\sup_{j,t} |(a_j^\omega(t)| \le 1$ for almost all ω. Then there exists a random constant $C^\omega(x)$ such that*

$$|\dot\gamma_{t,0}^x(0)| \le C^\omega(x), \tag{4.4}$$

uniformly in t, where $t < 0$ and $\gamma_{t,0}^x(\tau)$ is a minimizer on $[t,0]$ such that $\gamma_{t,0}^x(0) = x$.

If this conjecture holds true, then global solutions exist with probability 1 in the case of random potentials.

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Appendix A. Proof of Lemma 4

Eq. (3.2) is obtained by two integrations by parts.

To obtain (3.3), integrate the right-hand side of (3.1) by parts twice to get

$$\begin{aligned} g_T(s) &= Ks\big(\log(T/s)\big)^{2/\beta}\left(1+\frac{2}{\beta\log(T/s)}\right)\\ &\quad+\frac{2(2-\beta)K}{\beta^2}\int_0^s\big(\log(T/u)\big)^{(2/\beta)-2}du. \end{aligned}\tag{A.1}$$

Making the change of variable $v=\log(T/u)$ in the integral in the right-hand side of this formula, we get

$$\int_0^s\big(\log(T/u)\big)^{(2/\beta)-2}du = T\int_{\log(T/s)}^{\infty}v^{(2/\beta)-2}e^{-v}\,dv \le s\big(\log(T/s)\big)^{(2/\beta)-2}\tag{A.2}$$

because $v^{(2/\beta)-2}\le\big(\log(T/s)\big)^{(2/\beta)-2}$ for $v\ge\log(T/s)$ and $\beta>1$. Together with (A.1) this implies (3.3).

We now use (3.3) to obtain

$$\begin{aligned} s^{1-\beta}\left(g_T(s)\right)^{\beta} &\ge K^\beta s\big(\log(T/s)\big)^2\left(1+\frac{2}{\beta\log(T/s)}-\frac{2|2-\beta|}{(\beta\log(T/s))^2}\right)^{\beta}\\ &\ge K^\beta s\big(\log(T/s)\big)^2\left(1+\frac{2}{\log(T/s)}-\frac{2|2-\beta|}{\beta(\log(T/s))^2}\right); \end{aligned}\tag{A.3}$$

the last line here follows from the inequality $(1+z)^\beta\ge1+\beta z$ valid for $\beta>1$. Together with (3.2) this gives

$$\int_0^s\frac{1}{\beta}|\dot g_T(u)|^\beta\,du-\frac{s^{1-\beta}}{\beta}\left(g_T(s)\right)^\beta\le\frac{2K^\beta s}{\beta}\left(1+\frac{|2-\beta|}{\beta}\right),\tag{A.4}$$

which implies inequality (3.4) for $\beta>1$.

We finally notice that monotonicity of $g_T(\cdot)$ implies that for $s>1$

$$\frac{(g_T(s)-g_T(1))^\beta}{(s-1)^{\beta-1}}-\frac{(g_T(s))^\beta}{s^{\beta-1}}\le\left(\frac{g_T(s)}{s}\right)^\beta s\left(\frac{1}{(1-s^{-1})^{\beta-1}}-1\right).\tag{A.5}$$

Observe that

$$\frac{1}{(1-x)^{\beta-1}}-1\le3\left(\left(\frac{3}{2})\right)^{\beta-1}-1\right)x,\tag{A.6}$$

if $x\in[0,1/3]$ (note that the left-hand side of (A.6) is a convex function, whose graph therefore lies below its chord given by the right-hand side). Furthermore, notice that

$$\frac{g_T(s)}{s}=\frac{K}{s}\int_0^s\big(\log(T/u)\big)^{2/\beta}du=K\int_0^\infty\big(v+\log(T/s)\big)^{2/\beta}\mathrm{e}^{-v}\,dv,\tag{A.7}$$

where we performed the change of variable $v = \log(s/u)$. If $T/2 < s \le T$, then the right-hand side of this expression is bounded uniformly in T; for $s < s \le T/2$ we have

(A.8)

$$K \int_0^\infty \big(v + \log(T/s)\big)^{2/\beta} \mathrm{e}^{-v}\, dv = K\big(\log(T/s)\big)^{2/\beta} \int_0^\infty \left(\frac{v}{\log(T/s)} + 1\right)^{2/\beta} \mathrm{e}^{-v}\, dv \le K\big(\log(T/s)\big)^{2/\beta} \int_0^\infty \left(\frac{v}{\log 2}\right.$$

Therefore for T large enough

$$\frac{g_T(s)}{s} \le \tilde{K}\big(\log(T/s)\big)^{2/\beta} \tag{A.9}$$

with a suitable $\tilde{K} > 0$. Inequalities (A.5), (A.6), and (A.9) together give (3.5) for $3 \le s \le T$.

Appendix B. Absense of blow-up in the time-periodic case

In this appendix we present A. Fathi's proof that there is no blow-up if the potential $U(x,t)$ is periodic in time. Therefore, in addition to assumptions (1.11), we require $U(x,t) = U(x,t+1)$ for any $x \in \mathbf{R}^d$ and any $t \in \mathbf{R}$.

Let $x, y \in \mathbf{R}^d$, $t_1 < t_2$. Since the action functional (1.4) is bounded below, we can write, repeating definition (1.5),

$$A_{t_1,t_2}(y,x) = \inf_{\gamma(t_1)=y,\ \gamma(t_2)=x} A_{t_1,t_2}[\gamma]. \tag{B.1}$$

In what follows we assume that this infimum is attained, which is a standard result under the present hypotheses on the Lagrangian (see, e.g., [**Fat01**]). The following elementary lemma is also standard.

Lemma 8. *The function $A_{t_1,t_2}(y,x)$ is uniformly locally Lipschitz: for any $W > 0$, there exists $K = K(W)$ such that if $t_1 < t_2$ and $x_1, x_2, y \in \mathbf{R}^d$ are such that $|x_i - y| \le W(t_2 - t_1)$, $i = 1, 2$, then*

$$|A_{t_1,t_2}(y,x_2) - A_{t_1,t_2}(y,x_1)| \le K(W)|x_2 - x_1| \tag{B.2}$$

Moreover, the function $A_{t_1,t_2}(y,x)$ admits the following bound: for any $x, y \in \mathbf{R}^d$, $t_1 < t_2$

$$\frac{|x-y|^\beta}{\beta(t_2-t_1)^\beta} - C \le \frac{1}{t_2 - t_1} A_{t_1,t_2}(y,x) \le \frac{|x-y|^\beta}{\beta(t_2-t_1)^\beta}. \tag{B.3}$$

Proof. Let γ_0 be a minimizing curve and $w = |\gamma_0(t_2) - \gamma_0(t_1)|/(t_2 - t_1)$ be its average velocity defined as in (1.16) above. If $w \le W$, then, by an argument similar to that of Lemma 3, there exists $\tilde{K}(W) > 0$ such that $|\dot{\gamma}_0(t)| \le \tilde{K}(W)$ for all $t \in [t_1, t_2]$ (as in the main text, we do not write the dependence on the parameters β and C explicitly).

To establish (B.2), take now $x_1, x_2 \in \mathbf{R}^d$ such that $|x_i - y| \le W(t_2 - t_1)$, $i = 1, 2$, and let γ_1 be a minimizing trajectory for which $A_{t_1,t_2}[\gamma_1] = A_{t_1,t_2}(y, x_1)$.

Define also $\gamma(t) = \gamma_1(t) + \frac{t-t_1}{t_2-t_1}(x_2 - x_1)$. We have

$$\begin{aligned} A_{t_1,t_2}(y, x_2) - A_{t_1,t_2}(y, x_1) &\le A_{t_1,t_2}[\gamma] - A_{t_1,t_2}[\gamma_1] \\ &= \int_{t_1}^{t_2} \frac{1}{\beta}\left(\left|\dot\gamma_1(t) + \frac{1}{t_2-t_1}(x_2-x_1)\right|^\beta - |\dot\gamma_1(t)|^\beta \right) dt \\ &\quad + \int_{t_1}^{t_2} \left(U(\gamma_1(t), t) - U\left(\gamma_1(t) + \frac{t-t_1}{t_2-t_1}(x_2-x_1), t\right)\right) dt. \end{aligned} \tag{B.4}$$

Using Taylor's formula, we obtain

$$\begin{aligned} &\left|\dot\gamma_1(t) + \frac{1}{t_2-t_1}(x_2-x_1)\right|^\beta - |\dot\gamma_1(t)|^\beta \\ &\quad = \left|\dot\gamma_1(t) + \frac{\phi(t)}{t_2-t_1}(x_2-x_1)\right|^{\beta-2} \left(\dot\gamma_1(t) + \frac{\phi(t)}{t_2-t_1}(x_2-x_1), \frac{1}{t_2-t_1}(x_2-x_1)\right), \\ &U(\gamma_1(t), t) - U\left(\gamma_1(t) + \frac{t-t_1}{t_2-t_1}(x_2-x_1), t\right) \\ &\quad = -\nabla U\left(\gamma_1(t) + \frac{\psi(t)(t-t_1)}{t_2-t_1}(x_2-x_1), t\right)\frac{1}{t_2-t_1}(x_2-x_1), \end{aligned}$$

where $0 \le \phi(t), \psi(t) \le 1$ for all $t \in [t_1, t_2]$. Substituting this into (B.4) and observing that $\dot\gamma(t)$ and ∇U are both bounded, we get

$$A_{t_1,t_2}(y, x_2) - A_{t_1,t_2}(y, x_1) \le K|x_2 - x_1| \tag{B.5}$$

with a suitable $K = K(W)$. Interchanging the roles of x_1 and x_2, we get a reverse inequality and thus establish the desired Lipschitz property of $A_{t_1,t_2}(y, x)$.

The left inequality in (B.3) follows from Jensen's inequality (2.3) and condition (1.11). The right inequality follows in a similar way from the inequality $A_{t_1,t_2}[\gamma_0] \le A_{t_1,t_2}[\gamma]$ written for $\gamma(t) = \frac{t_2-t}{t_2-t_1}y + \frac{t-t_1}{t_2-t_1}x$. □

Following A. Fathi, we introduce two concepts now. A function $S\colon \mathbf{R}^d \to \mathbf{R}$ is said to be *L-dominated* for a fixed time interval $[t_1, t_2]$ a constant $L \in \mathbf{R}$ if

$$S(x) - S(y) \le A_{t_1,t_2}(y, x) + L(t_2 - t_1) \tag{B.6}$$

for any $x, y \in \mathbf{R}^d$, and *Lipschitz in the large* (with constant K) if

$$|S(x) - S(y)| \le K(|x - y| + 1) \tag{B.7}$$

for any $x, y \in \mathbf{R}^d$ with some $K > 0$.

Lemma 9. *An L-dominated function is Lipschitz in the large with constant K depending on L and t_1, t_2.*

Proof. For $x, y \in \mathbf{R}^d$ define the sequence (x_i), $0 \le i \le [|x-y|]$, by $x_i = y + ir$, where $r = |x-y|^{-1}(x-y)$ is a unit vector collinear with $x - y$. (Here, as above, $[\cdot]$ stands for the integer part.) We can write

$$S(x) - S(y) = \sum_{1 \le i \le [|x-y|]} (S(x_i) - S(x_{i-1})) + S(x) - S(x_{[|x-y|]}). \tag{B.8}$$

Using the property of L-domination and the right inequality (B.3), we get

$$S(x) - S(y) \le \left(\frac{1}{\beta(t_2-t_1)^{\beta-1}} + L(t_2 - t_1)\right)([|x-y|] + 1). \tag{B.9}$$

Together with the reverse inequality obtained by interchanging the roles of x and y, this implies (B.7). □

Denote the Lax–Oleĭnik solution operator over a time interval $[t_1, t_2]$ for the Cauchy problem for equation (1.8) by

$$T_{t_1,t_2} S(x) \equiv \inf_{y \in \mathbf{R}^d} (A_{t_1,t_2}(y, x) + S(y)). \tag{B.10}$$

LEMMA 10. *For any L and any $t_1 < t_2$, the operator T_{t_1,t_2} maps the set of L-dominated functions into itself.*

PROOF. If $S(x)$ is L-dominated, it follows from (B.6) that for any $x \in \mathbf{R}^d$

$$S(x) \le \inf_{y \in \mathbf{R}^d} (A_{t_1,t_2}(y, x) + S(y) + L(t_2 - t_1)) = T_{t_1,t_2} S(x) + L(t_2 - t_1). \tag{B.11}$$

Therefore for any $z \in \mathbf{R}^d$

$$\begin{aligned} T_{t_1,t_2} S(x) &= \inf_{y \in \mathbf{R}^d} (A_{t_1,t_2}(y, x) + S(y)) \\ &\le A_{t_1,t_2}(z, x) + S(z) \le A_{t_1,t_2}(z, x) + T_{t_1,t_2} S(z) + L(t_2 - t_1), \end{aligned} \tag{B.12}$$

which implies L-domination for $T_{t_1,t_2} S(x)$. □

LEMMA 11. *For any $K > 0$ and any $t_1 < t_2$, the operator T_{t_1,t_2} maps the set of functions that are Lipschitz in the large with constant K into the set of Lipschitz functions with constant $\bar{K} = \bar{K}(K, t_1, t_2)$.*

PROOF. Let $S(x)$ be a function that is Lipschitz in the large with constant K. Then for any $y \in \mathbf{R}^d$

$$A_{t_1,t_2}(y, x) + S(y) \ge A_{t_1,t_2}(y, x) + S(x) - K(|x - y| + 1). \tag{B.13}$$

On the other hand, by definition (B.10) of the operator T_{t_1,t_2} and the last inequality in (B.3), we have

$$T_{t_1,t_2} S(x) \le A_{t_1,t_2}(x, x) + S(x) \le S(x). \tag{B.14}$$

Therefore instead of (B.10) we can write

$$T_{t_1,t_2} S(x) = \inf_{y \in \mathbf{R}^d} (A^K_{t_1,t_2}(y, x) + S(y)), \tag{B.15}$$

where $A^K_{t_1,t_2}(y, x) = \min\{A_{t_1,t_2}(y, x), K(|x - y| + 1)\}$.

The first inequality in (B.3) implies that $A^K_{t_1,t_2}(y, x) = K(|x-y|+1)$ if $|x-y| > R$ with a suitable $R = R(K, t_1, t_2)$. Together with the first part of Lemma 8 this means that $A^K_{t_1,t_2}(y, x)$ is a Lipschitz function of x, with a constant $\bar{K} = \bar{K}(K, t_1, t_2)$ that does not depend on y. It now follows from (B.15) that $T_{t_1,t_2} S(x)$ is Lipschitz with the same constant. □

Now observe that by (B.3) any constant function is L-dominated with $L = C$. Using Lemmas 9–11 with $t_1 = n$, $t_2 = n + 1$ for integer $n \ge 0$, we see that the solution $S(x, t)$ of the Cauchy problem for equation (1.8) with the initial condition $S(x, 0) = 0$ stays C-dominated and therefore Lipschitz for all integer moments of time. Applying, for any noninteger $t > 0$, Lemma 11 again with $t_1 = [t]$, $t_2 = t$, we get Lipschitzness for all $t > 0$ with a suitable constant depending on the parameters of the problem.

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Isaac Newton Institute for Mathematical Sciences, 20 Clarkson Rd, CB3 0EH Cambridge, UK

Heriot-Watt University, Edinburgh, UK

Landau Institute, Moscow, Russia
E-mail address: kk262@newton.cam.ac.uk

Department of Mathematics, University of Toronto, M5S 3G3 Toronto, Ontario, Canada
E-mail address: dkhmelev@math.toronto.edu

Department of Physics, M.V. Lomonossov Moscow State University, Moscow, Russia
Current address: Laboratoire G.D. Cassini, Observatoire de la Côte d'Azur, BP 4229, 06304 Nice, France
E-mail address: ansobol@obs-nice.fr