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Abstract: The paper [1] contains description of pure (simple) spinors in a complex vector space; this particularly simple approach uses extensively Fock bases of the underlying vector space. The paper [2] describes pure spinors in a complexified vector space using a complexified version of Fock bases, better known in relativity theory as null bases, and introducing an important notion of real index. Both these papers concern spinors in an even–dimensional vector space. The present paper aims at descrption of pure spinors in odd–dimensional vector spaces; it is complementary to [1] and [2], however in one essential place (the sufficient conditions for spinor to be pure) it goes beyond [1], using the Pauli–Kofink type identity found in [3].

1. General — spinors in odd dimensions

Let $(W, \cdot)$ be a complex vector space of dimension $2m+1$ ($m = 1, 2, \ldots$) and the central dot denotes the scalar product in W. Let $\gamma : \mathrm{Cl}(W) \longrightarrow \mathrm{End}\, S$ be one of two irreducible representations of the Clifford algebra $\mathrm{Cl}(W)$ in the spinor space S of complex dimension 2^m. In contrast to [1], I have to make a distinction in notation between elements of $\mathrm{Cl}(W)$ and corresponding element of $\mathrm{End}\, S$, since γ is non–faithful. E.g. the letter u denotes a vector and also its image in the Clifford algebra, but not a corresponding endomorphism of the spinor space; the latter is denoted by $\gamma(u)$. The fundamental multiplication property in the Clifford algebra will be written in the form

$$u \vee v + v \vee u = 2u \cdot v,$$

where $u, v \in W \subset \mathrm{Cl}(W)$.

If $(e_1, \ldots, e_{2m+1})$ is an orthonormal basis of W,

$$e_\mu \vee e_\nu + e_\nu \vee e_\mu = 2\delta_{\mu\nu},$$

then we can introduce an *almost null basis* $(n_1, \ldots, n_m, p_1, \ldots, p_m, e)$ by the formulas:

$$n_\alpha = \frac{1}{2}\left(e_{2\alpha-1} - \mathrm{i}e_{2\alpha}\right), \quad p_\alpha = \frac{1}{2}\left(e_{2\alpha-1} + \mathrm{i}e_{2\alpha}\right), \quad \alpha = 1, \ldots, m; \qquad e = e_{2m+1}.$$

The elements of this basis satisfy the relations

$$\begin{array}{lll} n_\alpha \vee n_\beta + n_\beta \vee n_\alpha = 0, & p_\alpha \vee p_\beta + p_\beta \vee p_\alpha = 0, & e^2 = 1, \\ n_\alpha \vee p_\beta + p_\beta \vee n_\alpha = \delta_{\alpha\beta}, & n_\alpha \vee e + e \vee n_\alpha = 0, & p_\alpha \vee e + e \vee p_\alpha = 0. \end{array} \tag{1}$$

Null vectors of both kinds span a *maximal totally null* (MTN) subspaces of W: $N = \mathrm{span}\{n_1, \ldots, n_m\}$, $P = \mathrm{span}\{p_1, \ldots, p_m\}$, and we have the direct sum decomposition

$$W = N \oplus P \oplus \mathrm{span}\{e\}. \tag{2}$$

Conversely, given an MTN subspace N, one can find an MTN subspace P such that $N \cap P = \{0\}$ and a unit vector e orthogonal to N and P (determined up to a sign) such that the decomposition (2) holds. One can also find bases $(n_1, \ldots, n_m)$ in N and $(p_1, \ldots, p_m)$ in P such that the orthogonality conditions (1) hold. An arbitrary vector in W admits a unique decomposition

$$u = n + p + ze,$$

where $n \in N$, $p \in P$, $z \in \mathbf{C}$. Writing

$$n = \sum_{\alpha=1}^{m} x_\alpha n_\alpha, \qquad p = \sum_{\alpha=1}^{m} y_\alpha p_\alpha,$$

we have

$$u^2 \equiv u \cdot u = x_1 y_1 + \ldots + x_m y_m + z^2.$$

The orthogonal group $\mathrm{O}(V)$ (also $\mathrm{SO}(V)$) acts transitively on the set of all MTN subspaces of V; this set has a natural structure of a complex manifold of dimension $m(m+1)/2$.

Let us fix the orientation of an orthonormal basis (e_μ). The product of all $\gamma_\mu = \gamma(e_\mu)$ is proportional to the identity endomorphism of S. The two inequivalent representations of $\mathrm{Cl}(W)$ in S are distinguished by the sign below:

$$\gamma_1 \cdots \gamma_{2m+1} = \pm \mathrm{i}^m rmid_S.$$

This formula can be written in terms of the almost null basis as

$$\gamma(n_1 \wedge p_1) \cdots \gamma(n_m \wedge p_m)\gamma(e) = \pm 2^{-m}\, \mathrm{id}_S.$$

The transposing mapping $B : S \longrightarrow S^*$ (S^* is the dual of the spinor space S) is defined by

$${\gamma_\mu}^T = (-1)^m B \gamma_\mu B^{-1}. \tag{3}$$

It satisfies

$$B^T = (-1)^{m(m+1)/2}. \tag{4}$$

With any two spinors one bilinearly associate a k–vector $B_k(\phi, \psi) \in \Lambda^k W$ defined by

$$B_k(\phi, \psi) \cdot v = \langle B\phi, \gamma(v)\psi \rangle =: B(\phi, \gamma(v)\psi),$$

where v is an arbitrary k–vector. (The conventions for the scalar product of k–vectors are the same as in [2].) The formulae (3) and (4) lead to the statement that the symmetry properties of the blinear forms $B_k(\phi, \psi)$ are determined by

$$B_k(\phi, \psi) = (-1)^{(m-k)(m-k+1)/2}\, B_k(\psi, \phi). \tag{5}$$

Therefore $B_k(\psi, \psi) = 0$ for $m - k \equiv 1, 2 \pmod 4$.

2. Pure spinors

Following [3], I shall introduce two invariantly determined subspaces $M(\phi)$ and $V(\phi)$ associated with any spinor $\phi \neq 0$:

$$\begin{aligned} M(\phi) &= \{u \in W :\ \gamma(u)\phi = 0\}, \\ V(\phi) &= \{B_1(\psi, \phi) :\ \psi \in S\}. \end{aligned}$$

$M(\phi)$ is always totally null (TN), i.e. $M(\phi)^{-} \supset M(\phi)$. Consider $V(\phi)^{-}$. It consists of all $u \in W$ such that

$$B_1(\psi, \phi) \cdot u = 0 \qquad \forall \psi \in S,$$

but

$$B_1(\psi, \phi) \cdot u = \langle B\psi, \gamma(u)\phi \rangle = 0$$

implies $\gamma(u)\phi = 0$ (since B is an isomorphism), therefore $V(\phi)^{-} \subset M(\phi)$. Since it is quite obvious that $M(\phi) \subset V(\phi)^{-}$, we have $V(\phi)^{-} = M(\phi)$. It follows also that $M(\phi)^{-} = V(\phi)^{--} = V(\phi)$. These remarks apply also in the even–dimensional case. For economy of notation, I shall avoid writing $V(\phi)$ in the sequel.

The spinor ϕ is pure, if the subspace $M(\phi)$ is maximal totally null (MTN). In the even–dimensional cale it simply means $M(\phi)^{-} = M(\phi)$, in odd dimensions the situation seems to be slightly more complicated: $\dim(M(\phi)^{-}/M(\phi)) = 1$.

Do there exist pure spinors in dimensions $3, 5, \ldots$? Since the representation of the even Clifford algebra $\mathrm{Cl}_{+}(W)$ is faithful, there should exist $\chi \in S$ such that either $\omega = \gamma(n_1)\cdots\gamma(n_m)\chi$ or $\omega = \gamma(n_1)\cdots\gamma(n_m)\gamma(e)\chi$ (depending on the parity of m) is nonzero. Then $M(\omega) = N$ and ω is pure.

The spinor ω, similarly as in the even–dimensional case, can be thought as a "vacuum state" annihilated by the operators $\gamma(n_1), \ldots, \gamma(n_m)$. The collection of 2^m spinors

$$\begin{aligned} &\omega; \gamma(p_1)\omega, \ldots, \gamma(p_m); \\ &\gamma(p_1 \vee p_2)\omega, \ldots, \gamma(p_{m-1} \vee p_m)\omega; \\ &\ldots; \gamma(p_1 \vee \cdots \vee p_m)\omega \end{aligned} \tag{6}$$

is linearly independent in virtue of the anticommutation relations satisfied by n_α and p_β. Therefore this is a basis of S. All the spinors occuring in (6) are pure, e.g. the subspace

$$M(\gamma(p_{k+1})\cdots\gamma(p_m)\omega) = \mathrm{span}\{n_1,\ldots,n_k,p_{k+1},\ldots,p_m\}$$

is TN and of dimension m.

The direction of a pure spinor is determined by corresponding MTN subspace. Indeed, let $\omega' \in S$ be such that $M(\omega') = M(\omega)$ and let

$$\omega' = \xi_0\,\omega + \xi_1\,\gamma(p_1)\omega + \cdots + \xi_{1\ldots m}\,\gamma(p_1\cdots p_m)\omega.$$

Multiplication by succesive products of $\gamma(n_\alpha)$ gives $\xi_I = 0$ for any multiindex $I \neq 0$. Therefore, analogously like in the even–dimensional case, there holds

Proposition 1: There is a natural one–to–one correspondence between the set of all MTN subspaces of W and the set of directions of pure spinors.

What can be said about the spinor $\gamma(e)\omega$, which is not present in (6)? The anticommutation relations for e and n_α lead immediately to the conclusion that $M(\gamma(e)\omega) = M(\omega)$. It follows from Proposition 1. that $\gamma(e)\omega \sim \omega$ and, since $e^2 = 1$, we have $\gamma(e)\omega = \pm\omega$.

Proposition 2: Let ω and ϕ be linearly independent pure spinors. There is a basis of S of the form (5) such that ϕ is one of the basis vectors other than ω.

Proof mutatis mutandis can be transferred from [1].

Proposition 3: The group $\mathrm{Spin}(2m+1)$ acts transitively on the set of directions of all simple spinors.

Proof: Following Proposition 2, if ϕ and ω are two pure spinors having different directions, then there holds a relation of the form

$$\begin{aligned}\phi &= \gamma(p_{k+1})\cdots\gamma(p_m)\omega\\ &= \gamma(p_{k+1}+n_{k+1})\cdots\gamma(p_m+n_m)\omega.\end{aligned}$$

Since all factors above contain reflections, the group $\mathrm{Pin}(2m+1)$ acts transitively. At the same time holds

$$\phi = \gamma(p_{k+1}+n_{k+1})\cdots\gamma(p_m+n_m)\gamma(\pm e)\omega.$$

One of the products above must represent an element of the group $\mathrm{Spin}(2m+1)$. Q.E.D.

Proposition 4: A necessary and sufficient condition for a spinor ω to be simple is that the multivectors $B_k(\omega,\omega) = 0$ for $k = 0,1,\ldots,m-1,m+2,\ldots,2m+1$. Moreover, the only nonvanishing multivectors of this form are:

$$B_m(\omega,\omega) \sim n_1 \wedge \cdots n_m, \tag{7}$$

and

$$B_{m+1}(\omega,\omega) = \pm B_m(\omega,\omega)\wedge e. \tag{8}$$

Proof: To prove necessity, consider the action of the endomorphism $\gamma(n_1\wedge\cdots\wedge n_m) = \gamma(n_1)\cdots\gamma(n_m)$ on elements of a basis (6) of S constructed out of the pure spinor ω. It is clear the result will be nonvanishing only for the last element of the basis,

$$0 \neq \gamma(n_1\wedge\cdots\wedge n_m)\gamma(p_1)\cdots\gamma(p_m) \sim \omega.$$

Therefore $\mathrm{im}(\gamma(n_1 \wedge \cdots \wedge n_m)) = \mathrm{span}\{\omega\}$. This means that there exists a spinor ϕ such that

$$\gamma(n_1 \wedge \cdots \wedge n_m) = \omega \otimes B\phi. \tag{9}$$

But, according to (5), $B_m(\psi, \chi) = B_m(\chi, \psi)$, which gives

$$B(\psi, \gamma(n_1 \wedge \cdots \wedge n_m)\chi) = B(\chi, \gamma(n_1 \wedge \cdots \wedge n_m)\psi),$$

and this formula together with (9) yields

$$B(\phi, \chi)B(\psi, \omega) = B(\phi, \psi)B(\chi, \omega).$$

Since the spinors ψ and χ above are arbitrary, we get $\phi \sim \omega$, thus

$$\omega \otimes B\omega \sim \gamma(n_1 \wedge \cdots \wedge n_m). \tag{10}$$

The tensor product $\phi \otimes B\psi$ can always be expressed in terms of the basis $(\gamma_{\mu_1} \cdots \gamma_{\mu_k})$ (where $1 \leq \mu_1 < \ldots < \mu_k \leq 2m+1$ and $0 \leq k \leq m$)) of End S:

$$\phi \otimes B\psi = 2^{-m} \sum_{k=0}^{m} \gamma(B_k(\psi, \phi)). \tag{11}$$

Applying (11) to (10), we get the proof of necessity and moreover the result (7).

To prove (8), notice that, if u is a vector and v a $(p-1)$–vector, then [2]

$$v \wedge u = (-1)^p u \rfloor v + v \vee u.$$

For any m–vector v we have then

$$B_{m+1}(\omega, \omega) \cdot (v \wedge n_\mu) = (-1)^{m+1} B(\omega, (n_\mu \rfloor v)\omega) = (-1)^{m+1} B_{m-1}(\omega, \omega) \cdot (n_\mu \rfloor v) = 0,$$

because the first factor above is zero. Similarly

$$B_{m+1}(\omega, \omega) \cdot (v \wedge e) = B(\omega, \gamma(v)\gamma(e)\omega) = \pm B(\omega, \gamma(v)\omega) = \pm B_m(\omega, \omega) \cdot v.$$

¿From that follows immediately the formula (8).

The sufficiency proof follows closely that of Urbantke [3] and is based on his form of the identity of the Pauli–Kofink type:

$$B_1(\phi, \omega) \cdot B_1(\omega, \chi) - (-1)^m B(\phi, \omega) B(\omega, \chi) = \sum_{k=0}^{m-1} B_k(\omega, \omega) \cdot B_k(\phi, \chi) d_k, \tag{12}$$

where ϕ, ω and χ are arbitrary spinors and coefficients d_k (whose exact form can be derived from [3]) do not vanish for $m - k \equiv 0, 3 \pmod 4$.

We notice that the nullity conditions, $B_k(\omega,\omega)=0$, lead, consistently with the identity (12), to

$$B_1(\phi,\omega)\cdot B_1(\omega,\chi)-(-1)^m B(\phi,\omega)B(\omega,\chi)=0. \tag{13}$$

Consider the map $P_\omega : \phi \longmapsto B_1(\phi,\omega)$ and the subspace $\omega^- = \{\phi\in S : B(\phi,\omega)=0\}$ of S. Since B is non–degenerate, the subspace ω^- has codimension 1. Thus

$$\dim S = \dim \mathrm{im}\, P_\omega + \dim \ker P_\omega = \dim M(\omega)^- + \dim \ker P_\omega \tag{14}$$

and

$$\dim \omega^- = \dim \mathrm{im}\, P_\omega|_{\omega^\perp} + \dim \ker P_\omega|_{\omega^\perp}. \tag{15}$$

But if follows from (13) that $\mathrm{im}\, P_\omega|_{\omega^\perp} = M(\omega)$ and also that $\ker P_\omega|_{\omega^\perp} = \ker P_\omega$. Subtracting (15) from (14), we get

$$1 = \dim M(\omega)^- - \dim M(\omega),$$

which means that $M(\omega)$ is MTN. Q.E.D.

Proposition 5: If ϕ and ψ are pure spinors, then the dimension of the intersection $M(\phi)\cap M(\psi)$ is the least integer k such that $B_k(\phi,\psi)\neq 0$. The multivector $B_k(\phi,\psi)$ is proportional to the exterior product of the vectors of a basis of this intersection.

Proof: We can find an almost null basis such that

$$\begin{aligned} M(\phi) &= \mathrm{span}\{n_1,\dots,n_m\},\\ M(\psi) &= \mathrm{span}\{n_1,\dots,n_k,p_{k+1},\dots,p_m\},\\ \psi &= \gamma(p_{k+1}\vee\cdots\vee p_m)\phi, \end{aligned}$$

and $k = \dim\,(M(\phi)\cap M(\psi))$. Since

$$p_m\vee n_1\vee\cdots\vee n_m = (-1)^{m-1} n_1\vee\cdots n_{m-1}\left(\frac{1}{2}+p_m\wedge n_m\right),$$

the recursive procedure gives

$$\begin{aligned} p_{k+1}\vee\cdots p_m\vee n_1\vee\cdots\vee n_m &= (\frac{1}{2})^{m-k}(-1)^{k(2m+1-k)/2} n_1\wedge\cdots\wedge n_k\\ &+ (k+2)\text{–vector} + \ldots + (2m-k)\text{–vector}. \end{aligned} \tag{16}$$

On the other hand,

$$\begin{aligned} \gamma(p_{k+1})\cdots\gamma(p_m)\psi\otimes B\phi &= 2^{-m}\sum_{l=0}^{m}\gamma(B_l(\phi,\psi))\\ &\sim \gamma(p_{k+1}\cdots\gamma(p_m)\gamma(n_1)\cdots\gamma(n_m). \end{aligned}$$

The representation γ, when applied to the tail of the formula (16), gives images of multivectors of degrees higher then k, e.g. $\gamma((2m-k)$–vector) is a γ–image of a $(k+1)$–vector. This observation proves the proposition. Q.E.D.

In 3 and 5 dimensions the nullity conditions are empty, i.e. each spinor $\neq 0$ is pure. In 7 dimensions there appears the first constraint $B_0(\phi,\phi)=0$, in 9 dimensions we have two constraints $B_0(\phi,\phi)=0$ and $B_1(\phi,\phi)=0$ and so on.

3. Real index of a pure spinor

Let now W be a complexification of a real vector space V, $W=\mathbf{C}\otimes V$, and let the scalar product $\cdot$ in W be inherited from V, where it has the signature (k,l), $k+l=2m+1$.

If N is a TN subspace of W, it can be decomposed as

$$N=\mathbf{C}\otimes K\oplus M_0\oplus M, \qquad 17)$$

where K consists of all real vectors belonging to N, $\mathbf{C}\otimes K=N\cap\bar{N}$, M_0 is a subspace in ${}^{-}N=\{u\in N : u\cdot\bar{v}=0\ \forall v\in N\}$ complementary to $\mathbf{C}\otimes K$, and the Hermitian scalar product $(u,v)\longmapsto\bar{u}\cdot v$ in M is nondegenerate. Although the decomposition (14) is nonunique, the dimensions r, n_0 of $N\cap\bar{N}$ and M_0 respectively and the signature (n_+,n_-) of the Hermitian scalar product in M are determined uniquely by N. Following the argumentation of [2], we get the inequalities

$$r+2n_0+2n_+\leq k, \qquad r+2n_0+2n_-\leq l. \qquad (18)$$

If they are satisfied for some nonnegative integers r, n_0, n_+ and n_-, one can construct a suitable TN subspace N of W.

If N is MTN, then

$$r+n_0+n_++n_-=m. \qquad (19)$$

Taking into account the fact that $k+l=2m+1$, we get from (18) and (19)

$$n_0=0$$

and

$$2n_+=\begin{cases}k-1-r,\\ k-r,\end{cases}\quad \text{and respectively} \qquad 2n_-=\begin{cases}l-r,\\ l-1-r.\end{cases} \qquad (20)$$

For a given value of r there is only one integer solution (n_+,n_-) of (20). Therefore the *real index* r determines uniquely all interesting integer parameters characterizing an MTN subspace N and is restricted by

$$r=0,1,\ldots,\min(k,l). \qquad (21)$$

In contrast to the even–dimensional case [2], there are no empty places in the sequence (21).

Proposition 5: Let N and P be two MTN subspaces of $W=\mathbf{C}\otimes V$ such that $\dim(N\cap P)=m-1$. Then the following four conditions are equivalent:

I. ${}^{-}(N\cap P)=N\cap P\cap\overline{N\cap P}$,

II. N and P have the same real index,

III. There exists a real vector $u\in N+P$ such that $u^2=\pm1$,

IV. There exists a real reflection mapping N onto P, $P=u\vee N\vee u^{-1}$.

Proof is identical to that of Theorem 1 in [2].

Proposition 6: The groups $\mathrm{O}(k,l)$ and $\mathrm{SO}(k,l)$ act transitively on each set of all MTN subspaces of $W = \mathbf{C} \otimes V$ with a given real index.

Proof for the group $\mathrm{O}(k,l)$ is identical to that of Theorem 2 in [2].

If $\Re N$ denotes a subspace of N consisting of real parts of all vectors belonging to the MTN subspace N of W, then

$$\Re N = K \oplus \Re M,$$

where the signature of the scalar product in $\Re M$ is $(2n_+, 2n_-)$. The scalar product in $(\Re M)^-$ has signature $(k - 2n_+, l - 2n_-)$ which is either $(r-1, r)$ or $(r, r+1)$. Therefore, there exists a real MTN subspace L of $(\Re M)^-$ and a vector $e \in (\Re M)^-$ orthogonal to K and L and normalized ($e^2 = +1$ or -1) such that

$$(\Re M)^- = K \oplus L \oplus \mathrm{span}_{\mathbf{R}}\{e\}.$$

The vector e is determined up to a sign. We have also the decompositions

$$\begin{aligned} V &= K \oplus L \oplus \mathrm{span}_{\mathbf{R}}\{e\} \oplus \Re M, \\ W &= \mathbf{C} \otimes K \oplus \mathbf{C} \otimes L \oplus \mathrm{span}\{e\} \oplus M \oplus \overline{M}. \end{aligned} \tag{22}$$

It is clear that $e \vee N \vee e^{-1} = N$, thus the vector e provides us with a real reflection belonging to the stability subgroup of N in the orthogonal group $\mathrm{O}(k,l)$. Since there exists an element of $\mathrm{O}(k,l)$ mapping N onto P, there exists also an element of $\mathrm{SO}(k,l)$ performing this operation. Q.E.D.

Remark. For odd dimensions, only if $\min(k,l) = 0$ (the Euclidean case), the action of $\mathrm{O}(k,l)$ on the set of all MTN subspaces is transitive. For even dimensions, the fact occured also in the case $\min(k,l) = 1$.

Proposition 7: The connected component of unity $\mathrm{SO}_o(k,l)$ of the group $\mathrm{O}(k,l)$ acts transitively on each set of all MTN subspaces of $W = \mathbf{C} \otimes V$ with a given real index $r > 0$.

Proof: Mutatis mutandis that of Theorem 3 of Ref. [2].

Let us introduce a basis $(k_i, l_j, e, m_A, \bar{m}_{\dot{B}})$ of W adapted to the decomposition (22). Here and below $i, j, \ldots = 1, \ldots, r$, $A, B, \ldots, \dot{A}\dot{B}, \ldots = 1, \ldots, m - r$. The only nonvanishing scalar products of the vectors of this basis are

$$k_i \cdot l_j = \frac{1}{2}\delta_{ij}, \qquad m_A \cdot \bar{m}_{\dot{B}} = \eta_{A\dot{B}}, \qquad e \cdot e = \pm 1 =: \epsilon,$$

where

$$(\eta_{A\dot{B}}) = \mathrm{diag}(\underbrace{1, \ldots, 1}_{n_+}, \underbrace{-1, \ldots, -1}_{n_\perp}).$$

Any linear map $A : V \longrightarrow V$ which leaves N invariant, $AN = N$, can be described in terms of this basis as (c.f. [4])

$$\begin{aligned} Ak_i &= a^j{}_i k_j, \\ Am_A &= b^j{}_A k_j + c^B{}_A m_B, \\ Al_i &= d^j{}_i k_j + e^B{}_i m_B + \bar{e}^{\dot{B}}{}_i \bar{m}_{\dot{B}} + f^j{}_i l_j + g_i e, \\ Ae &= h^j k_j + i^B m_B + \bar{\imath}^{\dot{B}} \bar{m}_{\dot{B}} + j^j l_j + \alpha e, \end{aligned}$$

where a, d, f, g, h, j, α are real and $\det a \neq 0$. The map A belongs to $\mathrm{O}(k,l)$ if, and only if,

$$
\begin{aligned}
&c^\dagger \eta c = \eta, \qquad f = \left(a^T\right)^{-1}, \qquad b = -a e^\dagger \eta c,\\
&d^T f + f^T d + e^T \eta \bar{e} + e^\dagger \eta e + 2 g g^T \epsilon = 0,\\
&j = 0, \qquad i = 0, \qquad f^t h = -2 g k \epsilon, \qquad \alpha = \pm 1.
\end{aligned} \tag{23}
$$

Instead of d we can use $s = d^T f$, then (23) shows that its symmetric part is determined by e and g, whereas its skewsymmetric part $t = s - s^T$ is arbitrary. Therefore, an element A of the stability subgroup in $\mathrm{O}(k,l)$ of the subspace N with the real index equal to r is parametrized by: $(a, c, e, t, g, \alpha) \in \mathrm{GL}(r, \mathbf{R}) \times U(n_+, n_-) \times (\mathbf{C}^r \otimes \mathbf{C}^{n_+ + n_\perp}) \times (\mathbf{R}^r \wedge \mathbf{R}^r) \times \mathbf{R}^r \times \mathbf{Z}_2$. The dimension of the stability subgroup is

$$
r^2 + (m-r)^2 + 2r(m-r) + \frac{1}{2} r(r-1) + r = m^2 + \frac{1}{2} r(r+1).
$$

The dimension of the orbit is

$$
m(m+1) - \frac{1}{2} r(r+1). \tag{24}
$$

Notice that according to (24), only these MTN subspaces and pure spinors for which $r = 0$ are generic, i.e. they form open sets in the topological spaces of all MTN subspaces and of all directions of pure spinors, respectively. This situation is distinct from that in the even–dimensional case, when also MTN subspaces and pure spinors with $r = 0$ were generic.

Let us introduce the charge conjugation $C : S \longrightarrow \bar{S}$ ny the formula

$$
\bar{\gamma}_\mu = (-1)^{(k-l)(k-l-1)/2} C \gamma_\mu C^{-1}.
$$

The charge conjugate of the spinor ϕ is then $\phi_c = \bar{C}\bar{\phi} \in S$. The same argumentation as that in [2] leads to

Proposition 8: There always holds $M(\phi_c) = \overline{M(\phi)}$.

For any MTN subspace N, we can define its Kähler bivector $j \in \Lambda^2 W$, provided we distinguished in advance a subspace $M \subset N$:

$$
j \cdot (\bar{u} \wedge v) = \begin{cases} \mathrm{i}\bar{u} \cdot v, & \text{for any } u, v \in M; \\ 0, & \text{otherwise.} \end{cases}
$$

The Kähler bivector uniquely determines a complex structure in the space

$$
\tilde{N} = \Re(N/(N \cap \bar{N}))
$$

by mappings of the form $\tilde{N} \ni u \longmapsto n \rfloor \tilde{j} \in \tilde{N}$.

Proposition 9: The algebraic constraints for a pure spinor ϕ to have the real index equal to r are

$$
B_r(\phi_c, \phi) \neq 0, \qquad B_{r-1}(\phi_c, \phi) = 0. \tag{25}
$$

The only nonvanishing multivectors of the form $B_q(\phi_c, \phi)$ are

$$B_r(\phi_c, \phi) \sim k_1 \wedge \cdots \wedge k_r, \tag{26}$$

$$B_{r+2p}(\phi_c, \phi) = (\mathrm{i}^{-p}/p!)\, B_r(\phi_c, \phi) \wedge \wedge^p j, \tag{27}$$

for $p = 1, 2, \ldots, m - r$, and

$$B_{r+2p+1}(\phi_c, \phi) = \pm\epsilon B_{r+2p}(\phi_c, \phi), \tag{28}$$

for $p = 0, 1, \ldots, m - r$.

Proof: The formulas (25) and (26) follow from Prop. 8, Prop. 5 and its proof.

To prove (28) for $p = 0$, notice that

$$\begin{aligned} B_{r+1}(\phi_c, \phi) \cdot (l_1 \wedge \cdots \wedge l_r \wedge e) &= B(\phi_c, \gamma(l_1) \cdots \gamma(l_r)\gamma(e)\phi) \\ = \pm B(\phi_c, \gamma(l_1) \cdots \gamma(l_r)\phi) &= \pm B_r(\phi_c, \phi) \cdot (l_1 \wedge \cdots \wedge l_r). \end{aligned}$$

and from (26) it follows that

$$B_{r+1}(\phi_c, \phi) = \pm\epsilon B_r(\phi_c, \phi) \wedge e.$$

To prove (27), notice that the statements concerning the expression

$$B_{r+q}(\phi_c, \phi) \cdot \left(l_{i_1} \wedge \cdots \wedge l_{i_\lambda} \wedge \bar{m}_{\dot{A}_1} \wedge \cdots \bar{m}_{\dot{A}_\nu} \wedge m_{B_1} \wedge \cdots \wedge m_{B_\mu} \wedge k_{j_1} \wedge \cdots \wedge k_{j_\kappa}\right)$$

in the proof of Theorem 5 of Ref. [2] remain valid. In particular, the expression does not vanish only if $\kappa = 0$, $\lambda = r$ and $\mu = \nu = q/2 =$ an integer. Considering a similar expression,

$$B_{r+q}(\phi_c, \phi)\cdot\left(l_{i_1} \wedge \cdots \wedge l_{i_\lambda} \wedge \bar{m}_{\dot{A}_1} \wedge \cdots \bar{m}_{\dot{A}_\nu} \wedge e \wedge m_{B_1} \wedge \cdots \wedge m_{B_\mu} \wedge k_{j_1} \wedge \cdots \wedge k_{j_\kappa}\right) \tag{29}$$

we get, modulo a sign,

$$\begin{aligned} &B_{r+q-2\mu-2\kappa}(\phi_c, \phi) \cdot (m_{B_1} \rfloor (\ldots (m_{B_\mu} \rfloor (k_{j_1} \rfloor (\ldots (k_{j_\kappa} \rfloor \\ &\quad (l_{i_1} \wedge \cdots \wedge l_{i_\lambda} \wedge \bar{m}_{\dot{A}_1} \wedge \cdots \wedge \bar{m}_{\dot{A}_\nu} \wedge e)) \ldots))) \ldots)). \end{aligned} \tag{30}$$

If $2\kappa + 2\mu > q$, then in (30) we get $B_{r+q-2\mu-2\kappa}(\phi_c, \phi) = 0$; moreover the interior products vanish if $\mu > \nu$ or $\kappa > \lambda$. Since $(\phi_c)_c \sim \phi$, we have $B_p((\phi_c)_c, \phi_c) \sim B_p(\phi_c, \phi)$, and since $M(\phi_c) = \mathbf{C} \otimes K \oplus \overline{M}$, the remarks concerning vanishing of (29) $= \pm$(30) hold too, if the roles of μ and ν are reversed. To have nonvanishing expression (29) we need $\mu = \nu$ and, since now $r + q = \kappa + \lambda + \mu + \nu + 1$, we obtain $\lambda + 1 \geq r + \kappa$. These conditions lead to the following three solutions

$$\kappa = 0, \qquad \lambda = r, \qquad 2\mu = q; \tag{31}$$

$$\kappa = 0, \qquad \lambda = r, \qquad 2\mu = q - 1; \tag{32}$$

$$\kappa = 1, \qquad \lambda = r, \qquad 2\mu = q - 2. \tag{33}$$

The degree $r + q - 2\mu - 2\kappa$ of the B–multivector in (30) is r in the cases (31) and (33) and $r + 1$ in the case (32). Therefore, in virtue of (26), the expression (29) vanishes in the cases (31) and (33). Since in the case (32) q is odd, nonvanishing of (29) has no meaning for the value of the multivector on the left hand side of (27). It follows that the formula (27) can be proven using the same argumentation as in [2]. The simplest way of proving (28) for $p > 0$ is to repeat the argumentation for the case $p = 0$ using (27) instead of (26) as the starting point. Q.E.D.

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