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Multiplicative Structures of 2-dimensional Topological Quantum Field Theory[1]

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Abstract

Category theory provides a uniform method of encoding mathematical structures and universal constructions with them. In this article we apply the method of additional structures on the objects of a category to deform a comonoid structure, used implicitly in all categories. To deform this comultiplication we consider internal categories in a monoidal category with some special properties. Then we consider structures over comonoids and show that deformed internal categories form a 2-category. This provides the possibility to study, in a uniform way, different types of generalized multiplicative and comultiplicative structures of 2-dimensional Topological Quantum Field Theory.

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Contents

1 Introduction

For physics the language of category theory opens new possibilities, especially through new structures and constructions [32]. In recent years there has been an increasing interest in algebraic structures on a category motivated by coherence problems arising from topological quantum field theory (TQFT). The categories of representations of quantum groups are braided monoidal categories [6, 7]. Another motivation comes from developments in homotopy theory, in particular, models for the stable homotopy category. Monoidal categories correspond to loop spaces, and the group completion of the classifying space of a braided monoidal category is a two-fold loop space [18]. Modular categories are monoidal categories with additional structure (braiding, twist, duality, a finite set of dominating simple objects satisfying a non-degeneracy axiom). If we remove the last axiom, we get a pre-modular category. A pre-modular category provides invariants of links, tangles, and sometimes of 3-manifolds.

Any modular category yields a Topological Quantum Field Theory (TQFT) [19, 25, 17]. The interest in modular tensor categories comes e.g. from the fact that such a category contains the data needed for the construction of a two- and three-dimensional TQFTs.

Here we describe different structures connected with multiplicative and comultiplicative structures. Any structure type defines the category of these structures and its forgetful functor. We try to define properties of the forgetful functor that they define the structure type.

Previous works [28, 29, 32, 34] attempted to formulate the method of additional structures as a set of axioms for a category, which would be sufficient for an abstract expression of the basic concepts of the theory of structures on objects of a category. Then all main properties of a structure are properties of its forgetful functor. Additional (external) structures on objects of a category provide the possibility to construct new categories for physics.

In physics, interest in n-categories was sparked by developments in relating topology and quantum field theory. In 1985 Jones came across a invariant of knots while studying von Neumann operator algebras introduced for the mathematical foundations of quantum theory. Then this Jones polynomial was generalized to a family of knot invariants, which could be systematically derived from quantum groups, invented in exactly soluble 2-dimensional field theories. Then Witten arrived at a manifestly 3-dimensional approach to the new knot invariants, deriving them from a quantum field theory in 3-dimensional spacetime (Chern-Simons-Witten theory). This approach also gave invariants of 3-dimensional manifolds.

Atiyah in 1988 formulated an axiomatic setup for TQFTs. Independently

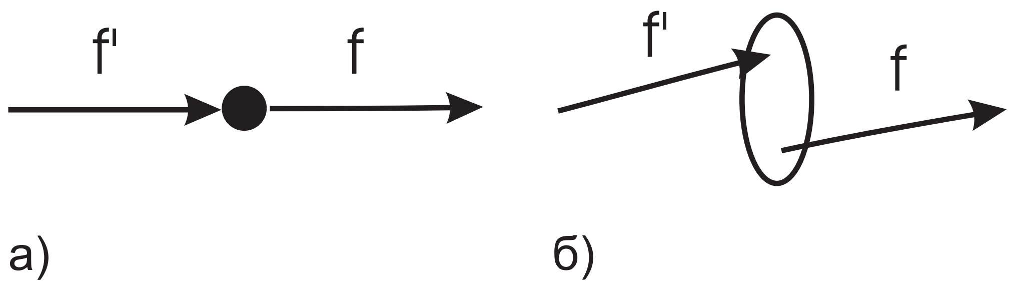

Figure 1:

and at about the same time G. Segal gave a mathematical definition of conformal field theories (CFTs), which is very similarly based on categories and functors.

Categories play a centre role in mathematical formulation of TQFT. An n-dimensional TQFT is defined as a monoidal functor from the category $\mathcal{C}\mathrm{ob}_{\,n+1}$ of oriented $(n+1)$-cobordisms with disjoint union as tensor product to the category $\mathcal{V}\mathrm{ect}$ of complex finite dimensional vector spaces with usual tensor product of vector spaces.

The main idea of the text is illustrated by the Figure 1:

a) Composition in classical case, for $f \circ f'$ we need the identification $s(f) = t(f')$, which is not natural for the quantum case.

b) Composition in the quantum case, for $f \circ f'$ we have the 'quantum' condition $(s(f), t(f')) \in \delta(C)$, where δ is a comultiplication on C.

The paper is organized as follows:

In Section 2 the general concept of additional (or external) structures on objects of a category is considered for application in terms of monoidal categories. In Section 3 some construction in monoidal categories are described in details. First we list the well known definitions of monoid and comonoid. In Subsection 3.4 we describe the monoidal category $\overline{\mathcal{S}\mathrm{et}}$ of sets and multi-valued maps. Then in Subsection 3.5 we define the cup-product of morphisms from a comonoid to a monoid. In Subsection 3.6 a monoidal structure on functors between monoidal categories is defined. In Subsection 3.7 constructions for operads are considered. At the end we consider the 2-category structure, which is deformed in the paper. In Section 4 at first two dualities between comonoid and monoid are described. In Subsection 4.2 we introduce multiplicative morphisms, which gives a generalization of multiplicative elements [30, 31]. Then we consider constructions which lead to the notion of a quantum category as a specific internal category wich contain the deformed comonoid structure on "the object of objects". The connection between a composition in a category and a comonoid structure on the 'object of objects' is the central notion in the paper. In the other subsections we give general description of such internal categories over comonoids. Section 5 contains a general description of the structure of TQFT to explore possible constructions for our deformations. In two- and three-dimensional quantum field theory modular tensor categories have become an indispensable tool for studying braid statistics, quantum symmetries and Turaev's type of TQFT etc. The analysis presented in this section is primarily inspired by problems in two-dimensional TQFT. Concrete applications of our analysis to TQFT form the subject of a forthcoming paper.

We use usual categorical notation (see, for example, [1, 2, 3, 4]). A category $\mathcal{C}$ has the set $\mathrm{Ob}\,(\mathcal{C})$ of objects, the set $\mathrm{Mor}\,(\mathcal{C})$ of morphisms, two maps $s,t:\mathrm{Mor}\,(\mathcal{C})\rightrightarrows\mathrm{Ob}\,(\mathcal{C})$, source and target, any morphism $f\in\mathrm{Mor}\,(\mathcal{C})$ we consider as an arrow $f:s(f)\to t(f)$, id_X is the identity morphism of the object $X\in\mathrm{Ob}\,(\mathcal{C})$,

$$\mathrm{id}\,:\mathrm{Ob}\,(\mathcal{C})\to\mathrm{Mor}\,(\mathcal{C}):X\to\mathrm{id}_X,$$

$\mathcal{C}(X,Y):=(s,t)^{-1}(X,Y)$ denotes all morphisms of the category $\mathcal{C}$ with source X and target Y, the composition $f\circ g$ of morphisms f and g is defined if and only if $s(f)=t(g)$. Note that the definition used a diagonal comonoid structure on $\mathrm{Ob}\,(\mathcal{C})$ which is defined by the diagonal map $\delta(x)=(x,x)$ and counit map $\varepsilon:\mathrm{Ob}\,(\mathcal{C})\to\star$, where $\star$ is an one-point set, a unit object in the monoidal category $\mathcal{S}\mathrm{et}$. A category is essentially small if it is equivalent to a small category, i.e. one whose class of objects is a set. A functor $F:\mathcal{C}\to\mathcal{C}'$ is faithful if it is injective on sets of morphisms $F(X,Y):\mathcal{C}(X,Y)\to\mathcal{C}'(F(X),F(Y))$ for all $X,Y\in\mathrm{Ob}\,(\mathcal{C})$. Let $\mathcal{S}\mathrm{et}$ be a small category of sets and functions, and $\mathcal{C}\mathrm{at}$ a category of small categories and functors. We deal with such categories only.

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2 Additional Structures

Categorical methods have proved to be particularly useful when studying various settings for handling the required structures. In this section we give the description of good categorical settings for the development general structure theory.

To use the categorical language more effectively we introduce the general concept of an additional structure of a given type on objects of a category, [5]. This is the concept of concrete category but over any category [28, 29, 32]. This method applied to a monoidal Kleisli category and to a Cayley-Klein category is presented, respectively, in [34] and [35, 36].

We consider any category $\mathcal{C}$ as an analogue of the category $\mathcal{S}\mathrm{et}$, but there is a difference: a set X is a set of points, then as an object $C \in \mathrm{Ob}\,(\mathcal{C})$ in general does not have an inner structure, more precisely it is defined by sets $\mathcal{C}(C, C')$, $\mathcal{C}(C', C)$ for all $C' \in \mathrm{Ob}\,(\mathcal{C})$, i.e. by relations with all other objects. Usually an additional structure arises on a set X connected with points of X, as by inner constructions.

In physics a disappearance of points under quantization requires to transfer structures onto category, in the algebra of observables. Thus, from structures on phase space we go to structures on an algebra of functions, observables, and on morphisms. Another problem is to find appropriate structures on categories of physical systems.

This is the fist step of general iterative constructions of the transferring structures in n-categories from j-morphisms to $(j+1)$-morphisms (see [22, 9]).

2.1 Definition

In categorical terms a type of a structure on objects of a category $\mathcal{C}'$ is given by a special functor $J : \mathcal{C} \to \mathcal{C}'$. Then an object $Y \in \mathrm{Ob}\,(\mathcal{C})$ is considered as an object $J(Y)$ of $\mathcal{C}'$ with the additional $\mathcal{C}$-structure Y and a morphism $f \in \mathcal{C}(X, Y)$ is considered as an morphism $J(f) \in \mathcal{C}(J(X), J(Y))$, compatible with $\mathcal{C}$–structures X and Y on $J(X)$ and $J(Y)$.

DEFINITION 1 *We say that a functor $J : \mathcal{C} \to \mathcal{C}'$ defines an additional $\mathcal{C}$-structure on objects of the category $\mathcal{C}'$ if*

(1) the functor J is faithful, i.e. $\forall X, Y \in Ob\,(\mathcal{C})$ the map $J : \mathcal{C}(X; Y) \to \mathcal{C}'(J(X), J(Y))$ is injective,

(2) for isomorphisms of $\mathcal{C}'$ transfer $\mathcal{C}$-structures, i.e. for all $X \in Ob\,(\mathcal{C})$, $Y' \in Ob(\mathcal{C}')$ and an isomorphism $u : Y' \to J(X)$ there is an object

$Y \in Ob(\mathcal{C})$ (it is denoted by u^*X) and an isomorphism $\tilde{u} : Y \to X$ such that $J(Y) = Y'$ and $J(\tilde{u}) = u$.

Such a functor J is called a forgetful functor.

Almost all usual mathematical structures are structures on sets in this sense and there are corresponding forgetful functors to the category $\mathcal{S}ets$ of sets.

EXAMPLE 1. Functor $J : \mathcal{C}\mathrm{at} \to \mathcal{S}\mathrm{et} : \mathcal{C} \mapsto \mathrm{Mor}\,\mathcal{C}$ is a forgetful functor.

EXAMPLE 2. Let $\mathcal{M}or(\mathcal{C})$ be a category with objects $f \in \mathrm{Mor}\,(\mathcal{C})$ and morphisms from f to f' are pairs (u, v) such that $f' \circ u = v \circ f$. There is the forgetful functor

$$J : \mathcal{M}or(\mathcal{C}) \to \mathcal{C} \times \mathcal{C} : (f : X \to Y) \mapsto (X, Y).$$

We say that a forgetful functor $F : \mathcal{D} \to \mathcal{M}or(\mathcal{C}))$ defines a $\mathcal{D}$-structure on morphisms of the category $\mathcal{C}$.

EXAMPLE 3. Let $B \in \mathrm{Ob}\,(\mathcal{C})$. The category $\mathcal{C}_{\downarrow B}$ of objects over B is a subcategory of $\mathcal{M}or(\mathcal{C})$ with target B and morphisms in the form $(u, \mathrm{id}\,_B)$. There is the forgetful functor

$$J : \mathcal{C}_{\downarrow B} \to \mathcal{C} : (f : X \to B) \mapsto X.$$

EXAMPLE 4. A forgetful functor $J : \mathcal{D} \to \mathcal{C}$ defines the forgetful functor

$$J_* : \mathcal{C}'\mathcal{D} \to \mathcal{C}'\mathcal{C} : F \to J \circ F.$$

2.2 Operations over Structures

For these general structures we can set up the usual construction:

— inverse and direct images of $\mathcal{C}$-structures;

— restrictions of $\mathcal{C}$-structures on subobjects,

— different products of structures.

We define the category $Str(\mathcal{C})$ of forgetful functors to the category $\mathcal{C}$. It is a full subcategory of the category $\mathcal{C}\mathrm{at}_{\downarrow\mathcal{C}}$ of all categories over $\mathcal{C}$.

Let $J : \mathcal{D} \to \mathcal{C}'$ be a forgetful functor. Then any functor $F : \mathcal{C} \to \mathcal{C}'$ defines a structure on objects of the category $\mathcal{C}$ (pullback)

$$\begin{array}{ccc} F^*\mathcal{D} & \longrightarrow & \mathcal{D} \\ {\scriptstyle J'}\downarrow & & \downarrow{\scriptstyle J} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{C}' \end{array}$$

Thus, the $F^*\mathcal{D}$-structure on $C \in \mathrm{Ob}\,(\mathcal{C})$ is simply a $\mathcal{D}$-structure on $F(C)$.

To transfer structures defined on sets onto objects of a category $\mathcal{C}$ we can use the (point) functors

$$h_C : \mathcal{C}^\circ \to \mathcal{S}\mathrm{et} \,: X \mapsto \mathcal{C}(X, C),$$
$$h : \mathcal{C} \to \mathcal{C}^\circ\mathcal{S}\mathrm{et} \,: C \mapsto h_C.$$

A forgetful functor $J : \mathcal{D} \to \mathcal{S}\mathrm{et}$ defines a $\hat{\mathcal{D}}$-structure on objects of category $\mathcal{C}$, which on $C \in \mathrm{Ob}\,(\mathcal{C})$ is given by functor $\hat{h}_C$ from the commutative diagram:

$$\begin{tikzcd} & \mathcal{D} \arrow[d, "J"] \\ \mathcal{C}^\circ \arrow[ur, "\hat{h}_C"] \arrow[r, "h_C"'] & \mathcal{S}\mathrm{et} \end{tikzcd}$$

The Yoneda embedding theorem shows how every category can be thought of as a category of 'sets with additional structure'. However, when we study categories of sets with additional structure, it turns out to be worthwhile to develop category theory as a subject in its own right. This idea works in n-categories too.

The category of $\hat{\mathcal{D}}$-structures on objects of $\mathcal{C}$ has the forgetful functor $\hat{J}$ in the following diagram

$$\hat{\mathcal{D}} = \qquad \begin{tikzcd} h^*(\mathcal{C}^\circ\mathcal{D}) \arrow[r] \arrow[d, "\hat{J}"'] & \mathcal{C}^\circ\mathcal{D} \arrow[d, "J_*"] \\ \mathcal{C} \arrow[r, "h"'] & \mathcal{C}^\circ\mathcal{S}\mathrm{et} \end{tikzcd}$$

In the general case to define a $\mathcal{D}$-structure on $C \in \mathrm{Ob}\,(\mathcal{C})$ is not equivalent to giving a structure on the functor h_C.

Another well known method is to transfer a structure from one category to others: define the structure by commutative diagrams. Then the definition may be used in other categories. One of these methods is the categorification, see [23, 35, 36].

Categorigication is a game with forgetful functors, when we used similar structures on $\mathcal{C}\mathrm{at}$ and others categories. A more general construction is connected with a forgetful functor $J : \mathcal{C} \to \mathcal{A} \times \mathcal{B}$. For example, we have the forgetful functor $(\mathrm{Ob}\,, \mathrm{Mor}\,) : \mathcal{C}\mathrm{at} \to \mathcal{S}\mathrm{et} \times \mathcal{S}\mathrm{et}$. And we can try to find an iterative definition of n-categories by a construction $J : n\mathcal{C}\mathrm{at} \to \mathcal{S}\mathrm{et} \times (n-1)\mathcal{C}\mathrm{at}$, which used additional structures on categories $\mathcal{S}\mathrm{et}$ and

$(n-1)\mathcal{C}\mathrm{at}$, and one can to apply it for other higher algebraic structures. The notion of the forgetful functor for n-categories gives new perspectives.

We consider quantization, which originates from physics, as a special operation over structures, which change structures with some commutative properties to structures with noncommutative ones. First, about 80 years ago, it was the product of functions (first quantization) and of functionals (second quantization). Now it is a composition of morphisms in a category, monoidal product in a monoidal category, commutativity of diagrams (up to isomorphism) in an n-category, and in this paper described as the coproduct of objects of a category.

2.3 Structures and Products

Sometimes a monoidal structure on a category $\mathcal{C}$ does not exist, but there is a forgetful functor $J : \mathcal{M} \to \mathcal{C}$ from a monoidal category $\mathcal{M}$.

Then we consider it as multiplication on $\mathcal{C}$ depending on other parameters. To define $C \otimes C'$ for $C, C' \in \mathrm{Ob}\,(\mathcal{C})$ we need to take $\mathcal{M}$-structures M on C and M' on C' and then take $J(M \otimes M')$ as the product of C and C' dependent on properties of M and M'. We have the diagram

$$\begin{tikzcd} \mathcal{M} \times \mathcal{M} \arrow[r, "\otimes"] \arrow[d, "J \times J"'] & \mathcal{M} \arrow[d, "J"] \\ \mathcal{C} \times \mathcal{C} & \mathcal{C} \end{tikzcd}$$

In the subsection 3.7 we introduce the forgetful functor $J : \overline{\mathcal{O}} \to \mathcal{O}$ to define such a multiplication for objects of $\mathcal{O}$.

2.4 Interaction and Products

If $\mathcal{P}$ is a category of physical systems, then for two systems ξ and ξ' we have new systems $\xi * \xi'$, which consist of interacting subsystems ξ and ξ'. Physicists do not always define explicitly the properties of a system, thus the multiplication $*$ may be defined only after an extension of a category $\mathcal{C}$ by a forgetful functor $J : \mathcal{M} \to \mathcal{C}$. In [15] the general concept of particle interactions is considered in terms of monoidal categories.

Symmetry in physics is based on group theory. The construction of a tensor product of representations allows to built states for composite systems from simple ones. It is known that we used a (diagonal) comultiplication on the group G for a tensor product of its representations. A hidden comonoid structure in group symmetry appears explicitly in quantum groups theory. A comultiplication does exist for a large class of deformed universal enveloping algebras. The category of representations of a Hopf algebra (or a bialgebra) is a monoidal category.

It is no surprise that the universal interactions in physics, gauge fields, principal bundles with connections, are monoidal functors from vector spaces to bundles. More precisely, in the usual differential geometric description, the (classical) interaction fields are connections on principal bundles and the matter fields are sections of the associated vector bundles.

3 Monoidal Structures

We have a sequences of products. The first of them is the direct product $\times$ in $\mathcal{C}$at , it defines a monoidal structure on $\mathcal{C}$at . Then we consider a monoidal category $\mathcal{M}$, an object of $\mathcal{C}$at with a product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and a unit object U, as a monoid in the monoidal category $\mathcal{C}$at . And then we define a monoid M, an object of $\mathcal{M}$ with a product $\varphi : M \otimes M \to M$ and a unit morphism $\iota : U \to M$. A dual notion is a comonoid C, an object of $\mathcal{M}$ with a coproduct $\varphi : C \to C \otimes C$ and a counit morphism $\varepsilon : C \to U$. Usually cocategories or comonoids in $\mathcal{S}$et are not considered because in the category $\mathcal{S}$et there are only trivial (diagonal) comonoids.

3.1 Monoidal Categories

For the definition of a monoidal category $(\mathcal{M}, U, \otimes, \alpha, \lambda, \varrho)$ we refer to [1]. Here U is an unit object, $\otimes$ is a multiplication, and isomorphisms $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$, $\lambda_X : U \otimes X \to X$, and $\varrho_X : X \otimes U \to X$. These isomorphisms satisfy certain equations, called coherence relations.

A monoidal category is said to be strict if the morphisms α, λ, ϱ are identities.

Thus we shall begin with categories $\mathcal{C}$ equipped with a suitable bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is required to be associative, but usually it is associative only "up to" an isomorphism α.
A monoidal category $\mathcal{M}$ is called *symmetric* if it equipped with isomorphisms

$$\sigma_{M,M'} : M \otimes M' \to M' \otimes M$$

natural in M, M', such that the diagrams

$$\gamma_{M,M'} \circ \gamma_{M',M} = \mathrm{id}_{M',M}, \;\; \rho_M = \lambda_M \circ \gamma_{M,U},$$

$$\begin{array}{ccccc}
M \otimes (M' \otimes M'') & \xrightarrow{\alpha} & (M \otimes M') \otimes M'' & \xrightarrow{\gamma} & M'' \otimes (M \otimes M') \\
{\scriptstyle \mathrm{id} \otimes \gamma}\downarrow & & & & \downarrow{\scriptstyle \alpha} \\
M \otimes (M'' \otimes M') & \xrightarrow[\alpha]{} & (M \otimes M'') \otimes M' & \xrightarrow[\gamma \otimes \mathrm{id}]{} & (M'' \otimes M) \otimes M'
\end{array}$$

all commute.

A closed category $\mathcal{M}$ is a symmetric, monoidal category in which each functor $\_ \otimes M : \mathcal{M} \to \mathcal{M}$ has a right adjoint $(\_)^M : \mathcal{M} \to \mathcal{M}$.

Definition 2 *A monoidal functor (F, γ, γ_0) from a monoidal category $(\mathcal{M}, \otimes, U)$ to a monoidal category $(\mathcal{M}', \otimes', U')$ consists of the following:*

(1) A functor $F : \mathcal{M} \to \mathcal{M}'$;

(2) A functor morphism $\gamma : F \otimes' F \to F \circ \otimes$, i.e. for each pair $X, Y \in Ob(\mathcal{M})$ a morphism

$$\gamma_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y) \tag{1}$$

in $\mathcal{M}'$.

(3) For the units U and U', a morphism in $\mathcal{M}'$

$$\gamma_0 : U' \to F(U), \tag{2}$$

together with some commutative diagrams (for details see [1]). A monoidal functor is said to be strong if γ and F_0 are isomorphisms, and strict if γ and F_0 are identities.

Thus, a monoidal functor preserves the tensor product up to a canonical coherent isomorphism.

Any monoidal category $\mathcal{M}$ is categorically equivalent, via a strong monoidal functor $G : \mathcal{M} \to \mathcal{S}$ and a strong monoidal functor $F : \mathcal{S} \to \mathcal{M}$, to a strict monoidal category $\mathcal{S}$ [1].

There are several results which assert that a monoidal category with some additional properties "is" the category of representations of some group: this is Tannaka-Krein duality. For a Hopf algebra H the category H-**mod** is a monoidal category [6].

3.2 Direct Products

A pullback is a direct product in the category $\mathcal{C}_{\downarrow C}$ of objects over $C \in \mathrm{Ob}(\mathcal{C})$.

General construction works as follows: let a diagram $d_{X,Y}$ depend on two objects X and Y and the inverse limit of $d_{X,Y}$ be associative and has a unit. In this case we additionally get morphisms, projections.

The direct product is obtained if a diagram $d_{X,Y}$ is simply a pair of objects X and Y, $X \times Y$ is a inverse limit of $d_{X,Y}$, the projections are $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. A Cartesian closed category $\mathcal{C}$ is a category equipped with finite products, including a terminal object, which is closed in the sense that for every object X, the functor $X \times \_$ has the right adjoint $(\_)^X$. Then $\mathcal{C}$ is a monoidal category with the product $\times$, projections π_1 and π_2, and a terminal object is a unit.

If the diagram $d_{f,f'}$ consists of two morphisms $f : M \to B$ and $f' : M' \to B$ then its inverse limit $\lim d_{X,Y}$ (if it exists) is called *fiber product* or *pullback* $\times_B$ and defines a monoidal structure on the category $\mathcal{C}_{\downarrow B}$ of objects over B. More generally, we need to take its subcategory, but for simplicity we consider the case of the whole category $\mathcal{C}_{\downarrow B}$. We obtain the functor $\Pi : \mathrm{Mor}(\mathcal{C}) \to \mathcal{C}$ with a monoidal structure on any fiber $\mathcal{C}_{\downarrow B}$ over B.

In a category $\mathcal{C}\mathrm{at}_{\downarrow \mathcal{C}}$ fiber products always exist (a pullback in $\mathcal{C}\mathrm{at}$).

If $\mathcal{M}$ is a category with finite direct products and a terminal object T then each object X of $\mathcal{M}$ has a unique comonoid structure with $\delta_X = (\mathrm{id}_X, \mathrm{id}_X)$ and $\varepsilon : X \to T$.

Example 5. In category $\mathcal{S}\mathrm{et}$ of sets with direct product $\times$ and one-point set unit object each set X has an unique comonoid structure, diagonal map $\Delta : X \to X \times X : x \mapsto (x, x)$. Indeed, let $\Delta(x) = (f(x), g(x))$, where $f, g : X \to X$. Then $\varepsilon_1 \circ \Delta = \varepsilon_2 \circ \Delta = \mathrm{id}_X$ and thus $f = g = \mathrm{id}_X$. Here the unit, one-point set, is a terminal object, thus it exists as a unique counit $\varepsilon : C \to U$. Sometimes this comonoid structure is used without mentioning explicitly. In other monoidal categories there are different comonoid structures and consequently we can deform them.

3.3 Monoids and Comonoids

We now turn to the general notion of a monoid and of a comonoid in a monoidal category. Let $(\mathcal{C}, U, \alpha, \lambda, \varrho)$ be a monoidal category.

A monoid in the category $\mathcal{S}\mathrm{et}$ (semigroup) is defined by the maps relative to the cartesian product $\times$ in $\mathcal{S}\mathrm{et}$.

Definition 3 *A monoid in $\mathcal{C}$ is an object $M \in Ob(\mathcal{C})$ with multiplication $\varphi : M \otimes M \to M$ and unit $\iota : U \to M$ such that*

(1) *Associativity condition: we have the equality of the two following morphisms* $(M \otimes M) \otimes M \to M$

$$\mu \circ (\mu \otimes id_M) = (id_M \otimes \mu) \circ \alpha_{M,M,M}$$

(2) *Unit condition (left and right):*

$$\mu \circ (\iota \otimes id_M) \circ \lambda_M^{-1} = id_M,$$
$$\mu \circ (id_M \otimes \iota) \circ \rho_M^{-1} = id_M.$$

DEFINITION 4 *A comonoid in $\mathcal{C}$ is an object $C \in Ob(\mathcal{C})$ with comultiplication $\delta : C \to C \otimes C$ and counit $\varepsilon : C \to U$ with the following properties:*

(1) *Coassociativity condition:* $\delta_2 \circ \delta = \delta_1 \circ \delta \circ \alpha_{C,C,C}$.

(2) *Counit condition: the compositions*

$$C \xrightarrow{\delta} C \otimes C \xrightarrow{\varepsilon_1} U \otimes C \xrightarrow{\lambda_C} C,$$
$$C \xrightarrow{\delta} C \otimes C \xrightarrow{\varepsilon_2} C \otimes U \xrightarrow{\varrho_C} C,$$

are equal to id_C, i.e. $id_C = \lambda_C \circ \varepsilon_1 \circ \delta$ and $id_C = \rho_C \circ \varepsilon_2 \circ \delta$.

EXAMPLE 6. In category $\mathcal{S}$et each object has a diagonal monoid structure and there are no other monoids. That is why we do not have in this case cocategories and comonoidal categories.

DEFINITION 5 *A bimonoid $(B, \mu, \iota, \delta, \varepsilon)$ is a monoid (B, μ, ι) and a comonoid (B, δ, ε) such that μ, ι are comonoid morphisms and δ, ε are monoid morphisms.*

EXAMPLE 7. In category $\mathcal{S}$et each monoid M has a natural bimonoid structure, defined by the diagonal map.

3.4 Comonoid Structures on Sets

We construct the extension $\overline{\mathcal{S}\mathrm{et}}$ of the category $\mathcal{S}$et and a monoidal structure on $\overline{\mathcal{S}\mathrm{et}}$ such that it has nontrivial monoids. Category $\overline{\mathcal{S}\mathrm{et}}$ is a monoidal category of sets with multi-valued maps and ordinary direct product of sets. We can consider internal and enriched categories over such comonoids in $\overline{\mathcal{S}\mathrm{et}}$. These are **quantum categories**.

Let $\overline{\mathcal{S}\mathrm{et}}$ be the category with the same objects as in $\mathcal{S}\mathrm{et}$ but morphisms from X to Y are $\mathcal{S}\mathrm{et}\,(X, 2^Y)$, where 2^Y denotes the set of all subsets in Y, or $\mathcal{S}\mathrm{et}\,(Y, \{0,1\})$. We have the inclusion

$$\overline{\mathcal{S}\mathrm{et}}\,(X,Y) = \mathcal{S}\mathrm{et}\,(X, 2^Y) \to \mathcal{S}\mathrm{et}\,(2^X, 2^Y) : f \mapsto (A \mapsto \bigcup_{x\in A} f(x))$$

and use the notation $f : X \rightsquigarrow Y$ for morphisms in $\overline{\mathcal{S}\mathrm{et}}$, which is the usual map $f : X \to 2^Y$.

Then the usual direct product in $\mathcal{S}\mathrm{et}$ on $\overline{\mathcal{S}\mathrm{et}}$ determines (after obvious extension on the morphisms) a monoidal structure with the same unit object, the one-point set $t = \{\star\}$.

Remark: This product is not the direct product in $\overline{\mathcal{S}\mathrm{et}}$. In $\overline{\mathcal{S}\mathrm{et}}$ the direct product of X and Y is $2^X \times 2^Y$.

In the monoidal category $\overline{\mathcal{S}\mathrm{et}}$ there are nontrivial comonoids. A comonoid (M, δ, ε) is defined by the following data:

(1) comultiplication $\delta : M \rightsquigarrow M \times M$, or $\delta : M \to 2^{M\times M}$,

(2) counit $\varepsilon : M \rightsquigarrow \star$, or $\varepsilon : M \to 2^\star = \{\star, \varnothing\}$

(3) coassociativity: $\delta_1 \circ \delta = \delta_2 \circ \delta$,

(4) unit relations: $\varepsilon_1 \circ \delta = \mathrm{id}_M$ and $\varepsilon_2 \circ \delta = \mathrm{id}_M$.

Example 8. Trivial (diagonal) comonoidal structure on C: comultiplication $\delta(x) = \{(x,x)\}$, counit $\varepsilon(x) = \star$, $\forall\ x \in C$.

Example 9. Let $C = \{x,y\}$ and $\delta(x) = \{(x,x)\}$, $\delta(y) = \{(y,x),(x,y)\}$, $\varepsilon(x) = \star$, $\varepsilon(y) = \varnothing$.

Example 10. The simplest noncocommutative comonoid: Let $C = \{e,x,y\}$ and $\delta(e) = \{(e,e)\}$, $\delta(x) = \{(x,x)\}$, $\delta(y) = \{(y,e),(x,y)\}$, $\varepsilon(e) = \star$, $\varepsilon(x) = \star$, $\varepsilon(y) = \varnothing$.

Example 11. 'Matrices': Comonoid $M = \{e_{ij}\}_{i,j=1,\dots n}$ and $\delta(e_{ij}) = \{(e_{ik}, e_{kj})\}_{k=1,\dots n} \subset M \times M$,

$$\varepsilon(e_{ij}) = \begin{cases} \varnothing, & \text{if } i \neq j, \\ \star, & \text{if } i = j. \end{cases}$$

The cases $n = \infty$ or $-\infty < i,j < \infty$ are well-defined too.

Example 12. A bimonoid in $\overline{\mathcal{S}\mathrm{et}}$. Let Q is a quiver with vertexes Q_0 and arrows Q_1, $Q_1 \rightrightarrows Q_0$. Let $B = O, E \coprod Q_0 \coprod Q_1$ are generators of the monoid $\hat{B}$ with usual relations appearing in quiver algebras (they do not include + or -), which have the form

$$\begin{aligned} e\cdot e = e,\ e\cdot e' = O,\ e\cdot f = f,\ f\cdot e = f, \\ f\cdot f' = 0,\ e\cdot f' = O,\ f'\cdot e = O, \\ E\cdot E = E,\ E\cdot e = e\cdot E = e,\ E\cdot f = f\cdot E = f, \\ O\cdot O = O,\ O\cdot e = e\cdot O = O, \ldots \end{aligned}$$

where $e \in Q_0,\ f, f' \in Q_1$ (O is a 'zero', E is a unit). The diagonal comultiplication

$$\Delta(e) = (e,e),\ \Delta(E) = (E,E),\ \Delta(O) = (O,O),$$

is compatible with these relations.
Using various comonoids in $\overline{\mathcal{S}\mathrm{et}}$ we can construct $\mathcal{C}\mathrm{at}\,(\overline{\mathcal{S}\mathrm{et}})$ instead $\mathcal{C}\mathrm{at}\,(\mathcal{S}\mathrm{et}\,)$, higher internal categories, monoidal internal categories.

3.5 Cup-product in $\mathcal{C}(C, M)$

For a group G we may define a group structure on $H = \mathcal{S}\mathrm{et}\,(X, G)$:

$$f * f'(x) = \mu_G \circ (f \otimes f') \circ \Delta(x) = f(x) \cdot f'(x),$$

where the sign $\cdot$ denotes the multiplication in G. To define $f * f'$ we used here a diagonal comonoid structure on the set X. In a generic monoidal category $\mathcal{M}$ we do not have a monoid structure on the set $\mathcal{M}(X, M)$ of morphisms from an object X to a monoid M. This is the case only if X has a comonoid structure.

Let $(\mathcal{M}, \otimes, U, \alpha, \lambda, \varrho)$ be a monoidal category, (M, μ, ι) be a monoid, and (C, δ, ε) be a comonoid.

The set $H = \mathcal{M}(C, M)$ has a natural monoid structure in $\mathcal{S}\mathrm{et}$. Let

$$\begin{aligned} &\smile\, : H \times H \to H : (f, f') \mapsto f \smile f' = \mu \circ (f \otimes f') \circ \delta, \\ &e = \iota \circ \varepsilon : C \to M. \end{aligned}$$

It is easy to see that $\smile$ is associative and e is a unit, because the composition

$$C \xrightarrow{\lambda_C^{-1}} U \otimes C \xrightarrow{\mathrm{id}_U \otimes f} U \otimes M \xrightarrow{\lambda_M} M$$

is equal to f, which follows from naturality of λ.

By duality the set $Q = \mathcal{M}(M, C)$ may have a comonoid structure. It is interesting, because in the category $\mathcal{S}\mathrm{et}$ there are only diagonal comonoids. Actually, in this case we have some problems.

To define a comonoid structure on Q we need to define a map $\Delta : Q \to Q \times Q$. But by duality to $\smile$ for $f \in Q$ we have only the morphism

$$\delta \circ f \circ \mu : M \otimes M \to C \otimes C. \tag{3}$$

In general case it is impossible to define an element from $Q \times Q$, as a pair of maps $M \to C$, using only (3).

3.6 Products on $\mathcal{M}\mathcal{M}'$

Let $(\mathcal{M}, \otimes, U)$ and $(\mathcal{M}', \otimes', U')$ are monoidal categories. Then we have the functors

$$\begin{aligned} &\mathcal{M}\mathcal{M}' \times \mathcal{M}\mathcal{M}' \to (\mathcal{M} \times \mathcal{M})\,\mathcal{M}' : (F, G) \mapsto F \otimes' G, \\ &\mathcal{M}\mathcal{M}' \to (\mathcal{M} \times \mathcal{M})\,\mathcal{M}' : F \mapsto F \circ \otimes, \end{aligned}$$

which makes it possible to define associativity for functor morphisms in the form $\gamma : F \otimes' F \to F \circ \otimes$.

We consider a morphism $\gamma : F \otimes' F \to F \circ \otimes$ as a multiplication on F with the associativity condition for γ

$$\begin{tikzcd} F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[r, "\gamma_{Y,Z}"] \arrow[d, "\alpha'_{F(X),F(Y),F(Z)}"'] & F(X) \otimes' F(Y \otimes Z) \arrow[r, "\gamma_{X,Y\otimes Z}"] & F(X \otimes (Y \otimes Z)) \arrow[d, "F(\alpha_{X,Y,Z})"] \\ (F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[r, "\gamma_{XY}"'] & F(X \otimes Y) \otimes' F(Z) \arrow[r, "\gamma_{X\otimes Y,Z}"'] & F((X \otimes Y) \otimes Z) \end{tikzcd}$$

where we omit the notation "id" at the horizontal arrows to simplify the diagram. Thus, associativity is an isomorphism of the two compositions

$$\begin{aligned} F \otimes (F \otimes F) &\xrightarrow{\mathrm{id}_F \otimes \gamma} F \otimes \hat{F} \xrightarrow{\gamma} F \circ \otimes^{(2)}, \\ (F \otimes F) \otimes F &\xrightarrow{\gamma \otimes \mathrm{id}_F} \hat{F} \otimes F \xrightarrow{\gamma} F \circ \otimes^{(2)}, \end{aligned}$$

where $\hat{F} = F \circ \otimes$.

A unit for γ is a morphism $\gamma_0 : U' \to F(U)$ with obvious properties in the form of commutative diagrams (cf. the Definition 2).

We have the category $Mon(\mathcal{M}, \mathcal{M}')$ with objects (F, γ, γ_0) and forgetful functor

$$J : Mon(\mathcal{M}, \mathcal{M}') \to \mathcal{M}\mathcal{M}' : (F, \gamma, \gamma_0) \mapsto F.$$

For a monoidal functor $m : \mathcal{M}' \to \mathcal{M}''$ there is functor

$$\begin{aligned} Mon(m) : Mon(\mathcal{M}, \mathcal{M}') &\to Mon(\mathcal{M}, \mathcal{M}'') \\ &: (F, \gamma, \gamma_0) \mapsto (m \circ F, m(\gamma), m(\gamma_0 \circ m_0)). \end{aligned}$$

If (M, φ, ι) is a monoid in $\mathcal{M}$ and (F, γ, γ_0) is our triple then $F(M)$ has a monoid structure in $\mathcal{M}$

$$F(M) \otimes' F(M) \xrightarrow{\gamma_{M,M}} F(M \otimes M) \xrightarrow{F(\varphi)} F(M),$$
$$U' \xrightarrow{\gamma_0} F(U) \xrightarrow{\iota} F(M).$$

Thus, we have a functor

$$Mon(\mathcal{M}\mathcal{M}') \times Mon(\mathcal{M}) \to Mon(\mathcal{M}')$$
$$: ((F, \gamma, F_0), (M, \varphi, \iota)) \mapsto (F(M), F(\varphi) \circ \gamma_{M,M}, F(\iota) \circ \gamma_0).$$

A dual construction of the structure on F, comonoid structure, is given by morphisms

$$\beta : F \circ \otimes \to F \otimes' F \quad \text{and} \quad \beta_0 : F(U) \to U' \tag{4}$$

with usual constrains in the form of commutative diagrams.

3.7 Operads

It is wellknown that an operad is a multicategory with one object, see [13], [14], [9]. We consider another construction, which use the structure of a product from the previous subsection and describe a particular strict monoidal category $\overline{\mathcal{O}}$ which plays a central role in the definition of operad. The category $\mathcal{O}$ has as objects all finite ordinal numbers $[n] = \{1, ..., n\}$ and as arrows $f : [m] \to [n]$ all bijective functions. Thus $\mathcal{O}([n], [n]) = S_n$, the symmetrical group, and $\mathcal{O}([m], [n]) = \varnothing$ if $m \neq n$.

Ordinal addition is a bifunctor $* : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ defined on ordinals n, m as the usual (ordered) sum $n + m$ with the obvious action on arrows. But for operads we used another multiplication on $\overline{\mathcal{O}}$, whose objects are pairs $([n], i)$ with $i = 1, \ldots, n$ and

$$\overline{\mathcal{O}}(([n], i), ([n], j)) = \{s \in S_n \mid s(i) = j\}$$

and forgetful functor $J : \overline{\mathcal{O}} \to \mathcal{O} : ([n], i) \mapsto [n]$.
Thus $\overline{\mathcal{O}}(([n], i), ([n], j)) \cong S_{n-1}$ and a monoidal structure $*$ on $\overline{\mathcal{O}}$ is given by

$$([m], i) * ([n], j) = ([m + n - 1], i + j - 1),$$
$$: (s, s') \mapsto s * s',$$

where a unit object is (1) [1], and

$$s * s'(k) = \begin{cases} s(k), & \text{if } k = 1, \ldots, i - 1, \\ s'(k - i + 1), & \text{if } k = i, \ldots, i + n - 1, \\ s'(k - n + 1), & \text{if } k = i + n, \ldots, m + n - 1. \end{cases}$$

An operad in a monoidal category $\mathcal{M}$ is a functor $D \in Mon(\overline{\mathcal{O}}, \mathcal{M})$, i.e.

$$\begin{aligned} D &: \mathcal{O} \to \mathcal{C} : [n] \mapsto D_n, \\ \gamma_{m,i,n,j} &: D_m \otimes D_n \to D_{m+n-1}, \\ D_{(0)} &: U \to D_1. \end{aligned}$$

The category of operads in $\mathcal{M}$ on $\mathcal{M}$ is $Mon(\overline{\mathcal{O}}, \mathcal{M})$.

3.8 2-categories

A structure on categories is given by a forgetful functor $J : \mathcal{C} \to \mathcal{C}\mathrm{at}$. There are lots of structures on categories. One of them is the structure of a monoidal category, another is a braided monoidal category.

A category $\mathcal{C}$ is linear if the set $\mathcal{C}(X, Y)$ of morphisms has the structure of a vector space, and the composition is bilinear. A linear category is semisimple if each object is a direct sum of simple objects (objects with no nontrivial sub- or quotient objects). A semisimple category is finitely generated if it has only finitely many inequivalent irreducible objects.

The structure of an enriched category over a monoidal category $\mathcal{C}$, in general, is not a structure on ordinary categories because $\mathcal{C}$ may be not a category of sets with additional structure or forgetful functor $J : \mathcal{C} \to \mathcal{S}\mathrm{et}$ may be not a monoidal.

The structure of higher category, in particular, 2-category is not a structure on ordinary categories. The notion n-category extends the notion category, which is an additional structure on two sets, objects and morphisms. An n-category $\mathcal{C}$ is algebraic structure consisting of a set C_0 of 'objects', a set C_1 of 'morphisms', '1-morphisms', between objects, a set C_2 of '2-morphisms' between 1-morphisms, and so on up to n, with various compositions of these j-morphisms and coherence constraints (see [13]). Thus, a 0-category is a set, while a 1-category is an ordinary category. In weak n-categories all rules governing the composition of j-morphisms hold only up to equivalence. Recently n-categories for arbitrarily large n have begun to play an increasingly important role in topological quantum field theory, which focuses on processes which have adjoints or duals. More details on higher categories can be found in [22, 9, 11].

Often an n-category is written in the form $n\mathcal{C}$ or n-$\mathcal{C}$. For each n there is an $(n+1)$-category of all n-categories, $n\mathcal{C}\mathrm{at}$. To understand n-categories we need to understand this $(n+1)$-category. This requires an understanding of $(n+1)$-categories in general, which then leads us to define $(n+1)\mathcal{C}\mathrm{at}$.

Using the 2-category structure on $\mathcal{C}\mathrm{at}$, one can show that every bicategory is equivalent to a strict 2-category in a certain precise sense. This follows from the Yoneda embedding for bicategories.

The categorification, invented by Crane, and some constructions for n-categories, which are seen as iterated categorifications of the natural numbers and integers, can be found in [23].

Now we consider 2-categories in more detail [27]. The classic example of a 2-category is $\mathcal{C}\mathrm{at}$, which has categories as objects, functors as morphisms, and natural transformations as 2-morphisms. Recall that a bicategory, also called lax or weak 2-category, and first defined by Bénabou, can be obtained from a category after performing the following two steps. First, enrich the sets $\mathrm{Hom}\,(X, Y)$ of morphisms with the category of small categories in the sense of Kelly [26], a category $\mathcal{H}\mathrm{om}(X, Y)$. Second, weaken the associativity and unit axioms on the composition by substituting 2-isomorphisms for the equations, with the consequent introduction of coherence relations.
A 2-category $\mathcal{C}$ consists of the following data:

(1) A set C_0 of objects which are called 0-cells, a set C_1 of arrows called 1-cells, a set C_2 of 2-arrows called 2-cells. The objects or 0-cells and arrows or 1-cells form a category, called the underlying category of $\mathcal{C}$, which we also denote by $\mathcal{C}$, with identities $1_X : X \to X$.

(2) For each pair $X, Y \in C_0$ there is a small category $\mathcal{C}_1(X, Y)$ whose objects or 1-cells are morphisms $f : X \to Y$, and arrows or 2-cells are morphisms of morphisms from X to Y. A 2-cell $\alpha : f \Rightarrow g$ is pictured as

$$\begin{tikzcd} X \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \alpha"] & \bullet Y \end{tikzcd}$$

We visualize the objects as 0-dimensional, i.e. points, the morphisms as 1-dimensional, i.e. arrows going from one point (source) to another (target). A composition of two morphisms corresponds to gluing together an arrow. Continuing in this spirit, we visualize the 2-morphisms as 2-dimensional, and compose 2-morphisms in a way that corresponds to gluing together 2-dimensional shapes. For any pair 2-cells α, β in $\mathcal{C}_1(X, Y)$ the 2-cells composition under which $\mathcal{C}_1(X, Y)$ form a category, is called vertical composition. A 2-cells vertical composition as displayed as

$$\begin{tikzcd} X \bullet \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\beta"] & \bullet Y \end{tikzcd}$$

The vertical composite $f \Rightarrow h$ is denoted by $\beta \circ \alpha$ and it is not shown on the picture. Identities of category $\mathcal{C}_1(X,Y)$ are denoted by

$$\begin{tikzcd} X \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & \bullet X \end{tikzcd}$$

We can compose 2-cells $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$ under another 2-cells operation known as *horizontal composition*. In this case we have

$$\begin{tikzcd} X \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow \alpha"] & \bullet Y & Y \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow \beta"] & \bullet Z & \mapsto & X \bullet \arrow[r, bend left, "g \circ f"] \arrow[r, bend right, "g' \circ f'"'] \arrow[r, phantom, "\Downarrow \beta \circ \alpha"] & \bullet Z \end{tikzcd}$$

Under this composition law $\beta \circ \alpha : g \circ f \to g' \circ f'$ the 2-cells form a category, with identities $1_{1_X} : 1_X \to 1_X$.

This structure also provides a horizontal composite of a 2-cell with 1-cell

$$\begin{tikzcd} X \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow \alpha"] & \bullet Y \arrow[r, "g"] & \bullet Z \end{tikzcd}$$

EXAMPLE 13. 2-category $\mathcal{C}$at , which has categories as objects, functors as morphisms, and natural transformations as 2-morphisms.

EXAMPLE 14. Another example of 2-categories is the category $2\mathcal{T}$op whose objects or 0-cells are topological spaces, 1-cells are continuous maps between spaces, 2-cells are homotopy classes of homotopies between continuous maps.

EXAMPLE 15. Given a topological space X, we form a 2-category $\Pi_2(X)$ called the 'fundamental 2-groupoid' of X. The objects of this 2-category are the points of X. The morphisms from $x \in X$ to $y \in X$ are the paths $f : [0,1] \to X$ starting at x and ending at y. We get the topological space $\hom(x,y)$. The 2-morphisms from f to g are the homotopy classes of paths in $\hom(x,y)$ starting at f and ending at g. The associative law for the composition of paths holds only up to homotopy, this 2-category is a weak 2-category.

There is the same for a vertical composite $h \circ \alpha$

$$\begin{tikzcd} X \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet Y \arrow[r, bend left, "g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow 1_g"] & \bullet Z \end{tikzcd}$$

A 2-functor $F : \mathcal{C} \to \mathcal{D}$ between 2-categories $\mathcal{C}$ and $\mathcal{D}$ is a triple of functions sending 0-cells, 1-cells, and 2-cells of $\mathcal{C}$ to items of the same types in $\mathcal{D}$, so as to preserve all the categorical structures: domain, codomain, identities, and composites.

The horizontal and the vertical compositions are related by the conditions

$$\begin{tikzcd} X \bullet \arrow[r, bend left=50, "f"] \arrow[r, "f'"'] \arrow[r, bend right=50, "f''"'] \arrow[r, phantom, shift left=2, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2, "\Downarrow\alpha'"] & \bullet Y \arrow[r, bend left=50, "g"] \arrow[r, "g'"'] \arrow[r, bend right=50, "g''"'] \arrow[r, phantom, shift left=2, "\Downarrow\beta"] \arrow[r, phantom, shift right=2, "\Downarrow\beta'"] & \bullet Z & \mapsto & X \bullet \arrow[r, bend left, "g\circ f"] \arrow[r, bend right, "g''\circ f''"'] \arrow[r, phantom, "\Downarrow\gamma"] & \bullet Z \end{tikzcd}$$

$$\text{where} \quad \gamma = (\beta' \circ \alpha') \circ (\beta \circ \alpha) = (\beta' \circ \beta) \circ (\alpha' \circ \alpha).$$

In the situation

$$\begin{tikzcd} X \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & \bullet Y \arrow[r, bend left, "g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow 1_g"] & \bullet Z \end{tikzcd}$$

the horizontal composite of two vertical identities is itself a vertical identity i.e. $1_g \circ 1_f = 1_{g\circ f}$.

EXAMPLE 16. The category $\mathcal{G}$rp whose objects are groups, 1-cells are homomorphism between two groups, and 2-cells $\alpha : f \Rightarrow g$ are the automorphims of the target of f and g (for details see [1]).

4 Products over Comonoids

4.1 Duality: Pairing

We consider two dualities between monoids and comonoids. The first is defined by a pairing ψ and the second is given by a multiplicative morphism.

This duality is as Takeuchi duality for bialgebras [7].

Definition 6 *We say that the morphism* $\psi : C \otimes M \to U$ *realizes a duality between a monoid* M *and a comonoid* C*, if*

(1) the diagram

$$\begin{tikzcd}
C \otimes (M \otimes M) \arrow[r, "\mu"] \arrow[d, "\delta \otimes id_{M \otimes M}"'] & C \otimes M \arrow[dd, "\psi"] \\
(C \otimes C) \otimes (M \otimes M) \arrow[d, "\beta_{C,M}"'] & \\
(C \otimes M) \otimes (C \otimes M) \arrow[r, "\psi^{(2)}"'] & U
\end{tikzcd}$$

is commutative. Here $\beta_{C,M}$ *is the composition*

$$(C \otimes M) \otimes (C \otimes M) \xrightarrow{\alpha^{-1}_{C,M,C \otimes M}} C \otimes ((M \otimes C) \otimes M) \xrightarrow{id_C \otimes (\sigma \otimes id_M)}$$
$$C \otimes ((C \otimes M) \otimes M) \xrightarrow{\sigma_{23}} C \otimes ((M \otimes C) \otimes M) \xrightarrow{id_C \otimes \alpha^{-1}_{M,C,M}}$$
$$C \otimes (M \otimes (C \otimes M)) \xrightarrow{\alpha^{-1}_{C,M,C \otimes M}} (C \otimes M) \otimes (C \otimes M))$$

and $\psi^{(2)}$ *is the composition*

$$(C \otimes M) \otimes (C \otimes M) \xrightarrow{\psi \otimes \psi} U \otimes U \xrightarrow{\lambda_U = \varrho_U} U;$$

(2) the composition

$$C \xrightarrow{\varrho_C} C \otimes U \xrightarrow{id_C \otimes \iota} C \otimes M \xrightarrow{\psi} U$$

coincides with the counit ε*.*

4.2 Duality: Multiplicative Morphisms

Let (M, μ, ι) be a monoid and (C, δ, ε) a comonoid in a symmetrical monoidal category $(\mathcal{M}, \otimes, \alpha, \lambda, \varrho, \sigma)$.

To permute two middle factors in $(C \otimes M) \otimes (C \otimes M)$ and then apply φ to the two last right factors we use the isomorphisms α and σ. Then $*_2$ is the composition

$$(C \otimes M) \otimes (C \otimes M) \xrightarrow{\alpha^{-1}_{C,M,C \otimes M}} C \otimes ((M \otimes C) \otimes M) \xrightarrow{\mathrm{id}_C \otimes (\sigma \otimes \mathrm{id}_M)}$$
$$C \otimes ((C \otimes M) \otimes M) \xrightarrow{\mathrm{id}_C \otimes \alpha_{C,M,M}} C \otimes (C \otimes (M \otimes M))$$
$$\xrightarrow{\mu_{34}} C \otimes (C \otimes M) \xrightarrow{\alpha^{-1}_{C,C,M}} (C \otimes C) \otimes M,$$

where $\sigma_{M,C} : M \otimes C \to C \otimes M$. It is a multiplication $*_2$ on $C \otimes M$

$$*_2 : (C \otimes M) \otimes (C \otimes M) \to (C \otimes C) \otimes M.$$

As γ in $Man(\mathcal{M}\mathcal{M}')$ with $F = \_ \otimes M : \mathcal{M} \to \mathcal{M}$, then $F \otimes F(C, C') = (C \otimes M) \otimes (C' \otimes M)$ and $F \circ \otimes(C, C') = (C \otimes C') \otimes M$, but here $C = C'$, i.e. γ is restricted on the diagonal.

Definition 7 *A morphism $X : U \to C \otimes M$ defines a duality between a monoid M and a comonoid C if $\delta_1 X = X *_2 X$, where these morphisms are the compositions*

$$\delta_1(X) = \mu_{34} \circ \sigma \circ X \otimes X : U \to (C \otimes C) \otimes M,$$
$$X *_2 X : U \xrightarrow{\lambda_U = \varrho_U} U \otimes U \xrightarrow{X \otimes X} (C \otimes M) \otimes (C \otimes M) \xrightarrow{*_2} (C \otimes C) \otimes M,$$

and unit property: the composition

$$U \xrightarrow{X} C \otimes M \xrightarrow{\varepsilon_1} U \otimes M \xrightarrow{\lambda_M} M$$

is equal to ι.

Example 17. In the category $\mathcal{S}$et for any monoid M and comonoid C let $X = (c, x) \in C \times M$. We have $\delta_1(c, x) = (c, c, x) \in C \times C \times M$ and $(c, x) *_2 (c, x) = (c, c, x^2)$. Thus $\{z\} \to C \times M : z \to X = (c, x)$ is a multiplicative morphisms if and only if x is an idempotent in M, i.e. if $x^2 = x$.

For a monoid M the set $\mathcal{C}(U, M)$ is a monoid in $\mathcal{S}$et .

If M and M' are monoids then $M \otimes M'$ has a natural monoid structure. For a monoid M the set $\mathcal{C}(U, M)$ is a monoid in $\mathcal{S}$et . Thus, $\mathcal{C}(U, M \otimes M)$ is a monoid too.

A multiplicative morphisms $X : U \to C \otimes M$ defines a monoid homomorphism

$$\hat{X} : \mathcal{C}(C, M) \to \mathcal{C}(U, M \otimes M) : f \mapsto (U \xrightarrow{X} C \otimes M \xrightarrow{f \otimes \mathrm{id}_M} M \otimes M).$$

4.3 Pullback over C

Let $(\mathcal{M}, \otimes, \alpha, \lambda, \varrho)$ be a monoidal category and (C, δ, ε) a comonoid in $\mathcal{M}$. We apply the construction (4) to the functor $J : \mathcal{M}_{\downarrow C} \to \mathcal{M}$. Suppose that in $\mathcal{M}_{\downarrow C}$ there is a multiplication $\otimes_C$.

In the category $\mathcal{M}_{\downarrow C}$ of objects over C (i.e. morphisms $\pi : M \to C$) we consider the following multiplication $\otimes_C$:

$$(\pi : M \to C) \otimes_C (\pi' : M' \to C) = (\pi \otimes_C \pi : M \otimes_C M' \to C),$$

where $M \otimes_C M$ may be $\delta^*(M \otimes M')$, a pullback in the diagram:

$$\begin{tikzcd}
M \otimes M' \arrow[d, "\pi\otimes\pi'"'] & \delta^*(M \otimes M') \arrow[l] \arrow[d, "\pi\otimes_C\pi'"] \\
C \otimes C & C \arrow[l, "\delta_1"]
\end{tikzcd}$$

Associativity: the diagram

$$\begin{tikzcd}
(M \otimes M') \otimes M'' \arrow[d, "(\pi\otimes\pi')\otimes\pi''"'] & (\delta_1 \circ \delta)^*((M \otimes M') \otimes M'') \arrow[l] \arrow[d] \\
(C \otimes C) \otimes C & C \arrow[l, "\delta_1\circ\delta"]
\end{tikzcd}$$

with $(M\otimes_C M')\otimes_C M'' \cong (\delta_1\circ\delta)^*((M\otimes M')\otimes M'')$ and the analogous diagram for $M \otimes_C (M' \otimes_C M'')$ show that $(M \otimes_C M') \otimes_C M'' \cong M \otimes_C (M' \otimes_C M'')$ over C.

As an unit object we can try to use $\lambda_C : U \otimes C \to C$, which is isomorphic to $\mathrm{id}_C : C \to C$.

4.4 Pullback over $C \otimes C$

Let $(\mathcal{M}, \otimes, \alpha, \lambda, \varrho)$ be a monoidal category, (C, δ, ε) a comonoid. We apply the construction (4) to functor $J : \mathcal{M}_{\downarrow C\otimes C} \to \mathcal{M}$. Suppose that in $\mathcal{M}_{\downarrow C\otimes C}$ there is a multiplication $\otimes_C$.

In the category $\mathcal{M}_{\downarrow C\otimes C}$ of objects over $C \otimes C$, i.e. with objects $\pi : M \to C \otimes C$, we consider the following multiplication $*$:

$$(\pi : M \to C \otimes C) * (\pi' : M' \to C \otimes C) = (\pi \otimes_C \pi' : M \otimes_C M' \to C \otimes C),$$

where $M \otimes_C M$ may be $\delta^*_{(2)}(M \otimes M')$, a pullback in the diagram:

$$\begin{tikzcd}
M \otimes M' \arrow[d, "\pi\otimes\pi'"'] & \delta^*_{(2)}(M \otimes M') \arrow[l] \arrow[d] \arrow[dr, "\pi\otimes_C\pi'"] & \\
C^{\otimes 2} \otimes C^{\otimes 2} & C^{\otimes 3} \arrow[l, "\delta_{(2)}"] \arrow[r, "\varepsilon_2"'] & C \otimes C
\end{tikzcd}$$

where $C^{\otimes n} := C \otimes C^{\otimes(n-1)}$. As a unit it is natural to try here $\delta : C \to C \otimes C$ but this does not work in a general category $\mathcal{M}$, because $C \otimes_C M$ may be not isomorphic to M.

4.5 Internal Categories

The *internal categories* are described by diagrams within an ambient category, which has all products, pullbacks, and a terminal object. In this framework there are *higher-dimensional categories* [1, 9]. To construct internal categories in a monoidal category without these restrictions needs the explicit using of a comonoid structure on the 'object' of objects.

An enriched category in a monoidal category $\mathcal{M}$ has a set $\mathrm{Ob}\,(\mathcal{M})$ but the morphisms $\mathcal{M}(X,Y)$ are objects of the category $\mathcal{M}$, thus, in general, we do not have arrows. There is no composition of arrows, but we have a composition morphisms $\varphi_{X,Y,Z} : \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$ and identities morphisms $\iota_X : U \to \mathcal{C}(X,X)$ [10, 26, 3].

There is the natural modification (deformation) of the notion of a enriched category by using a comonoid structure on the set of objects instead of the diagonal comonoid structure.

To consider a general notion of an internal category $\mathcal{D}$ in $\mathcal{C}$ we suppose that both morphisms $M = \mathrm{Mor}\,(\mathcal{D})$ and objects $C = \mathrm{Ob}\,(\mathcal{D})$ are objects of a monoidal category $\mathcal{M}$ with a morphism $\pi : M \to C \otimes C$, identities morphism $C \to M$ and composition morphism $\varphi : M \otimes_C M \to M$, both over $C \otimes C$. To define the composition morphism we used a comonoid structure on C and some additional condition on the multiplication $\otimes_C$ as in previous subsection.

The source and target morphisms are $s = \varrho_M \circ \varepsilon_2 \circ \pi$ and $t = \lambda_M \circ \varepsilon_1 \circ \pi$. Roughly speaking (in general, M is not a set) the composition $f \circ f'$ of $f \otimes f'$ is defined only if $s(f) \otimes t(f') \in \delta(C)$. For the usual situation in $\mathcal{S}\mathrm{et}$ with $\otimes = \times$ we have $\delta(x) = (x,x)$ and thus the composition $f \circ f'$ of (f, f') is defined only if $s(f) = t(f')$.

If the product $\otimes_C$ is defined as pullback $M \otimes_C M' := (\delta_2)^*(M \otimes M')$, it is associative but usually there is no a unit. For example in the category $\mathcal{V}\mathrm{ect}_f(k)$ of finite dimensional vector spaces over a field k homomorphism δ is injective,

$$M \otimes_C M' = (\pi \otimes \pi')^{-1}(C \otimes \delta(C) \otimes C),$$

and thus $\pi \otimes_C \pi'$ has the kernel $M \otimes \mathrm{Ker}(\pi') + \mathrm{Ker}(\pi) \otimes M$. In particular, for $M' = C$ the kernel $\mathrm{Ker}(\pi \otimes_C \delta) = \mathrm{Ker}(\pi) \otimes C$ is larger than $Ker(\pi)$. Thus, $\delta : C \to C \otimes C$ is not a unit for $\otimes_C$.

4.5.1 Internal Categories

We can define monoid, group, graph, and other structures in a category $\mathcal{C}$. Objects with such a structure form a corresponding category, which in some sense extends the category $\mathcal{C}$. We can also define a category, so-called $\mathcal{C}$-category, with object of objects ('set of objects') Ob and morphisms object Mor in $\mathcal{C}$. Let $\mathcal{C}$ be a monoidal category whose multiplication $\otimes$ is given a direct product and suppose that in $\mathcal{C}$ there exist pullbacks. An internal category $\mathcal{D}$ in a monoidal category $\mathcal{C}$ consists of:

(1) an object $C_0 \in \mathrm{Ob}(\mathcal{C})$, called object of objects;

(2) an object of morphisms $C_1 \in \mathrm{Ob}(\mathcal{C})$, called object of arrows;

together with four morphisms in $\mathcal{C}$:

(1) source or domain morphism$s : C_1 \to C_0$ and target or codomain morphism $t : C_1 \to C_0$;

(2) an identity arrow $i : C_0 \to C_1$; (as in $\mathcal{S}$et where one-point set $\star$ is a unit for direct product $\otimes = \times$);

(3) a composition morphism $\varphi : C_1 \otimes_{C_0} C_1 \to C_1$, here composition φ is defined on the following pullback $(C_1 \otimes_{C_0} C_1, p, q)$ of morphisms s and t:

$$\begin{tikzcd} C_1 \otimes_{C_0} C_1 \arrow[r, "q"] \arrow[d, "p"'] & C_1 \arrow[d, "t"] \\ C_1 \arrow[r, "s"'] & C_0 \end{tikzcd}$$

Here the pullback used a diagonal morphism $\Delta = (\mathrm{id}_{C_0}, \mathrm{id}_{C_0}) : C_0 \to C_0 \times C_0$, comonoid structure on C_0.

This is equal to the following two conditions : $s \circ \varphi = s \circ p$, and $t \circ \varphi = s \circ q$.

These data must satisfy the following commutative conditions, which simply express the well known axiom for a category :

(1) $s \circ i = 1_{C_0} = t \circ i$ specifies domain and codomain of the identity arrows;

(2) $s \circ \varphi = s \circ p$, and $t \circ \varphi = t \circ q$ assigns the domain and codomain of composite morphisms, here we used a comultiplication of the comonoid structure on C_0;

(3) $\varphi \circ (\varphi \otimes_{C_0} \mathrm{id}_{C_1}) = \varphi \circ (\mathrm{id}_{C_1} \otimes_{C_0} \varphi)$ expresses that associative law for composition in terms of triple pullback, here we used the coassociativity of Δ on C_0;

(4) $p = \varphi \circ (i \otimes_{C_0} \mathrm{id}_{C_1})$, and $(\mathrm{id}_{C_1} \otimes_{C_0} i) = q$ gives the left and right unit laws for composition of morphisms.

When $\mathcal{C} = \mathcal{S}\mathrm{et}$ the pullback is the set of composable pairs (g, f) of arrows. When there is a forgetful functor $J : \mathcal{C} \to \mathcal{S}\mathrm{et}$ the internal $\mathcal{C}$-category has objects and arrows but composable pairs have sense only if J is monoidal.

An internal category in $\mathcal{S}\mathrm{et}$ is just an ordinary small category which is same as an object in $\mathcal{C}\mathrm{at}$, $\mathcal{C}\mathrm{at} = \mathcal{C}\mathrm{at}(\mathcal{S}\mathrm{et})$. An internal category in $\mathcal{G}rp$ (category of groups and homomorphisms [1]) is a category in which both C_0 and C_1 are groups, and all the maps i, s, t and φ are homomorphisms of groups. Then an internal category in $\mathcal{G}rp$ is same as a group object in $\mathcal{C}\mathrm{at}$.

An internal functor (morphism in $\mathcal{C}\mathrm{at}(\mathcal{C})$) $F : \mathcal{D} \to \mathcal{D}'$ between two internal categories $\mathcal{D}$ and $\mathcal{D}'$ of $\mathcal{C}$ consists of:

(1) morphisms $F_0 : D_0 \to D_0'$, and $F_1 : D_1 \to D_1'$ of $\mathcal{C}$; such that following holds:

(2) $F_0 \circ s = s' \circ F_1$, and $F_0 \circ t = t' \circ F_1$ preservation of domain and codomain;

(3) $F_1 \circ i = i' \circ F_0$ preservation of identity arrows;

(4) $F_1 \circ \varphi = \varphi' \circ (F_1 \otimes_{D_0} F_1)$ preservation of composite arrows.

Using a similar procedure, an internal natural transformation between two internal functors F and G from $\mathcal{D}$ to $\mathcal{D}'$ in $\mathcal{C}$, say $\alpha : F \Rightarrow G$, is a morphism $\alpha : D_0 \to D_1'$ which satisfies the following conditions : $s \circ \alpha = F_0$, and $t \circ \alpha = G_0$ and

$$\varphi' \circ (\alpha \circ t \otimes_{D_0} F_1) \circ \Delta = \varphi' \circ (G_1 \otimes_{D_0} \alpha \circ s) \circ \Delta,$$

where Δ is a diagonal morphism $D_1 \to D_1 \otimes_{D_0} D_1$. Here the 2-category structure of these internal categories depends on the comonoid structure on D_1.

4.5.2 Internal Categories over Comonoids

We describe here only a general scheme of the construction. In general, an internal categories $\mathcal{C}$ in $\mathcal{M}$ does not have objects and morphisms, it has only two $\mathcal{M}$-objects $C = \mathrm{Ob}(\mathcal{C})$ and $M = \mathrm{Mor}(\mathcal{C})$ and some $\mathcal{M}$-morphisms.

Let $\mathcal{M}$ be a monoidal category, suppose that it is strict for simplicity, $\mathcal{C}omon\mathcal{M}$ the category of comonoids in $\mathcal{M}$.

Let $\mathcal{M}_\downarrow$ be a category of some objects over $\otimes$-squares of comonoids in $\mathcal{M}$, i.e. its objects are morphisms $\pi : M \to C \otimes C$, where C is a comonoid, and its morphism from $\pi : M \to C \otimes C$ to $\pi' : M' \to C' \otimes C'$ is a pair (u, v) of $\mathcal{M}$-morphisms $u : M \to M'$ and $v : C \to C'$ such that v is a comonoid morphism and $\pi' \circ u = (v \otimes v) \circ \pi$. There is the projection

$$\Pi : \mathcal{M}_{\downarrow} \to \mathcal{C}omon\mathcal{M} : (\pi : M \to C \otimes C) \mapsto C.$$

Then we suppose that there is a monoidal structure $\otimes_{\downarrow C}$ in each fiber $\mathcal{M}_{\downarrow C} = \Pi^{-1}(C, \mathrm{id}_C)$

$$\otimes_{\downarrow} : \mathcal{M}_{\downarrow} \times_{\Pi} \mathcal{M}_{\downarrow} \to \mathcal{M}_{\downarrow}$$

with two functor morphisms (natural transformations) p and q

$$\begin{tikzcd}
M \otimes M' \arrow[d, "\pi\otimes\pi'"'] & & M \otimes_{\downarrow} M' \arrow[ll, "q"'] \arrow[dl, "p"'] \arrow[d, "\pi\otimes_{\downarrow}\pi'"] \\
C^{\otimes 2} \otimes C^{\otimes 2} & C^{\otimes 3} \arrow[l, "\delta_2"] \arrow[r, "\varepsilon_2"'] & C \otimes C
\end{tikzcd}$$

with some coherence conditions for p and q. A unit object for $\otimes_C$ is a comultiplication $\delta : C \to C \otimes C$. We write $\otimes_{\downarrow}$ or $\otimes_C$ when it is needed to indicate a comonoid C.

An internal category in $\mathcal{M}$ is a monoidal object in $\mathcal{M}_{\downarrow}$ with respect to the product $\otimes_{\downarrow}$. On $\pi : M \to C \otimes C$ an internal category structure is given by two morphisms $\varphi : M \otimes_{\downarrow} M \to M$, a composition morphism, and $\iota : C \to M$, a unit morphism, both over $C \otimes C$.

Definition 8 *A functor between internal categories from $(\pi : M \to C \otimes C, \varphi, \iota)$ to category $(\pi' : M' \to C' \otimes C', \varphi', \iota')$ is a pair $F = (u, v)$ of $\mathcal{M}$-morphisms $u : M \to M'$ and $v : C \to C'$ such that*

(1) v is a comonoid morphism,

(2) units condition: $\iota' \circ v = u \circ \iota$,

(3) commutative diagram

$$\begin{tikzcd}
& M \arrow[rr, "u"] \arrow[dd, "\pi"'] & & M' \arrow[dd, "\pi'"] \\
M \otimes_C M \arrow[ur, "\varphi"] \arrow[rr, "F\otimes_C F"] \arrow[dr, "\pi\otimes_C\pi"'] & & M' \otimes_{C'} M' \arrow[ur, "\varphi'"] \arrow[dr, "\pi'\otimes_{C'}\pi'"'] & \\
& C \otimes C \arrow[rr, "v\otimes v"'] & & C' \otimes C'
\end{tikzcd}$$

Let $F = (u, v)$ and $F' = (u', v')$ are functors from an internal category $(\pi : M \to C \otimes C, \varphi, \iota)$ to an internal category $(\pi' : M' \to C' \otimes C', \varphi', \iota')$.

Definition 9 *A functor morphism $\alpha : F \Rightarrow F'$ is a morphism $\alpha : C \to M'$ such that the following diagrams commute*

$$\begin{tikzcd}
C \arrow[r, "\alpha"] \arrow[d, "\delta"'] & M' \arrow[d, "\pi'"] \\
C \otimes C \arrow[r, "v \otimes v'"'] & C' \otimes C'
\end{tikzcd}$$

$$\begin{tikzcd}
 & C \otimes_C M \arrow[r, "\alpha \otimes_C u"] & M' \otimes_{C'} M' \arrow[dr, "\varphi'"] & \\
M \arrow[ur, "\lambda^{-1}"] \arrow[dr, "\varrho^{-1}"'] & & & M' \\
 & M \otimes_C C \arrow[r, "u' \otimes_C \alpha"'] & M' \otimes_{C'} M' \arrow[ur, "\varphi'"'] &
\end{tikzcd}$$

Now we can define a 2-category $2\mathcal{C}\mathrm{at}\,(\mathcal{M})$ of internal categories over a monoidal category $\mathcal{M}$ with additional structure $\otimes_{\downarrow}, p, q$. There are the following results, which are similar to the ones for usual 2-categories of internal categories (see for example, [16]).

Internal categories, internal functors, and internal natural transformations form a strict 2-category $2\mathcal{C}\mathrm{at}\,(\mathcal{M})$.

If we define now a 2-vector space as an internal category in $\mathcal{V}\mathrm{ect}_{\,k}$ we get a structure with additional parameters, a comonoid structure on the space of objects.

5 2-dimensional TQFTs

Let us consider first the notation and terminology [8, 16, 17, 19, 20, 24, 25, 27]. Here we give a general description of the structure of TQFT to trace possible constructions for our deformations. But we are not ready suggest a concrete deformation of TQFTs.

5.1 Categories in Quantum Theory

We need to work in a category of physical systems but what are the morphisms between systems? Usually physical systems have not enough morphisms compatible with all structures of objects. In this case it necessary to transfer some of the structure on the category. Here we consider the category of Hilbert spaces and show how an additional structure on a category comes from additional structure on the objects.

Quantum physics is based on the theory of Hilbert spaces. The inner product in quantum physics which gives complex amplitudes $\langle\psi_1, \psi_2\rangle$ lacks the intuitive immediacy of probabilities. How should this the inner product be transformed to categories? Here we transfer a structure from objects to the category, more precisely to an additional structure on category.

From the category theoretic point of view, part of the problem is to understand the structures on the category of Hilbert spaces. Thus in topological quantum field theory we take all linear operators as morphisms. However, if we define a category $\mathcal{H}\mathrm{ilb}$ in this way, then $\mathcal{H}\mathrm{ilb}$ is equivalent to the category $\mathcal{V}\mathrm{ect}_{\mathbb{C}}$ of complex vector spaces. This then raises the question: how does $\mathcal{H}\mathrm{ilb}$ really differ from $\mathcal{V}ect_{\mathbb{C}}$, if as categories they are equivalent? The answer is to determine on $\mathcal{H}\mathrm{ilb}$ an additional structure, a $*$-category by the adjoint functor.

To transfer the inner product in Hilbert spaces to the category we consider the adjoint as making $\mathcal{H}\mathrm{ilb}$ into a $*$-category: a category $\mathcal{C}$ equipped with a contravariant functor $* : \mathcal{C} \to \mathcal{C}$ fixing objects and satisfying $*^2 = \mathrm{id}_{\mathcal{C}}$. While $\mathcal{H}\mathrm{ilb}$ and $\mathcal{V}ect_{\mathbb{C}}$ are equivalent as categories, only $\mathcal{H}\mathrm{ilb}$ is a $*$-category.

This is particularly important in topological quantum field theory. The relationships between topology, algebra and physics exploited by this subject amount in a large part to the existence of functors from various topologically defined categories to the category $\mathcal{H}\mathrm{ilb}$. These topologically defined categories are $*$-categories, and the really interesting functors from them to $\mathcal{H}ilb$ are always $*$-functors, functors preserving the $*$-structure.

Example 18. There is a $*$-category whose objects are finite sets of points and whose morphisms are 'tangles', category $\mathcal{T}$ of tangles [7]. This example involves 1-dimensional curves in 3-dimensional spacetime.

More generally, topological quantum field theory studies n-dimensional manifolds embedded in $(n+k)$-dimensional spacetime, which in the $k \to \infty$ limit appear as abstract n-dimensional manifolds. It appears that these are best described using certain n-categories with duals, meaning n-categories in which every j-morphism f has a dual f^*.

Example 19. One class of n-categories with duals should be the n-groupoids; they explain many relationships between TQFT and homotopy theory. However, the novel aspects of TQFT should arise from n-categories with duals that are not n-groupoids.

The analogy between adjoint functors and adjoint linear operators relies upon a deeper analogy: just as in quantum theory the inner product $\langle\phi, \psi\rangle$ represents the amplitude to pass from ψ to ϕ, in category theory $\mathrm{Hom}\,(X, Y)$

represents the set of ways to go from X to Y. These are to Hilbert spaces as categories are to sets. The analogues of adjoint linear operators between Hilbert spaces are certain adjoint functors between 2-Hilbert spaces.

Just as the basic example of a category is $\mathcal{S}\mathrm{et}$, the basic example of a 2-Hilbert space is $\mathcal{H}\mathrm{ilb}$. Also, just as the 2-category $\mathcal{C}\mathrm{at}$ is a 3-category, it appears that the 2-category $2\mathcal{H}\mathrm{ilb}$ is an example of a '3-Hilbert space'. More generally, it appears that $n\mathcal{H}\mathrm{ilb}$ is an n-category with duals, and that 'n-Hilbert spaces' are needed for the proper treatment of n-dimensional topological quantum field theories.

An approach to quantum field theory based on operator algebras in Hilbert spaces has been presented in [21].

5.2 Cobordism Category

Let us recall the structure of a cobordism category. A manifold we always understand as a compact, oriented, differentiable manifold. In particular, it admits at least one structure of a finite CW-complex with cells up to the dimension of the manifold.

The category $\mathcal{C}\mathrm{ob}_n$ of oriented n-dimensional cobordisms has oriented compact $n-1$ dimensional manifolds as objects and cobordisms as morphisms.

A cobordism from M to N is an oriented n-dimensional manifold P with boundary, together with an oriented diffeomorphism between the boundary of P and disjoin union $M^* \coprod N$. Composition of morphisms results from gluing of manifolds along shared boundary components.

The category of oriented n-cobordisms has the natural structure of a tensor category with duality. The tensor product is direct sum (disjoint union) and the duality is the reversal of orientation.

Definition 10 *A cobordism $W : M \to N$ between closed manifolds M and N is a manifold W such that $\partial W = M \coprod N$ where M and N have chosen orientations via their normal bundles, and such that the orientation induced from W agrees with that of N and disagrees with the one on M. This state of affairs is described symbolically by $W = -M \coprod N$. The standard notations are $\partial^- = M$ and $\partial^+ = N$ to denote the incoming and the outgoing components of the boundary.*

A diffeomorphism between cobordisms (W, F) and (V, G) is a diffeomorphism $\Psi : W \to V$ such that $\Psi(\partial^\pm W) = \partial^\pm V$ and $F = \Psi G$, where equality is understood as equality of homotopy classes of maps.

Let Cob_{n+1} be a category of $(n+1)$-dimensional cobordisms, i.e. has compact oriented n-dimensional manifolds as objects and compact oriented cobordisms, which are equivalence classes of $(n+1)$-manifolds with boundary, between them as morphisms, and it has a monoidal structure with the product given by disjoint union of manifolds.

5.3 TQFT

Let us recall the structure of a TQFT to show the role of an additional structure and to show a possible deformation of categorical structures using a nontrivial comonoid structure on the set of objects. An n-dimensional TQFT is a symmetric monoidal functor $F : Cob_{n+1} \to \mathcal{V}\mathrm{ect}$.

Roughly speaking, if a lagrangian (the theory) is invariant under topological isomorphisms (homeomorphisms), then we get a TQFT.

The structures of generators and relations for the construction of low dimensional TQFTs by various combinatorial methods are equivalent to the structures of various fundamental objects in abstract algebra. Thus, 2D-TQFTs can be constructed from commutative Frobenius algebras or from semisimple associative algebras; while 3D theories can be constructed either from braided monoidal categories or from Hopf algebras.

The subject of TQFT began with the study of path integrals for lagrangians with topological invariance. Now we do not have a rigorous theory of this approach. It is possible to make formal manipulations of path integrals to deduce that the theories derived from them should have certain properties. The properties come from two aspects of the theory of path integrals. One is the idea that since a path integral is a sort of integral for each point in a space, we can separate it into integrals over parts of a space, by a sort of Fubini's theorem. The other is that the topological lagrangians possess a large gauge symmetry, with respect to which physical states (and the whole theory) must be invariant. If we cut space up along submanifolds of codimensions one and two, we get states with boundary attached to codimension one submanifolds with codimension two boundaries which transform non-trivially under the quantum version of the gauge symmetry on the codimension two submanifolds. This gauge symmetry is responsible for the appearance of tensor categories or Hopf algebras in the structure of a TQFT. A formal derivation from a path integral would serve no mathematical purpose, since path integrals themselves are not rigorously defined.

The definition of an n-dimensional TQFT is that it is a monoidal functor F from the category $\mathcal{C}\mathrm{ob}_{n+1}$ of oriented $(n+1)$-cobordisms to the category $\mathcal{V}\mathrm{ect}$ of complex finite dimensional vector spaces with the usual tensor product. Then a manifold with opposite orientation is sent to the dual space of the image with the given orientation, $F(M^*) = F(M)^*$, and $F(M \coprod N) = F(M) \otimes F(N)$.

The Hilbert space $E = E(S^1)$ attached to the circle S^1 in a two-dimensional TQFT is a commutative associative algebra A with unit and inner product $\langle\ ,\ \rangle$ such that the trilinear form $x, y, z \mapsto \langle xy, z\rangle$ is totally symmetric. An algebra A with this structure, called a Frobenius algebra, is derived from the functional integral over simple surfaces — spheres with disks removed. If this algebra is semisimple it can be diagonalized and then there is an explicit formula for the partition function of an oriented surface. It is also true that the field theory is determined by the Frobenius algebra.

Let us note that since the boundary of a cobordism is a disjoint union of two manifolds, one with reversed orientation, it is equivalent to assign a linear map to a cobordism, or a vector in the vector space on the boundary to a manifold with boundary. It follows from this observation that the invariant of closed 3-manifolds arising from a 3D-TQFT can be viewed as the dual pairing of vectors associated to 3-manifolds with (common, but oppositely oriented) boundary. This is, of course, Atiyah's original view.

5.4 Modifications

There are modifications of the definition of TQFTs by modifying the cobordism category by using additional structures on manifolds. For instance, we can specify a framing of the tangent bundle of the cobordisms and of a formal neighborhood of the closed manifolds. Another possibility is to include insertions of submanifolds in the manifolds and matching insertions in the cobordisms. Tensor and duality preserving functors from such modified cobordism categories to $\mathcal{V}\mathrm{ect}$ are called TQFTs too.

Any modification in cobordism category may lead to a modification in TQFT. This modification can be thought of as an extended version of TQFT. For example in Chern-Simons-Witten TQFT cobordisms are supplied with some additional structures.

The role of higher-dimensional algebra is clear from the various constructions of extended TQFTs. Baez and Dolan [23] outline a program in which n-dimensional TQFTs are described as n-category representation. They describe an n-dimensional extended TQFT as a weak n-functor from the free stable weak n-category with duals of one objects to n-$\mathcal{H}\mathrm{ilb}$ the category of n-Hilbert spaces, which preserve all levels of duality.
Homotopy theory methods were used to build examples of TQFT's. Homotopy quantum field theories (HQFT) are defined as topological quantum field theories for manifolds endowed with the additional structure in the form of a map into some background space X, that is a theory of objects over X. All these theories do is to fix a background space X and to compute a weighted sum over homotopy classes of maps $f : M \to X$ for a closed manifold M.

There is the important Turaev's theorem that the HQFT's only depend on the n-homotopy type of X. There are papers which discussed HQFT's in dimension $1+1$, but for a simply connected target space. There is the opinion that TQFT's may be considered a first approximation to full-blown quantum gravity, HQFT's are a first approximation to gravity coupled with matter.

In the n-categorical set up, one of the examples of monoidal 2-categories is the category $n\mathcal{C}\mathrm{ob}$, which has 0-manifolds as 0-cells, 1-manifolds with corners, i.e. cobordism between 0-manifolds as 1-cells, and 2-manifolds with corners as 2-cells.

Instead of taking 0-cells as 0-manifolds, one can also start with objects as 1-manifolds with or without corners to obtain a Atiya-Segal-style TQFT.

A 2-dimensional TQFT is a particular case of the construction. Here the category $\mathcal{C}\mathrm{ob}_{1+1}$ or $\mathcal{C}\mathrm{ob}_2$ has compact oriented 1-manifolds as objects and compact oriented cobordism between them as morphisms.

Extended TQFTs constructed by Kerler and Lyubashenko [33] involves higher category theory, namely double categories and double functors. Their construction of an extended version of TQFTs is quite different from the n-categorical version of extended TQFTs proposed by Baez and Dolan. It is not a generalized version of the Turaev construction of a TQFT functor, actually both constructions are different because of the different base categories.

Baez and Dolan's hypothesis for extended TFQTs shows that the TQFT functor which produces 2-dimensional extended TQFTs cannot be easily generalized to a 3-dimensional extended TQFT functor. Either they do not have a nice structure in higher dimension or their structure is very complicated, e.g. the enriched n-categorical version of $\mathcal{V}\mathrm{ect}$ is not very clear in dimension $n \geq 2$.

For $n = 2$, one can think 2-vector spaces as a vector space over the category $\mathcal{V}\mathrm{ect}_k$ of vector spaces over k.

5.5 Conclusions

Thus a monoidal category $\mathcal{M}$ with an tensor product $\oplus$ and a functor $\otimes : \mathcal{V}\mathrm{ect}_k \times \mathcal{M} \to \mathcal{M}$ must satisfy various conditions. It necessitates to construct different TQFT functors at different dimensional level. This suggests that in most of the higher dimensional cases these TQFTs functors will be independent from each other.

For the higher dimensional extended TQFTs, one needs to generalize the internal categories structure for higher dimensions in such a way that existing base category structures remain preserved, e.g. as in the case of $2\mathcal{V}\mathrm{ect}$, which contains ordinary vector spaces as objects. If we consider $3\mathcal{V}\mathrm{ect}$ to be the category having objects as internal categories of $2\mathcal{V}$ect and arrows are internal functors, then under suitable conditions $3\mathcal{V}$ect can gives a higher category version of $2\mathcal{V}$ect which also contain 2-vector spaces as objects.

The structure n-category results of iterative using ordinary categorical structure with weaken modified coherence conditions. Our deformation of category structure is similar. We modify diagonal comultiplication, but save all diagrams from the categorical axioms.

We describe a deformation of categories which gives new structures. But their theory is similar to the category theory because we deform only comultiplications which are in compositions on all levels in n-category. Our deformation can be applied to n-categories on different levels. Such a deformation of j-level induces deformation of structure on all higher levels.

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