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Exact Solution of Noncommutative Field Theory in Background Magnetic Fields

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Abstract

We obtain the exact non-perturbative solution of a scalar field theory defined on a space with noncommuting position and momentum coordinates. The model describes non-locally interacting charged particles in a background magnetic field. It is an exactly solvable quantum field theory which has non-trivial interactions only when it is defined with a finite ultraviolet cutoff. We propose that small perturbations of this theory can produce solvable models with renormalizable interactions.

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Quantum field theories on noncommutative spaces have received a surge of interest in recent years, primarily because they can be obtained as limits of string theory with background magnetic fields in which the massive string modes decouple (see [1] for reviews and exhaustive lists of references). They capture many of the non-local effects possessed by string theory but in a much simpler setting, and have attained a fundamental level of interest as examples of non-local field theories which may be well-defined. Various versions of them have also been proposed as effective field theory descriptions of some planar condensed matter systems in strong magnetic fields, such as quantum Hall models. Because of their embedding into string theory, these models are sometimes believed to be unitary and renormalizable.[§] However, they possess several unusual aspects which continue to challenge the conventional wisdom of quantum field theory, and question the renormalizability and overall consistency of these field theories.

On a canonical noncommutative space, the usual pointwise product of fields is replaced by the star-product

$$\Phi \star \Phi'(x) = \int \frac{d^D k \, d^D q}{(2\pi)^D} \tilde{\Phi}(k) \tilde{\Phi}'(q) e^{i k_\mu \theta^{\mu\nu} q_\nu} e^{i(k+q)\cdot x}, \quad (1)$$

where the tildes denote Fourier transforms and $\theta^{\mu\nu}$ is a constant antisymmetric matrix. The noncommutativity of space is encoded in the fact that the commutators of coordinates computed with this product are non-vanishing, as $x^\mu \star x^\nu = x^\mu x^\nu + i\theta^{\mu\nu}$. In perturbation theory, the phases in (1) produce momentum dependent vertices in Feynman diagrams which affect the interactions of the quantum field theory at energy scales below the scale $1/\sqrt{|\theta|}$ set by the dimensionful noncommutativity parameter $\theta^{\mu\nu}$. The most drastic example of this is known as ultraviolet/infrared (UV/IR) mixing. If one uses Fourier expansion of fields in a basis of plane waves $e^{ip\cdot x}$, as in (1), then the natural regularization of the quantum field theory is the restriction of momenta p to an annulus $\Lambda_0 < |p| < \Lambda$, where Λ_0 is an IR cutoff and Λ a UV cutoff. Removing the cutoffs amounts to taking the limits $\Lambda_0 \rightarrow 0$ and $\Lambda \rightarrow \infty$. Planar diagrams essentially coincide with those of ordinary quantum field theory, while non-planar graphs are modified by phases containing internal and external line momenta and are generically convergent. The rapid phase oscillations in (1) imply that a high-momentum cutoff Λ generates an effective IR cutoff $\Lambda_0 = 1/|\theta|\Lambda$. This appears to ruin the usual Wilsonian renormalization procedure which would require a clear separation of high and low momentum scales.

However, the puzzling UV/IR mixing properties may simply be an artifact of perturbation theory which disappears when summed to all orders. This is a non-perturbative issue which is in general difficult to address. In this letter we will formulate a noncommutative scalar field theory which is exactly solvable and obtain its non-perturbative solution explicitly. The model describes charged scalar particles in a background magnetic field with a four-point interaction defined by the star-product [2]. We will circumvent the problems set in by UV/IR mixing by using a basis for the expansion of fields on \mathbb{R}^D which differs from the more conventional plane wave basis and which will allow us to make sense of the field theory at a fully non-perturbative level. This expansion provides a natural non-perturbative regularization of the quantum field theory, producing both a short-distance and low-momentum cutoff simultaneously. We will show how to extract from this the exact expressions for Green's functions of the quantum field theory.

[§] Disclaimer: Views and opinions mentioned in this letter do not necessarily reflect those of the authors.

The model is defined by the Euclidean action

$$S = \int d^2x \left[\Phi^* (H_B + m^2) \Phi + \frac{g}{2} \Phi^* \star \Phi \star \Phi^* \star \Phi \right] , \quad (2)$$

where Φ is a charged scalar field on flat space \mathbb{R}^2 and

$$H_B = (-i \partial_\mu - B \epsilon_{\mu\nu} x^\nu)^2 \quad (3)$$

is the Landau Hamiltonian for a charged particle moving in two dimensions under the influence of a constant perpendicularly applied magnetic field $2B > 0$. For brevity, we will only work in $D = 2$ dimensions. The noncommutativity parameter is then given by $\theta^{\mu\nu} = \theta \epsilon^{\mu\nu}$. Because the commutators of the covariant momentum operators $-i \partial_\mu - B \epsilon_{\mu\nu} x^\nu$ are equal to $-2i B \epsilon_{\mu\nu}$, we may interpret the field theory (2) as being defined on a noncommutative space whose corresponding momentum space is also given by noncommuting coordinates. However, the ensuing analysis carries through to arbitrary even dimensionality [3, 4], and remarkably most of our conclusions hold quite generally. This follows from the fact that in any even dimension D there is a choice of coordinates which skew-diagonalizes the problem into a product of $D/2$ two-dimensional ones, to which the analysis of this letter applies with the appropriate changes. The details will be presented in a separate publication [4].

We shall find that the UV fixed point of this theory is trivial. The only scaling limit possible is one in which the coupling constant g vanishes as the UV cutoff is removed. We shall find that there is no intermediate scale in between the two natural UV and IR cutoffs in this model, consistent with the UV/IR duality found in [2], and the field theory is not renormalizable because the fields are correlated on the scale of the cutoff. This result is similar in spirit to earlier observations that asymptotically-free noncommutative field theories are trivial [5], and that generic ones are only well-defined when they contain both a finite UV and IR cutoff [6]. The renormalized propagator as an exact function of external momentum is given in the scaling limit by

$$\tilde{G}(p) = \frac{\sqrt{(p^2 + m^2)^2 + 4M^4} - (p^2 + m^2)}{2M^4} , \quad (4)$$

where m is the bare scalar particle mass and M is a dynamically generated mass scale. The non-free form of (4) takes into account a non-perturbative resummation of leading power divergences in the scaling limit which are generated by the degeneracies of the Landau levels. Such a renormalization procedure, though formally consistent, is physically meaningless and has little chance to produce an interacting quantum field theory in the scaling limit. The scalar field theory is thereby an example of a noncommutative field theory, with a finite cutoff, which is exactly solvable. The exact propagator at finite cutoff, which is computed below, produces (4) in the scaling limit and has a qualitatively similar but somewhat more complicated form. It exhibits a novel oscillatory behaviour in position space on top of its long-distance exponential decay, which may be attributed to the appearance of an Aharonov-Bohm phase acquired by the charged particles in the magnetic background, similar to those observed numerically in [7]. Slight modifications of the model, such as the inclusion of a background harmonic oscillator potential that lifts the Landau level degeneracy, may produce good scaling limits.

Because of the magnetic field in the action, a natural basis of normal modes is comprised of the orthonormal eigenfunctions $\phi_{\ell,n}$ of the Landau Hamiltonian (3),

$$H_B \phi_{\ell,n} = 4B \left(\ell - \frac{1}{2} \right) \phi_{\ell,n} , \quad (5)$$

with ℓ, n positive integers. Some properties of the Landau eigenfunctions $\phi_{\ell,n}$ are briefly described in an appendix at the end of this letter. These wavefunctions form the position space representation of the occupation number states $|\ell, n\rangle$ of two decoupled harmonic oscillators, and with them we can expand the complex scalar fields of (2) as

$$\Phi(x) = \sqrt{4\pi\theta} \sum_{\ell,n} A_{\ell n} \phi_{\ell,n}^*(x) \quad (6)$$

with $\phi_{\ell,n}^* = \phi_{n,\ell}$ and $A_{\ell n}$ dimensionless complex numbers.

In this basis, the free part of the action (2) is diagonal, but the four-point star-product interaction term is rather complicated. However, a special simplification occurs when the parameters of the model are related through $B = 1/\theta$. In this case, the Landau wavefunctions have a remarkably simple behaviour under star-products, $\phi_{\ell,n} \star \phi_{\ell',n'} = \delta_{n\ell'} \phi_{\ell,n'} / \sqrt{4\pi\theta}$, which can be derived by an explicit calculation and reflects the fact that the one-particle wavefunctions $\sqrt{4\pi\theta} \phi_{\ell,n}$ form the Wigner representations of the Fock space operators $|\ell\rangle\langle n|$. As we show below, the quantum field theory defined by (2) is exactly solvable precisely when the magnetic field and noncommutativity parameter are related in this way, and we shall assume this relation for the remainder of this letter. The action (2) then takes the simple form

$$S = \text{Tr} \left[E A^\dagger A + 2\pi\theta g (A^\dagger A)^2 \right] , \quad (7)$$

where we have naturally assembled the expansion coefficients of (6) into an infinite complex matrix $A = (A_{\ell n})$ and defined $E_{\ell n} = 4\pi(4\ell - 2 + \theta m^2) \delta_{\ell n}$. The noncommutativity of space is now manifested in the noncommutativity of matrix multiplication in (7).

This suggests that we may define the regularized quantum field theory with action (2) by restricting the quantum numbers of the Landau wavefunctions to $\ell, n = 1, \dots, N$ with $N < \infty$. The path integral is then defined as the $N \rightarrow \infty$ limit of the $N \times N$ matrix integral

$$Z_N = \int \prod_{\ell,n=1}^N dA_{\ell n} dA_{\ell n}^* e^{-\text{Tr} [E A^\dagger A + 2\pi\theta g (A^\dagger A)^2]} . \quad (8)$$

The finite matrix dimension N provides both a short-distance and low-momentum cutoff simultaneously, because in matrix regularizations of noncommutative field theory the UV and IR divergences are not clearly separated and one needs to regulate them both at the same time [2, 6]. There are, however, many different ways to take the large N limit of the matrix model with partition function (8), and we need to decide which is the appropriate one that captures the true non-perturbative physics of the original continuum field theory.

The large N limit is meaningful only when the entropy from the growth in the number of integration variables is compensated by a large action. In matrix models, an action of order N^2 , typically of the form $N \text{Tr} (\dots)$, is necessary to balance quantum fluctuations [8].

Hence we need to require that $\theta \sim N$ as $N \rightarrow \infty$ in (8). In other words, we must take the large N limit while keeping fixed the ratio

$$\Lambda^2 = N/4\pi\theta , \quad (9)$$

which simply defines the $\theta \rightarrow \infty$ limit of the noncommutative field theory. This limit is a generic feature of the matrix regularization of noncommutative field theories [6, 9]. The important feature of the model with a background magnetic field is that the whole action has a nice matrix representation, in contrast to other noncommutative field theories, in which the kinetic term has no matrix representation [9], and to ordinary field theories with background fields, in which the kinetic term is simple in the Landau basis, but the interactions are complicated [10].

There are two important consequences of this correlated large N and large θ limit. First of all, this limit is just the standard 't Hooft planar limit of the matrix model. Secondly, the natural UV cutoff of the original noncommutative field theory is the energy of the N^{th} Landau level, which is $2B(2N - 1) = 16\pi\Lambda^2 - 2B$ and stays *finite* as N goes to infinity. Thus the quantity (9) is the true UV cutoff of the quantum field theory. This will be confirmed below by explicit calculations. Since $B \rightarrow 0$ as $N \rightarrow \infty$, the spacing between Landau levels also vanishes. Thus taking the limit described above is equivalent to filling the finite energy interval $[0, 16\pi\Lambda^2]$ with infinitely many Landau levels and an infinite density of states.

Thus the large θ limit of the model defined by (2) is a quantum field theory with a finite cutoff whose exact solution is given by the 't Hooft limit of the complex external field matrix model (8). As an example, we will explicitly compute the exact two-point function defined by

$$G(x, y) = \left\langle \Phi^*(x) \Phi(y) \right\rangle = 4\pi\theta \sum_{\ell, n, \ell', n'} \left\langle A_{\ell n}^* A_{\ell' n'} \right\rangle \phi_{\ell, n}(x) \phi_{n', \ell'}(y) . \quad (10)$$

Both the action and integration measure in the path integral (8) are invariant under unitary transformations $A \rightarrow U \cdot A$ with $U \in U(N)$. This is just a consequence of the degeneracy of Landau levels. We can make this transformation explicit in the matrix integral and then integrate over the unitary group. We then use the well-known properties of the Haar measure of $U(N)$ and the fact that, by $U(N)$ invariance, the partition function (8) depends generically only on the N eigenvalues λ_ℓ of the external field E/N and is symmetric under permutation of them. It follows that the matrix averages appearing in (10) are given as $\langle A_{\ell n}^* A_{\ell' n'} \rangle = -\frac{1}{N} \delta_{nn'} \delta_{\ell\ell'} W(\lambda_\ell)$, where

$$W(\lambda_\ell) = \frac{1}{N} \frac{\partial \ln Z_N}{\partial \lambda_\ell} \quad (11)$$

and after differentiation the eigenvalues should be set equal to $\lambda_\ell = 16\pi \frac{\ell}{N} + \frac{m^2}{\Lambda^2}$. In what follows it will prove convenient to shift $\lambda_\ell \rightarrow \lambda_\ell - m^2/\Lambda^2$.

The computation of (10) thereby boils down to the calculation of the function (11) and the sum over Landau levels $\sum_n \phi_{\ell, n}(x) \phi_{n, \ell}(y)$ in the limit $N \rightarrow \infty$, $\ell \rightarrow \infty$ with ℓ/N fixed. Using known properties of the Landau wavefunctions, the sum can be evaluated in this scaling limit in terms of the Bessel function J_0 of the first kind of order 0 (see the

appendix). In the large N limit, we replace sums over Landau levels by integrals in the standard way according to the rule $\frac{1}{N} \sum_\ell \rightarrow \int_0^{16\pi} d\lambda/16\pi$, so that (10) becomes

$$G(x, y) = - \int_0^{16\pi} \frac{d\lambda}{4\pi} W \left(\lambda + \frac{m^2}{\Lambda^2} \right) J_0 \left(\Lambda \sqrt{\lambda} |x - y| \right) . \quad (12)$$

After a change of variables $\lambda = p^2/\Lambda^2$ and by using the angular integral representation of the Bessel function, we can express (12) as a two-dimensional integral

$$G(x, y) = - \frac{1}{\Lambda^2} \int_{|p| < 4\sqrt{\pi}\Lambda} \frac{d^2 p}{(2\pi)^2} W \left(\frac{p^2 + m^2}{\Lambda^2} \right) e^{i p \cdot (x - y)} . \quad (13)$$

This result has several remarkable implications. First of all, it demonstrates that the limit of large noncommutativity, in which the underlying space is expected to degenerate and all symmetries to be maximally violated, yields rotationally and translationally invariant Green's functions. There are remnants of UV/IR mixing in the far IR at $|x - y| \sim \sqrt{\theta}$, but these distances have been scaled out and all results here are valid at length scales far below the noncommutativity scale. Secondly, we see that the quantity $4\sqrt{\pi}\Lambda$ is a sharp cutoff in the momentum integral (13), showing clearly that (9) is the UV cutoff of the field theory. Finally, the matrix model partition function has the physical interpretation of providing the exact propagator in momentum space through the function (11),

$$\tilde{G}(p) = - \frac{1}{\Lambda^2} W \left(\frac{p^2 + m^2}{\Lambda^2} \right) , \quad p^2 < 16\pi \Lambda^2 . \quad (14)$$

It is instructive to consider a simple instance of this identification. At zero coupling, the matrix integral (8) can be explicitly evaluated to $Z_N = e^{-N \text{Tr} \ln E}$, so that $W(\lambda) = -1/\lambda$. This recovers the expected free propagator $\tilde{G}(p) = (p^2 + m^2)^{-1}$.

It remains to compute (11) in the general case. This can be done rather explicitly, because this function satisfies in the large N limit a closed equation, which is the Schwinger-Dyson equation of the matrix model given by

$$\frac{g}{\Lambda^2} \left(W^2(\xi) + \int_{m^2/\Lambda^2}^{16\pi + m^2/\Lambda^2} \frac{d\lambda}{16\pi} \frac{W(\xi) - W(\lambda)}{\xi - \lambda} \right) = \xi W(\xi) + 1 . \quad (15)$$

The loop equation (15) gives a straightforward way to generate the perturbative expansion to arbitrary orders of the original noncommutative field theory as an iterative solution of (15) in the coupling constant g . By using (14), the propagator up to one-loop order is easily determined in this way as

$$\tilde{G}(p) = \frac{1}{p^2 + m^2} - \frac{g}{16\pi} \frac{\ln(16\pi \Lambda^2/m^2)}{(p^2 + m^2)^2} - \frac{g \Lambda^2}{(p^2 + m^2)^3} + O(g^2) . \quad (16)$$

The second term in (16) recovers the usual one-loop logarithmic UV divergence of Φ^4 theory in two dimensions which is generated by the planar (field theoretical) bubble diagram and would lead to the renormalization group running of the mass. The third term is an additional quadratic UV divergence which is the non-planar (field theoretical) contribution. The additional divergences in Λ are even worse at higher loop orders. They

arise from the summations over degenerate Landau levels, whose degree of divergence grows with the order of perturbation theory and differs from that of usual scalar field theory.

The Schwinger-Dyson equation (15) can be solved by means of the methods developed in [11] to give

$$W(\lambda) = \frac{\Lambda^2}{2g} \left(\lambda - \sqrt{\lambda^2 + a\lambda + b} \right) + \frac{1}{2} \int_{m^2/\Lambda^2}^{16\pi+m^2/\Lambda^2} \frac{d\xi}{16\pi} \frac{1}{\sqrt{\xi^2 + a\xi + b}} \frac{\sqrt{\lambda^2 + a\lambda + b} - \sqrt{\xi^2 + a\xi + b}}{\lambda - \xi} . \quad (17)$$

The parameters a and b are unambiguously determined by substituting (17) into (15), which determines them through the algebraic equations

$$\int_{m^2/\Lambda^2}^{16\pi+m^2/\Lambda^2} \frac{d\xi}{16\pi} \frac{1}{\sqrt{\xi^2 + a\xi + b}} = \frac{a\Lambda^2}{2g} , \quad (18)$$

$$\int_{m^2/\Lambda^2}^{16\pi+m^2/\Lambda^2} \frac{d\xi}{16\pi} \frac{\xi}{\sqrt{\xi^2 + a\xi + b}} = \frac{\Lambda^2}{2g} \left(b - \frac{3}{4}a^2 \right) - 1 . \quad (19)$$

The solution (17) with these constraints matches the perturbation expansion of (15) and has the correct asymptotic behaviour $W(\lambda) \simeq -1/\lambda$ for $\lambda \rightarrow \infty$. The loop amplitude $W(\lambda)$ is an analytic function of λ on the complex plane with a square-root branch cut. The two branch points are the roots of the polynomial $\lambda^2 + a\lambda + b$ and are always complex, as follows from the constraints (18) and (19).

From (13) it follows that the long-distance asymptotics of the propagator are determined by the singularities of $W(\lambda)$. Since the two branch points occur at complex λ , the two-point function oscillates on top of its exponential decay. Consequently, for $|x - y| \gg 1/\Lambda$ we may write $G(x, y) \simeq e^{-|x-y|/L}$, where $1/L = \text{Im } p_0$ is determined by the condition that $z = (p_0^2 + m^2)/\Lambda^2$ solves the quadratic equation $z^2 + az + b = 0$. Careful inspection of the loop equations shows that the correlation length L is always of order of the cutoff scale unless the coupling g is very small, $g \sim 1/\Lambda^2$, and thus we define

$$g = M^4/\Lambda^2 . \quad (20)$$

From the constraint equations (18) and (19) we find that $b = 2M^4/\Lambda^4$ and $a = O(M^4/\Lambda^4)$ in this scaling limit. As a consequence, the renormalized two-point function, which is the $\Lambda \rightarrow \infty$ limit of (14), reduces to (4). By examining (16) one may infer that the scaling limit (20) resums the leading power divergences arising in the perturbation series. The explicit form of the propagator (14) at finite cutoff Λ is given by (17)–(19).

The power divergences arising in perturbation theory spoil the renormalizability of this field theory, but there can be many ways to get rid of them. For instance, we can replace the Landau Hamiltonian (3) in (2) by the combination $H_B + \sigma H_{-B}$, with σ a small parameter. Physically, this corresponds to the addition of a confining electric potential to the background of the charged scalar fields. This extension lifts the degeneracy of the Landau levels, yet the regulated version of the field theory still reduces to the matrix model (8) with an additional term $\sigma \text{Tr } E A A^\dagger$ in the action. While this term spoils

the $U(N)$ invariance of the matrix model, the latter still has a regular large N limit and is potentially solvable by an extension of the techniques presented in this letter by perturbative expansion in σ . The special case $\sigma = 1$ corresponds to charged particles in a harmonic oscillator potential alone and is closest to the conventional noncommutative field theories with no background magnetic field.

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Appendix: Here we collect some pertinent properties of Landau eigenfunctions. By introducing two sets of creation and annihilation operators $a = \partial/\sqrt{B} + \sqrt{B} \bar{z}/2$, $a^\dagger = -\bar{\partial}/\sqrt{B} + \sqrt{B} z/2$ and $b = \bar{\partial}/\sqrt{B} + \sqrt{B} z/2$, $b^\dagger = -\partial/\sqrt{B} + \sqrt{B} \bar{z}/2$, with $z = x^1 + i x^2$ and $\partial = (\partial_1 - i \partial_2)/2$, the Landau Hamiltonian (3) can be written as $H_B = 4B (a^\dagger a + \frac{1}{2})$. The eigenfunctions $\phi_{\ell,n}(z, \bar{z})$ of this Hamiltonian are characterized by the occupation numbers associated with the a and b oscillators and can be conveniently written in terms of the generating function (see [3], for example)

$$F_{s,t}(z, \bar{z}) \equiv \sum_{\ell,n=1}^{\infty} \frac{s^{\ell-1} t^{n-1}}{\sqrt{(\ell-1)!(n-1)!}} \phi_{\ell,n}(z, \bar{z}) = \sqrt{\frac{B}{\pi}} e^{-B|z|^2/2 + \sqrt{B}(sz + t\bar{z}) - st}. \quad (21)$$

A straightforward calculation of the star-product with $\theta = 1/B$ yields the identity $F_{s,t} \star F_{s',t'} = e^{s't} F_{s,t'}/\sqrt{4\pi\theta}$, from which the formula for the star-product of Landau wavefunctions used in the main text may be easily deduced.

The sum over Landau levels that was encountered in the calculation of the two-point function can be found as follows. To compute $g_\ell(x, y) = 4\pi\theta \sum_n \phi_{\ell,n}(x) \phi_{n,\ell}(y)$, we introduce the generating function $g(x, y; r) = \sum_\ell g_\ell(x, y) \frac{r^{2\ell}}{\ell!}$. This can be calculated as

$$\begin{aligned} g(x, y; r) &= 4\pi\theta \int \frac{d^2 u}{\pi} e^{-|u|^2} \int_0^{2\pi} \frac{d\varphi}{2\pi} F_{r e^{i\varphi}, u}(x) F_{\bar{u}, r e^{-i\varphi}}(y) \\ &= 4 e^{-\frac{1}{2\theta} |x-y|^2 + \frac{i}{\theta} x \times y + r^2} J_0 \left(2r |x-y|/\sqrt{\theta} \right), \end{aligned} \quad (22)$$

where $x \times y = \epsilon_{\mu\nu} x^\mu y^\nu$. By extracting the Taylor coefficients of (22) using contour integration, we get in the limit of large ℓ and large θ the result $g_\ell(x, y) = 4 J_0(2|x-y|/\sqrt{\ell/\theta})$ that was used in the main text.

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