Hitchin–Kobayashi Correspondence,
Quivers, and Vortices

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ABSTRACT. A twisted quiver bundle is a set of holomorphic vector bundles over a complex manifold, labelled by the vertices of a quiver, linked by a set of morphisms twisted by a fixed collection of holomorphic vector bundles, labelled by the arrows. When the manifold is Kähler, quiver bundles admit natural gauge-theoretic equations, which unify many known equations for bundles with extra structure. In this paper we prove a Hitchin–Kobayashi correspondence for twisted quiver bundles over a compact Kähler manifold, relating the existence of solutions to the gauge equations to a stability criterion, and consider its application to a number of situations related to Higgs bundles and dimensional reductions of the Hermitian–Einstein equations.

INTRODUCTION

A quiver Q consists of a set Q_0 of vertices $v, v', \ldots$, and a set Q_1 of arrows $a : v \to v'$ connecting the vertices. Given a quiver and a compact Kähler manifold, a *quiver bundle* is defined by assigning a holomorphic vector bundle $\mathcal{E}_v$ to a finite number of vertices and a homomorphism $\phi_a : \mathcal{E}_v \to \mathcal{E}_{v'}$ to a finite number of arrows. A *quiver sheaf* is defined by replacing the term 'holomorphic vector bundle' by 'coherent sheaf' in this definition. If we fix a collection of holomorphic vector bundles M_a parametrized by the set of arrows, and the morphisms are $\phi_a : \mathcal{E}_v \otimes M_a \to \mathcal{E}_{v'}$, twisted by the corresponding bundles, we have a *twisted quiver bundle* or a *twisted quiver sheaf*. In this paper we define natural gauge-theoretic equations, that we call *quiver vortex equations*, for a collection of hermitian metrics on the bundles associated to the vertices of a twisted quiver bundle (for this, we need to fix hermitian metrics on the twisting vector bundles). To solve these equations, we introduce a stability criterion for twisted quiver sheaves, and prove a Hitchin–Kobayashi correspondence, relating the existence of (unique) hermitian metrics satisfying the quiver vortex equations to the stability of the quiver bundle. The equations and the stability criterion depend on some real numbers, the *stability parameters* (cf. Remarks 2.6 for the exact number of parameters). It is relevant to point out that our results cannot be derived from the general Hitchin–Kobayashi correspondence scheme developed by Banfield [Ba] and further generalized by Mundet [M]. This is due not only to the presence of twisting vector bundles, but also to the deformation of the Hermitian–Einstein terms in the equations. This deformation is naturally explained by the symplectic interpretation of the equations, and accounts for extra parameters in the stability condition for the twisted quiver bundle.

This correspondence provides a unifying framework to study a number of problems that have been considered previously. The simplest situation occurs when the quiver has a single vertex and no arrows, in which case a quiver bundle is just a holomorphic bundle $\mathcal{E}$, and the gauge equation is the *Hermitian–Einstein equation*. A theorem of Donaldson, Uhlenbeck and Yau [D1, D2, UY], establishes that a (unique) solution to the Hermitian–Einstein equation exists if and only if $\mathcal{E}$ is polystable. The bundle $\mathcal{E}$ is called *stable* (in the sense of Mumford–Takemoto) if $\mu(\mathcal{F}) < \mu(\mathcal{E})$ for each proper coherent subsheaf $\mathcal{F} \subset \mathcal{E}$, where the slope $\mu(\mathcal{F})$ is the degree divided by the rank; a finite direct sum of stable bundles with the same slope is called *polystable*. A correspondence of this type is usually known as a *Hitchin–Kobayashi correspondence*. A Hitchin–Kobayashi correspondence, where some extra structure is added to the bundle $\mathcal{E}$, appears in the theory of *Higgs bundles*, consisting of pairs

$(\mathcal{E}, \Phi)$ formed by a holomorphic vector bundle $\mathcal{E}$ and a morphism $\Phi : \mathcal{E} \to \mathcal{E} \otimes \Omega$, where Ω is the sheaf of holomorphic differentials (sometimes the condition $\Phi \wedge \Phi = 0$ is added as part of the definition). Higgs bundles were first studied by Hitchin [H] (when X is a compact Riemann surface), and Simpson [S] (when X is higher dimensional), who introduced a natural gauge equation for them, and proved a Hitchin–Kobayashi correspondence. Higgs bundles are twisted quiver bundles, for a quiver formed by one vertex and one arrow whose head and tail coincide, and the twisting bundle is the holomorphic tangent bundle (i.e. the dual to Ω). Another class of quiver bundles are *holomorphic triples* $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$, consisting of two holomorphic bundles $\mathcal{E}_1$ and $\mathcal{E}_2$, and a morphism $\Phi : \mathcal{E}_2 \to \mathcal{E}_1$. The quiver has two vertices, say 1 and 2, and one arrow $a : 2 \to 1$ (the twisting sheaf is $\mathcal{O}_X$). The corresponding equations are called the coupled vortex equations [G2, BG]. When $\mathcal{E}_2 = \mathcal{O}_X$, holomorphic triples are *holomorphic pairs* $(\mathcal{E}, \Phi)$, where $\mathcal{E}$ is a bundle and $\Phi \in H^0(X, \mathcal{E})$ (cf. [B]).

There are other examples of quiver vortex equations that come out naturally from the study of the moduli of solutions to the Higgs bundle equation. Combining a theorem of Donaldson and Corlette [D3, C] with the Hitchin–Kobayashi correspondence for Higgs bundles [H, S], one has that the set of isomorphism classes of semisimple complex representations of the fundamental group of X in $\mathrm{GL}(r, \mathbb{C})$ is in bijection with the moduli space of polystable Higgs bundles with vanishing Chern classes. When X is a compact Riemann surface, this generalizes a theorem of Narasimhan and Seshadri [NS], which provides an interpretation of the unitary representations of the fundamental group as degree zero polystable vector bundles, up to isomorphism. Now, if X is a compact Riemann surface of genus $g \geq 2$, the Morse methods introduced by Hitchin [H] reduce the study of the topology of the moduli $\mathcal{M}$ of Higgs bundles to the study of the topology of the moduli of complex variations of the Hodge structure — the critical points of the Morse function in this case. These are twisted quiver bundles, called *twisted holomorphic chains*, for a quiver whose vertex set is the set $\mathbb{Z}$ of integer numbers, and whose arrows are $a_i : i \to i + 1$, for each $i \in \mathbb{Z}$; the twisting bundle associated to each arrow is the holomorphic tangent bundle. The twisted holomorphic chains that appear in these critical submanifolds are polystable for particular values of the stability parameters. Using Morse theory, Hitchin [H] computed the Poincaré polynomial of $\mathcal{M}$ for the rank 2 case. Gothen [Go] obtained similar results for rank 3: the critical submanifolds are moduli spaces of stable twisted holomorphic chains formed by a line bundle and a rank 2 bundle (i.e. twisted holomorphic triples), and by three line bundles. To use these methods for higher rank, one needs to study moduli spaces of other twisted holomorphic chains. A possible strategy is to proceed as in [Th], studying the moduli of twisted holomorphic chains in the whole parameter space. Another interesting type of quiver bundles arise in the study of semisimple representations of the fundamental group of X in $\mathrm{U}(p, q)$, the unitary group for a hermitian inner product of indefinite signature. Here, the quiver has two vertices, say 1 and 2, and two arrows, $a : 1 \to 2$ and $b : 2 \to 1$, and the twisting bundle associated to each arrow is the holomorphic tangent bundle. These are studied in [BGG1, BGG2].

Another context in which quiver bundles appear naturally is in the study of dimensional reductions of the Hermitian–Einstein equation over the product of a Kähler manifold X and a flag manifold. In this case, the parabolic subgroup defining the flag manifold entirely determines the structure of the quiver [AG1, AG2]. The dimensional reduction for this kind of manifolds has provided insight in the general theory of quiver bundles, and was actually the first method used to prove a Hitchin–Kobayashi correspondence for holomorphic triples [G2, BG], holomorphic chains [AG1], and quiver bundles for more general quivers with relations [AG2]. In these examples, the quiver bundles are not twisted, however, there are other examples for which a generalization of the method of dimensional reduction has produced *twisted holomorphic triples* [BGK1, BGK2].

An important feature of the stability of quiver sheaves is that it generally depends on several real parameters. When X is an algebraic variety, the ranks and degrees appearing in the numerical condition defining the stability criterion are integral, and the parameter space is partitioned into chambers. Strictly semistable quiver sheaves can occur when the parameters are on a wall separating the chambers, and the stability condition only depends on the chamber in which the parameters are. In the

case of holomorphic triples [BG], there is a chamber (actually an interval in $\mathbb{R}$) where the stability of the triple is related to the stability of the bundles. This can be used to obtain existence theorems for stable triples when the parameters are in this chamber, while the methods of [Th] can be used to prove existence results for other chambers (see [BGG2] for recent work in the case of triples). The geography of the resulting convex polytope for other quivers is an interesting issue to which we wish to return in a future paper. To approach this problem, one should study the homological algebra of quiver bundles. This has been developed by Gothen and King in a paper [GK] that appeared after we submitted this paper.

When the manifold X is a point, a quiver bundle is just a quiver module (over $\mathbb{C}$; cf. e.g. [ARS]). For arbitrary X, a quiver bundle can be regarded as a family of quiver modules (the fibres of the quiver bundle), parametrized by X. One can thus transfer to our setting many constructions of the theory of quiver modules. In the last part of the paper we introduce a more algebraic point of view by considering the path algebra bundle of the twisted quiver and looking at twisted quiver bundles as locally free modules over this bundle of algebras. This point of view is inspired by a similar construction for quiver modules [ARS], and suggests a generalization to other algebras that appear naturally in other problems. This is something to which we plan to come back in the future.

The Hitchin–Kobayashi correspondence for quiver bundles combines in one theory two different versions, in some sense, of the theorem of Kempf and Ness [KN] identifying the symplectic quotient of a projective variety by a compact Lie group action, with the geometric invariant theory quotient. The first one is the classical Hitchin–Kobayashi correspondence for vector bundles, and the second one occurs when the manifold X is a point, in which case the equations and the stability condition reduce to the moment map equations and the stability condition for quiver modules introduced by King [K]. As we prove in Theorem 4.3, there is in fact a very tight relation between the quiver vortex equations and the moment map equations for quiver modules: when the twisting sheaves are $\mathcal{O}_X$ and the bundles have vanishing Chern classes, the existence of solutions to the quiver vortex equations is equivalent to the existence of flat metrics on the bundles which fibrewise satisfy the moment map equations for quiver modules.

1. TWISTED QUIVER BUNDLES

In this section we define the basic objects that we shall study: *twisted quiver bundles* and *twisted quiver sheaves*. They are representations of quivers in the categories of holomorphic vector bundles and coherent sheaves, respectively, twisted by some fixed holomorphic vector bundles, as explained in §1.2. Thus, many results about quiver modules, i.e. quiver representations in the category of vector spaces, can be tranferred to our setting. A good reference for quivers and their linear representations is [ARS].

1.1. **Quivers.**
A quiver, or directed graph, is a pair of sets $Q = (Q_0, Q_1)$ together with two maps $h, t : Q_1 \to Q_0$. The elements of Q_0 (resp. Q_1) are called the vertices (resp. arrows) of the quiver. For each arrow $a \in Q_1$, the vertex ta (resp. ha) is called the tail (resp. head) of the arrow a. The arrow a is sometimes represented by $a : v \to v'$ when $v = ta$ and $v' = ha$.

1.2. **Twisted quiver sheaves and bundles.**
Throughout this paper, X is a connected compact Kähler manifold, Q is a quiver, and M is a collection of finite rank locally free sheaves M_a on X, for each arrow $a \in Q_1$. By a sheaf on X, we shall will mean an analytic sheaf of $\mathcal{O}_X$-modules. Our basic objects are given by the following:

Definition 1.1. *An M-twisted Q-sheaf on X is a pair $\mathcal{R} = (\mathcal{E}, \phi)$, where $\mathcal{E}$ is a collection of coherent sheaves $\mathcal{E}_v$ on X, for each $v \in Q_0$, and ϕ is a collection of morphisms $\phi_a : \mathcal{E}_{ta} \otimes M_a \to \mathcal{E}_{ha}$, for each $a \in Q_1$, such that $\mathcal{E}_v = 0$ for all but finitely many $v \in Q_0$, and $\phi_a = 0$ for all but finitely many $a \in Q_1$.*

Remark 1.2. Given a quiver $Q = (Q_0, Q_1)$, as defined in §1.1, the sets Q_0 and Q_1 can be infinite, but for each M-twisted Q-sheaf $\mathcal{R} = (\mathcal{E}, \phi)$, the subset $Q_0' \subset Q_0$ of vertices v such that $\mathcal{E}_v \neq 0$, and the subset $Q_1' \subset Q_1$ of arrows a such that $\phi_a \neq 0$, are both finite. Thus, to any M-twisted Q-sheaf $\mathcal{R} = (\mathcal{E}, \phi)$, we can associate the subquiver $Q' = (Q_0', Q_1')$ of Q, and $\mathcal{R}$ can be seen as an M'-twisted Q'-sheaf, where Q_0', Q_1' are finite sets, and $M' \subset M$ is the collection of sheaves M_a with $a \in Q_1'$.

As usual, we identify a holomorphic vector bundle $\mathcal{E}$, with the locally free sheaf of sections of $\mathcal{E}$. Accordingly, a *holomorphic M-twisted Q-bundle* is an M-twisted Q-sheaf $\mathcal{R} = (\mathcal{E}, \phi)$ such that the sheaf $\mathcal{E}_v$ is a holomorphic vector bundle, for each $v \in Q_0$. For the sake of brevity, in the following the terms 'Q-sheaf' or 'Q-bundle' are to be understood as 'M-twisted Q-sheaf' or 'M-twisted Q-bundle', respectively, often suppressing the adjective 'M-twisted'.

A *morphism* $f : \mathcal{R} \to \mathcal{R}'$ between two Q-sheaves $\mathcal{R} = (\mathcal{E}, \phi)$, $\mathcal{R}' = (\mathcal{E}', \phi')$, is given by a collection of morphisms $f_v : \mathcal{E}_v \to \mathcal{E}_v'$, for each $v \in Q_0$, such that $\phi_a' \circ (f_v \otimes \mathrm{id}_{M_a}) = f_{v'} \circ \phi_a$, for each arrow $a : v \to v'$ in Q. If $f : \mathcal{R} \to \mathcal{R}'$ and $g : \mathcal{R}' \to \mathcal{R}''$ are two morphisms between representations $\mathcal{R} = (\mathcal{E}, \phi)$, $\mathcal{R}' = (\mathcal{E}', \phi')$, $\mathcal{R}'' = (\mathcal{E}'', \phi'')$, then the composition $g \circ f$ is defined as the collection of composed morphisms $g_v \circ f_v : \mathcal{E}_v \to \mathcal{E}_v''$, for each $v \in Q_0$. We have thus defined the category of M-twisted Q-sheaves on X, which is abelian. Important concepts in relation to stability and semistability (defined in §2.3) are the notions of Q-subsheaves and quotient Q-sheaves, as well as indecomposable and simple Q-sheaves. They are defined as for any abelian category. In particular, an M-twisted Q-subsheaf of $\mathcal{R} = (\mathcal{E}, \phi)$ is another M-twisted Q-sheaf $\mathcal{R}' = (\mathcal{E}', \phi')$ such that $\mathcal{E}_v' \subset \mathcal{E}_v$, for each $v \in Q_0$, $\phi_a(\mathcal{E}_{ta}' \otimes M_a) \subset \mathcal{E}_{ha}'$, for each $a \in Q_1$, and $\phi_a' : \mathcal{E}_{ta}' \otimes M_a \to \mathcal{E}_{ha}'$ is the restriction of ϕ_a to $\mathcal{E}_{ta}' \otimes M_a$, for each $a \in Q_0$.

2. GAUGE EQUATIONS AND STABILITY

2.1. **Gauge equations.** Throughout this paper, given a smooth bundle E on X, $\Omega^k(E)$ (resp. $\Omega^{i,j}(E)$) is the space of smooth E-valued complex k-forms (resp. (i, j)-forms) on X, ω is a fixed Kähler form on X, and $\Lambda : \Omega^{i,j}(E) \to \Omega^{i-1,j-1}(E)$ is contraction with ω (we use the same notation as e.g. in [D1]). The gauge equations will also depend on a fixed collection q of hermitian metrics q_a on M_a, for each $a \in Q_1$, which we fix once and for all. Let $\mathcal{R} = (\mathcal{E}, \phi)$ be a holomorphic M-twisted Q-bundle on X. A *hermitian metric* on $\mathcal{R}$ is a collection H of hermitian metrics H_v on $\mathcal{E}_v$, for each $v \in Q_0$ with $\mathcal{E}_v \neq 0$. To define the gauge equations on $\mathcal{R}$, we note that $\phi_a : \mathcal{E}_{ta} \otimes M_a \to \mathcal{E}_{ha}$ has a smooth adjoint morphism $\phi_a^{*H_a} : \mathcal{E}_{ha} \to \mathcal{E}_{ta} \otimes M_a$ with respect to the hermitian metrics $H_{ta} \otimes q_a$ on $\mathcal{E}_{ta} \otimes M_a$, and H_{ha} on $\mathcal{E}_{ha}$, for each $a \in Q_0$, so it makes sense to consider the composition $\phi_a \circ \phi_a^{*H_a} : \mathcal{E}_{ha} \to \mathcal{E}_{ta} \otimes M_a \to \mathcal{E}_{ha}$. Moreover, ϕ_a and $\phi_a^{*H_a}$ can be seen as morphisms $\phi_a : \mathcal{E}_{ta} \to \mathcal{E}_{ha} \otimes M_a^*$ and $\phi_a^{*H_a} : \mathcal{E}_{ha} \otimes M_a^* \to \mathcal{E}_{ta}$, so $\phi_a^{*H_a} \circ \phi_a : \mathcal{E}_{ta} \to \mathcal{E}_{ta}$ makes sense too.

Definition 2.1. *Let σ and τ be collections of real numbers σ_v, τ_v, with σ_v positive, for each $v \in Q_0$. A hermitian metric H satisfies the M-twisted quiver (σ, τ)-vortex equations if*

$$(2.2) \qquad \sigma_v \sqrt{-1}\, \Lambda F_{H_v} + \sum_{a \in h^{-1}(v)} \phi_a \circ \phi_a^{*H_a} - \sum_{a \in t^{-1}(v)} \phi_a^{*H_a} \circ \phi_a = \tau_v \, \mathrm{id}_{\mathcal{E}_v},$$

for each $v \in Q_0$ such that $\mathcal{E}_v \neq 0$, where F_{H_v} is the curvature of the Chern connection A_{H_v} associated to the metric H_v on the holomorphic vector bundle $\mathcal{E}_v$, for each $v \in Q_0$ with $\mathcal{E}_v \neq 0$.

2.2. **Moment map interpretation.** The twisted quiver vortex equations appear as a symplectic reduction condition, as we explain now. Let E be a collection of smooth vector bundles E_v, for each $v \in Q_0$, with $E_v = 0$ for all but finitely many $v \in Q_0$. By removing the vertices $v \in Q_0$ with $E_v = 0$ and all but finitely many arrows $a \in Q_1$, we obtain a finite subquiver, which we still call $Q = (Q_0, Q_1)$, such that $E_v \neq 0$ for each $v \in Q_0$ (see Remark 1.2). Let H_v be a hermitian metric on E_v, for each $v \in Q_0$. Let $\mathscr{A}_v$ and $\mathscr{G}_v$ be the corresponding spaces of unitary connections and their

unitary gauge groups, and let $\mathscr{A}_v^{1,1} \subset \mathscr{A}_v$ be the space of unitary connections A_v with $(\bar{\partial}_{A_v})^2 = 0$, for each $v \in Q_0$. The group

$$\mathscr{G} = \prod_{v \in Q_0} \mathscr{G}_v$$

acts on the space $\mathscr{A}$ of unitary connections, and on the representation space Ω^0, defined by

$$(2.3) \qquad \mathscr{A} = \prod_{v \in Q_0} \mathscr{A}_v, \quad \Omega^0 = \Omega^0(\mathscr{R}(Q, E)), \text{ with } \mathscr{R}(Q, E) = \bigoplus_{a \in Q_1} \mathrm{Hom}(E_{ta} \otimes M_a, E_{ha}),$$

where $\mathrm{Hom}(E_{ta} \otimes M_a, E_{ha})$ is the vector bundle of homomorphisms $E_{ta} \otimes M_a \to E_{ha}$. An element $g \in \mathscr{G}$ is a collection of group elements $g_v \in \mathscr{G}_v$, for each $v \in Q_0$, and an element $A \in \mathscr{A}$ (resp. $\phi \in \Omega^0$) is a collection of unitary connections $A_v \in \mathscr{A}_v$ (resp. smooth morphisms $\phi_a : E_{ta} \otimes M_a \to E_{ha}$), for each $v \in Q_0$ (resp. $a \in Q_1$). The $\mathscr{G}$-actions on $\mathscr{A}$ and Ω^0 are $\mathscr{G} \times \mathscr{A} \to \mathscr{A}, (g, A) \mapsto A' = g \cdot A$, with $\mathrm{d}_{A'_v} = g_v \circ \mathrm{d}_{A_v} \circ g_v^{-1}$, for each $v \in Q_0$; $\mathscr{G} \times \Omega^0 \to \Omega^0, (g, \phi) \mapsto \phi' = g \cdot \phi$, with $\phi'_a = g_{ha} \circ \phi_a \circ (g_{ta}^{-1} \otimes \mathrm{id}_{M_a})$, for each $a \in Q_1$, respectively. The induced $\mathscr{G}$-action on the product $\mathscr{A} \times \Omega^0$ leaves invariant the subset $\mathscr{N}$ of pairs (A, ϕ) such that $A_v \in \mathscr{A}_v^{1,1}$, for each $v \in Q_0$, and $\phi_a : E_{ta} \otimes M_a \to E_{ha}$ is holomorphic with respect to $\bar{\partial}_{A_{ta}}$ and $\bar{\partial}_{A_{ha}}$, for each $a \in Q_0$. Let ω_v be the $\mathscr{G}_v$-invariant symplectic form on $\mathscr{A}_v$, for each $v \in Q_0$, as given in [AB] for a compact Riemann surface, or e.g. in [DK, Proposition 6.5.8] for any compact Kähler manifold, that is,

$$\omega_v(\xi_v, \eta_v) = \int_X \Lambda \, \mathrm{tr}(\xi_v \wedge \eta_v), \quad \text{for } \xi_v, \eta_v \in \Omega^1(\mathrm{ad}(E_v)),$$

where $\mathrm{ad}(E_v)$ is the vector bundle of H_v-antiselfadjoint endomorphisms of E_v. The corresponding moment map $\mu_v : \mathscr{A}_v \to (\mathrm{Lie}\,\mathscr{G}_v)^*$ is given by $\mu_v(A_v) = \Lambda F_{A_v}$ (we use implicitly the inclusion of $\mathrm{Lie}\,\mathscr{G}_v$ in its dual space by means of the metric H_v on E_v). The symplectic form $\omega_{\mathscr{R}}$ on Ω^0 associated to the L^2-metric induced by the hermitian metrics on the spaces $\Omega^0(\mathrm{Hom}(E_{ta} \otimes M_a, E_{ha}))$ is $\mathscr{G}$-invariant, and has associated moment map $\mu_{\mathscr{R}} : \Omega^0 \to (\mathrm{Lie}\,\mathscr{G})^*$ given by $\mu_{\mathscr{R}} = \sum_{v \in Q_0} \mu_{\mathscr{R},v}$, with $\mu_{\mathscr{R},v} : \Omega^0 \to \mathrm{Lie}\,\mathscr{G}_v \subset \mathrm{Lie}\,\mathscr{G} \subset \mathrm{Lie}\,\mathscr{G})^*$ given by

$$(2.4) \qquad \sqrt{-1}\,\mu_{\mathscr{R},v}(\phi) = \sum_{a \in h^{-1}(v)} \phi_a \circ \phi_a^{*H_a} - \sum_{a \in t^{-1}(v)} \phi_a^{*H_a} \circ \phi_a, \quad \text{for } \phi \in \Omega^0,$$

(this follows as in [K, §6], which considers the action of a unitary group on a representation space of quiver modules). Given a collection σ of real numbers $\sigma_v > 0$, for each $v \in Q_0$, $\sum_{v \in Q_0} \sigma_v \omega_v + \omega_{\mathscr{R}}$ is obviously a $\mathscr{G}$-invariant symplectic form on $\mathscr{A} \times \Omega^0$. A moment map for this symplectic form is $\mu_\sigma = \sum_{v \in Q_0} \sigma_v \mu_v + \mu_{\mathscr{R}}$, where we are omitting pull-backs to $\mathscr{A} \times \Omega^0$ in the notation. Any collection τ of real numbers τ_v, for each $v \in Q_0$ defines an element $\sqrt{-1}\,\tau \cdot \mathrm{id} = \sqrt{-1}\sum_{v \in Q_0} \tau_v \, \mathrm{id}_{\mathcal{E}_v}$ in the center of $\mathrm{Lie}\,\mathscr{G}$. The points of the symplectic reduction $\mu_\sigma^{-1}(-\sqrt{-1} \cdot \tau)/\mathscr{G}$ are precisely the orbits of pairs (A, ϕ) such that the hermitian metric H satisfies the M-twisted (σ, τ)-vortex quiver equations on the corresponding holomorphic quiver bundle $\mathcal{R} = (\mathcal{E}, \phi)$. Thus, Definition 2.1 picks up the points of $\mu_\sigma^{-1}(-\sqrt{-1}\,\tau)$ in the Kähler submanifold (outside its singularities) $\mathscr{N}$. For convenience in the Hitchin–Kobayashi correspondence, it is formulated in terms of hermitian metrics.

2.3. **Stability.** To define stability, we need some preliminaries and notation. Let n be the complex dimension of X. Given a torsion-free coherent sheaf $\mathcal{E}$ on X, the double dual sheaf $\det(\mathcal{E})^{**}$ is a holomorphic line bundle, and we define the first Chern class $c_1(\mathcal{E})$ of $\mathcal{E}$ as the first Chern class of $\det(\mathcal{E})^{**}$. The degree of $\mathcal{E}$ is the real number

$$\deg(\mathcal{E}) = \frac{2\pi}{\mathrm{Vol}(X)} \frac{1}{(n-1)!} \left\langle c_1(\mathcal{E}) \smile [\omega^{n-1}], [X] \right\rangle,$$

where $\mathrm{Vol}(X)$ is the volume of X, $[\omega^{n-1}]$ is the cohomology class of ω^{n-1}, and $[X]$ is the fundamental class of X. Note that the degree depends on the cohomology class of ω. Given a holomorphic

vector bundle $\mathcal{E}$ on X, by Chern-Weil theory, its degree equals

$$\deg(\mathcal{E}) = \frac{1}{\text{Vol}(X)} \int_X \text{tr}(\sqrt{-1}\,\Lambda F_H),$$

where F_H is the curvature of the Chern connection associated to a hermitian metric H on $\mathcal{E}$.

Let Q be a quiver, and σ, τ be collections of real numbers σ_v, τ_v, with $\sigma_v > 0$, for each $v \in Q_0$; σ and τ are called the *stability parameters*. Let $\mathcal{R} = (\mathcal{E}, \phi)$ be a Q-sheaf on X.

Definition 2.5. *The (σ, τ)-degree and (σ, τ)-slope of $\mathcal{R}$ are*

$$\deg_{\sigma,\tau}(\mathcal{R}) = \sum_{v \in Q_0} \left(\sigma_v \deg(\mathcal{E}_v) - \tau_v \,\text{rk}(\mathcal{E}_v) \right), \quad \mu_{\sigma,\tau}(\mathcal{R}) = \frac{\deg_{\sigma,\tau}(\mathcal{R})}{\sum_{v \in Q_0} \sigma_v \,\text{rk}(\mathcal{E}_v)},$$

respectively. The Q-sheaf $\mathcal{R}$ is called (σ, τ)-stable (resp. (σ, τ)-semistable) if for all proper Q-subsheaves $\mathcal{R}'$ of $\mathcal{R}$, $\mu_{\sigma,\tau}(\mathcal{R}') < \mu_{\sigma,\tau}(\mathcal{R})$ (resp. $\mu_{\sigma,\tau}(\mathcal{R}') \leq \mu_{\sigma,\tau}(\mathcal{R})$). A (σ, τ)-polystable Q-sheaf is a direct sum of (σ, τ)-stable Q-sheaves, all of them with the same (σ, τ)-slope.

As for coherent sheaves, one can prove that any (σ, τ)-stable Q-sheaf is simple, i.e. its only endomorphisms are the multiples of the identity.

Remarks 2.6.   (i) If a holomorphic Q-bundle $\mathcal{R}$ admits a hermitian metric satisfying the (σ, τ)-vortex equations, then taking traces in (2.2), summing for $v \in Q_0$, and integrating over X, we see that the parameters σ, τ are constrained by $\deg_{\sigma,\tau}(\mathcal{R}) = 0$.

(ii) If we transform the parameters σ, τ, multiplying by a global constant $c > 0$, obtaining $\sigma' = c\sigma$, $\tau' = c\tau$, then $\mu_{\sigma',\tau'}(\mathcal{R}) = \mu_{\sigma,\tau}(\mathcal{R})$. Furthermore, if we transform the parameters τ by $\tau'_v = \tau_v + d\sigma_v$ for some $d \in \mathbb{R}$, and let $\sigma' = \sigma$, then $\mu_{\sigma',\tau'}(\mathcal{R}) = \mu_{\sigma,\tau}(\mathcal{R}) - d$. Since the stability condition does not change under these two kinds of transformations, the 'effective' number of stability parameters of a quiver sheaf $\mathcal{R} = (\mathcal{E}, \phi)$ is $2N(\mathcal{R}) - 2$, where $N(\mathcal{R})$ is the (finite) number of vertices $v \in Q_0$ with $\mathcal{E}_v \neq 0$. ¿From the point of view of the vortex equations (2.2), the first type of transformations, $\sigma' = c\sigma$, $\tau' = c\tau$, corresponds to a redefinition of the sections $\phi' = c^{1/2}\phi$ (note that the stability condition is invariant under this transformation), while the second type corresponds to the constraint $\deg_{\sigma,\tau}(\mathcal{R}) = 0$ in (i).

(iii) As usual with stability criteria, in Definition 2.5, to check (σ, τ)-stability of a Q-sheaf $\mathcal{R}$, it suffices to check $\mu_{\sigma,\tau}(\mathcal{R}') < \mu_{\sigma,\tau}(\mathcal{R})$ for the proper Q-subsheaves $\mathcal{R}' \subset \mathcal{R}$ such that $\mathcal{E}'_v \subset \mathcal{E}_v$ is saturated, i.e. such that the quotient $\mathcal{E}_v/\mathcal{E}'_v$ is torsion-free, for each $v \in Q_0$.

3. HITCHIN–KOBAYASHI CORRESPONDENCE

In this section we will prove a Hitchin–Kobayashi correspondence between the twisted quiver vortex equations and the stability condition for holomorphic twisted quiver bundles:

Theorem 3.1. *Let σ and τ be collections of real numbers σ_v and τ_v, respectively, with $\sigma_v > 0$, for each $v \in Q_0$. Let $\mathcal{R} = (\mathcal{E}, \phi)$ be a holomorphic M-twisted Q-bundle such that $\deg_{\sigma,\tau}(\mathcal{R}) = 0$. Then $\mathcal{R}$ is (σ, τ)-polystable if and only if it admits a hermitian metric H satisfying the quiver (σ, τ)-vortex equations (2.2). This hermitian metric H is unique up to an automorphism of the Q-bundle, i.e. up to a multiplication by a constant $\lambda_j > 0$ for each (σ, τ)-stable summand $\mathcal{R}_j$ of $\mathcal{R} = \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_l$.*

Remark 3.2. This theorem generalizes previous theorems, mainly Donaldson–Uhlenbeck–Yau theorem [D1, D2, UY], the Hitchin–Kobayashi correspondence for Higgs bundles [H, S], holomorphic triples and chains [AG1, BG], twisted holomorphic triples [BGK2], etc. It should be mentioned that Theorem 3.1 does not follow from the general theorems proved in [Ba, M] for the following two reasons. First, the symplectic form $\sum_{v \in Q_0} \sigma_v \omega_v + \omega_\mathscr{R}$ on $\mathscr{A} \times \Omega^0$ (cf. §2.2) has been deformed by the parameters σ whenever $\sigma_v \neq \sigma_{v'}$ for some $v, v' \in Q_0$; as a matter of fact, the vortex equations (2.2) depend on new parameters even for holomorphic triples or chains [AG1, BG], hence generalizing

their Hitchin–Kobayashi correspondences (in the case of a holomorphic pair $(\mathcal{E}, \phi)$, consisting of a holomorphic vector bundle $\mathcal{E}$ and a holomorphic section $\phi \in H^0(X, \mathcal{E})$, as considered in [B], which can be understood as a holomorphic triple $\phi : \mathcal{O}_X \to \mathcal{E}$, the new parameter can actually be absorbed in ϕ, so no new parameters are really present). Second, the twisting bundles M_a, for $a \in Q_1$, are not considered in [Ba, M]. Our method of proof combines the moment map techniques developed in [B, D2, S, UY] for bundles with a proof of a similar correspondence for quiver modules in [K, §6].

3.1. **Preliminaries and general notation.**

Throughout Section 3, $\mathcal{R} = (\mathcal{E}, \phi)$ is a fixed holomorphic (M-twisted) Q-bundle with $\deg_{\sigma, \tau}(\mathcal{R}) = 0$. To prove Theorem 3.1, we can assume that $Q = (Q_0, Q_1)$ is a finite quiver, with $\mathcal{E}_v \neq 0$, for $v \in Q_0$, and $\phi_a \neq 0$, for $a \in Q_1$ (if this is not the case, we remove the vertices v with $\mathcal{E}_v = 0$, and the arrows a with $\phi_a = 0$, see Remark 1.2). The technical details of the proof largely simplify by introducing the following notation. Unless otherwise stated, $v, v', \ldots$ (resp. $a, a', \ldots$) stand for elements of Q_0 (resp. Q_1), while sums, direct sums and products in $v, v', \ldots$ (resp. $a, a', \ldots$) are over elements of Q_0 (resp. Q_1). Thus, the condition $\deg_{\sigma, \tau}(\mathcal{R}) = 0$ is equivalent to $\sum_v \sigma_v \deg(\mathcal{E}_v) = \sum_v \tau_v \operatorname{rk}(\mathcal{E}_v)$. Let

$$(3.3) \qquad \mathcal{E} = \oplus_v \mathcal{E}_v;$$

a vector u in the fibre $\mathcal{E}_x$ over $x \in X$, is a collection vectors u_v in the fibre $\mathcal{E}_{v,x}$ over, for each $v \in Q_0$. Let $\bar{\partial}_{\mathcal{E}_v} : \Omega^0(\mathcal{E}_v) \to \Omega^{0,1}(\mathcal{E}_v)$ be the $\bar{\partial}$-operator of the holomorphic vector bundle $\mathcal{E}_v$, and let

$$(3.4) \qquad \bar{\partial}_{\mathcal{E}} = \oplus_v \bar{\partial}_{\mathcal{E}_v}$$

be the induced $\bar{\partial}$-operator on $\mathcal{E}$. A hermitian metric H_v on $\mathcal{E}_v$ defines a unique Chern connection A_{H_v} compatible with the holomorphic structure $\bar{\partial}_{\mathcal{E}_v}$; the corresponding covariant derivative is $\mathrm{d}_{H_v} = \partial_{H_v} + \bar{\partial}_{\mathcal{E}_v}$, where $\partial_{H_v} : \Omega^0(\mathcal{E}_v) \to \Omega^{1,0}(\mathcal{E}_v)$ is its $(1,0)$-part. Thus, given $u \in \Omega^{i,j}(\mathcal{E})$, $\bar{\partial}_{\mathcal{E}}(u) \in \Omega^{i,j+1}(\mathcal{E}) = \oplus_v \Omega^{i,j+1}(\mathcal{E}_v)$ is the collection of $\mathcal{E}_v$-valued $(i, j+1)$-forms $(\bar{\partial}_{\mathcal{E}}(u))_v = \bar{\partial}_{\mathcal{E}_v}(u_v)$, for each $v \in Q_0$.

3.1.1. **Metrics and associated bundles.**

Let Met_v be the space of hermitian metrics on $\mathcal{E}_v$. A hermitian metric $(\cdot, \cdot)_{H_v}$ on $\mathcal{E}_v$ is determined by a smooth morphism $H_v : \mathcal{E}_v \to \overline{\mathcal{E}_v^*}$, by $(u_v, u_v')_{H_v} = H_v(u_v)(u_v')$, with u_v, u_v' in the same fibre of $\mathcal{E}_v$. The *right* action of the complex gauge group $\mathscr{G}_v^c$ on Met_v is given, by means of this correspondence, by $Met_v \times \mathscr{G}_v^c \to Met_v$, $(H_v, g_v) \mapsto H_v \circ g_v$. Let $S_v(H_v)$ be the space of H_v-selfadjoint smooth endomorphisms of $\mathcal{E}_v$, for each $H_v \in Met_v$. We choose a fixed hermitian metric $K_v \in Met$ such that the hermitian metric $\det(K_v)$ induced by K_v on the determinant bundle $\det(\mathcal{E}_v)$ satisfies $\sqrt{-1}\,\Lambda F_{\det(K_v)} = \deg(\mathcal{E}_v)$, for each $v \in Q_0$ (such hermitian metric K_v exists by Hodge theory). Any other metric on $\mathcal{E}_v$ is given by $H_v = K_v e^{s_v}$ for some $s_v \in S_v$, or equivalently, by $(u_v, u_v')_{H_v} = (e^{s_v} u_v, u_v')_{K_v}$, where $S_v = S_v(K_v)$. Let Met be the space of hermitian metrics on $\mathcal{E}$ such that the direct sum $\mathcal{E} = \oplus_v \mathcal{E}_v$ is orthogonal. A metric $H \in Met$ is given by a collection of metrics $H_v \in Met_v$, by $(u, u')_H = \sum_v (u_v, u_v')_{H_v}$. Let $S(H) = \oplus_v S_v(H_v)$, for each $H \in Met$, and $S = S(K) = \oplus_v S_v$. A vector $s \in S(H)$ is given by a collection of vectors $s_v \in S_v(H_v)$, for each $v \in Q_0$, while a metric $H \in Met$ is given by $H = K e^s$ for some $s \in S$, i.e. $H_v = K_v e^{s_v}$. The (fibrewise) norm on $\mathcal{E}_v$ (resp. $\mathcal{E}$) corresponding to H_v (resp. H), is given by $|u_v|_{H_v} = (u_v, u_v)_{H_v}^{1/2}$ (resp. $|u|_H = (u, u)_H^{1/2}$). The corresponding L^2-metric and L^2-norm on the space of sections of $\mathcal{E}_v$ (resp. $\mathcal{E}$), is defined by

$$(u_v, u_v')_{L^2, H_v} = \int_X (u_v, u_v')_{H_v}, \quad \|u_v\|_{L^2, H_v} = (u_v, u_v)_{L^2, H_v}^{1/2}, \quad \text{for } u_v, u_v' \in \Omega^0(\mathcal{E}_v),$$

(resp. $(u, u')_{L^2, H} = \sum_v (u_v, u_v')_{L^2, H_v}$, $\|u\|_{L^2, H} = (u, u)_{L^2, H}^{1/2}$). The L^p-norm on the space of sections of $\mathcal{E}$, given by

$$\|u\|_{L^p, H} = \left(\int_X |u|_H^p \right)^{\frac{1}{p}} \quad \text{for } u \in \Omega^0(\mathcal{E}),$$

will also be useful. These metrics and norms induce canonical metrics on the associated bundles, which will be denoted with the same symbols. For instance, $H_v \in Met_v$ (resp. $H \in Met$) induces an L^p-norm $\| \cdot \|_{L^p, H_v}$ on $S_v(H_v)$ (resp. $\| \cdot \|_{L^p, H}$ on $S(H)$). To simplify the notation, we set $(u_v, u'_v) = (u_v, u'_v)_{K_v}, |u_v| = |u_v|_{K_v}, (u, u') = (u, u')_K, |u| = |u|_K$; and $(u_v, u'_v)_{L^2} = (u_v, u'_v)_{L^2, K_v}, \|u_v\|_{L^2} = \|u_v\|_{L^2, K_v}, (u, u')_{L^2} = (u, u')_{L^2, K}, \|u\|_{L^p} = \|u\|_{L^p, K}$.

The morphisms $\phi_a : \mathcal{E}_{ta} \otimes M_a \to \mathcal{E}_{ha}$ induce a section $\phi = \oplus_a \phi_a$ of the *representation bundle*, defined as the smooth vector bundle over X

$$\mathscr{R} = \bigoplus_a \operatorname{Hom}(\mathcal{E}_{ta} \otimes M_a, \mathcal{E}_{ha}).$$

A metric $H \in Met$ induces another metric H_a on each term $\operatorname{Hom}(\mathcal{E}_{ta} \otimes M_a, \mathcal{E}_{ha})$ of $\mathscr{R}$, by $(\phi_a, \phi'_a)_{H_a} = \operatorname{tr}(\phi_a \circ \phi'^{*H_a}_a)$ for ϕ_a, ϕ'_a in the same fibre of $\operatorname{Hom}(\mathcal{E}_{ta}, \mathcal{E}_{ha})$, where $\phi'^{*H_a}_a : \mathcal{E}_{ha} \to \mathcal{E}_{ta} \otimes M_a$ is defined as in §2.1. Thus, H defines a hermitian metric on $\mathscr{R}$, which we shall also denote H, by $(\phi, \phi')_H = \sum_a (\phi_a, \phi'_a)_{H_a}$, where ϕ, ϕ' are in a fibre of $\mathscr{R}$. The corresponding fibrewise norm $| \cdot |_H$ is given by $|\phi|_H = (\phi, \phi)^{1/2}_H$. By integrating the hermitian metric over X, $(\cdot, \cdot)_{H_a}$ and $(\cdot, \cdot)_H$ induce L^2-inner products $(\cdot, \cdot)_{H_a, L^2}$ and $(\cdot, \cdot)_{H, L^2}$ on $\Omega^0(\mathcal{E}_{ta} \otimes M_a, \mathcal{E}_{ha})$ and $\Omega^0 = \Omega^0(\mathscr{R})$ respectively, given by $(\phi_a, \phi'_a)_{H_a, L^2} = \int_X (\phi_a, \phi_a)_{H_a}$, for $\phi_a, \phi'_a \in \Omega^0(\mathcal{E}_{ta} \otimes M_a, \mathcal{E}_{ha})$, and $(\phi, \phi')_{H, L^2} = \sum_a (\phi_a, \phi_a)_{L^2, H_a}$, for $\phi, \phi' \in \Omega^0$, with associated L^2-norms $\| \cdot \|_{H_a, L^2}, \| \cdot \|_{H, L^2}$ given by $\|\phi_a\|_{L^2, H} = (\phi_a, \phi_a)^{1/2}_{L^2, H}$ and $\|\phi\|_{L^2, H} = (\phi, \phi)^{1/2}_{L^2, H}$. We set $(\phi, \phi') = (\phi, \phi')_K, |\phi| = |\phi|_K$, for each ϕ, ϕ' in the same fibre of $\mathscr{R}$; and $(\phi, \phi')_{L^2} = (\phi, \phi')_{L^2, K}, \|\phi\|_{L^2} = \|\phi\|_{L^2, K}$, for each ϕ, ϕ' smooth sections of $\mathscr{R}$.

3.1.2. The vortex equations.

Composition of two endomorphisms $s, s' \in S$ is defined by $(s \circ s')_v = s_v \circ s'_v$ for $v \in Q_0$. The identity endomorphism id of $\mathcal{E}$ is given by $\operatorname{id}_v = \operatorname{id}_{\mathcal{E}_v}$. Given a vector bundle F on X, we define the endomorphisms $\sigma, \tau : F \otimes \operatorname{End}(\mathcal{E}) \to F \otimes \operatorname{End}(\mathcal{E})$, where $\operatorname{End}(\mathcal{E})$ is the bundle of smooth endomorphisms of $\mathcal{E}$, by fibrewise multiplication, i.e. $(\sigma \cdot (f \otimes s))_v = f \otimes \sigma_v s_v$ and $(\tau \cdot (f \otimes s))_v = f \otimes \tau_v s_v$, for $f \in F$ and $s \in \operatorname{End}(\mathcal{E})$ in the fibres over the same point $x \in X$. For instance, if $s \in S$, then $(\sigma \cdot \bar{\partial}_{\mathcal{E}}(s))_v = \sigma_v \bar{\partial}_{\mathcal{E}_v}(s_v)$. Given $H \in Met$ and sections ϕ, ϕ' of $\mathscr{R}$, we define the endomorphisms $\phi \circ \phi'^{*H}, \phi^{*H} \circ \phi', [\phi, \phi'^{*H}] \in \Omega^0(\operatorname{End}(\mathcal{E}))$, using §2.1, by

$$(\phi \circ \phi'^{*H})_v = \sum_{v \in h^{-1}(a)} \phi_a \circ \phi'^{*H_a}_a, \quad (\phi^{*H} \circ \phi')_v = \sum_{v \in t^{-1}(a)} \phi^{*H_a}_a \circ \phi'_a,$$

$$[\phi, \phi'^{*H}] = \phi \circ \phi'^{*H} - \phi^{*H} \circ \phi'.$$

Note that $[\phi, \phi^{*H}] \in S(H)$. The quiver vortex equations (2.2) can now be written in a compact form

$$(3.5) \qquad \sigma \cdot \sqrt{-1}\, \Lambda F_H + [\phi, \phi^{*H}] = \tau \cdot \operatorname{id}, \quad \text{for } H \in Met.$$

Given $s \in S$ and $\phi \in \Omega^0 = \Omega^0(\mathscr{R})$, $s \circ \phi, \phi \circ s, [s, \phi], [\phi, s] \in \Omega^0$ are defined by

$$(s \circ \phi)_a = s_{ha} \circ \phi_a, \quad (\phi \circ s)_a = \phi_a \circ (s_{ta} \otimes \operatorname{id}_{M_a}), \quad [s, \phi] = s \circ \phi - \phi \circ s, \quad [\phi, s] = \phi \circ s - s \circ \phi.$$

3.1.3. The trace and trace free parts of the vortex equations.

The trace map is defined by $\operatorname{tr} : \operatorname{End}(\mathcal{E}) \to \mathbb{C}$, $s \mapsto \operatorname{tr}(s) = \sum_v \operatorname{tr}(s_v)$. Let $S^0(H)$ be the space of 'σ-trace free' H-selfadjoint endomorphisms $s \in S(H)$, i.e. such that $\operatorname{tr}(\sigma \cdot s) = 0$, or more explicitly, $\sum_v \sigma_v \operatorname{tr}(s_v) = 0$, for each $H \in Met$; let $S^0 = S^0(K) \subset S$. Let Met^0 be the space of metrics $H = Ke^s$ with $s \in S^0$. The metrics $H \in Met^0$ satisfy the trace part of equation (3.5), i.e.

$$(3.6) \qquad \operatorname{tr}(\sigma \cdot \sqrt{-1}\, \Lambda F_H) = \operatorname{tr}(\tau \cdot \operatorname{id}).$$

To prove this, let $H = Ke^s \in Met$ with $s \in S$. Then $\det(H_v) = \det(K_v)e^{\operatorname{tr} s_v}$ so $\operatorname{tr} F_{H_v} = F_{\det(H_v)} = F_{\det(K_v)} + \bar{\partial}\partial \operatorname{tr} s_v = \operatorname{tr} F_{K_v} + \bar{\partial}\partial \operatorname{tr} s_v$ (since the operators induced by $\bar{\partial}_{\det(\mathcal{E}_v)}$ and $\partial_{\det(K_v)}$ on the trivial bundle of endomorphisms of $\det(\mathcal{E}_v)$ are $\bar{\partial}$ and ∂, resp.). Adding for all v,

$\operatorname{tr}(\sigma \cdot \sqrt{-1}\,\Lambda F_H) = \operatorname{tr}(\sigma \cdot \sqrt{-1}\,\Lambda F_K) + \sqrt{-1}\,\Lambda \bar{\partial}\partial \operatorname{tr}(\sigma \cdot s)$, where $\operatorname{tr}(\sqrt{-1}\,\Lambda F_{K_v}) = \deg(\mathcal{E}_v)$ by construction (cf. §3.1.1), so $\operatorname{tr}(\sigma \cdot \sqrt{-1}\,\Lambda F_K) = \sum_v \sigma_v \deg(\mathcal{E}_v) = \sum_v \tau_v \operatorname{rk}(\mathcal{E}_v) = \operatorname{tr}(\tau \cdot \mathrm{id})$. Thus,

$$(3.7) \qquad \operatorname{tr}(\sigma \cdot \sqrt{-1}\,\Lambda F_H - \tau \cdot \mathrm{id}) = \sqrt{-1}\,\Lambda \bar{\partial}\partial \operatorname{tr}(\sigma \cdot s),$$

which is zero if $s \in S^0$. This proves (3.6). Therefore, a metric $H = Ke^s \in Met^0$ satisfies the quiver (σ, τ)-vortex equations (3.5) if and only if it satisfies the trace free part, i.e.

$$p_H^0\left(\sigma \cdot \sqrt{-1}\,\Lambda F_H + [\phi, \phi^{*H}] - \tau \cdot \mathrm{id}\right) = 0,$$

where $p_H^0 : S(H) \to S(H)$ is the H-orthogonal projection onto $S^0(H)$.

3.1.4. Sobolev spaces.
Following [UY, S, B], given a smooth vector bundle E, and any integers $k, p \geq 0$, $L_k^p \Omega^{i,j}(E)$ is the Sobolev space of sections of class L_k^p, i.e. E-valued (i,j)-forms whose derivatives of order $\leq k$ have finite L^p-norm. Throughout the proof of Theorem 3.1, we fix an even integer $p > \dim_{\mathbb{R}}(X) = 2n$. Note that there is a compact embedding of $L_2^p \Omega^{i,j}(E)$ into the space of continuous E-valued (i,j)-forms on X, for $p > 2n$. This embedding will be used in §3.1.6. Particularly important are the collection $L_2^p S = \oplus_v L_2^p S_v$ of Sobolev spaces $L_2^p S_v$ of K_v-selfadjoint endomorphisms of $\mathcal{E}_v$ of class L_2^p; the collection $Met_2^p \cong \prod_v Met_{2,v}^p$ of Sobolev metrics, with

$$Met_{2,v}^p = \{K_v e^{s_v} | s_v \in L_2^p S_v\}, \quad \text{for each } v \in Q_0;$$

the subspace $L_2^p S^0 \subset L_2^p S$ of sections $s \in L_2^p S$ such that $\operatorname{tr}(\sigma \cdot s) = 0$ almost everywhere in X; and

$$Met_2^{p,0} = \{Ke^s | s_v \in L_2^p S^0\} \subset Met_2^p.$$

Given $H = Ke^s \in Met_2^p$, with $s \in L_2^p S$, we define the H-adjoint of ϕ, generalizing the case where s_v is smooth, i.e. $\phi^{*H} = e^{-s} \circ \phi^{*K} \circ e^s$. Similar generalizations apply to the other constructions in §§3.1.2, 3.1.3, to define $L_2^p S_v(H_v)$ and $L_2^p S(H) = \oplus_v L_2^p S_v(H)$, as well as the subspace $L_2^p S^0(H) \subset L_2^p S(H)$, for each $H \in Met_2^p$. If $H_v = K_v e^{s_v} \in Met_{2,v}^p$ with $s_v \in L_2^p S_v$, we define the connection A_{H_v}, with L_1^p coefficients, and its curvature $F_{H_v} \in L^p \Omega^{1,1}(\mathrm{End}(\mathcal{E}_v))$, with L^p coefficients, generalizing the case where s_v is smooth:

$$(3.8) \qquad \mathrm{d}_{H_v} := \mathrm{d}_{K_v} + e^{-s_v}\partial_{K_v}(e^{s_v}), \quad F_{H_v} = F_{K_v} + \bar{\partial}_{\mathcal{E}_v}(e^{-s_v}\partial_{K_v}(e^{s_v})),$$

(where d_{H_v} is the covariant derivative associated to the connection A_{H_v}).

3.1.5. The degree of a saturated subsheaf.
A saturated coherent subsheaf $\mathcal{F}'$ of a holomorphic vector bundle $\mathcal{F}$ on X (i.e., a coherent subsheaf with $\mathcal{F}/\mathcal{F}'$ torsion-free), is reflexive, hence a vector subbundle outside of codimension 2. Given a hermitian metric H on $\mathcal{F}$, the H-orthogonal projection π' from $\mathcal{F}$ onto $\mathcal{F}'$, defined outside codimension 2, is an L_1^2-section of the bundle of endomorphisms of $\mathcal{F}$, so $\beta = \bar{\partial}_{\mathcal{F}}(\pi')$ is of class L^2, where $\bar{\partial}_{\mathcal{F}}$ is the $\bar{\partial}$-operator of $\mathcal{F}$. The degree of $\mathcal{F}'$ is

$$\deg(\mathcal{F}') = \frac{1}{\mathrm{Vol}(X)}\left(\int_X \operatorname{tr}(\pi' \sqrt{-1}\,\Lambda F_H) - \|\beta\|_{L^2,H}^2\right),$$

(cf. [UY, S, B]).

3.1.6. Some constructions involving hermitian matrices.
The following definitions slightly generalize [S, §4]. Let $\varphi : \mathbb{R} \to \mathbb{R}$ and $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be smooth functions. Given $s \in S$, we define $\varphi(s) \in S$ and linear maps $\Phi(s) : S \to S$ and $\Phi(s) : \Omega^0(\mathscr{R}) \to \Omega^0(\mathscr{R})$ (we denote the last two maps with the same symbol since there will not be possible confusion between them). Actually, we define maps of fibre bundles $\Phi : S \to S(\mathrm{End}\,\mathcal{E})$ and $\Phi : S \to S(\mathrm{End}\,\mathscr{R})$, for certain spaces $S(\mathrm{End}\,\mathcal{E})$ and $S(\mathrm{End}\,\mathscr{R})$, which we first define. Let $S(\mathrm{End}\,\mathcal{E}) = \oplus_v S(\mathrm{End}\,\mathcal{E}_v)$, where $S(\mathrm{End}\,\mathcal{E}_v)$ is the space of smooth sections of the bundle $\mathrm{End}(\mathrm{End}\,\mathcal{E}_v)$ which are selfadjoint w.r.t. the metric induced by K_v. Let $\mathrm{End}\,\mathscr{R}$ be the endomorphism bundle of the vector bundle $\mathscr{R}$; $S(\mathrm{End}\,\mathscr{R})$ is the space of smooth sections of $\mathrm{End}\,\mathscr{R}$ which are selfadjoint w.r.t. the metric induced by K_v and q_a. We

define $\varphi(s_v) \in S_v$ for $s_v \in S_v$ and a linear map $\Phi : S_v \to S(\operatorname{End} \mathcal{E}_v)$ as follows. Let $s_v \in S_v$. If $x \in X$, let $(u_{v,i})$ be an orthonormal basis of $\mathcal{E}_{v,x}$ (w.r.t. K_v), with dual basis $(u^{v,i})$, such that $s_v = \sum_i \lambda_{v,i} u_{v,i} \otimes u^{v,i}$. Furthermore, let $(m^{a,k})$ be the dual of an orthonormal basis of $M_{a,x}$ (w.r.t. q_a). The value of $\varphi(s_v) \in S_v$ at the point $x \in X$ is defined as in [S, §4], by

$$\varphi(s_v)(x) := \sum_i \varphi(\lambda_{v,i}) u_{v,i} \otimes u^{v,i}. \tag{3.9}$$

We define $\varphi(s) \in S$, for $s \in S$, by $\varphi(s)_v := \varphi(s_v)$. Given $f_v \in S_v$ with $f_v(x) = \sum_{i,j} f_{v,ij} u_{v,i} \otimes u^{v,j}$, the value of $\Phi(s_v) f_v \in S_v$ at the point $x \in X$ is

$$\Phi(s_v) f_v(x) := \sum_{i,j} \Phi(\lambda_{v,i}, \lambda_{v,j}) f_{v,ij} u_{v,i} \otimes u^{v,j}, \tag{3.10}$$

and we define $\Phi : S \to S(\operatorname{End} \mathcal{E})$ and $\Phi : S \to S(\operatorname{End} \mathcal{R})$ as follows. Let $s \in S$. First, if $f \in S$, $(\Phi(s) f)_v := \Phi(s_v) f_v$. Second, given a section ϕ of $\mathcal{R}$ such that the value of $\phi_a : \mathcal{E}_{ta} \otimes M_a \to \mathcal{E}_{ha}$ at $x \in X$ is $\phi_a(x) = \sum_{i,j,k} \phi_{a,ijk}(x) u_{ha,j} \otimes u^{ta,i} \otimes m^{a,k}$ for each $a \in Q_1$, the value of $\Phi(s) \phi \in \Omega^0(\mathcal{R})$ at $x \in X$ is

$$(\Phi(s) \phi(x))_a := \sum_{i,j,k} \Phi(\lambda_{ha,j}, \lambda_{ta,i}) \phi_{a,ijk}(x) u_{ha,j} \otimes u^{ta,i} \otimes m^{a,k}, \quad \text{for each } a \in Q_1. \tag{3.11}$$

Note that if Φ is given by $\Phi(x,y) = \varphi_1(x) \varphi_2(y)$ for certain functions $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$, then $(\Phi(s) \phi)_a = \varphi_1(s_{ha}) \circ \phi_a \circ (\varphi_2(s_{ha}) \otimes \operatorname{id}_{M_a})$, that is,

$$\Phi(s) \phi = \varphi_1(s) \circ \phi \circ \varphi_2(s). \tag{3.12}$$

Finally, given a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, we define $\operatorname{d} \varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as in [S, §4]:

$$\operatorname{d} \varphi(x,y) = \frac{\varphi(y) - \varphi(x)}{y - x}, \text{ if } x \neq y, \text{ and } \operatorname{d} \varphi(x,y) = \varphi'(x) \text{ if } x = y.$$

Thus,

$$\bar{\partial}_\mathcal{E}(\varphi(s)) = \operatorname{d} \varphi(s)(\bar{\partial}_\mathcal{E}(s)) \quad \text{for } s \in S. \tag{3.13}$$

The following lemma will be especially important in the proof of Lemma 3.44. Given a number b, $L^2_{k,b} S \subset L^p_k S$ is the closed subset of sections $s \in L^2_k S$ such that $|s| \leq b$ a.e. in X; $L^2_{0,b} S(\operatorname{End} \mathcal{R})$ is similarly defined.

Lemma 3.14.   (i) $\varphi : S \to S$ extends to a continous map $\varphi : L^2_{0,b} S \to L^2_{0,b'} S$ for some b'.

   (ii) $\varphi : S \to S$ extends to a map $\varphi : L^2_{1,b} S \to L^q_{1,b'} S$ for some b', for $q \leq 2$, which is continuous for $q < 2$. Formula (3.13) holds in this context.

   (iii) $\Phi : S \to S(\operatorname{End} \mathcal{E})$ extends to a map $\Phi : L^2_{0,b} S \to \operatorname{Hom}(L^2 \Omega^0(\operatorname{End} \mathcal{E}), L^q \Omega^0(\operatorname{End} \mathcal{E}))$ for $q \leq 2$, which is continous in the norm operator topology for $q < 2$.

   (iv) $\Phi : S \to S(\operatorname{End} \mathcal{R})$ extends to a continuous map $\varphi : L^2_{0,b} S \to L^2_{0,b'} S(\operatorname{End} \mathcal{R})$ for some b'.

   (v) The previous maps extend to smooth maps $\varphi : L^p_2 S \to L^p_2 S$, $\Phi : L^p_2 S \to L^p_2 S(\operatorname{End} \mathcal{E})$ and $\Phi : L^p_2 S \to L^p_2 S(\operatorname{End} \mathcal{R})$ between Banach spaces of Sobolev sections. Formulas (3.9)-(3.13) hold everywhere in X.

Proof. This follows as in [B, S]. For (v), $p > 2n$, so there is a compact embedding $L^p_2 \subset C^0$. $\square$

3.2. **Existence of special metric implies polystability.**

Let H be a hermitian metric on $\mathcal{R}$ satisfying the quiver (σ, τ)-vortex equations. To prove that $\mathcal{R}$ is (σ, τ)-polystable, we can assume that it is indecomposable —then we have to prove that it is actually (σ, τ)-stable. Let $\mathcal{R}' = (\mathcal{E}', \phi') \subset \mathcal{R}$ be a proper Q-subsheaf. We can assume that $\mathcal{E}'_v \subset \mathcal{E}_v$ is saturated for each $v \in Q_0$ (cf. Remark 2.6(iii)). Let π'_v be the H_v-orthogonal projection from $\mathcal{E}_v$ onto $\mathcal{E}'_v$, defined outside codimension 2,

$\pi_v'' = \mathrm{id} - \pi_v'$, and $\beta_v = \bar{\partial}_{\mathcal{E}}(\pi_v')$. The collections of sections π_v', π_v'', β_v define elements $\pi', \pi'' \in L_1^2 \Omega^0(\mathrm{End}\,\mathcal{E})$, $\beta \in L^2 \Omega^{0,1}(\mathrm{End}\,\mathcal{E})$, respectively. Taking the L^2-product with π' in (3.5),

$$(\sigma \cdot \sqrt{-1}\,\Lambda F_H, \pi')_{L^2, H} + ([\phi, \phi^{*H}], \pi')_{L^2, H} = (\tau \cdot \mathrm{id}, \pi')_{L^2, H}.$$

We now evaluate the three terms of this equation. The first term in the left hand side is

$$(\sigma \cdot \sqrt{-1}\,\Lambda F_H, \pi')_{L^2, H} = \sum_v \sigma_v (\sqrt{-1}\,\Lambda F_{H_v}, \pi_v')_{L^2, H_v} = \mathrm{Vol}(X) \sum_v \sigma_v \deg(\mathcal{E}_v) + \sum_v \sigma_v \|\beta_v\|^2_{L^2, H_v}$$

(cf. §3.1.5). Let $\phi' = \pi' \circ \phi \circ \pi'$, $\phi'' = \pi'' \circ \phi \circ \pi'$, $\phi^\perp = \pi' \circ \phi \circ \pi''$. Then $\phi = \phi' \circ \pi' + \phi^\perp \circ \pi'' + \phi'' \circ \pi''$ outside of codimension 2, for $\mathcal{R}' \subset \mathcal{R}$. Thus, $[\pi', \phi] = \phi^\perp \circ \pi''$, and the second term is

$$([\phi, \phi^{*H}], \pi')_{L^2, H} = (\phi, [\pi', \phi])_{L^2, H} = (\phi, \phi^\perp)_{L^2, H} = \|\phi^\perp\|^2_{L^2, H}.$$

Finally, the right hand side is

$$(\tau \cdot \mathrm{id}, \pi')_{L^2, H} = \int_X \sum_v \tau_v \,\mathrm{tr}(\pi_v') = \mathrm{Vol}(X) \sum_v \tau_v \,\mathrm{rk}(\mathcal{E}_v'),$$

(since $\mathrm{tr}(\pi_v') = \mathrm{rk}(\mathcal{E}_v')$ outside of codimension 2). Therefore

$$\mathrm{Vol}(X)\deg_{\sigma,\tau}(\mathcal{R}') = -\sum_{v \in Q_0} \sigma_v \|\beta_v\|^2_{L^2, H_v} - \sum_{a \in Q_1} \|\phi_a^\perp\|^2_{L^2, H_a}.$$

The indecomposability of $\mathcal{R}$ implies that either $\beta_v \neq 0$ for some $v \in Q_0$ or $\phi_a^\perp \neq 0$ for some $a \in Q_1$; thus, $\deg_{\sigma,\tau}(\mathcal{R}') < 0$, so $\mu_{\sigma,\tau}(\mathcal{R}') < 0 = \mu_{\sigma,\tau}(\mathcal{R})$, hence $\mathcal{R}$ is (σ, τ)-stable. $\square$

3.3. **The modified Donaldson lagrangian.** To define the modified Donaldson Lagrangian, we first recall the definition of the Donaldson lagrangian (cf. [S, §5]). Let $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by

$$(3.15) \qquad\qquad \Psi(x, y) = \frac{e^{y-x} - (y-x) - 1}{(y-x)^2}.$$

The *Donaldson lagrangian* $M_{D,v} = M_D(K_v, \cdot) : Met_{2,v}^p \to \mathbb{R}$ is given by

$$M_{D,v}(H_v) = (\sqrt{-1}\,\Lambda F_{K_v}, s_v)_{L^2} + (\Psi(s_v)(\bar{\partial}_{\mathcal{E}_v} s_v), \bar{\partial}_{\mathcal{E}_v} s_v)_{L^2}, \text{ for } H_v = K_v e^{s_v} \in Met_{2,v}^p, \ s_v \in L_2^p S_v.$$

The Donaldson lagrangian $M_{D,v} = M_D(K_v, \cdot)$ is additive in the sense that

$$(3.16) \qquad M_{D,v}(K_v, H_v) + M_{D,v}(H_v, J_v) = M_{D,v}(K_v, J_v), \quad \text{for } H_v, J_v \in Met_2^p.$$

Another important property is that the Lie derivative of $M_{D,v}$ at $H_v \in Met_2^p$, in the direction of $s_v \in L_2^p S_v(H_v)$, is given by the moment map (cf. §2.2), i.e.

$$(3.17) \qquad \frac{\mathrm{d}}{\mathrm{d}\varepsilon} M_{D,v}(H_v e^{\varepsilon s_v})\Big|_{\varepsilon=0} = (\sqrt{-1}\,\Lambda F_{H_v}, s_v)_{L^2, H_v}, \quad \text{with } H_v \in Met_2^p, s_v \in L_2^p S_v(H_v).$$

Higher order Lie derivatives can be easily evaluated. Thus, from (3.8),

$$(3.18) \qquad \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F_{H_v e^{\varepsilon s_v}} = \bar{\partial}_{\mathcal{E}_v}\partial_{H_v e^{\varepsilon s_v}} s_v, \quad \text{for each } H_v \in Met_2^p \text{ and } s_v \in L_2^p S(H_v)$$

so the second order Lie derivative is

$$(3.19) \qquad \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} M_{D,v}(H_v e^{\varepsilon s_v})\Big|_{\varepsilon=0} = (\sqrt{-1}\,\Lambda \bar{\partial}_{\mathcal{E}_v}\partial_{H_v} s_v, s_v)_{L^2, H_v} = \|\bar{\partial}_{\mathcal{E}_v} s_v\|_{L^2, H_v}$$

(the second equality is obtained by integrating $\mathrm{tr}(s_v \sqrt{-1}\,\Lambda \bar{\partial}_{\mathcal{E}_v}\partial_{H_v} s_v) = \sqrt{-1}\,\Lambda \bar{\partial}\,\mathrm{tr}(s_v \partial_{H_v} s_v) + |\bar{\partial}_{\mathcal{E}_v} s_v|^2_{H_v}$ over X, where $|\bar{\partial}_{\mathcal{E}_v} s_v|^2_{H_v} = -\sqrt{-1}\,\Lambda\,\mathrm{tr}(\bar{\partial}_{\mathcal{E}_v} s_v \wedge \partial_{H_v} s_v)$ by the Kähler identities, and $\int_X \Lambda \bar{\partial}\,\mathrm{tr}(s_v \partial_{H_v} s_v) = \int_X \bar{\partial}\,\mathrm{tr}(s_v \partial_{H_v}(s_v)) \wedge \omega^{n-1}/(n-1)! = 0$ by Stokes theorem — cf. e.g. [S, Lemma 3.1(b) and the proof Proposition 5.1]).

Definition 3.20. *The* modified Donaldson lagrangian $M_{\sigma,\tau} = M_{\sigma,\tau}(K, \cdot) : Met_2^p \to \mathbb{R}$ *is*

$$M_{\sigma,\tau}(H) = \sum_v \sigma_v M_{D,v}(H_v) + \|\phi\|_{L^2,H}^2 - \|\phi\|_{L^2,K}^2 - (s, \tau \cdot \mathrm{id})_{L^2}, \quad \text{for } H = Ke^s \in Met_2^p, \, s \in L_2^p S.$$

Using the constructions of §3.1.6, the modified Donaldson lagrangian can be expressed in terms of the functions $\Psi, \psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, with Ψ given by (3.15) and ψ defined by

$$(3.21) \qquad\qquad\qquad\qquad\qquad \psi(x,y) = e^{x-y}.$$

In the following, we use the notation $(\cdot, \cdot)_{L^2} = (\cdot, \cdot)_{L^2,K}$, $\|\cdot\|_{L^2} = \|\cdot\|_{L^2,K}$, as defined in §3.1.1.

Lemma 3.22. *If* $H = Ke^s \in Met_2^p$, *with* $s \in L_2^p S$, *then*

$$M_{\sigma,\tau}(H) = (\sigma \cdot \sqrt{-1}\,\Lambda F_K, s)_{L^2} + (\sigma \cdot \Psi(s)(\bar{\partial}_{\mathcal{E}} s), \bar{\partial}_{\mathcal{E}} s)_{L^2} + (\psi(s)\phi, \phi)_{L^2} - \|\phi\|_{L^2}^2 - (\tau \cdot \mathrm{id}, s)_{L^2}.$$

Proof. The first two terms follow from the definitions of $M_{D,v}$ and $M_{\sigma,\tau}$. To obtain the third term, we note that $\phi_a^{*H_a} = (e^{-s_{ta}} \otimes \mathrm{id}_{M_a}) \circ \phi_a^{*K_a} \circ e^{s_{ha}}$ and $(\psi(s)\phi)_a = e^{s_{ha}} \circ \phi_a \circ (e^{-s_{ta}} \otimes \mathrm{id}_{M_a})$ (cf. (3.12)), so $|\phi_a|_{H_a}^2 = \mathrm{tr}(\phi_a \circ \phi_a^{*H_a}) = \mathrm{tr}(e^{s_{ha}} \circ \phi_a \circ (e^{-s_{ta}} \otimes \mathrm{id}_{M_a}) \circ \phi_a^{*K_a}) = \mathrm{tr}((\psi(s)\phi)_a \circ \phi_a^{*K_a}) = ((\psi(s)\phi)_a, \phi_a)_{K_a}$. The last two terms follow directly from the definition of $M_{\sigma,\tau}$. $\qquad\square$

3.4. **Minima of $M_{\sigma,\tau}$, the main estimate, and the vortex equations.** Let $m_{\sigma,\tau} : Met_2^p \to L^p\Omega^0(\mathrm{End}\,\mathcal{E})$ be defined by

$$(3.23) \qquad m_{\sigma,\tau}(H) = \sigma \cdot \sqrt{-1}\,\Lambda F_H + [\phi, \phi^{*H}] - \tau \cdot \mathrm{id}, \quad \text{for } H = Ke^s \in Met_2^p, \, s \in L_2^p S.$$

Thus, $m_{\sigma,\tau}(H) \in L^p S(H)$ for each $H \in Met_2^p$, and actually $m_{\sigma,\tau}(H) \in L^p S^0(H)$ if $H \in Met_2^{p,0}$, by (3.6). Let $B > \|m_{\sigma,\tau}(K)\|_{L^p}^p$ be a positive real number. We are interested in the minima of $M_{\sigma,\tau}$ in the closed subset of $Met_2^{p,0}$ defined by

$$Met_{2,B}^{p,0} := \{H \in Met_2^{p,0} | \|m_{\sigma,\tau}(H)\|_{L^p,H}^p \le B\}$$

(the restriction to this subset will be necessary to apply Lemma 3.33 below).

Proposition 3.24. *If* $\mathcal{R}$ *is simple, i.e. its only endomorphisms are multiples of the identity, and* $H \in Met_{2,B}^{p,0}$ *minimises* $M_{\sigma,\tau}$ *on* $Met_{2,B}^{p,0}$, *then* $m_{\sigma,\tau}(H) = 0$.

The minima are thus the solutions of the vortex equations. To prove this, we need a lemma about the first and second order Lie derivatives of $M_{\sigma,\tau}$. Given $H \in Met_2^p$, $L_H : L_2^p S(H) \to L^p S(H)$ is defined by

$$(3.25) \qquad\qquad\qquad L_H(s) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} m_{\sigma,\tau}(He^{\varepsilon s})\big|_{\varepsilon=0}, \quad \text{for each } s \in L_2^p S(H).$$

Since $\phi^{*H_\varepsilon} = e^{-\varepsilon s}\phi^{*H}e^{\varepsilon s}$, with $H_\varepsilon = He^{\varepsilon s}$, we have

$$(3.26) \qquad\qquad\qquad\qquad \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\phi^{*H_\varepsilon}\big|_{\varepsilon=0} = [s, \phi]^{*H},$$

so $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}[\phi, \phi^{*H_\varepsilon}]\big|_{\varepsilon=0} = [\phi, [s, \phi]^{*H}]$. Together with (3.18), this implies that

$$(3.27) \qquad\qquad\qquad L_H(s) = \sigma \cdot \sqrt{-1}\,\bar{\partial}_{\mathcal{E}} \partial_H s + [\phi, [s, \phi]^{*H}].$$

Lemma 3.28. (i) $M_{\sigma,\tau}(K, H) + M_{\sigma,\tau}(H, J) = M_{\sigma,\tau}(K, J)$, *for* $H, J \in Met_2^p$;

(ii) $\dfrac{\mathrm{d}}{\mathrm{d}\varepsilon} M_{\sigma,\tau}(He^{\varepsilon s})\big|_{\varepsilon=0} = (m_{\sigma,\tau}(H), s)_{L^2,H}$, *for each* $H \in Met_2^p$ *and* $s \in L_2^p S(H)$;

(iii) $\dfrac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} M_{\sigma,\tau}(He^{\varepsilon s})\big|_{\varepsilon=0} = (L_H(s), s)_{L^2,H} = \sum_v \sigma_v \|\bar{\partial}_{\mathcal{E}_v} s_v\|_{L^2,H_v}^2 + \|[s, \phi]\|_{L^2,H}^2$, *for each* $H \in Met_2^p$ *and* $s \in L_2^p S(H)$.

Proof. Part (i) follows immediately from (3.16) and $(Ke^s)e^{s'} = Ke^{s+s'}$. To prove (ii) and (iii), let $H_\varepsilon = He^{\varepsilon s}$, for $\varepsilon \in \mathbb{R}$. From (3.26) we get $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|\phi|^2_{H_\varepsilon}\big|_{\varepsilon=0} = \mathrm{tr}\left(\phi\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\phi^{*H_\varepsilon}\big|_{\varepsilon=0}\right) = \mathrm{tr}(\phi[s,\phi]^{*H}) = ([\phi,\phi^{*H}],s)_H$, which together with (3.17), proves (ii) (the last term in (3.23) is trivially obtained). The first equality in (iii) follows from (ii), the H_ε-selfadjointness of s (since $s^{*H_\varepsilon} = e^{-\varepsilon s}s^{*H}e^{\varepsilon s} = e^{-\varepsilon s}se^{\varepsilon s} = s$), and (3.25):

$$\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}M_{\sigma,\tau}(H_\varepsilon)\big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(m_{\sigma,\tau}(H_\varepsilon),s)_{L^2,H_\varepsilon}\big|_{\varepsilon=0} = \int_X \mathrm{tr}\left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}m_{\sigma,\tau}(H_\varepsilon)\big|_{\varepsilon=0}s\right) = \int_X \mathrm{tr}(L_H(s)s),$$

which equals $(L_H(s),s)_{L^2,H}$. To prove the second equality in (iii), we first notice that if ϕ' is a smooth section of $\mathscr{R}$, then $(s,\phi'\circ\phi^{*H})_H = (s\circ\phi,\phi')_H$ and $(s,\phi^{*H}\circ\phi')_H = (\phi\circ s,\phi')_H$, so $(s,[\phi',\phi^{*H}])_H = ([s,\phi],\phi')_H$. The second equality in (iii) is now obtained using (3.27), (3.18) and taking $\phi' = [s,\phi]$ in the previous formula. $\qquad\square$

Proof of Proposition 3.24. We start proving that if $\mathcal{R}$ is simple and $H \in Met_2^{p,0}$, then the restriction of L_H to $L_2^p S^0(H)$, which we also denote by $L_H : L_2^p S^0(H) \to L^p S^0(H)$, is surjective. To do this, we only have to show that L_H is a Fredholm operator of index zero and that it has no kernel. First, for each vertex v, $k_v : L_2^p S_v(H_v) \to L^p S_v(H_v)$, defined by $k_v = \sqrt{-1}\Lambda\bar{\partial}_{\mathcal{E}_v}\partial_{H_v} - \sqrt{-1}\Lambda\bar{\partial}_{\mathcal{E}_v}\partial_{K_v}$, is obviously a compact operator (cf. §3.1.4), and by the Kähler identities, $\sqrt{-1}\Lambda\bar{\partial}_{\mathcal{E}_v}\partial_{K_v}$ acting on $L_2^p S$ is the $(1,0)$-laplacian $\Delta'_{K_v} = \partial_{K_v}\partial^*_{K_v} + \partial^*_{K_v}\partial_{K_v}$, which is elliptic and selfadjoint, hence Fredholm, and has index zero. Now, L_H equals $\sum_v \sigma_v \sqrt{-1}\Lambda\bar{\partial}_{\mathcal{E}}\partial_{H_v}$, up to a compact operator, so it is also a Fredholm operator of index zero. To prove that it has no kernel, we notice that if $s \in L_2^p S^0(H)$ satisfies $L_H(s) = 0$, then $(s,L_H(s))_{L^2,H} = 0$, so Lemma 3.28(iii) implies $\bar{\partial}_{\mathcal{E}_v}s_v = 0$ and $[s,\phi] = 0$; i.e. s is actually an endomorphism of $\mathcal{R}$, so $s_v = c\,\mathrm{id}_{\mathcal{E}_v}$, for certain constant c. Since $\mathrm{tr}(\sigma\cdot s) = 0$, the constant is $c = 0$, so $s_v = 0$.

Let H minimise $M_{\sigma,\tau}$ in $Met_{2,B}^{p,0}$. To prove that $m_{\sigma,\tau}(H) = 0$, we assume the contrary. Since $L_H : L_2^p S^0(H) \to L^p S^0(H)$ is surjective, and $m_{\sigma,\tau}(H) \in S^0(H)$ is not zero, there exists a non-zero $s \in L_2^p S^0(H)$ with $L_H(s) = -m_{\sigma,\tau}(H)$. We shall consider the values of $M_{\sigma,\tau}$ along the path $H_\varepsilon = He^{\varepsilon s} \in Met_2^{p,0}$ for small $|\varepsilon|$. First,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|m_{\sigma,\tau}(H_\varepsilon)|^2_{H_\varepsilon}\big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathrm{tr}(m_{\sigma,\tau}(H_\varepsilon)^2)\big|_{\varepsilon=0} = 2(m_{\sigma,\tau}(H),L_H(s))_H = -2|m_{\sigma,\tau}(H)|^2_H,$$

(cf. (3.25)), and since p is even,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\|m_{\sigma,\tau}(H_\varepsilon)\|^p_{L^p,H_\varepsilon}\big|_{\varepsilon=0} = \frac{p}{2}\int_X |m_{\sigma,\tau}(H)|^{p-2}_H\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|m_{\sigma,\tau}(H_\varepsilon)|^2_{H_\varepsilon}\big|_{\varepsilon=0} = -p\|m_{\sigma,\tau}(H)\|^p_{L^p,H} < 0,$$

so the path H_ε is in $Met_{2,B}^{p,0}$ for small $|\varepsilon|$. Thus, $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}M_{\sigma,\tau}(H_\varepsilon)\big|_{\varepsilon=0} = 0$, as H minimises $M_{\sigma,\tau}$ in $Met_{2,B}^{p,0}$. Now, Lemma 3.28(ii) applied to $s \in L_2^p S(H)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}M_{\sigma,\tau}(H_\varepsilon)\big|_{\varepsilon=0} = (m_{\sigma,\tau}(H),s)_{L^2,H} = -(L_H(s),s)_{L^2,H}.$$

As in the first paragraph of this proof, if $\mathcal{R}$ is simple and $s \in L_2^p S^0(H)$ satisfies $(s,L_H(s))_{L^2,H} = 0$, then Lemma 3.28(iii) implies that s is zero. This contradicts the assumption $m_{\sigma,\tau}(H) \neq 0$. $\qquad\square$

Definition 3.29. *We say that $M_{\sigma,\tau}$ satisfies the main estimate in $Met_{2,B}^{p,0}$ if there are constants $C_1, C_2 > 0$, which only depend on B, such that $\sup|s| \leq C_1 M_{\sigma,\tau}(H) + C_2$, for all $H = Ke^s \in Met_{2,B}^{p,0}$, $s \in L_2^p S$.*

Proposition 3.30. *If $\mathcal{R}$ is simple and $M_{\sigma,\tau}$ satisfies the main estimate in $Met_{2,B}^{p,0}$, then there is a hermitian metric on $\mathcal{R}$ satisfying the (σ,τ)-vortex equations. This hermitian metric is unique up to multiplication by a positive constant.*

Proof. This result is proved in exactly the same way as in [B, §3.14], so here we only sketch the proof. One first shows that if $M_{\sigma,\tau}(Ke^s)$ is bounded above, then the Sobolev norms $\|s\|_{L_2^p}$ are bounded. One then takes a minimising sequence $\{Ke^{s_j}\}$ for $M_{\sigma,\tau}$, with $s_j \in L_2^p S^0$; then $\|s_j\|_{L_2^p}$ are uniformly bounded, so after passing to a subsequence, $\{s_j\}$ converges weakly in L_2^p to some s. One then sees that $M_{\sigma,\tau}$ is continuous in the weak topology on $Met_{2,B}^{p,0}$, so $M_{\sigma,\tau}(Ke^{s_j})$ converges to $M_{\sigma,\tau}(Ke^s)$. Thus, $H = Ke^s$ minimises $M_{\sigma,\tau}$. By Proposition 3.24, $m_{\sigma,\tau}(H) = 0$, i.e. H satisfies the vortex equations. By elliptic regularity, H is smooth. The uniqueness of the solution H follows from the convexity of $M_{\sigma,\tau}$ (cf. Lemma 3.28(iii)) and the simplicity of $\mathcal{R}$. □

The proof of Theorem 3.1 is therefore reduced to show that if $\mathcal{R}$ is (σ,τ)-stable, then $M_{\sigma,\tau}$ satisfies the main estimate in $Met_{2,B}^{p,0}$ (this is the content of §3.6).

3.5. **Equivalence of C^0 and L^1 estimates.** The following proposition will be used in §3.6.

Proposition 3.31. *There are two constants $C_1, C_2 > 0$, depending on B and σ, such that for all $H = Ke^s \in Met_{2,B}^{p,0}$, $s \in L_2^p S^0$, $\sup |s| \leq C_1 \|s\|_{L^1} + C_2$.*

Corollary 3.32. *$M_{\sigma,\tau}$ satisfies the main estimate in $Met_{2,B}^{p,0}$ if and only if there are constants $C_1, C_2 > 0$, which only depend on B, such that $\|s\|_{L^1} \leq C_1 M_{\sigma,\tau}(H) + C_2$, for all $H = Ke^s \in Met_{2,B}^{p,0}$, $s \in L_2^p S^0$.* □

Corollary 3.32 is immediate from Proposition 3.31. To prove Proposition 3.31, we need three lemmas. The first one is due to Donaldson [D3] (see also the proof of [S, Proposition 2.1]).

Lemma 3.33. *There exists a smooth function $a : [0,\infty) \to [0,\infty)$, with $a(0) = 0$ and $a(x) = x$ for $x > 1$, such that the following is true: For any $\widetilde{B} \in \mathbb{R}$, there is a constant $C(\widetilde{B})$ such that if f is a positive bounded function on X and $\Delta f \leq b$, where b is a function in $L^p(X)$ $(p > n)$ with $\|b\|_{L^p} \leq \widetilde{B}$, then $\sup |f| \leq C(\widetilde{B}) a(\|f\|_{L^1})$. Furthermore, if $\Delta f \leq 0$, then $\Delta f = 0$.* □

Lemma 3.34. *If $s \in L_2^p S$ and $H = Ke^s \in Met_2^p$, then $([\phi, \phi^{*H}], s) \geq ([\phi, \phi^{*K}], s)$.*

Proof. The function $f(\varepsilon) = ([\phi, \phi^{*H_\varepsilon}], s)$ for $\varepsilon \in \mathbb{R}$, where $H_\varepsilon = Ke^{\varepsilon s}$, is increasing, as $\mathrm{d}f(\varepsilon)/\mathrm{d}\varepsilon = |[s,\phi]|^2_{H_\varepsilon} \geq 0$ (cf. (3.26)). Now, $f(0) = ([\phi, \phi^{*K}], s)$, $f(1) = ([\phi, \phi^{*H}], s)$, so we are done. □

Lemma 3.35. *If $H = Ke^s \in Met_2^p$, with $s \in L_2^p S$, then*

$$(m_{\sigma,\tau}(H) - m_{\sigma,\tau}(K), s) \geq \frac{1}{2} |\sigma^{1/2} \cdot s| \, \Delta |\sigma^{1/2} \cdot s|,$$

where $\sigma^{1/2} \cdot s \in L_2^p S$ is of course defined by $(\sigma^{1/2} \cdot s)_v = \sigma_v^{1/2} s_v$, for $v \in Q_0$.

Proof. This lemma, and its proof, are similar to (but not completely immediate from) [B, Proposition 3.7.1]. First, Lemma 3.34 and (3.8) imply

$$(3.36) \quad (m_{\sigma,\tau}(H) - m_{\sigma,\tau}(K), s) \geq \sqrt{-1}\,\Lambda(\sigma \cdot F_H - \sigma \cdot F_K, s) = \sqrt{-1}\,\Lambda(\sigma \cdot \bar{\partial}_{\mathcal{E}}(e^{-s}\partial_K e^s), s),$$

where

$$(3.37) \qquad (\sigma \cdot \bar{\partial}_{\mathcal{E}}(e^{-s}\partial_K e^s), s) = \bar{\partial}(\sigma \cdot e^{-s}\partial_K e^s, s) + (\sigma \cdot e^{-s}\partial_K e^s, \partial_K s)$$

(for A_K is the Chern connection corresponding to the metric K). To make some local calculations, we choose a local K_v-orthogonal basis $\{u_{v,i}\}$ of eigenvectors of s_v, for each vertex v, with corresponding eigenvalues $\{\lambda_{v,i}\}$, and let $\{u^{v,i}\}$ be the corresponding dual basis; thus,

$$s_v = \sum_i \lambda_{v,i} u_{v,i} \otimes u^{v,i}.$$

As in [B, (3.36)], a local calculation gives $(e^{-s_v}\partial_{K_v} e^{s_v}, s_v) = \frac{1}{2}\partial|s_v|^2$; multiplying by σ_v and adding for $v \in Q_0$, we get $(\sigma \cdot e^{-s}\partial_K e^s, s) = \frac{1}{2}\partial|s'|^2$, where $s' = \sigma^{1/2} \cdot s$. Thus,

$$(3.38) \qquad \bar{\partial}(\sigma \cdot e^{-s}\partial_K e^s, s) = \frac{1}{2}\bar{\partial}\partial|s'|^2 = |s'|\bar{\partial}\partial|s'| + \bar{\partial}|s'| \wedge \partial|s'|$$

¿From (3.36), (3.37), (3.38) and the equality $\Delta = 2\sqrt{-1}\Lambda\bar{\partial}\partial$ for the action of the laplacian on 0-forms in a Kähler manifold, we get

$$(m_{\sigma,\tau}(H) - m_{\sigma,\tau}(K), s) \geq \frac{1}{2}|s'|\Delta|s'| + \sqrt{-1}\,\Lambda(\bar{\partial}|s| \wedge \partial|s'|) + \sqrt{-1}\,\Lambda(\sigma \cdot e^{-s}\partial_K e^s, \partial_K s).$$

In the proof of [B, Proposition 3.7.1], there are several local calculations which, although there they are only used for the section $s \in L_2^p S$ defining the metric $H = Ke^s$, are actually valid for any K-selfadjoint section, in particular for $s' \in L_2^p S$. Thus, [B, (3.42)] applied to s_v is

$$\sqrt{-1}\,\Lambda(e^{-s_v}\partial_{K_v} e^{s_v}, \sigma \cdot \partial_{K_v} s_v) \geq \sum_i \sqrt{-1}\,\Lambda(\partial\lambda_{v,i} \wedge \bar{\partial}\lambda_{v,i}),$$

and multiplying by σ_v and adding for $v \in Q_0$, we get

$$(3.39) \qquad \sqrt{-1}\,\Lambda(\sigma \cdot e^{-s}\partial_K e^s, \sigma \cdot \partial_K s) \geq \sum_{v,i} \sqrt{-1}\,\Lambda(\partial\lambda'_{v,i} \wedge \bar{\partial}\lambda'_{v,i}),$$

where $\lambda'_{v,i} := \sigma_v^{1/2}\lambda_{v,i}$ are the eigenvalues of $s'_v = \sigma_v^{1/2}s_v$; similarly, [B, (3.43)] applied to s' is

$$(3.40) \qquad \sum_{v,i} \sqrt{-1}\,\Lambda(\partial\lambda'_{v,i} \wedge \bar{\partial}\lambda'_{v,i}) \geq \sqrt{-1}\,\Lambda(\partial|s'| \wedge \bar{\partial}|s'|) = -\sqrt{-1}\,\Lambda(\bar{\partial}|s'| \wedge \partial|s'|).$$

¿From (3.38), (3.39), (3.40), we obtain $(m_{\sigma,\tau}(H) - m_{\sigma,\tau}(K), s) \geq \frac{1}{2}|s'|\Delta|s'|$. $\square$

Proof of Proposition 3.31. Let $\sigma_{\min} = \min\{\sigma_v | v \in Q_0\}$, $\sigma_{\max} = \max\{\sigma_v | v \in Q_0\}$. Given $H = Ke^s \in Met_{2,B}^{p,0}$, with $s \in L_2^p S^0$, let $f = |\sigma^{1/2} \cdot s|$ and $b = 2\sigma_{\min}^{-1/2}(|m_{\sigma,\tau}(H)| + |m_{\sigma,\tau}(K)|)$. We now verify that f and b satisfy the hypotheses of Lemma 3.33, for a certain $\widetilde{B}$ which only depends on B. First, $\|b\|_{L^p} \leq 2\sigma_{\min}^{-1/2}(\|m_{\sigma,\tau}(H)\|_{L^p} + \|m_{\sigma,\tau}(K)\|_{L^p}) \leq \widetilde{B} := 2\sigma_{\min}^{-1/2}2B^{1/p}$. Second, we prove that

$$(3.41) \qquad \Delta f \leq b.$$

At the points where f does not vanish, $f^{-1} \leq \sigma_{\min}^{-1/2}|s|^{-1}$, so Lemma 3.35 gives

$$\Delta f \leq 2\sigma_{\min}^{-1/2}|s|^{-1}(m_{\sigma,\tau}(H) - m_{\sigma,\tau}(K), s) \leq 2\sigma_{\min}^{-1/2}|m_{\sigma,\tau}(H) - m_{\sigma,\tau}(K)| \leq b,$$

while to consider the points where f vanishes, we just take into account that $\Delta f = 0$ almost everywhere (a.e.) in $f^{-1}(0) \subset X$, and that $b \geq 0$ by its definition, so (3.41) actually holds a.e. in X. The hypotheses of Lemma 3.33 are thus satisfied, so there exists a constant $C(B) > 0$ such that $\sup f \leq C(B)a(\|f\|_{L^1})$, with with $a : [0,\infty) \to [0,\infty)$ as in Lemma 3.33. This estimate can also be written as $\sup f \leq C_1\|f\|_{L^1} + C_2$, where $C_1, C_2 > 0$ only depend on B. Now, $|s| \leq \sigma_{\min}^{-1/2}f$ and $f \leq \sigma_{\max}^{1/2}|s|$, so

$$\sup|s| \leq \sigma_{\min}^{-1/2}(C_1\|f\|_{L^1} + C_2) \leq \sigma_{\min}^{-1/2}(C_1\sigma_{\max}^{1/2}\|s\|_{L^1} + C_2)$$

The estimate is obtained by redefining the constants C_1, C_2. $\square$

3.6. **Stability implies the main estimate.** The following proposition, together with Proposition 3.30, are the key ingredients to complete the proof of Theorem 3.1 (cf. Definition 3.29 for the main estimate).

Proposition 3.42. *If $\mathcal{R}$ is (σ, τ)-stable, then $M_{\sigma,\tau}$ satisfies the main estimate in $Met^{p,0}_{2,B}$.*

To prove this, we need some preliminaries (Lemmas 3.43-3.46). Let $\{C_j\}_{j=1}^{\infty}$ be a sequence of constants with $\lim_{j\to\infty} C_j = \infty$.

Lemma 3.43. *If $M_{\sigma,\tau}$ does not satisfy the main estimate in $Met^{p,0}_{2,B}$, then there is a sequence $\{s_j\}_{j=1}^{\infty}$ in $L_2^p S^0$ with $Ke^{s_j} \in Met^{p,0}_{2,B}$ (which we can assume to be smooth), such that*

(i) $\lim_{j\to\infty} \|s_j\|_{L^1} = \infty$,

(ii) $\|s_j\|_{L^1} \geq C_j M(Ke^{s_j})$.

Proof. Let $b > \|m_{\sigma,\tau}(K)\|^p_{L^p}$ with $b < B$, so $Met^{p,0}_{2,b} \subset Met^{p,0}_{2,B}$. Thus, if $M_{\sigma,\tau}$ does not satisfy the main estimate in $Met^{p,0}_{2,B}$, then it does not satisfy the main estimate in $Met^{p,0}_{2,b}$ either. We shall prove that for any positive constant C', if there are positive constants C'' and N such that $\|s\|_{L^1} \leq C' M_{\sigma,\tau}(Ke^s) + C''$ whenever $s \in L_2^p S^0$ with $Ke^s \in Met^{p,0}_{2,b}$ and $\|s\|_{L^1} \geq N$, then $M_{\sigma,\tau}$ satisfies the main estimate in $Met^{p,0}_{2,b}$. The lemma follows from this claim by choosing a sequence of constants $\{N_j\}_{j=1}^{\infty}$ with $N_j \to \infty$, and taking C_j'' and $s_j \in L_2^p S^0$ with $Ke^{s_j} \in Met^{p,0}_{2,b} \subset Met^{p,0}_{2,B}$, $\|s_j\|_{L^1} \geq N_j$, and $\|s\|_{L^1} > C_j M_{\sigma,\tau}(Ke^{s_j}) + C_j''$. Let C', C'', N be such that

$$\|s\|_{L^1} \leq C' M_{\sigma,\tau}(Ke^s) + C'' \text{ for } \|s\|_{L^1} \geq N.$$

Let $S_N = \{s \in L_2^p S^0 | Ke^s \in Met^{p,0}_{2,b} \text{ and } \|s\|_{L^1} \leq N\}$. By Proposition 3.31, if $s \in S_N$, then $\sup |s_v| \leq \sup |s| \leq C_1 \|s\|_{L^1} + C_2 \leq C_1 N + C_2$ (here C_1 and C_2 are not the first elements of the sequence $\{C_j\}_{j=1}^{\infty}$ but constants as in Proposition 3.31), so by Lemma 3.22, $M_{\sigma,\tau}$ is bounded below on S_N, i.e. $M_{\sigma,\tau}(Ke^s) \geq -\lambda$ for each $s \in S_N$, for some constant $\lambda > 0$. Thus, $\|s\|_{L^1} \leq C'(M_{\sigma,\tau}(Ke^s) + \lambda) + N$ for each $s \in S_N$. Replacing C'' by $\max\{C'', C'\lambda + N\}$, we see that $\|s\|_{L^1} \leq C' M_{\sigma,\tau}(Ke^s) + C''$, for each $s \in L_2^p S^0$ with $Ke^s \in Met^{p,0}_{2,b}$. By Corollary 3.32, $M_{\sigma,\tau}$ satisfies the main estimate in $Met^{p,0}_{2,b}$. Finally, since the set of smooth sections is dense in $L_2^p S^0$, we can always assume that s_j is smooth (we made the choice $b < B$ so that if Ke^{s_j} is in the boundary $\|m_{\sigma,\tau}(H)\|^p_{L^p,H} = b$ of $Met^{p,0}_{2,b}$, we can still replace s_j by a smooth s_j' with $Ke^{s_j'} \in Met^{p,0}_{2,B}$). $\square$

Lemma 3.44. *Assume that $M_{\sigma,\tau}$ does not satisfy the main estimate in $Met^{p,0}_{2,B}$. Let $\{s_j\}_{j=1}^{\infty}$ be a sequence as in Lemma 3.43, $l_j = \|s_j\|_{L^1}$, $C(B) = C_1 + C_2$, where C_1, C_2 are as in Proposition 3.31, and $u_j = s_j/l_j$. Thus, $\|u_j\|_{L^1} = 1$ and $\sup |u_j| \leq C(B)$. After going to a subsequence, $u_j \to u_\infty$ weakly in $L_1^2 S^0$, for some nontrivial $u_\infty \in L_2^p S^0$ such that if $\mathscr{F} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth non-negative function such that $\mathscr{F}(x, y) \leq 1/(x - y)$ whenever $x > y$, and $\mathscr{F}_\varepsilon : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth non-negative function with $\mathscr{F}_\varepsilon(x, y) = 0$ whenever $x - y \leq \varepsilon$, for some fixed $\varepsilon > 0$, then*

$$(\sigma \cdot \sqrt{-1} \Lambda F_K, u_\infty)_{L^2} + (\sigma \cdot \mathscr{F}(u_\infty)\bar{\partial}_{\mathcal{E}} u_\infty, \bar{\partial}_{\mathcal{E}} u_\infty)_{L^2} + (\mathscr{F}_\varepsilon(s)\phi, \phi)_{L^2} - (\tau \cdot \mathrm{id}, u_\infty)_{L^2} \leq 0.$$

Proof. To prove this inequality, we can assume that $\mathscr{F}$ and $\mathscr{F}_\varepsilon$ have compact support (for $\sup |u_j|$ are bounded, by Lemma 3.31, and the definitions of $\mathscr{F}(s)\bar{\partial}_{\mathcal{E}} u_\infty$ and $\mathscr{F}_\varepsilon(s)\phi$ only depend on the values of $\mathscr{F}$ and $\mathscr{F}_\varepsilon$ at the pairs (λ_i, λ_j) of eigenvalues, as seen in §3.1.6). Now, if $\mathscr{F}$ and $\mathscr{F}_\varepsilon$ have compact support then, for large enough l,

$$\mathscr{F}(x, y) \leq l \Psi(lx, ly), \quad \mathscr{F}_\varepsilon(x, y) \leq l^{-1} \psi(lx, ly),$$

where Ψ and ψ are defined as in (3.15) and (3.21) (cf. the proof of [B, Proposition 3.9.1]). Since $l_j \to \infty$, from these inequalities we obtain that for large enough j,

$$(\mathscr{F}(u_{j,v})\bar{\partial}_{\mathcal{E}} u_{j,v}, \bar{\partial}_{\mathcal{E}} u_{j,v})_{L^2} \leq l(\Psi(l_{j,v} u_{j,v})\bar{\partial}_{\mathcal{E}} u_{j,v}, \bar{\partial}_{\mathcal{E}} u_{j,v})_{L^2},$$

$$(\mathscr{F}_\varepsilon(u_j)\phi, \phi)_{L^2} \leq l^{-1}(\psi(l_j u_j)\phi, \phi)_{L^2},$$

so Lemma 3.43(iii) applied to $s_i = l_j u_j$, together with lemma 3.22, give an upper bound

$$\frac{1}{C_j} + \frac{\|\phi\|_{L^2}^2}{l_j} \geq l_j^{-1} M_{\sigma,\tau}(K e^{l_j u_j}) + l_j^{-1}\|\phi\|_{L^2}^2 \geq (\sigma \cdot \sqrt{-1}\, \Lambda F_K, u_j)_{L^2}$$

$$+ (\sigma \cdot \mathscr{F}(u_j)\bar{\partial}_{\mathcal{E}} u_j, \bar{\partial}_{\mathcal{E}} u_j)_{L^2} + (\mathscr{F}_\varepsilon(u_j)\phi, \phi)_{L^2} - (\tau \cdot \mathrm{id}, u_j)_{L^2}.$$

As in the proof of [B, Proposition 3.9.1], one can use this upper bound to show that the sequence $\{u_j\}_{j=1}^\infty$ is bounded in L_1^2. Thus, after going to a subsequence, $u_j \to u_\infty$ in L_1^2, for some $u_\infty \in L_1^2 S$ with $\|u_\infty\|_{L^1} = 1$, so u_∞ is non-trivial.

We now prove the estimate for u_∞. First, since $\sup |u_j| \leq b := C(B)$, $u_j \to u_\infty$ in $L_{0,b}^2$; applying Lemma 3.14(iii) , one can show (as in the proof of [S, Lemma 5.4]) that $(\sigma \cdot \sqrt{-1}\, \Lambda F_K, u_j)_{L^2} + (\sigma \cdot \mathscr{F}(u_j)\bar{\partial}_{\mathcal{E}} u_j, \bar{\partial}_{\mathcal{E}} u_j)_{L^2}$ approaches $(\sigma \cdot \sqrt{-1}\, \Lambda F_K, u_\infty)_{L^2} + (\sigma \cdot \mathscr{F}(u_\infty)\bar{\partial}_{\mathcal{E}} u_\infty, \bar{\partial}_{\mathcal{E}} u_\infty)_{L^2}$ as $j \to \infty$. Second, since $L_1^2 \subset L^2$ is a compact embedding and actuallly $u_j \in L_{1,b}^2 S \subset L_{0,b}^2 S$, applying Lemma 3.14(iv) (as in the proof of [B, Proposition 3.9.1]), $\mathscr{F}_\varepsilon : L_{0,b}^2 S \to L_{0,b'}^2 S(\mathrm{End}\,\mathscr{R})$, $u \mapsto \mathscr{F}_\varepsilon(u)$, is continuous on $L_{0,b}^2 S$, so $\lim_{j\to\infty} \mathscr{F}_\varepsilon(u_j) = \mathscr{F}_\varepsilon(u_\infty)$. Since $\sup |u_j|$ are bounded, this implies that $(\mathscr{F}_\varepsilon(u_j)\phi, \phi)_{L^2}$ converges to $(\mathscr{F}_\varepsilon(u_\infty)\phi, \phi)_{L^2}$ as $j \to \infty$. Finally, it is clear that $(\tau \cdot \mathrm{id}, u_j)_{L^2} \to (\tau \cdot \mathrm{id}, u_\infty)_{L^2}$ as $j \to \infty$. This completes the proof. $\square$

Lemma 3.45. *If $M_{\sigma,\tau}$ does not satisfy the main estimate in $Met_{2,B}^{p,0}$, and $u_\infty \in L_2^p S^0$ is an in Lemma 3.44, then the following happens:*

(i) *The eigenvalues of u_∞ are constant almost everywhere.*
(ii) *Let the eigenvalues of u_∞ be $\lambda_1, \ldots, \lambda_r$. If $\mathscr{F} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies $\mathscr{F}(\lambda_i, \lambda_j) = 0$ whenever $\lambda_i > \lambda_j$, $1 \leq i, j \leq r$, then $\mathscr{F}(u_\infty)(\bar{\partial}_{\mathcal{E}} u_\infty) = 0$.*
(iii) *If $\mathscr{F}_\varepsilon$ is an in Proposition 3.44, then $\mathscr{F}_\varepsilon(u_\infty)\phi = 0$.*

Proof. Parts (i) and (ii) of are proved as in [UY, appendix], [S, §§6.3.4 and 6.3.5], or [B, §§3.9.2 and 3.9.3], using Lemma3.14(ii) for part (i) and the estimate in Lemma 3.44 for part (ii). Part (iii) is similar to [B, Lemma 3.9.4], and again uses the estimate in Lemma 3.44. $\square$

We now construct a filtration of quiver subsheaves of $\mathcal{R}$ using L_2^p-subsystems, as in [B, §3.10].

Lemma 3.46. *Assume that $M_{\sigma,\tau}$ does not satisfy the main estimate in $Met_{2,B}^{p,0}$. Let $u_\infty \in L_2^p S^0$ be as in Lemma 3.44. Let the eigenvalues of u_∞, listed in ascending order, be $\lambda_0 < \lambda_1 < \cdots < \lambda_r$. Since u_∞ is 'σ-trace free' (cf. §3.1.3), there are at least two different eigenvalues, i.e. $r \geq 1$. Let $p_0, \ldots, p_r : \mathbb{R} \to \mathbb{R}$ be smooth functions such that, for $j < r$, $p_j(x) = 1$ if $x \leq \lambda_j$, $p_j(x) = 0$ if $x \geq \lambda_{j+1}$, and $p_r(x) = 1$ if $x \leq \lambda_r$. Let $\pi_v : \mathcal{E} \to \mathcal{E}_v$ be the canonical projections (cf. (3.3)) and $\bar{\partial}_{\mathcal{E}}$ be as in (3.4). The operators $\pi_r' = p_j(u_\infty)$ and $\pi_{j,v}' = \pi_j' \circ \pi_v$, for $0 \leq j \leq r$, satisfy:*

(i) $\pi_j' \in L_1^2 S$, $\pi_j'^2 = \pi_j' = \pi_j'^{*K}$ *and* $(1 - \pi_j')\bar{\partial}_{\mathcal{E}} \pi_j' = 0$.
(ii) $(\mathrm{id} - \pi_{j,ha}') \circ \phi_a \circ (\pi_{j,ta}' \otimes \mathrm{id}_{M_a}) = 0$ *for each $v \in Q_0$.*
(iii) *Not all the eigenvalues of u_∞ are positive.*

Proof. The proof of (i) is as in [S] (right below Lemma 5.6; see also [B, Proposition 3.10.2(i)-(iii)]). Part (ii) is similar to, but more involved than, [B, Proposition 3.10.2(iv)], so we now give a detailed proof of this part. For each j, let $\varepsilon > 0$ be such that $\varepsilon \leq (\lambda_{j+1} - \lambda_j)/2$, and $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$ be smooth non-negative functions such that $\varphi_1(x) = 0$ if $x \leq \lambda_{j+1} - \varepsilon/2$ and $\varphi_1(x) = 1$ if $x \geq \lambda_{j+1}$,

in the case of φ_1; and $\varphi_2(y) = 1$ if $y \leq \lambda_j$ and $\varphi_2(y) = 0$ if $y \geq \lambda_j + \varepsilon/2$, in the case of φ_2. Let $\mathscr{F}_\varepsilon : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by

$$\mathscr{F}_\varepsilon(x, y) = \varphi_1(x)\varphi_2(y).$$

If $\mathscr{F}_\varepsilon(x, y) \neq 0$, then $x > \lambda_{j+1} - \varepsilon/2$ and $y < \lambda_j + \varepsilon/2$, so $x - y > \lambda_{j+1} - \lambda_j - \varepsilon \geq \varepsilon$; thus, $\mathscr{F}_\varepsilon$ satisfies the hypothesis of Lemma 3.45(iii), so $\mathscr{F}_\varepsilon(u_\infty)\phi = 0$. But $\mathscr{F}_\varepsilon(u_\infty)\phi = \varphi_1(u_\infty) \circ \phi \circ \varphi_2(u_\infty)$ (cf. (3.12)), where $\varphi_1(u_\infty) = \mathrm{id} - \pi'_j$ and $\varphi_2(u_\infty) = \pi'_j$, which completes the proof of part (ii). Finally, part (iii) follows from $\mathrm{tr}(\sigma \cdot u_\infty) = 0$ and the non-triviality of u_∞. $\square$

Proof of Proposition 3.42. Assume that $M_{\sigma,\tau}$ does not satisfy the main estimate in $Met_{2,B}^{p,0}$. We have to prove that $\mathcal{R}$ is not (σ, τ)-stable. By Lemma 3.46(i), the operators $\pi'_{j,v}$ are weak holomorphic vector subbundles of $\mathcal{E}_v$, for $v \in Q_0$ [UY, §4]. Applying Uhlenbeck–Yau regularity theorem [UY, §7], they represent reflexive subsheaves $\mathcal{E}'_{j,v} \subset \mathcal{E}_v$, and by Lemma 3.46(ii), the inclusions $\mathcal{E}'_{j,v} \subset \mathcal{E}_v$ are compatible with the morphisms ϕ_a, hence define Q-subsheaves $\mathcal{R}'_j = (\mathcal{E}'_j, \phi'_j)$ of $\mathcal{R} = (\mathcal{E}, \phi)$. We thus get a filtration of Q-subsheaves

$$0 \hookrightarrow \mathcal{R}'_0 \hookrightarrow \mathcal{R}'_1 \hookrightarrow \cdots \hookrightarrow \mathcal{R}'_r = \mathcal{R}.$$

As in [B, (3.7.2)],

$$u_\infty = \lambda_0 \pi'_0 + \sum_{j=1}^{r} \lambda_j (\pi'_j - \pi'_{j-1}) = \lambda_r \, \mathrm{id}_\mathcal{E} - \sum_{j=0}^{r-1}(\lambda_{j+1} - \lambda_j)\pi'_j,$$

so the v-component $u_{\infty,v} = u_\infty \circ \pi_v$ of u_∞ is

$$(3.47) \qquad u_{\infty,v} = \lambda_r \, \mathrm{id}_{\mathcal{E}_v} - \sum_{j=0}^{r-1}(\lambda_{j+1} - \lambda_j)\pi'_{j,v},$$

(note that it may happen that $\pi'_{j,v} = \pi'_{j+1,v}$ for some v and j). From (3.13) and $\pi'_{j,v} = p_j(u_{\infty,v})$, $\bar{\partial}_{\mathcal{E}_v}\pi'_{j,v} = \mathrm{d}\, p_j(u_{\infty,v})(\bar{\partial}_{\mathcal{E}_v} u_{\infty,v})$, so

$$(3.48) \qquad \sum_{j=0}^{r-1}(\lambda_{j+1} - \lambda_j)|\bar{\partial}_{\mathcal{E}_v}\pi'_{j,v}|^2 = \sum_{j=0}^{r-1}(\lambda_{j+1} - \lambda_j)((\mathrm{d}\,p_j)^2(u_{\infty,v})\bar{\partial}_{\mathcal{E}_v}(u_{\infty,v}), \bar{\partial}_{\mathcal{E}_v}(u_{\infty,v}))$$

$$= (\mathscr{F}(u_{\infty,v})(\bar{\partial}_{\mathcal{E}_v} u_{\infty,v}), \bar{\partial}_{\mathcal{E}_v} u_{\infty,v}),$$

where $\mathscr{F} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, defined by $\mathscr{F} = \sum_{j=0}^{l-1}(\lambda_{j+1} - \lambda_j)(\mathrm{d}\,p_j)^2$, satisfies the conditions of Lemma 3.44 (cf. e.g. the proof of [S, Lemma 5.7]). We make use of the previous calculations to estimate the number

$$\chi = \mathrm{Vol}(X)\left(\lambda_r \deg_{\sigma,\tau}(\mathcal{R}) - \sum_{j=0}^{r-1}(\lambda_{j+1} - \lambda_j)\deg_{\sigma,\tau}(\mathcal{R}'_j)\right).$$

On the one hand, the degree of the subsheaf $\mathcal{E}'_{j,v} \subset \mathcal{E}_v$ is given by (3.1.5),

$$\mathrm{Vol}(X)\deg(\mathcal{E}'_{j,v}) = (\sqrt{-1}\,\Lambda F_{K_v}, \pi'_{j,v})_{L^2} - \|\bar{\partial}_{\mathcal{E}_v}\pi'_{j,v}\|^2_{L^2},$$

and this formula, together with equations (3.47) and (3.48), imply

$$\chi = \sum_{v\in Q_0} \sigma_v \left(\sqrt{-1}\,\Lambda F_{K_v}, \lambda_r\,\mathrm{id}_{\mathcal{E}_v} - \sum_{j=0}^{r-1}(\lambda_{j+1}-\lambda_j)\pi'_{j,v} \right)_{L^2} + \sum_{v\in Q_0}\sigma_v\sum_{j=0}^{r-1}(\lambda_{j+1}-\lambda_j)\|\bar{\partial}_{\mathcal{E}_v}\pi'_{j,v}\|^2_{L^2}$$

$$- \sum_{v\in Q_0}\tau_v\,\mathrm{Vol}(X)\left(\lambda_r\,\mathrm{rk}(\mathcal{E}_v) - \sum_{j=0}^{r-1}(\lambda_{j+1}-\lambda_j)\,\mathrm{rk}(\mathcal{E}'_{j,v})\right)$$

$$= (\sigma\cdot\sqrt{-1}\,\Lambda F_K, u_\infty)_{L^2} + (\sigma\cdot\mathscr{F}(u_\infty)(\bar{\partial}_{\mathcal{E}}u_\infty),\bar{\partial}_{\mathcal{E}}u_\infty)_{L^2} - (\tau\cdot\mathrm{id}, u_\infty)_{L^2}.$$

It follows from Lemma 3.44 (with $\mathscr{F}_\varepsilon = 0$, cf. Lemma 3.45(iii)), that $\chi \leq 0$. On the other hand, if $\mathcal{R}$ is (σ,τ)-stable, then $\mu_{\sigma,\tau}(\mathcal{R}) > \mu_{\sigma,\tau}(\mathcal{R}'_j)$, for $0 \leq j < r$, and since $\sigma\cdot u_\infty \in L_2^p S^0$ is trace free,

$$\mathrm{tr}(\sigma\cdot u_\infty) = \sum_v \sigma_v\,\mathrm{tr}(u_\infty\circ\pi_v) = \lambda_r\sum_{v\in Q_0}\sigma_v\,\mathrm{rk}(\mathcal{E}_v) - \sum_{j=0}^{r-1}(\lambda_{j+1}-\lambda_j)\sum_{v\in Q_0}\sigma_v\,\mathrm{rk}(\mathcal{E}'_{j,v}) = 0,$$

so we get

$$\chi = \frac{\mathrm{Vol}(X)}{\sum_{v\in Q_0}\sigma_v\,\mathrm{rk}(\mathcal{E}_v)}\sum_{j=0}^{r-1}(\lambda_{j+1}-\lambda_j)\left(\sum_{v\in Q_0}\sigma_v\,\mathrm{rk}(\mathcal{E}'_{j,v})\,\deg_{\sigma,\tau}(\mathcal{R}) - \sum_{v\in Q_0}\sigma_v\,\mathrm{rk}(\mathcal{E}_v)\,\deg_{\sigma,\tau}(\mathcal{R}'_j)\right)$$

$$= \mathrm{Vol}(X)\sum_{j=0}^{r-1}(\lambda_{j+1}-\lambda_j)\sum_{v\in Q_0}\sigma_v\,\mathrm{rk}(\mathcal{E}'_{j,v})(\mu_{\sigma,\tau}(\mathcal{R}) - \mu_{\sigma,\tau}(\mathcal{R}'_j)) > 0.$$

Therefore, if $M_{\sigma,\tau}$ does not satisfy the main estimate in $Met_{2,B}^{p,0}$, then $\mathcal{R}$ cannot be (σ,τ)-stable. $\square$

3.7. Stability implies existence and uniquenes of special metric.

Let $\mathcal{R} = (\mathcal{E},\phi)$ be a (σ,τ)-polystable holomorphic Q-bundle on X. To prove that it admits a hermitian metric satisfying the quiver (σ,τ)-vortex equations, we can assume that $\mathcal{R}$ is (σ,τ)-stable, which in particular implies that it is simple. The existence and uniqueness of a hermitian metric satisfying the quiver (σ,τ)-vortex equations is now immediate from Propositions 3.30 and 3.42. $\square$

Sections 3.2 and 3.7 prove Theorem 3.1.

4. Yang–Mills–Higgs functional and Bogomolov inequality

Let σ,τ be collections of real numbers σ_v, τ_v, with $\sigma_v > 0$, for $v \in Q_0$. Given a smooth complex vector bundle E, let $c_1(E)$ and $ch_2(E)$ be its first Chern class and second Chern character, respectively. By Chern–Weil theory, if A is a connection on E then $c_1(E)$ (resp. $ch_2(E)$) is represented by the closed form $\frac{\sqrt{-1}}{2\pi}\mathrm{tr}(F_A)$ (resp. $-\frac{1}{8\pi^2}\mathrm{tr}(F_A^2)$). Define the topologial invariants of E

$$(4.1) \qquad C_1(E) = \int_X c_1(E)\wedge\frac{\omega^{n-1}}{(n-1)!} = \frac{1}{2\pi}\int_X \mathrm{tr}(\sqrt{-1}\,\Lambda F_A)\frac{\omega^n}{n!}$$

and

$$(4.2) \qquad Ch_2(E) = \int_X ch_2(E)\wedge\frac{\omega^{n-2}}{(n-2)!} = -\frac{1}{8\pi^2}\int_X \mathrm{tr}(F_A^2)\wedge\frac{\omega^{n-2}}{(n-2)!}$$

(thus, $C_1(E)$ is the degree of E, up to a normalisation factor). Given a holomorphic vector bundle $\mathcal{E}$ on X, we denote by $C_1(\mathcal{E})$ and $Ch_2(\mathcal{E})$ the corresponding topological invariants of its underlying smooth vector bundle.

Theorem 4.3. *If $\mathcal{R} = (\mathcal{E}, \phi)$ is a (σ, τ)-stable holomorphic Q-bundle on X, and the q_q-selfadjoint endomorphism $\sqrt{-1}\,\Lambda F_{q_a}$ of M_a is positive semidefinite, for each $a \in Q_0$, then*

$$(4.4) \qquad \sum_{v \in v} \tau_v C_1(\mathcal{E}_v) \geq 2\pi \sum_{v \in Q_0} \sigma_v Ch_2(\mathcal{E}_v).$$

If $C_1(\mathcal{E}_v) = 0$, $Ch_2(\mathcal{E}_v) = 0$ for all $v \in Q_0$, then the connections A_{H_v} are flat for each $v \in Q_0$, and

$$(4.5) \qquad \sum_{a \in h^{-1}(v)} \phi_a \circ \phi_a^{*H} - \sum_{a \in t^{-1}(v)} \phi_a^{*H} \circ \phi_a = \tau_v \operatorname{id}_{E_v}$$

for each $v \in Q_0$, where H is a solution of the M-twisted quiver (σ, τ)-vortex equations on $\mathcal{R}$.

Thus, quiver bundles can be useful to construct flat connections. Note that when X is an algebraic variety, (4.5) means that $\mathcal{R}$ is a family of τ-stable Q-modules parametrized by X (cf. [K, §§5, 6]).

This theorem is an immediate consequence of the Hitchin–Kobayashi correspondence for holomorphic Q-bundles and Proposition 4.7 below. We shall use the notation introduced in §2.2.

Definition 4.6. *The Yang–Mills–Higgs functional $YMH_{\sigma,\tau} : \mathscr{A} \times \Omega^0 \to \mathbb{R}$ is defined by*

$$YMH_{\sigma,\tau}(A, \phi) = \sum_{v \in Q_0} \sigma_v \|F_{A_v}\|_{L^2}^2 + \sum_{a \in Q_1} \|\,\mathrm{d}_{A_a}\,\phi_a\|_{L^2}^2$$

$$+ 2 \sum_{v \in Q_0} \sigma_v^{-1} \left\| \sum_{a \in h^{-1}(v)} \phi_a \circ \phi_a^{*H} - \sum_{a \in t^{-1}(v)} \phi_a^{*H} \circ \phi_a - \tau_v \operatorname{id}_{E_v} \right\|_{L^2}^2,$$

where A_a is the connection induced by A_{ta}, A_{q_a} and A_{ha} on the vector bundle $\operatorname{Hom}(E_{ta} \otimes M_a, E_{ha})$.

In the following, $\|\cdot\|$ will mean the L^2-norm in the appropiate space of sections. Note that in Theorem 4.3 it is assumed that $\sqrt{-1}\,\Lambda F_{q_a}$ is semidefinite positive for each $a \in Q_0$, so it defines a semidefinite positive sesquilinear form on $\Omega^0(\operatorname{Hom}(E_{ta} \otimes M_a, E_{ha}))$ by

$$(\phi_a, \phi_a')_{q_a} = \int_X \operatorname{tr}\left(\phi_a \circ (\operatorname{id}_{E_{ta}} \otimes \sqrt{-1}\,\Lambda F_{q_a}) \circ \phi_a^{*H_a}\right), \text{ for each } \phi_a, \phi_a' \in \Omega^0(\operatorname{Hom}(E_{ta} \otimes M_a, E_{ha})).$$

Adding together, we thus get a semidefinite positive sesquilinear form on Ω^0, defined by

$$(\phi, \phi')_{\mathscr{R}, M} = \sum_{a \in Q_1} (\phi_a, \phi_a')_{L^2, q_a}, \text{ for each } \phi, \phi' \in \Omega^0.$$

Thus, $\|\phi\|_{\mathscr{R}, M}^2 := (\phi, \phi)_{\mathscr{R}, M} \geq 0$ for each $\phi \in \Omega^0$.

Proposition 4.7. *If $(A, \phi) \in \mathcal{A}' \times \Omega^0$, with $A_v \in \mathcal{A}_v^{1,1}$ for all $v \in Q_0$, then*

$$YMH_{\sigma,\tau}(A, \phi) = 4 \sum_{a \in Q_1} \|\bar{\partial}_{A_a}\phi_a\|^2 + 4\pi \sum_{v \in Q_0} \tau_v C_1(E_v) - 8\pi^2 \sum_{v \in Q_0} \sigma_v Ch_2(E_v) - \|\phi\|_{\mathscr{R}, M}^2$$

$$+ \sum_{v \in Q_0} \sigma_v^{-1} \left\| \sigma_v \sqrt{-1}\,\Lambda F_{A_v} + \sum_{a \in h^{-1}(v)} \phi_a \circ \phi_a^{*H} - \sum_{a \in t^{-1}(v)} \phi_a^{*H} \circ \phi_a - \tau_v \operatorname{id}_{E_v} \right\|^2.$$

Proof. Before giving the proof, we need several preliminaries. First, note that for any $A_v \in \mathcal{A}_v^{1,1}$,

$$(4.8) \qquad \|F_{A_v}\|^2 = \|\Lambda F_{A_v}\|^2 - 8\pi^2 Ch_2(E_v)$$

(cf. e.g. [B, Theorem 4.2]). Secondly, we notice that the curvature of A_a, for $A \in Q_1$, is given by

$$(4.9) \qquad F_{A_a}(\phi_a) = F_{A_{ha}} \circ \phi_a - \phi_a \circ (F_{A_{ta}} \otimes \operatorname{id}_{M_a} + \operatorname{id}_{E_{ta}} \otimes F_{q_a})$$

where ϕ_a is a section of $\mathrm{Hom}(E_{ta}, E_{ha})$. Finally, since the $(0, 1)$-parts of the unitary connections A_{ta}, A_{ha} define holomorphic structures, A_a also defines a holomorphic structure on the smooth vector bundle $\mathrm{Hom}(E_{ta}, E_{ha})$, so it satisfies the Kähler identities

$$\sqrt{-1}[\Lambda, \partial_{A_a}] = -\bar{\partial}_{A_a}^*, \quad \sqrt{-1}[\Lambda, \bar{\partial}_{A_a}] = \partial_{A_a}^*.$$

In particular, the commutator of $\sqrt{-1}\,\Lambda$ with the curvature $F_{A_a} = \partial_{A_a}\bar{\partial}_{A_a} + \bar{\partial}_{A_a}\partial_{A_a}$ is $\sqrt{-1}[\Lambda, F_{A_a}] = \Delta_{A_a}' - \Delta_{A_a}''$, where $\Delta_A' = \partial_A^*\partial_A + \partial_A\partial_A^*$ and $\Delta_A'' = \bar{\partial}_A^*\bar{\partial}_A + \bar{\partial}_A\bar{\partial}_A^*$. When acting on sections ϕ_a of $\mathrm{Hom}(E_{ta}, E_{ha})$, this simplifies to

$$\sqrt{-1}\,\Lambda F_{A_a}\phi_a = \Delta_{A_a}'\phi_a - \Delta_{A_a}''\phi_a.$$

so that

$$(4.10) \qquad (\sqrt{-1}\,\Lambda F_{A_a}\phi_a, \phi_a)_{L^2} = \|\partial_{A_a}\phi_a\|^2 - \|\bar{\partial}_{A_a}\phi_a\|^2.$$

To prove the proposition, we define

$$U_v(\phi) = \sum_{a\in h^{-1}(v)} \phi_a \circ \phi_a^{*H} - \sum_{a\in t^{-1}(v)} \phi_a^{*H} \circ \phi_a$$

for $\phi \in \Omega^0$ and $v \in Q_0$. Then

$$\sum_{v\in Q_0} \sigma_v^{-1}\|\sigma_v\sqrt{-1}\,\Lambda F_{A_v} + U_v(\phi) - \tau_v\,\mathrm{id}_{E_v}\|^2 = \sum_{v\in Q_0} \sigma_v\|\Lambda F_{A_v}\|^2$$

$$+ \sum_{v\in Q_0} \sigma_v^{-1}\|U_v(\phi) - \tau_v\,\mathrm{id}_{E_v}\|^2 + 2\sum_{v\in Q_0}(\sqrt{-1}\,\Lambda F_{A_v}, U_v(\phi))_{L^2} - 2\sum_{v\in Q_0}\sigma_v^{-1}(\sqrt{-1}\,\Lambda F_{A_v}, \tau_v\,\mathrm{id}_{E_v})_{L^2},$$

where (4.9), (4.10) give

$$\sum_{v\in Q_0}(\sqrt{-1}\,\Lambda F_{A_v}, U_v(\phi))_{L^2} = \sum_{a\in Q_1}(\sqrt{-1}\,\Lambda F_{A_{ha}}\circ\phi_a - \phi_a\circ(\sqrt{-1}\,\Lambda F_{A_{ta}}\otimes\mathrm{id}_{M_a}), \phi_a)_{L^2}$$

$$= \sum_{a\in Q_1}(\sqrt{-1}\,\Lambda F_{A_a}\phi_a, \phi_a)_{L^2} - \|\phi\|_{\mathscr{R},M} = \sum_{a\in Q_1}\|\partial_{A_a}\phi_a\|^2 - \sum_{a\in Q_1}\|\bar{\partial}_{A_a}\phi_a\|^2 - \|\phi\|_{\mathscr{R},M}.$$

The proposition now follows from the previous equation, (4.8), and the definition of $C_1(E_v)$. $\qquad\square$

Proof of Theorem 4.3. Let $\mathcal{R} = (\mathcal{E}, \phi)$ be (σ, τ)-stable, H the hermitian metric on $\mathcal{R}$ satisfying the (σ, τ)-vortex equations (cf. Theorem 3.1), and $A \in \mathscr{A}$ the corresponding Chern connection. By Definition 4.6, $YMH_{\sigma,\tau}(A, \phi) \geq 0$, while from Proposition 4.7, this is $2\pi\sum_{v\in Q_0}\tau_v C_1(E_v) - 8\pi^2\sum_{v\in Q_0}\sigma_v Ch_2(E_v) - \|\phi\|_{\mathscr{R},M}^2$, as $\bar{\partial}_{A_a}\phi_a = 0$ for each $a \in Q_1$. Since we are assuming $\|\phi\|_{\mathscr{R},M}^2 \geq 0$, we obtain (4.4). Furthermore, if $C_1(\mathcal{E}_v) = Ch_2(\mathcal{E}_v) = 0$ for each $v \in Q_0$, then $YMH_{\sigma,\tau}(A, \phi) = -\|\phi\|_{\mathscr{R},M}^2 \leq 0$, but this functional is non-negative by Definition 4.6, so $YMH_{\sigma,\tau}(A, \phi) = 0$. Thus, $F_{A_v} = 0$ and we also obtain (4.5) for each $v \in Q_0$, again by Definition 4.6. $\qquad\square$

5. TWISTED QUIVER SHEAVES AND PATH ALGEBRAS

The category of M-twisted Q-sheaves is equivalent to the category of coherent sheaves of right $\mathcal{A}$-modules, where $\mathcal{A}$ is certain locally free $\mathcal{O}_X$-sheaf associated to Q and M —the so-called M-*twisted path algebra* of Q. This provides an alternative point of view of twisted quiver sheaves which, in certain cases, gives a more algebraic understanding of certain properties of Q-sheaves (cf. e.g. §5.2 below). In particular, it may be a better point of view to study the moduli space problem, which we will not address in this paper. To fix terminology, a locally free (resp. free, coherent) $\mathcal{O}_X$-algebra is a sheaf $\mathcal{S}$ of rings which at the same time is a locally free (resp. free, coherent) $\mathcal{O}_X$-module. Given such an $\mathcal{O}_X$-algebra $\mathcal{S}$, a locally free (resp. free, coherent) $\mathcal{S}$-algebra is a sheaf $\mathcal{A}$ of (not

necessarily commutative) rings over $\mathcal{S}$ which at the same time is a locally free (resp. free, coherent) $\mathcal{O}_X$-module. A coherent right $\mathcal{A}$-module is a sheaf of right $\mathcal{A}$-modules which at the same time is a coherent $\mathcal{O}_X$-module.

5.1. Coherent sheaves of right $\mathcal{A}$-modules.

Throughout §5.1, we assume that Q is a finite quiver, that is, Q_0 and Q_1 are both finite. Let M be as in §1.2.

5.1.1. Twisted path algebra.

Let $\mathcal{S} = \oplus_{v \in Q_0} \mathcal{O}_X \cdot e_v$ be the free $\mathcal{O}_X$-module generated by Q_0, where e_v are formal symbols, for $v \in Q_0$. We consider a structure of commutative $\mathcal{O}_X$-algebra on $\mathcal{S}$, defined by $e_v \cdot e_{v'} = e_v$ if $v = v'$, and $e_v \cdot e_{v'} = 0$ otherwise, for each $v, v' \in Q_0$. Let

$$\mathcal{M} = \bigoplus_{a \in Q_1} M_a$$

be a locally free sheaf of $\mathcal{S}$-bimodules, whose left (resp. right) $\mathcal{S}$-module structure is given by $e_v \cdot m = m$ if $m \in M_a$ and $v = ha$ (resp. $m \cdot e_v = m$ if $m \in M_a$ and $v = ta$), and $e_v \cdot m = 0$ otherwise (resp. $m \cdot e_v = 0$ otherwise), for each $v \in Q_0$, $a \in Q_1$, $m \in M_a$. The M-*twisted path algebra* of Q is the tensor $\mathcal{S}$-algebra of the $\mathcal{S}$-bimodule $\mathcal{M}$, that is,

$$\mathcal{A} = \bigoplus_{\ell \geq 0} \mathcal{M}^{\otimes_{\mathcal{S}} \ell}.$$

Note that $\mathcal{A}$ is a locally free $\mathcal{O}_X$-algebra. Furthermore, since Q is finite, $\mathcal{A}$ has a unit

$$(5.1) \qquad\qquad\qquad 1_{\mathcal{A}} = \oplus_{v \in Q_0} e_v.$$

5.1.2. Coherent $\mathcal{A}$-modules.

We will show now that the category of M-twisted Q-sheaves is equivalent to the category of coherent sheaves of right $\mathcal{A}$-modules, or *coherent right $\mathcal{A}$-modules*. This result is a direct generalisation of the corresponding equivalence of categories for quiver modules (cf. e.g. [ARS]). We define an equivalence functor from the first to the second category. Let $\mathcal{R} = (\mathcal{E}, \phi)$ be an M-twisted Q-sheaf. Let $E = \oplus_{v \in Q_0} \mathcal{E}_v$ as a coherent $\mathcal{O}_X$-module. The structure of right $\mathcal{A}$-module on E is given by a morphism of $\mathcal{O}_X$-modules $\mu_{\mathcal{A}} : E \otimes_{\mathcal{O}_X} \mathcal{A} \to E$ satisfying the usual axioms defining right modules over an algebra. Let $\pi_v : E \otimes_{\mathcal{O}_X} \mathcal{S} = \oplus_{v,v' \in Q_0} \mathcal{E}_v \otimes_{\mathcal{O}_X} \mathcal{O}_X \cdot e_{v'} \to \mathcal{E}_v \otimes_{\mathcal{O}_X} \mathcal{O}_X \cdot e_v \cong \mathcal{E}_v$, be the canonical projection, and $\iota_v : \mathcal{E}_v \hookrightarrow E$ the inclusion map, for each $v \in Q_0$. Let $\mu_v = \iota_v \circ \pi_v : E \otimes_{\mathcal{O}_X} \mathcal{S} \to E$. The morphism $\mu_{\mathcal{S}} = \sum_{v \in Q_0} \mu_v : E \otimes_{\mathcal{O}_X} \mathcal{S} \to E$ defines a structure of right $\mathcal{S}$-module on E. The tensor product of E and $\mathcal{M}$ over $\mathcal{S}$ is $E \otimes_{\mathcal{S}} \mathcal{M} \cong \otimes_{a \in Q_1} E_{ta} \otimes_{\mathcal{O}_X} M_a$; let $\pi_a : E \otimes_{\mathcal{S}} \mathcal{M} \to E_{ta} \otimes_{\mathcal{O}_X} M_a$ be the canonical projection, for each $a \in Q_1$. The morphism $\mu_{\mathcal{M}} = \sum_{a \in Q_1} \iota_{ha} \circ \phi_a \circ \pi_a : E \otimes_{\mathcal{S}} \mathcal{M} \to E$ is a morphism of $\mathcal{S}$-modules. Since $\mathcal{A}$ is the tensor $\mathcal{S}$-algebra of $\mathcal{M}$, $\mu_{\mathcal{M}}$ induces a morphism of $\mathcal{O}_X$-modules $\mu_{\mathcal{A}} : E \otimes_{\mathcal{O}_X} \mathcal{A} \to E$ defining a structure of right $\mathcal{A}$-module on E. This defines the action of the equivalence functor on the objects of the category of M-twisted Q-sheaves. It is straightforward to construct an action of the functor on morphisms of M-twisted Q-sheaves, so this defines a functor from the category of M-twisted Q-sheaves to the category of coherent right $\mathcal{A}$-modules. We now define a functor from the category of coherent right $\mathcal{A}$-modules to the category of M-twisted Q-sheaves, and see that this new functor is an inverse equivalence of the previous functor. Let E be a coherent right $\mathcal{A}$-module, with right $\mathcal{A}$-module structure morphism $\mu_{\mathcal{A}} : E \otimes_{\mathcal{O}_X} \mathcal{A} \to E$. The decomposition (5.1) is a sum of orthogonal idempotents in $\mathcal{A}$ (i.e. $e_v^2 = e_v$, $e_v \cdot e_{v'} = 0$ for $v, v' \in Q_0$ with $v \neq v'$), so $E = \oplus_{v \in Q_0} \mathcal{E}_v$ with $\mathcal{E}_v = \mu_{\mathcal{A}}(E \otimes_{\mathcal{O}_X} \mathcal{O}_X \cdot e_v) \subset E$, for each $v \in Q_0$, and the tensor product of E and $\mathcal{M}$ over $\mathcal{S}$ is $E \otimes_{\mathcal{S}} \mathcal{M} = \otimes_{a \in Q_1} \mathcal{E}_{ta} \otimes_{\mathcal{O}_X} M_a$. The restriction of $\mu_{\mathcal{A}}$ to $E \otimes_{\mathcal{O}_X} \mathcal{M}$ induces a morphism of $\mathcal{S}$-modules $\mu_{\mathcal{M}} : E \otimes_{\mathcal{S}} \mathcal{M} \to E$. The image of $\mathcal{E}_{ta} \otimes_{\mathcal{O}_X} M_a$ under $\mu_{\mathcal{M}}$ is therefore in $\mathcal{E}_{ha}$, hence defines a morphism of $\mathcal{O}_X$-modules $\phi_a : \mathcal{E}_{ta} \otimes_{\mathcal{O}_X} M_a \to \mathcal{E}_{ha}$, for each $a \in Q_1$. This defines a functor from the category of coherent right $\mathcal{A}$-modules to the category of M-twisted Q-sheaves. It is straightforward to define the action of this functor on morphisms and to prove that this functor,

together with the previous one, are inverse equivalences of categories (actually, of $\mathcal{O}_X$-categories, cf. e.g. [ARS]). This completes the proof of the following:

Proposition 5.2. *The category of coherent right $\mathcal{A}$-modules is equivalent to the category of M-twisted Q-sheaves on X.*

5.2. **Tensor products of stable twisted quiver bundles.** As a simple application of Proposition 5.2, we now prove that the tensor product of two polystable twisted holomorphic quiver bundles is polystable as well. To do this, we first define the appropriate notion of tensor product of quiver sheaves. Let $Q = (Q_0, Q_1)$ and $Q' = (Q'_0, Q'_1)$ be two finite quivers with the same vertex set $Q_0 = Q'_0$, and tail and head maps $t, h : Q_1 \to Q_0$, $t', h' : Q'_1 \to Q'_0$, respectively. Let M (resp. M') be a collection of finite rank locally free sheaves M_a (resp. $M'_{a'}$) on X, for each $a \in Q_1$ (resp. $a' \in Q'_1$). Let $\mathcal{S} = \oplus_{v \in Q_0} \mathcal{O}_X \cdot e_v$ be a free sheaf of $\mathcal{O}_X$-algebras as in §5.1.1. Let $\mathcal{M} = \oplus_{a \in Q_1} M_a$, $\mathcal{M}' = \oplus_{a' \in Q'_1} M_{a'}$, be locally free sheaves of $\mathcal{S}$-bimodules defined as in §5.1.1, and

$$\mathcal{A} = \bigoplus_{\ell=0}^{\infty} \mathcal{M}^{\otimes_{\mathcal{S}} \ell}, \quad \mathcal{A}' = \bigoplus_{\ell=0}^{\infty} \mathcal{M}'^{\otimes_{\mathcal{S}} \ell},$$

the M-twisted and M'-twisted path algebras of Q and Q', resp. Thus, the category of coherent right $\mathcal{A}$-modules (resp. $\mathcal{A}'$-modules) is equivalent to the category of M-twisted Q-sheaves (resp. M'-twisted Q'-sheaves) on X. Let Q'' be the quiver which has the same vertices as Q and Q', and has the arrows of Q and Q', i.e. $Q'' = (Q''_0, Q''_1)$ is the quiver, with tail and head maps $t'', h'' : Q''_1 \to Q''_0$, defined by

$$Q''_0 = Q_0 = Q'_0, \; Q''_1 = Q_1 \amalg Q'_1,$$

$$t'' a = ta, \; h'' a = ha \text{ if } a \in Q_1, \text{ and } t'' a' = t' a', \; h'' a' = h' a' \text{ if } a' \in Q'_1.$$

Let M'' be the collection of finite rank locally free sheaves M''_a on X, for each $a \in Q''_1$, given by $M''_a = M_a$ if $a \in Q_1$ and $M''_{a'} = M'_{a'}$ if $a' \in Q'_1$. Let

$$\mathcal{M}'' = \mathcal{M} \oplus \mathcal{M}' = \bigoplus_{a \in Q_0} M''_a.$$

The M''-twisted path algebra of Q'' is

$$\mathcal{A}'' = \bigoplus_{\ell=0}^{\infty} \mathcal{M}''^{\otimes_{\mathcal{S}} \ell} \cong \mathcal{A} \otimes_{\mathcal{S}} \mathcal{A}'.$$

The category of coherent $\mathcal{A}''$-modules is equivalent to the category of M''-twisted Q''-sheaves on X. Let now E (resp. E') be a coherent right $\mathcal{A}$-module (resp. $\mathcal{A}'$-module). Since $\mathcal{S}$ is a commutative $\mathcal{O}_X$-sheaf and E, E' are coherent $\mathcal{S}$-modules, their tensor product $E'' = E \otimes_{\mathcal{S}} E'$ is well defined and is again a coherent $\mathcal{S}$-module. We define the structure of a coherent right $\mathcal{A}''$-module on E'' by the isomorphism $\mathcal{A}'' \cong \mathcal{A} \otimes_{\mathcal{S}} \mathcal{A}'$: the action of $a \otimes a' \in (\mathcal{A} \otimes_{\mathcal{S}} \mathcal{A}')_x$ on $e \otimes e' \in E''_x$, for each $x \in X$, is $(e \otimes e') \cdot (a \otimes a') = e \cdot a \otimes e' \cdot a'$. Let now $\mathcal{R} = (\mathcal{E}, \phi)$ be the M-twisted Q-sheaf corresponding to E, and $\mathcal{R}' = (\mathcal{E}', \phi')$ the M'-twisted Q'-sheaf corresponding to E', by the equivalences of categories of Propositin 5.2. The M''-twisted Q''-sheaf corresponding to their tensor product E'' is then $\mathcal{R}'' = (\mathcal{E}'', \phi'')$, where $\mathcal{E}''_v = \mathcal{E}_v \otimes_{\mathcal{O}_X} \mathcal{E}'_v$ for each $v \in Q_0$, and $\phi''_a = \phi_a \otimes_{\mathcal{O}_X} \text{id}$ if $a \in Q_1$, $\phi''_{a'} = \text{id} \otimes_{\mathcal{O}_X} \phi'_{a'}$ if $a' \in Q'_1$. Thus, $\mathcal{R}''$ is the *tensor product* of $\mathcal{R}$ and $\mathcal{R}'$.

Proposition 5.3. *Let σ, τ, τ' be collections of real numbers σ_v, τ_v and τ'_v, respectively, with $\sigma_v > 0$, for each $v \in Q_0$, and let $\tau'' = \tau + \tau'$. If $\mathcal{R}$ is a (σ, τ)-polystable holomorphic M-twisted Q-bundle and $\mathcal{R}'$ is a (σ, τ')-polystable M'-twisted holomorphic Q'-bundle, then their tensor product $\mathcal{R}''$ is a (σ, τ'')-polystable holomorphic M-twisted Q''-bundle.*

Proof. To define the vortex equations on $\mathcal{R}$ and $\mathcal{R}'$, resp., we fix a family q of hermitian metrics q_a on M_a, for each $a \in Q_1$, and a family q' of hermitian metrics $q'_{a'}$ on $M'_{a'}$, for each $a' \in Q'_1$, resp. By Theorem 3.1, there is a hermitian metric H on $\mathcal{R}$ satisfying the (σ, τ)-vortex equations, and a hermitian metric H' on $\mathcal{R}'$ satisfying the (σ, τ')-vortex equations. The Chern connection associated to the metric $H''_v = H_v \otimes H'_v$ on $\mathcal{E}''_v = \mathcal{E}_v \otimes \mathcal{E}'_v$, for each $v \in Q_0$ with nonzero $\mathcal{E}_v$ and $\mathcal{E}'_v$, has curvature $F_{H''_v} = F_{H_v} \otimes \mathrm{id} + \mathrm{id} \otimes F_{H'_v}$. It is now straightforward to prove that the collection H'' of hermitian metrics H''_v on $\mathcal{E}''_v$, for each $v \in Q_0$, is a hermitian metric on $\mathcal{R}''$ satisfying the (σ, τ'')-vortex equations. Thus, $\mathcal{R}''$ is (σ, τ'')-polystable, again by Theorem 3.1. $\square$

6. EXAMPLES

6.1. **Higgs bundles.** Let X be a Riemann surface. A *Higgs bundle* on X is a pair (E, Φ), where E is a holomorphic vector bundle over X and $\Phi \in H^0(\mathrm{End}(E) \otimes K)$ is a holomorphic endomorphism of E twisted by the canonical bundle K of X. The quiver here consists of one vertex and one arrow whose head and tail coincide and the twisting bundle is dual of the canonical line bundle of X, i.e. the holomorphic tangent bundle $T'X$ of X. This quiver, and the twisting bundle attached to its arrow, is represented in Fig. 1.

The Higgs bundle (E, Φ) is *stable* if the usual slope stability condition $\mu(E') < \mu(E)$ is satisfied for all proper Φ-invariant subbundles E' of E. The existence theorem of Hitchin and Simpson [H, S] says that (E, Φ) is polystable if and only if there exists a hermitian metric H on E satisfying

$$(6.1) \qquad F_H + [\Phi, \Phi^*] = -\sqrt{-1}\mu \, \mathrm{id}_E \, \omega,$$

where ω is the Kähler form on X, id_E is the identity on E, and μ is a constant. Note that taking the trace in the first equation and integrating over X we get $\mu = \mu(E)$.

There are many reasons why Higgs bundles are of interest, one of the most important of which is the fact that there is a bijective correspondence between isomorphism classes of poly-stable Higgs bundles of degree zero on X and isomorphism classes of semisimple complex representations of the fundamental group of X. This important fact is derived from a combination of the theorem of Hitchin and Simpson mentioned above and an existence theorem for equivariant harmonic metrics proved by Donaldson [D3] and Corlette [C]. This correspondence can also be used to study representations of $\pi_1(X)$ in non-compact real Lie groups. In particular, by considering the group $\mathrm{U}(p, q)$ one obtains another interesting example of a twisted quiver bundle. To identify this quiver we observe that there is a homeomorphism between the moduli space of semisimple representation of $\pi_1(X)$ in $\mathrm{U}(p, q)$ and the moduli space of poly-stable zero degree Higgs bundles (E, Φ) of the form

$$(6.2) \qquad \begin{aligned} E &= V \oplus W, \\ \Phi &= \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}, \end{aligned}$$

where V and W are holomorphic vector bundles on X of rank p and q, respectively,

$$\beta \in H^0(\mathrm{Hom}(W, V) \otimes K) \quad \text{and} \quad \gamma \in H^0(\mathrm{Hom}(V, W) \otimes K).$$

The corresponding quiver, with the twisting bundle attached to each arrow, is represented in Fig. 2. Now, for this twisted quiver bundle one can consider the general quiver equations. Although they only coincide with Hitchin's equations (6.1) for a particular choice of the parameters, it turns out that the other values are very important to study the topology of the moduli of representations of $\pi_1(X)$ into $\mathrm{U}(p, q)$ [BGG1].
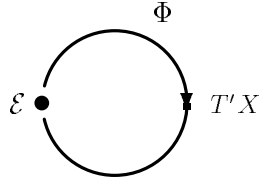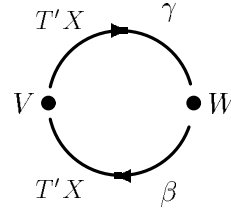
Fig. 1.

Fig. 2.

A very important tool to study topological properties of Higgs bundle moduli spaces and hence moduli spaces of representations of the fundamental group is to consider the $\mathbb{C}^*$-action on the moduli space given by multiplying the Higgs field Φ by a non-zero scalar. A point (E, Φ) is a fixed point of the $\mathbb{C}^*$-action action if and only if it is a variation of Hodge structure, that is,

$$(6.3) \qquad E = F_1 \oplus \cdots \oplus F_m$$

for holomorphic vector bundles F_i such that the restriction

$$\Phi_i := \Phi_{|F_i} \in H^0(\operatorname{Hom}(F_i, F_{i+1}) \otimes K).$$

A variation of Hodge structure is therefore a twisted quiver bundle, whose twisting bundles are $M_a = T'X$, and the infinite quiver represented in Fig. 3.
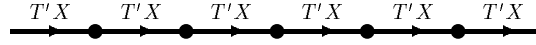
Fig. 3: Variations of Hodge structure.

One can generalize the notion of Higgs bundle to consider twistings by a line bundle other than the canonical bundle. These have also very interesting geometry [GR].

6.2. **Quiver bundles and dimensional reduction.** Quiver bundles and their vortex equations appear naturally in the context of dimensional reduction. To explain this, consider the manifold $X \times G/P$, where X is a compact Kähler manifold, G is a connected simply connected semisimple complex Lie group and $P \subset G$ is a parabolic subgroup, i.e. G/P is a flag manifold. The group G (and hence, its maximal compact subgroup $K \subset G$) act trivially on X and in the standard way on G/P. The Kähler structure on X together with a K-invariant Kähler structure on G/P define a product Kähler structure on $X \times G/P$. We can now consider a G-equivariant vector bundle over $X \times G/P$ and study K-invariant solutions to the Hermitian–Einstein equations. It turns that these invariant solutions correspond to special solutions to the quiver vortex equations on a certain quiver bundle over X, where the quiver is determined by the parabolic subgroup P. In [AG1] we studied the case in which $G/P = \mathbb{P}^1$, the complex projective line, which is obtained as the quotient of $G = \mathrm{SL}(2, \mathbb{C})$ by the subgroup of lower triangular matrices, generalizing previous work by [G1, G2, BG]. The general case has been studied in [AG2]. We will just mention here some of the main results and refer the reader to the above mentioned papers.

A key fact is the existence of a quiver Q with relations $\mathcal{K}$ naturally associated to the subgroup P. A *relation* of the quiver is a formal complex linear combination $r = \sum_j c_j p_j$ of paths p_j of the quiver (i.e. $c_j \in \mathbb{C}$), and a path in Q is a sequence $p = a_0 \cdots a_m$ of arrows $a_j \in Q_j$ which compose, i.e. with $ta_{j-1} = ha_j$ for $1 \leq j \leq m$:

$$(6.4) \qquad p: \quad \bullet \xrightarrow{a_m} \bullet \xrightarrow{a_{m-1}} \cdots \xrightarrow{a_0} \bullet$$

The set of vertices of the quiver associated to P coincides with the set of irreducible representations of P. The arrows and relations are obtained by studying certain isotopical decompositions related to the nilradical of the Lie algebra of P. For example, for $\mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$, the quiver is the disjoint union of two copies of the quivers in Fig. 4, 5 and 6, respectively.
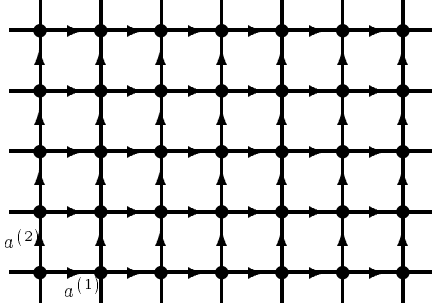

Fig. 4: $G/P = \mathbb{P}^1$.
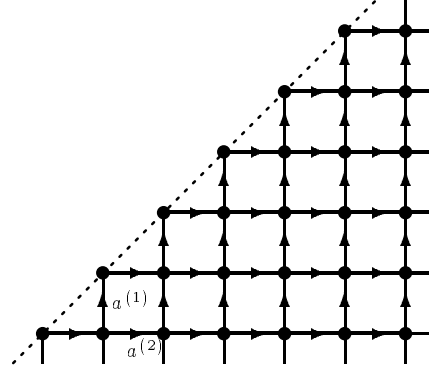
Fig. 5: $G/P = \mathbb{P}^1 \times \mathbb{P}^1$.   Fig. 6: $G/P = \mathbb{P}^2$.

In the case of the quiver associated to $\mathbb{P}^1$, the set of relations is empty, while for the quivers associated to $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$, the relations r_λ are given by

$$r_\lambda = a^{(2)}_{\lambda - L_1} a^{(1)}_{\lambda} - a^{(1)}_{\lambda - L_2} a^{(2)}_{\lambda},$$

where $\lambda \in \mathbb{Z}^2$ is a vertex, L_1 and L_2 are the canoncial basis of $\mathbb{C}^2$, and $a^{(j)}_\lambda : \lambda \to \lambda - L_j$ are the arrows going out from λ, for $j = 1, 2$. Given a set $\mathcal{K}$ of relations of the quiver Q, a holomorphic $(Q, \mathcal{K})$-bundle (with no twisting bundles M_a) is defined as a holomorphic Q-bundle $\mathcal{R} = (\mathcal{E}, \phi)$ which satisfies the relations $r = \sum_j c_j p_j$ in $\mathcal{K}$, i.e. such that $\sum_j c_j \phi(p_j) = 0$, where $\phi(p) : \mathcal{E}_{t a_m} \to \mathcal{E}_{h a_0}$ is defined for any path (6.4) as the composition $\phi(p) := \phi_{a_0} \circ \cdots \circ \phi_{a_m}$.

Let $(Q, \mathcal{K})$ be the quiver with relations associated to P. One has an equivalence of categories

$$\left\{ \begin{array}{c} \text{coherent } G-\text{equivariant} \\ \text{sheaves on } X \times G/P \end{array} \right\} \longleftrightarrow \left\{ \ (Q, \mathcal{K})-\text{sheaves on } X \ \right\}.$$

The holomorphic G-equivariant vector bundles on $X \times G/P$ and the holomorphic $(Q, \mathcal{K})$-bundles on X are in correspondence by this equivalence. Thus, the category of G-equivariant holomorphic vector bundles on $X \times (\mathbb{P}^1)^2$ and $X \times \mathbb{P}^2$ is equivalent to the category of commutative diagrams of holomorphic quiver bundles on X for the corresponding quiver Q. If we now fix a total order in the set of vertices, any coherent G-equivariant sheaf $\mathcal{F}$ on $X \times G/P$ admits a G-equivariant sheaf filtration

(6.5)
$$\mathcal{F} : 0 \hookrightarrow \mathcal{F}_0 \hookrightarrow \mathcal{F}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_m = \mathcal{F},$$
$$\mathcal{F}_s / \mathcal{F}_{s-1} \cong p^* \mathcal{E}_{\lambda_s} \otimes q^* \mathcal{O}_{\lambda_s}, \quad 0 \leq s \leq m,$$

where $\{\lambda_0, \lambda_1, \ldots, \lambda_m\}$ is a finite subset of vertices, listed in ascending order, $\mathcal{E}_0, \ldots, \mathcal{E}_m$ are non-zero coherent sheaves on X with trivial G-action, and $\mathcal{O}_{\lambda_s}$ is the homogeneous bundle over G/P corresponding to the representation λ_s. The maps p and q are the canonical projections from $X \times G/P$ to X and G/P, respectively. If $\mathcal{F}$ is a holomorphic G-equivariant vector bundle, then $\mathcal{E}_0, \ldots, \mathcal{E}_m$ are holomorphic vector bundles.

The appropriate equation to consider on a filtered bundle [AG1] is a deformation of Hermite–Einstein equation which involves as many parameters $\tau_0, \tau_1, \ldots, \tau_m \in \mathbb{R}$ as steps are in the filtration, and has the form

$$
(6.6) \qquad \sqrt{-1}\,\Lambda F_h = \begin{pmatrix} \tau_0 I_0 & & & \\ & \tau_1 I_1 & & \\ & & \ddots & \\ & & & \tau_m I_m \end{pmatrix},
$$

where the RHS is a diagonal matrix, written in blocks corresponding to the splitting which a hermitian metric h defines in the filtration $\mathcal{F}$. If $\tau_0 = \cdots = \tau_m$, then (6.6) reduces to the Hermite–Einstein equation. As in the ordinary Hermite–Einstein equation, the existence of invariant solutions to the τ-Hermite–Einstein equation on an equivariant holomorphic filtration is related to a stability condition for the equivariant holomorphic filtration which naturally involves the parameters.

Let $\mathcal{F}$ be a G-equivariant holomorphic vector bundle on $X \times G/P$. Let $\mathbf{\mathcal{F}}$ be the G-equivariant holomorphic filtration associated to $\mathcal{F}$ and $\mathcal{R} = (\mathcal{E}, \phi)$ be its corresponding holomorphic $(Q, \mathcal{K})$-bundle on X, where $(Q, \mathcal{K})$ is the quiver with relations associated to P. Then $\mathbf{\mathcal{F}}$ has a K-invariant solution to the τ-deformed Hermite–Einstein equations if and only if the vector bundles $\mathcal{E}_\lambda$ in $\mathcal{R}$ admit hermitian metrics H_λ on $\mathcal{E}_\lambda$, for each vertex λ with $\mathcal{E}_\lambda \neq 0$, satisfying

$$
(6.7) \qquad \sqrt{-1}\,n_\lambda \Lambda F_{H_\lambda} + \sum_{a \in h^{-1}(\lambda)} \phi_a \circ \phi_a^* - \sum_{a \in t^{-1}(\lambda)} \phi_a^* \circ \phi_a = \tau_\lambda' \, \mathrm{id}_{\mathcal{E}_\lambda},
$$

where n_λ is the multiplicity of the irreducible representation corresponding to the vertex λ and τ_λ' are related to τ_λ by the choice of the K-invariant metric on G/P. It is not difficult to show that the stability of the filtration coincides with the stability of the quiver bundle where the parameters σ_λ in the general stability condition for a quiver bundle equal the integers n_λ. This, together with the dimensional reduction obtainment of the equations, provides with an alternative proof of the Hitchin–Kobayashi correspondence for these special quiver bundles.

Although the quiver bundles obtained by dimensional reduction on $X \times G/P$ are not twisted, it seems that twisting may appear if one considers dimensional reduction on more general G-manifolds — this is something to which we plan to come back in the future.

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