

Periodic Orbits near Symplectic Submanifolds

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SYMPLECTIC HOMOLOGY AND PERIODIC ORBITS NEAR SYMPLECTIC SUBMANIFOLDS

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ABSTRACT. We show that a small neighborhood of a closed symplectic submanifold in a geometrically bounded aspherical symplectic manifold has non-vanishing symplectic homology. As a consequence, we establish the existence of contractible closed characteristics on any thickening of the boundary of the neighborhood. When applied to twisted geodesic flows on compact symplectically aspherical manifolds, this implies the existence of contractible periodic orbits for a dense set of low energy values.

1. INTRODUCTION

The symplectic homology of an open subset of a symplectic manifold is intimately connected with closed characteristics on the shells near the boundary of the subset, [FH]. In this paper we show that for a small neighborhood of a compact symplectic submanifold in a geometrically bounded symplectically aspherical manifold the symplectic homology does not vanish. As a consequence, we establish the existence of contractible closed characteristics on any thickening of the boundary of such a neighborhood.

Symplectic homology was originally introduced and studied in the series of papers [CFH, CFHW, FH, FHW]. Here we follow more recent treatments from [BPS] and, more indirectly, from [Vi]. In particular, we consider the (direct limit) symplectic homology of bounded open sets in a geometrically bounded, symplectically aspherical manifold (W, ω) , and we prove the following:

Theorem 1.1. *Let U be a sufficiently small neighborhood of a compact symplectic submanifold M of W . Then $\mathrm{SH}^{[a, b]}(U) \neq 0$ for $a < b < 0$, when $|a|$ is sufficiently large and $|b|$ is sufficiently small.*

This yields the following existence result along the lines of the Weinstein–Moser theorem.

Theorem 1.2. *Let $H: W \rightarrow \mathbb{R}$ be a smooth function which attains an isolated minimum on a compact symplectic submanifold M (say, $H|_M = 0$). Then the levels $\{H = \epsilon\}$ carry contractible periodic orbits for a dense set of small values $\epsilon > 0$.*

As an application we obtain a new existence theorem for twisted geodesic flows.

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Corollary 1.3. Let (M, σ) be a compact symplectically aspherical manifold and let $H: T^*M \rightarrow \mathbb{R}$ be the standard kinetic energy Hamiltonian for a Riemannian metric on M , i.e., $H(q, p) = \|p\|^2$. Then the Hamiltonian flow of H with respect to the twisted form $\omega = \omega_0 + \pi^*\sigma$ has contractible periodic orbits on the levels $\{H = \epsilon\}$ for a dense set of small values $\epsilon > 0$.

Theorem 1.2 strengthens or complements some other recent existence results. In particular, it is shown in [GK2], with no assumptions on the symplectic manifold (W, ω) but when M is a Morse–Bott non-degenerate minimum, that there exists a sequence of energy values $\epsilon_k \rightarrow 0$ such that all the levels $\{H = \epsilon_k\}$ carry contractible periodic orbits. Furthermore, under suitable additional conditions, a nontrivial topological lower bound for the number of periodic orbits on every low energy level was established in [GK1, Ke].

For twisted geodesic flows, Corollary 1.3 augments a large family of existence results beyond the applications of the results in [GK1, GK2, Ke] described above. For instance, it was shown in [Mac, Po] that there are contractible periodic orbits on a sequence of energy levels $\{H = \epsilon_k\}$ with $\epsilon_k \rightarrow 0$ for any compact manifold M and any closed form $\sigma \neq 0$ with $\sigma|_{\pi_2(M)} = 0$. Also, when M is a torus, there are periodic orbits (not necessarily contractible, e.g., if $\sigma = 0$) on almost all energy levels for any σ ; see [GK1] and references therein, cf. [Ji]. We refer the reader to the survey [Gi] for a discussion of the results on twisted geodesic flows obtained prior to 1995.

Remark 1.4. In Theorem 1.1, the assumption that W is geometrically bounded can be replaced by other hypotheses that ensure that the symplectic homology of U is well defined. In particular, the theorem also holds for symplectic manifolds with symplectically convex boundary (see Remark 2.3). The choice to consider geometrically bounded manifolds was motivated by our interest in the application to twisted geodesic flows.

The paper is organized as follows. In Section 2, we recall the definition of geometrically bounded manifolds and show that twisted cotangent bundles are geometrically bounded. In Section 3, we briefly review the definitions and properties of Floer and symplectic homology which are essential for the proofs of the main theorems. We also show that the non-vanishing property of symplectic homology implies the “nearby” existence of periodic orbits. In Section 4 we show that the symplectic homology in question is non-vanishing and we complete the proofs of the main theorems. Finally, in Section 5 we discuss the notion of a relative homological capacity related to these results.

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2. GEOMETRICALLY BOUNDED SYMPLECTIC MANIFOLDS

2.1. Definition of geometrically bounded symplectic manifolds. In what follows, we will need to use the Floer homology of compactly supported functions on non-compact symplectically aspherical manifolds. In order for this to be well

defined, certain conditions on the manifold at infinity must be imposed to ensure that the necessary compactness theorems hold. One standard way to achieve this is to require the symplectic manifold to be in a certain sense convex. However, the manifolds we are most interested in – twisted cotangent bundles – do not generally meet this requirement. Hence convexity is replaced here by the less restrictive requirement that the manifold is geometrically bounded. Let us recall the definition.

Definition 2.1. A symplectic manifold (W, ω) is said to be *geometrically bounded* if W admits an almost complex structure J and a complete Riemannian metric g such that

GB1. J is uniformly ω -tame, i.e., for some positive constants c_1 and c_2 we have

$$\omega(X, JX) \geq c_1 \|X\|^2 \quad \text{and} \quad |\omega(X, Y)| \leq c_2 \|X\| \|Y\|$$

for all tangent vectors X and Y to W .

GB2. The sectional curvature of (W, g) is bounded from above and the injectivity radius of (W, g) is bounded away from zero.

Observe that if W is compact then condition (GB1) holds automatically when g , ω and J are compatible, i.e.,

$$(1) \quad \omega(\cdot, J\cdot) = g(\cdot, \cdot).$$

Clearly, for compact manifolds condition (GB2) also holds for any metric, and so every compact symplectic manifold is geometrically bounded. Other examples of geometrically bounded manifolds are given in the next section. We refer the reader to Chapters V (by J.-C. Sikorav) and X (by M. Audin, F. Lalonde and L. Polterovich) in [AL] for a more detailed discussion of this concept.

2.2. Example: twisted cotangent bundles. Let us recall the definition of a twisted cotangent bundle. Consider a closed manifold M and a closed two-form (magnetic field) σ on M . Set $W = T^*M$. The form $\omega = \omega_0 + \pi^*\sigma$ is called a twisted symplectic form, where ω_0 is the standard symplectic form on T^*M and $\pi: T^*M \rightarrow M$ is the natural projection.

One reason these forms are of interest is because the Hamiltonian flow of the standard kinetic energy Hamiltonian $H: T^*M \rightarrow \mathbb{R}$, defined by a Riemannian metric on M , describes the motion of a charge on M in the magnetic field σ . This is also called a twisted geodesic flow.

Proposition 2.2. *A twisted cotangent bundle (T^*M, ω) is geometrically bounded for any closed manifold M and any closed two-form σ on M .*

This proposition is, of course, well known. The fact that T^*M satisfies condition (GB2) for the natural metric induced by a metric on M , is stated without a proof in [AL]; see p. 96 and p. 286. This assertion is used by Lu in [Lu] to show that a twisted cotangent bundle (again, with a natural metric) is geometrically bounded. For the sake of completeness, we outline a proof of Proposition 2.2 below (for a different choice of metric on T^*M).¹

Proof. Let φ_t be the flow on $W = T^*M$ formed by fiberwise dilations by the factor e^t . The standard symplectic form ω_0 is homogeneous of degree one with respect to the dilations: $\varphi_t^*\omega_0 = e^t\omega_0$. Pick a fiberwise convex hypersurface Σ in W , enclosing

¹The authors are grateful to Jean-Claude Sikorav for suggesting to us the idea of this proof.

the zero section M . Note that Σ has contact type for ω_0 , but not necessarily for the twisted form ω . Denote by U the closure of the unbounded component of the complement to Σ in W , i.e., $U = \cup_{t \geq 0} \varphi_t(\Sigma)$.

On the vector bundle $TW|_\Sigma$, pick any fiberwise metric g and a complex structure which are compatible with ω_0 in the sense of equation (1). (We also require the radial vectors to be g -orthogonal to Σ .) Let us extend these structures to U so that

$$(2) \quad \varphi_t^* g = e^t g \quad \text{for } t \geq 0,$$

i.e., g , just as ω_0 , is homogeneous of degree one with respect to the dilations, and

$$J \circ (\varphi_t)_* = (\varphi_t)_* \circ J.$$

Then the metric g , the almost complex structure J , and ω_0 are compatible on U and hence the condition (GB1) holds for these structures. We will show that this condition also holds for g , J , and ω for, perhaps, a dilated hypersurface Σ .

The metric g is obviously complete. Indeed, identifying U with $\Sigma \times [1, \infty)$, we see that the metric g has the form

$$g = \frac{dt^2}{t} + \frac{g|_\Sigma}{t}.$$

It is then clear that the integral curves $\varphi_t(x)$, for $t > 0$ and $x \in \Sigma$, are minimizing geodesics of g . Thus, the distance from x to $\varphi_t(x)$ is equal to $\ln t$ and goes to ∞ as $t \rightarrow \infty$. Therefore, every bounded subset of U is contained in some shell $\Sigma \times [1, t]$ and is, hence, relatively compact. This is equivalent to completeness.

It follows readily from (2) that the sectional curvature of g goes to zero as $x \rightarrow \infty$ in U . As a consequence, the sectional curvature of g is bounded from above on U . Combined with completeness, this implies that the injectivity radius is bounded away from zero. Thus, condition (GB2) holds.

It is also easy to verify condition (GB1). Note that $\pi^* \sigma$ is homogeneous of degree zero for φ_t , i.e., $\varphi_t^* \pi^* \sigma = \pi^* \sigma$. Then, by a straightforward calculation, it is clear that for any positive constants $c_1 < 1$ and $c_2 > 1$, this condition holds in the smaller set $\varphi_t(U)$, provided that t is large enough.

To complete the proof it suffices to extend g and J to the bounded part $W \setminus \varphi_t(U)$. \square

Remark 2.3. Recall that an open symplectic manifold (W, ω_0) is said to be *convex at infinity* if there exists: a hypersurface $\Sigma \subset W$ which separates W into one set with compact closure and another, U , with non-compact closure; and a flow of symplectic dilations on U , φ_t (for $t \geq 0$), which is transversal to $\partial U = \Sigma$. (See [EG].) An argument similar to the proof above (with the exception of the step dealing with $\pi^* \sigma$) shows that a symplectic manifold (W, ω_0) which is convex at infinity is also geometrically bounded. This reasoning together with the fact that $\pi^* \sigma$ is “small” compared to ω_0 is the main point of the proof above.

3. SYMPLECTIC HOMOLOGY

3.1. Floer theory. In this section we briefly recall the definitions of Floer homology and symplectic homology as well as some of their properties. All the results here are stated without proof. The reader interested in a detailed treatment of Floer homology should consult, for example, [HZ, Sa] or the original sources [F11, F12, F13].

For the definition and properties of symplectic homology the reader is referred to [BPS, CFH, CFHW, FH, FHW, Vi].²

3.1.1. Floer homology for negative actions. Let (W, ω) be a symplectic manifold, possibly open, which is geometrically bounded (see Definition 2.1). Assume, as well, that (W, ω) is symplectically aspherical, i.e.,

$$\omega|_{\pi_2(M)} = 0 \quad \text{and} \quad c_1(TM)|_{\pi_2(M)} = 0.$$

Denote by \mathcal{H} the space of smooth, compactly supported functions on $S^1 \times W$. To each $H \in \mathcal{H}$ we can associate the time-dependent Hamiltonian vector field X_H which is defined by the equation $dH = -i_{X_H}\omega$. The set of contractible periodic orbits of X_H with period equal to one is denoted by $\mathcal{P}(H)$.

Let $\mathcal{L}W$ be the space of smooth contractible loops in W . We can also associate to each $H \in \mathcal{H}$ the action functional $\mathcal{A}_H : \mathcal{L}W \rightarrow \mathbb{R}$ given by

$$\mathcal{A}_H(x) = - \int_{D^2} \bar{x}^* \omega + \int_{S^1} H(t, x) dt,$$

where $\bar{x} : D^2 \rightarrow W$ is any map which restricts to $S^1 = \partial D^2$ as x . This functional is well defined since $\omega|_{\pi_2(W)} = 0$, and the critical points of \mathcal{A}_H are exactly the elements of $\mathcal{P}(H)$. The set of critical values of \mathcal{A}_H is called the *action spectrum* and we denote it by

$$\mathcal{S}(H) = \{\mathcal{A}_H(x) \mid x \in \mathcal{P}(H)\}.$$

In general terms, the Floer homology of H is the homology of the (relative) Morse–Smale–Witten complex of \mathcal{A}_H on $\mathcal{L}W$. However, when W is not compact, every point in the complement of $\text{supp } H$ is a degenerate 1-periodic orbit (critical point of \mathcal{A}_H) with zero action. To avoid this set, we will only consider the homology generated by the contractible 1-periodic orbits with negative action.

More precisely, for a fixed $a \in (-\infty, 0)$, set

$$\mathcal{P}^a(H) = \{x \in \mathcal{P}(H) \mid \mathcal{A}_H(x) < a\}$$

and assume that H satisfies the following condition:

($*$ ^a): Every 1-periodic orbit $x \in \mathcal{P}^a(H)$ is nondegenerate.

Since $c_1(TM)|_{\pi_2(M)} = 0$, the elements of $\mathcal{P}^a(H)$ are graded by the Conley–Zehnder index μ_{CZ} ; see [Sa]. With this, the Floer complex of H for actions less than a is the graded \mathbb{Z}_2 -vector space

$$\text{CF}^a(H) = \bigoplus_{x \in \mathcal{P}^a(H)} \mathbb{Z}_2 x.$$

To define the Floer boundary operator, we first fix an almost complex structure J_{gb} for which (W, ω) is geometrically bounded as in Definition 2.1. Let \mathcal{J} be the set of smooth t -dependent ω -tame almost complex structures which are ω -compatible near $\text{supp } (H)$ and are equal to J_{gb} outside some compact set. Each $J \in \mathcal{J}$ defines a positive-definite bilinear form on $\mathcal{L}W$. We can then consider the moduli space $\mathcal{M}(x, y, H, J)$ of *downward* gradient-like trajectories of \mathcal{A}_H which go from x to y and have finite energy. For a dense subset, $\mathcal{J}_{\text{reg}}(H) \subset \mathcal{J}$, each moduli space $\mathcal{M}(x, y, H, J)$ is a smooth manifold of dimension $\mu_{\text{CZ}}(x) - \mu_{\text{CZ}}(y)$.

²The definitions of symplectic homology vary considerably from paper to paper. Here we adopt the approach of [BPS].

The Floer boundary operator is then defined by

$$\partial^{H,J} x = \sum_{y \in \mathcal{P}^a(H) \text{ with } \mu_{\text{CZ}}(x) - \mu_{\text{CZ}}(y) = 1} \tau(x, y) y,$$

where $\tau(x, y)$ stands for the number (mod 2) of elements in $\mathcal{M}(x, y, H, J)/\mathbb{R}$ and \mathbb{R} acts (freely) by translation on the gradient-like trajectories. The operator $\partial^{H,J}$ satisfies $\partial^{H,J} \circ \partial^{H,J} = 0$ and the resulting Floer homology groups $\text{HF}^a(H)$ are independent of the choice of $J \in \mathcal{J}_{\text{reg}}(H)$.

Remark 3.1. Since (W, ω) with J_{gb} is geometrically bounded and H is compactly supported, there is a uniform C^0 -bound for the elements of $\mathcal{M}(x, y, H, J)$ (see, for example, Chapter V in [AL]). Hence, the compactness of the appropriate moduli spaces follows from the usual arguments.

Remark 3.2. In this definition of $\partial^{H,J}$ we ignore matters of orientation by considering only coefficients in \mathbb{Z}_2 .

Remark 3.3. It is unclear whether $\text{HF}^a(H)$ depends on the choice of J_{gb} . It is independent of this choice if the set of almost complex structures for which W is geometrically bounded is connected.

It will also be useful to consider Floer homology restricted to smaller negative action intervals. More precisely, for constants $-\infty \leq a < b < 0$ let $H \in \mathcal{H}^{a,b}$, where

$$\mathcal{H}^{a,b} = \{H \in \mathcal{H} \mid a, b \notin \mathcal{S}(H)\}.$$

Assume, as well, that H has property (\ast^b) . Then $\text{CF}^a(H)$ is a subcomplex of $\text{CF}^b(H)$ and $\text{HF}^{[a,b]}(H)$ is the homology of the quotient complex $\text{CF}^{[a,b]}(H) = \text{CF}^b(H)/\text{CF}^a(H)$ with the induced boundary operator.

In fact, the set $\mathcal{H}^{a,b}$ is open in \mathcal{H} with respect to the strong Whitney C^∞ -topology. Moreover, in each component of $\mathcal{H}^{a,b}$, the functions with property (\ast^b) form a dense set and have identical Floer homology groups. Therefore, we can define $\text{HF}^{[a,b]}(H)$ for any $H \in \mathcal{H}^{a,b}$ as the restricted Floer homology of a nearby function in $\mathcal{H}^{a,b}$ with property (\ast^b) .

3.1.2. Morse–Bott Floer homology. The extension of the definition of $\text{HF}^{[a,b]}(H)$ to every $H \in \mathcal{H}^{a,b}$ is particularly useful in the Morse–Bott case which we now describe.

A subset $P \subset \mathcal{P}(H)$ is said to be a *Morse–Bott manifold of periodic orbits* if the set $C_0 = \{x(0) \mid x \in P\}$ is a compact submanifold of W and $T_{x_0}C_0 = \ker(D\phi_H^1(x_0) - \text{id})$ for every $x_0 \in C_0$. Here ϕ_H^1 is the time-1 flow of X_H .

For such sets of periodic orbits we have the following result which holds for geometrically bounded, symplectically aspherical manifolds.

Theorem 3.4. (Poźniak, [Po, Corollary 3.5.4]; Biran–Polterovich–Salamon, [BPS, Section 5.2]) *Let $-\infty \leq a < b < 0$ and $H \in \mathcal{H}^{a,b}$. Suppose that the set $P = \{x \in \mathcal{P}(H) \mid a < \mathcal{A}_H < b\}$ is a connected Morse–Bott manifold of periodic orbits. Then $\text{HF}^{[a,b]}(H)$ is isomorphic to $H_*(P; \mathbb{Z}_2)$.*

3.1.3. Monotone homotopies. Let $H, K \in \mathcal{H}^{a,b}$ be two functions with $H(t, x) \geq K(t, x)$ for all $(t, x) \in S^1 \times W$. Then there exists a monotone homotopy $s \mapsto K_s$ from H to K , i.e., a family of functions K_s such that

$$K_s = \begin{cases} H & \text{for } s \in (-\infty, -1] \\ K & \text{for } s \in [1, \infty) \end{cases}$$

and $\partial_s K_s \leq 0$. For each such homotopy there is a well-defined Floer chain map

$$\sigma_{KH}: \text{CF}^b(H) \rightarrow \text{CF}^b(K).$$

In fact, the map σ_{KH} takes $\text{CF}^a(H)$ to $\text{CF}^a(K)$, and so it defines a chain map for the quotient complexes,

$$\sigma_{KH}: \text{CF}^{[a,b]}(H) \rightarrow \text{CF}^{[a,b]}(K).$$

This induces a homomorphism of Floer homology,

$$\sigma_{KH}: \text{HF}^{[a,b]}(H) \rightarrow \text{HF}^{[a,b]}(K).$$

The following results concerning these homomorphisms are well known ; see, e.g., [CFH, FH] and [BPS, Sections 4.4 and 4.5].

Lemma 3.5. *The homomorphism σ_{KH} is independent of the choice of the monotone homotopy K_s and satisfies the following identities*

$$\begin{aligned} \sigma_{KH} \circ \sigma_{HG} &= \sigma_{KG} \text{ for } G \geq H \geq K, \\ \sigma_{HH} &= \text{id for every } H \in \mathcal{H}^{a,b}. \end{aligned}$$

Lemma 3.6. *If $K_s \in \mathcal{H}^{a,b}$ for all $s \in [0, 1]$, then σ_{KH} is an isomorphism.*

This last result states that the only way in which the map σ_{KH} can fail to be an isomorphism is if periodic orbits, with action equal to a or b , are created during the homotopy. In fact, this is a particular instance of the following more general phenomenon.

Let K_s be a monotone homotopy from H to K , as above. Assume, for some $c < a$, that $K \in \mathcal{H}^{c,b}$ and

$$\sigma_{KH}(\text{CF}^a(H)) \subset \text{CF}^c(K).$$

Then σ_{KH} induces a homomorphism $\hat{\sigma}_{KH}$ from $\text{HF}^{[a,b]}(H)$ to $\text{HF}^{[c,b]}(K)$. (We use the “ $\hat{}$ ” to denote the fact that the map goes between Floer homology groups restricted to different intervals of actions.) In this case, we get the following generalization of Lemma 3.6.

Lemma 3.7. *For $s \in [-1, 1]$, let a_s be a continuous family of numbers, less than b , such that $a_{-1} = a$ and $a_1 = c$. If $K_s \in \mathcal{H}^{a_s,b}$ for every $s \in [-1, 1]$, then*

$$\hat{\sigma}_{KH}: \text{HF}^{[a,b]}(H) \rightarrow \text{HF}^{[c,b]}(K)$$

is an isomorphism.

Remark 3.8. To define the chain maps above and to prove that the homomorphisms σ_{KH} and $\hat{\sigma}_{KH}$ are independent of the choice of the monotone homotopy (as long as $K_s \in \mathcal{H}^{a_s,b}$), one needs to consider moduli spaces which are defined using generic parameterized families in \mathcal{J} . Each such family is constant and equal to J_{gb} outside a compact set in W . Hence, as in Remark 3.1, the necessary compactness statements follow from the standard arguments.

3.2. Symplectic homology.

3.2.1. *Direct limits and exhausting sequences.* Following closely the discussion in [BPS, Sections 4.6 and 4.7], we recall here the algebraic constructions needed to define symplectic homology. Let (I, \preceq) be a partially ordered set which is thought of as a category with precisely one morphism from i to j whenever $i \preceq j$. A *partially ordered system of R -modules over I* is a functor from this category to the category of R -modules. This is written as a pair (G, σ) where G assigns to each $i \in I$ an R -module G_i and σ assigns to each $i \preceq j$ an R -module homomorphism $\sigma_{ji}: G_i \rightarrow G_j$ such that $\sigma_{kj} \circ \sigma_{ji} = \sigma_{ki}$ for $i \preceq j \preceq k$ and σ_{ii} is the identity on G_i .

The partially ordered set (I, \preceq) is said to be *upwardly directed* if for each $i, j \in I$ there is an $l \in I$ such that $i \preceq l$ and $j \preceq l$. In this case, the functor (G, σ) is called a *directed system of R -modules* and its *direct limit* is defined as

$$\varinjlim G = \{(i, x) \mid i \in I, x \in G_i\} / \sim,$$

where $(i, x) \sim (j, y)$ if and only if there exists an $l \in I$ such that $i \preceq l$, $j \preceq l$ and $\sigma_{li}(x) = \sigma_{lj}(y)$.

Note also that for each $i \in I$ there is a natural homomorphism

$$\iota_i: G_i \rightarrow \varinjlim G$$

which takes x to the equivalence class $[i, x]$.

It is possible to actually compute a direct limit given an *exhausting sequence*. This is a sequence $\{i_\nu\}_{\nu \in \mathbb{Z}^+} \subset I$ with the following two properties

- For every $\nu \in \mathbb{Z}^+$ we have $i_\nu \preceq i_{\nu+1}$ and $\sigma_{i_{\nu+1}i_\nu}: G_{i_\nu} \rightarrow G_{i_{\nu+1}}$ is an isomorphism.
- For every $i \in I$ there exists a $\nu \in \mathbb{Z}^+$ such that $i \preceq i_\nu$.

Lemma 3.9. *For such a sequence the map*

$$\iota_{i_\nu}: G_{i_\nu} \rightarrow \varinjlim G$$

is an isomorphism for all $\nu \in \mathbb{Z}^+$.

3.2.2. *Construction of $\text{SH}^{[a,b]}(U)$.* Let $U \subset W$ be a bounded open subset. We are now in a position to define an invariant $\text{SH}^{[a,b]}(U)$ of U , called the *symplectic homology of U for the action interval $[a, b]$* . This is done by taking the direct limit of certain Floer homology groups.

Let

$$\mathcal{H}^{a,b}(U) = \{H \in \mathcal{H}^{a,b} \mid \text{supp}(H) \subset S^1 \times U, H|_{S^1 \times U} \leq 0\}.$$

This set has a partial order given by

$$H \preceq K \iff H \geq K,$$

and it is clearly upwardly directed. Note that $\mathcal{H}^{a,b}(U) \neq \emptyset$ for any $-\infty \leq a < b < 0$.

To each $H \in \mathcal{H}^{a,b}(U)$ we can associate the \mathbb{Z}_2 -vector space given by the Floer homology groups $\text{HF}^{[a,b]}(H)$. Furthermore, for $H \preceq K$ we have, by Lemma 3.5, a homomorphism σ_{KH} which satisfies the required identities to make this a directed system. Passing to the direct limit, we set

$$\text{SH}^{[a,b]}(U) = \varinjlim \text{HF}^{[a,b]}(H).$$

As, above, for each $H \in \mathcal{H}^{a,b}(U)$ we have the natural homomorphism

$$\iota_H: \mathrm{HF}^{[a,b]}(H) \rightarrow \mathrm{SH}^{[a,b]}(U).$$

Remark 3.10. The condition that $\mathrm{supp}(H) \subset S^1 \times U$ is sometimes replaced by the weaker requirement $\mathrm{supp}(H) \subset S^1 \times \bar{U}$, see [CFH, FH], where the bar denotes the closure. This results in a symplectic homology with properties similar to the ones considered here and in [BPS]. However, it is likely that the different definitions can yield different symplectic homology groups when ∂U fails to have contact type. It is also worth noticing that the symplectic homology defined with $\mathrm{supp}(H) \subset S^1 \times \bar{U}$ cannot be calculated by passing to exhausting sequences (but rather exhausting families of functions). For example, the functions H_ν from Section 3.3 could not be used to calculate such symplectic homology.

We now recall the definitions and properties of some of the natural maps introduced in [FH] for symplectic homology. The reader interested in further details and the proofs of these properties should consult [BPS, CFH, FH].

3.2.3. Monotonicity maps. Let $V \subset U$ also be a bounded open subset of W . Then we have the natural inclusion

$$\mathcal{H}^{a,b}(V) \subset \mathcal{H}^{a,b}(U)$$

and the induced natural homomorphism in symplectic homology

$$\phi_{UV}: \mathrm{SH}^{[a,b]}(V) \rightarrow \mathrm{SH}^{[a,b]}(U),$$

which is called a *monotonicity map*.

Lemma 3.11. *For a nested sequence $U_1 \subset U_2 \subset U_3$ of bounded open subsets of W , $\phi_{U_3 U_2} \circ \phi_{U_2 U_1} = \phi_{U_3 U_1}$.*

The relationship of the monotonicity map and the map ι_H is described, for $H \in \mathcal{H}^{a,b}(V)$, by the following commutative diagram

$$(3) \quad \begin{array}{ccc} \mathrm{SH}^{[a,b]}(V) & \xrightarrow{\phi_{UV}} & \mathrm{SH}^{[a,b]}(U) \\ & \searrow \iota_H & \nearrow \iota_H \\ & \mathrm{HF}^{[a,b]}(H) & \end{array}$$

3.2.4. Exact triangles. For $-\infty \leq a \leq b \leq c < 0$ and a function H we have the short exact sequence of complexes

$$0 \rightarrow \mathrm{CF}^{[a,b]}(H) \rightarrow \mathrm{CF}^{[a,c]}(H) \rightarrow \mathrm{CF}^{[b,c]}(H) \rightarrow 0.$$

If $a, b, c \notin \mathcal{S}(H)$, this sequence yields the exact homology triangle $\Delta_{a,b,c}(H)$ given by

$$\begin{array}{ccc} \mathrm{HF}^{[a,b]}(H) & \longrightarrow & \mathrm{HF}^{[a,c]}(H) \\ & \searrow \partial^\bullet & \swarrow \Pi \\ & \mathrm{HF}^{[b,c]}(H) & \end{array}$$

Passing to the direct limit, we obtain an exact triangle $\Delta_{a,b,c}(U)$ for the symplectic homology for U :

$$\begin{array}{ccc} \mathrm{SH}^{[a,b]}(U) & \xrightarrow{\quad} & \mathrm{SH}^{[a,c]}(U) \\ & \nwarrow \partial^* & \nearrow \Pi \\ & \mathrm{SH}^{[b,c]}(U) & \end{array}$$

Note that the map ∂^* has degree -1 in both diagrams.

Finally, let us describe a relation between the exact triangles and the homomorphisms induced by monotone homotopies which will be crucial to us later on. Let $\{K_s\}$ be a monotone homotopy between functions $H, K \in \mathcal{H}^{a,b}(U)$ with $H \geq K$. As mentioned above, this induces a homomorphism

$$\sigma_{KH}: \mathrm{HF}^{[a,b]}(H) \rightarrow \mathrm{HF}^{[a,b]}(K).$$

If, for some $c < a$, we have $K \in \mathcal{H}^{c,b}$ and $\sigma(\mathrm{CF}^a(H)) \subset \mathrm{CF}^c(K)$, then K_s also induces a homomorphism

$$\hat{\sigma}_{KH}: \mathrm{HF}^{[a,b]}(H) \rightarrow \mathrm{HF}^{[c,b]}(K).$$

Lemma 3.12. *The following diagram commutes*

$$\begin{array}{ccc} & & \mathrm{HF}^{[c,b]}(K) \\ & \nearrow \hat{\sigma}_{KH} & \downarrow \Pi \\ \mathrm{HF}^{[a,b]}(H) & \xrightarrow{\sigma_{KH}} & \mathrm{HF}^{[a,b]}(K) \end{array}$$

where Π is the map from the exact triangle $\Delta_{c,a,b}(K)$.

Proof. Using the explicit definition of the maps σ_{KH} and $\hat{\sigma}_{KH}$ (see [CFH, FH]), it is not hard to see that the diagram commutes even on the chain level. \square

3.3. Symplectic homology and “nearby existence” of periodic orbits. Let U be a bounded open subset of W with smooth boundary ∂U . Then the symplectic homology $\mathrm{SH}^{[a,b]}(U)$ contains information about the symplectic geometry of the hypersurface ∂U . In particular, it detects the contractible closed characteristics on or near ∂U .

To make this idea more precise, set $\Sigma = \partial U$ and define a *thickening* of Σ to be a map $\Psi: [-1, 0] \times \Sigma \rightarrow M$ which is a diffeomorphism on its image and satisfies $\Psi(\{0\} \times \Sigma) = \Sigma$ and $\Psi([-1, 0] \times \Sigma) \subset U$. The following result, in a slightly different version, essentially goes back to [FH].

Proposition 3.13. *Suppose that $\mathrm{SH}^{[a,b]}(U) \neq 0$ for $-\infty < a \leq b \leq 0$. Then for any thickening Ψ of $\Sigma = \partial U$ there is a sequence $\{t_i\} \subset [-1, 0)$, converging to 0, such that the corresponding hypersurfaces $\Sigma_{t_i} = \Psi(t_i \times \Sigma)$ all contain contractible closed characteristics.*

Proof. Assume that no such sequence exists. Then there exists a number $\lambda \in [-1, 0)$ such that none of the hypersurfaces Σ_t for $t \in [\lambda, 0)$ contains a contractible closed characteristic. Consider a sequence of smooth functions $f_\nu: [\lambda, 0] \rightarrow (-\infty, 0]$ with the following properties

- $f_\nu(t) = \begin{cases} 0 & \text{for } t \in [\lambda/4\nu, 0], \\ a - \nu & \text{for } t \in [\lambda, \lambda(1 - 1/4\nu)]. \end{cases}$
- $f'_\nu(t) > 0$ for $t \in (\lambda(1 - 1/4\nu), \lambda/4\nu)$.
- $f_\nu(t) \geq f_{\nu+1}(t)$ for all $\nu \in \mathbb{Z}^+$ and $t \in [\lambda, 0]$.

Using these functions, we construct a sequence of Hamiltonians $H_\nu: M \rightarrow \mathbb{R}$ by setting

$$H_\nu(x) = \begin{cases} 0 & \text{for } x \in M \setminus \bar{U}, \\ f_\nu(t) & \text{for } x \in \Sigma_t, t \in [-1, 0], \\ a - \nu & \text{for } x \in U \setminus \Psi([-1, 0) \times \Sigma). \end{cases}$$

The functions $\{H_\nu\}$ clearly have the second property of an exhausting sequence and satisfy $H_\nu \geq H_{\nu+1}$. We claim that the maps $\sigma_{H_{\nu+1}H_\nu}$ are (trivially) isomorphisms and the functions $\{H_\nu\}$ indeed form an exhausting sequence. By construction, the Hamiltonian vector fields X_{H_ν} are only nonzero on the level sets corresponding to the hypersurfaces $\{\Sigma_t\}_{t \in [\lambda, 0]}$. On these level sets, the trajectories of the X_{H_ν} correspond to characteristics and, by assumption, none of these can be closed and contractible. Away from the level sets $\{\Sigma_t\}_{t \in [\lambda, 0]}$, the trajectories are all constant and have action either equal to zero or less than a . Thus, the Floer homology groups $\text{HF}^{[a, b]}(H_\nu)$ are trivial for all ν and the functions H_ν form an exhausting sequence. This implies that

$$\text{SH}^{[a, b]}(U) = \iota_{H_\nu}(\text{HF}^{[a, b]}(H_\nu)) = 0,$$

which is a contradiction. \square

4. PROOFS OF THE MAIN THEOREMS

We are now in a position to prove Theorem 1.1 and Theorem 1.2. Since most of this section is devoted to the proof of the first of these theorems, we recall its assertion for the sake of convenience.

Theorem 4.1. *Let M be a compact symplectic submanifold of (W, ω) , a geometrically bounded symplectically aspherical manifold. Let U be a sufficiently small neighborhood of M in W . Then $\text{SH}^{[a, b]}(U) \neq 0$ for all sufficiently large negative a and small negative b .*

Combining Theorem 4.1 and Proposition 3.13, we immediately obtain Theorem 1.2, i.e., the existence of periodic orbits near any compact oriented hypersurface bounding a small neighborhood of M in W .

4.1. Proof of Theorem 4.1. Let $E \rightarrow M$ be the symplectic normal bundle to M , i.e., the symplectic orthogonal complement to TM in $TW|_M$. We first recall that a neighborhood of the zero section in E has a natural symplectic structure. Moreover, on this neighborhood, there exists a fiberwise quadratic Hamiltonian whose flow is periodic. This can be seen as follows.

Let us equip E with a Hermitian metric which is compatible with the fiberwise symplectic structure on E . We denote by $\rho: E \rightarrow \mathbb{R}$ the square of the fiberwise norm, i.e., $\rho(z) = \|z\|^2$. Recall that E has a canonical fiberwise one-form whose differential is the fiberwise symplectic form. (The value of this form at $z \in E$ is equal to the contraction of the fiberwise symplectic form by z .)

Fixing a Hermitian connection on E we extend this fiberwise one-form to a genuine one-form θ on E . Then the form

$$\omega_E = \frac{1}{2}d\theta + \sigma$$

is symplectic on a neighborhood of the zero section in E . Here we have identified $\sigma = \omega|_M$ with its pull-back to E .

It is not hard to see that all the orbits of the Hamiltonian flow of the function $\rho: E \rightarrow \mathbb{R}$ are periodic with period π , just as for the square of the standard norm on \mathbb{R}^{2n} . (In fact, $i_X \omega_E = z/2$ everywhere on E , where X is the vector field generating the standard fiberwise Hopf action.) This fact will be essential for the calculation of the Floer homology of a small tubular neighborhood of M in W .

By the symplectic neighborhood theorem, a sufficiently small neighborhood of M in (W, ω) is symplectomorphic to a small neighborhood of M in (E, ω_E) . From now on, we will assume that this identification has been made. Hence, in what follows, $\omega = \omega_E$ and ρ is regarded as a function on a neighborhood of M in W . Sometimes, we will write $\rho(z)$ as $\|z\|^2$.

Denote by B_r the disc bundle of radius r in E . The key to the proof of Theorem 4.1 is the following result.

Proposition 4.2. *Let $0 < r < R$ be sufficiently small and assume that*

$$a \in [-\infty, -\pi R^2) \quad \text{and} \quad b \in [-\pi r^2, 0).$$

Then

$$\mathrm{SH}_{n_0}^{[a, b]}(B_r) = \mathrm{SH}_{n_0}^{[a, b]}(B_R) = \mathbb{Z}_2,$$

where

$$n_0 = \frac{1}{2}(\dim M - \mathrm{codim} M),$$

and the monotonicity map $\phi_{B_R B_r}: \mathrm{SH}_{n_0}^{[a, b]}(B_r) \rightarrow \mathrm{SH}_{n_0}^{[a, b]}(B_R)$ is an isomorphism.

Remark 4.3. In [FHW], Floer, Hofer and Wysocki compute the symplectic homology of ellipsoids in \mathbb{R}^{2n} and prove results which are similar to Proposition 4.2 for $E = \mathbb{R}^{2n}$ (see Corollary 2 and Proposition 6). Our proof of Proposition 4.2 is similar in spirit to that in [FHW]. However, instead of using explicitly perturbed non-degenerate Hamiltonians as in [FHW], we use Morse–Bott Floer theory (see Theorem 3.4) to calculate the relevant Floer homology groups. This is motivated by the successful use of this technique in [BPS].

Remark 4.4. The hypothesis that $R > 0$ is small is only needed to guarantee that B_R is contained in W for which the Floer homology is defined. This hypothesis can sometimes be relaxed. For example, if ω_E is symplectic on the entire total space of E and (E, ω) is geometrically bounded, we can take arbitrarily large radii R and r .

Theorem 4.1 follows immediately from Proposition 4.2. Indeed, when U is small enough, we may assume that $U \subset B_R$ and, since $M \subset U$, the disc bundle B_r is contained in U for sufficiently small $r > 0$, i.e.,

$$B_r \subset U \subset B_R.$$

By Lemma 3.11, the inclusion map $\phi_{B_R B_r}$ factors as

$$\mathrm{SH}^{[a, b]}(B_r) \rightarrow \mathrm{SH}^{[a, b]}(U) \rightarrow \mathrm{SH}^{[a, b]}(B_R),$$

and by Proposition 4.2, the map $\phi_{B_R B_r}$ is a nontrivial isomorphism for suitable negative constants a and b . Thus, we conclude that $\mathrm{SH}^{[a, b]}(U) \neq 0$.

4.2. Proof of Proposition 4.2.

4.2.1. *Outline of the proof.* First we consider the following diagram which allows us to work in the setting of Floer homology:

$$(4) \quad \begin{array}{ccc} \mathrm{SH}^{[a,b]}(B_r) & \xrightarrow{\phi_{B_R B_r}} & \mathrm{SH}^{[a,b]}(B_R) \\ \uparrow \iota_{H_\nu^r} & & \uparrow \iota_{H_\nu^R} \\ \mathrm{HF}^{[a,b]}(H_\nu^r) & \xrightarrow{\sigma_{H_\nu^R H_\nu^r}} & \mathrm{HF}^{[a,b]}(H_\nu^R). \end{array}$$

Here, $\{H_\nu^r\}_{\nu \in \mathbb{Z}^+}$ and $\{H_\nu^R\}_{\nu \in \mathbb{Z}^+}$ are chosen to be upward exhausting sequences for the symplectic homology groups $\mathrm{SH}^{[a,b]}(B_r)$ and $\mathrm{SH}^{[a,b]}(B_R)$, respectively. Hence, the two vertical arrows are isomorphisms (for sufficiently large ν). There is also a natural monotone homotopy H_ν^s from H_ν^r to H_ν^R , for each ν , so the map $\sigma_{H_\nu^R H_\nu^r}$ is well-defined. The fact that the diagram commutes then follows easily from diagram (3) and the definition of the direct limit.

By diagram (4), Proposition 4.2 will follow from the two results below which concern the restricted Floer homology of the functions H_ν^s . In particular, these results imply that the groups $\mathrm{HF}_{n_0}^{[a,b]}(H_\nu^r)$ and $\mathrm{HF}_{n_0}^{[a,b]}(H_\nu^R)$ are nontrivial, and that $\sigma_{H_\nu^R H_\nu^r}$ is an isomorphism between them.

Claim 4.5. Let $n_0 = \frac{1}{2}(\dim M - \mathrm{codim} M)$. For every $s \in [r, R]$ and any $c < a$ which is not a negative integer multiple of πs^2 , the map

$$\Pi : \mathrm{HF}_{n_0}^{[c,b]}(H_\nu^s) \rightarrow \mathrm{HF}_{n_0}^{[a,b]}(H_\nu^s)$$

is (eventually) an isomorphism and

$$\mathrm{HF}_{n_0}^{[c,b]}(H_\nu^s) = \mathrm{HF}_{n_0}^{[a,b]}(H_\nu^s) = \mathbb{Z}_2.$$

Here, Π is the map from the exact triangle $\Delta_{c,a,b}(H_\nu^s)$.

Claim 4.6. Let k' be the largest integer in $(0, -a/\pi r^2)$ and let $s_0, s_1 \in [r, R]$ satisfy

$$s_0 < s_1 < \sqrt{1 + 1/(k' + 1)} s_0.$$

Then there are constants c , satisfying $c < a$, such that the map

$$\hat{\sigma}_{H_\nu^{s_1} H_\nu^{s_0}} : \mathrm{HF}^{[a,b]}(H_\nu^{s_0}) \rightarrow \mathrm{HF}^{[c,b]}(H_\nu^{s_1})$$

is well-defined and an isomorphism. Moreover, c can be chosen so that $c \notin -\pi s_1^2 \mathbb{Z}^+$.

Claim 4.5 implies that the groups $\mathrm{HF}_{n_0}^{[a,b]}(H_\nu^r)$ and $\mathrm{HF}_{n_0}^{[a,b]}(H_\nu^R)$ are both isomorphic to \mathbb{Z}_2 . It just remains to show that

$$\sigma_{H_\nu^R H_\nu^r} : \mathrm{HF}_{n_0}^{[a,b]}(H_\nu^r) \rightarrow \mathrm{HF}_{n_0}^{[a,b]}(H_\nu^R)$$

is an isomorphism. To see this, let s_0 and s_1 satisfy the assumptions of Claim 4.6 and consider the following version of the commutative diagram from Lemma 3.12:

$$(5) \quad \begin{array}{ccc} & & \mathrm{HF}^{[c,b]}(H_\nu^{s_1}) \\ & \nearrow \hat{\sigma}_{H_\nu^{s_1} H_\nu^{s_0}} & \downarrow \Pi \\ \mathrm{HF}^{[a,b]}(H_\nu^{s_0}) & \xrightarrow{\sigma_{H_\nu^{s_1} H_\nu^{s_0}}} & \mathrm{HF}^{[a,b]}(H_\nu^{s_1}) \end{array}$$

Together, Claims 4.5 and 4.6 imply that the map

$$\sigma_{H_\nu^{s_1} H_\nu^{s_0}} : \mathrm{HF}_{n_0}^{[a, b]}(H_\nu^{s_0}) \rightarrow \mathrm{HF}_{n_0}^{[a, b]}(H_\nu^{s_1})$$

is an isomorphism. Now, by Lemma 3.5, for any sequence $\{s_i\}_{i=1, \dots, k}$ with

$$r = s_0 < s_1 < \dots < s_{k-1} < s_k = R,$$

we can decompose $\sigma_{H_\nu^R H_\nu^r}$ as

$$\sigma_{H_\nu^R H_\nu^r} = \sigma_{H_\nu^R H_\nu^{s_{k-1}}} \circ \sigma_{H_\nu^{s_{k-1}} H_\nu^{s_{k-2}}} \circ \dots \circ \sigma_{H_\nu^{s_1} H_\nu^{s_0}}.$$

Choosing the s_i so that $s_i < s_{i+1} < \sqrt{1 + 1/(k' + 1)} s_i$, it follows that $\sigma_{H_\nu^R H_\nu^r}$ is an isomorphism from $\mathrm{HF}_{n_0}^{[a, b]}(H_\nu^r)$ to $\mathrm{HF}_{n_0}^{[a, b]}(H_\nu^R)$.

4.2.2. Rescaling. For simplicity, we will assume that $r = 1$ and $R > 1$. This can always be achieved by scaling the symplectic structure on W and the metric on E . In this case, the fixed constants a and b satisfy $a < -\pi R^2 < -\pi < b < 0$, and k' is the largest integer in $(0, -a/\pi)$.

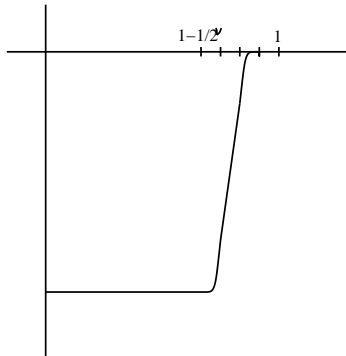
4.2.3. The functions H_ν^s . Here, we construct the functions H_ν^s , describe their closed orbits with period equal to one, and prove that for each $s \in [1, R]$ the sequence $\{H_\nu^s\}$ is (eventually) upwardly exhausting for $\mathrm{SH}^{[a, b]}(B_s)$.

We begin by constructing the exhausting sequence $\{H_\nu\} \equiv \{H_\nu^1\}$ for $\mathrm{SH}^{[a, b]}(B_1)$. These functions depend only on the norm of the fibre variable, i.e., $H_\nu = H_\nu(\|z\|)$. The functions H_ν^s are then obtained by rescaling the argument as $H_\nu^s(\|z\|) = H_\nu(\|z\|/s)$.

Consider a sequence of smooth nondecreasing functions $f_\nu : [0, \infty) \rightarrow (-\infty, 0]$, for $\nu \in \mathbb{Z}^+$, with the following properties (see Figure 1):

- (F1) $f_\nu(t) = \begin{cases} 0 & \text{for } t \in [1 - 1/2^{\nu+2}, \infty), \\ f_\nu(0) & \text{for } t \in [0, 1 - 1/2^\nu]. \end{cases}$
- (F2) $f_\nu(t) \in [-1/2^{\nu+2}, 0]$ for $t \in (1 - 1/2^{\nu+1}, 1 - 1/2^{\nu+2})$.
- (F3) $f'_\nu(t) = 2^{\nu+3}$ for all $t \in [1 - 3/2^{\nu+2}, 1 - 1/2^{\nu+1}]$.
- (F4) $f''_\nu > 0$ for $t \in (1 - 1/2^\nu, 1 - 3/2^{\nu+2})$ and $f''_\nu < 0$ for $t \in (1 - 1/2^{\nu+1}, 1 - 1/2^{\nu+2})$.

FIGURE 1. The functions f_ν



On $B = B_1$, set

$$H_\nu(z) = f_\nu(\|z\|^2),$$

and extend this function to be identically zero on $W \setminus B$.

Recall that on the neighborhood of M where ω_E is symplectic, the Hamiltonian flow of $\rho(z) = \|z\|^2$ is totally periodic with period π . Hence, for each ν , the flow of X_{H_ν} is totally periodic and the trajectories on each level set of H_ν all have the same period. The level sets which consist of non-constant closed orbits with period equal to one correspond to the solutions of the equations

$$(6) \quad f'_\nu(\|z\|^2) = k\pi,$$

where k is a positive integer. These solutions are easy to classify. When k is in the interval $[1, 2^{\nu+3}/\pi)$, there are exactly two levels of H_ν where equation (6) holds:

$$\begin{aligned} \|z\|^2 &= t_{\nu,k}^-, \quad \text{where } t_{\nu,k}^- \in [1 - 1/2^\nu, 1 - 3/2^{\nu+2}), \text{ and} \\ \|z\|^2 &= t_{\nu,k}^+, \quad \text{where } t_{\nu,k}^+ \in (1 - 1/2^{\nu+1}, 1 - 1/2^{\nu+2}]. \end{aligned}$$

Since $2^{\nu+3} \notin \pi\mathbb{Z}^+$, there are no other solutions to (6).

Remark 4.7. As ν increases, one just gets new solutions, $t_{\nu,k}^\pm$, of equation (6) for increasingly large values of k .

We now show that for large enough ν that only the orbits on the levels $\|z\|^2 = t_{\nu,k}^+ \in (1 - 1/2^{\nu+1}, 1 - 1/2^{\nu+2})$ with $k \in [1, -a/\pi)$ have action in $(a, b]$. By a simple calculation, the action of each of the orbits on the level set $\|z\|^2 = t_{\nu,k}^\pm$ is equal to

$$\mathcal{A}_{\nu,k}^\pm(1) = f_\nu(t_{\nu,k}^\pm) - t_{\nu,k}^\pm \pi k.$$

Since $f_\nu(t) < -2^\nu$ for all $t \in [0, 1 - 3/2^{\nu+2}]$, the actions $\mathcal{A}_{\nu,k}^-$ decrease exponentially to negative infinity as $\nu \rightarrow \infty$. Hence, the orbits on the levels $\|z\|^2 = t_{\nu,k}^-$ will eventually have action less than a . It is also easy to check that the actions $\mathcal{A}_{\nu,k}^+(1)$ decrease monotonically to $-\pi k$ as $\nu \rightarrow \infty$. Thus, because $b > -\pi$, all the levels $\|z\|^2 = t_{\nu,k}^+$ will eventually have action less than b . Similarly, the levels $\|z\|^2 = t_{\nu,k}^+$, with $k \in (-a/\pi, 2^{\nu+3}/\pi)$, will eventually have action less than a .

From this discussion we also see that the functions $\{H_\nu\}$ belong to $\mathcal{H}^{a,b}$ for all sufficiently large ν .

Lemma 4.8. *The sequence $\{H_\nu\}$ is (eventually) upwardly exhausting for the symplectic homology $\text{SH}^{[a,b)}(B)$.*

Proof. The only property that is not immediately obvious is that $\sigma_{H_{\nu+1}H_\nu}$ is an isomorphism for all sufficiently large ν . To see this, consider ν as a continuous parameter and look at the monotone homotopy $\{H_{\nu+\tau}\}_{\tau \in [0,1]}$ between H_ν and $H_{\nu+1}$. When ν is large enough, the new periodic orbits with period one which appear as τ goes from zero to one will all have action less than a . (This follows from Remark 4.7 and the fact that $\mathcal{A}_{\nu,k}^+(1) \rightarrow -k\pi$ as $\nu \rightarrow \infty$.) Hence, $\{H_{\nu+\tau}\} \in \mathcal{H}^{a,b}$ for all $\tau \in [0, 1]$ and Lemma 4.8 immediately follows from Lemma 3.6. \square

Remark 4.9. The same argument shows that $\{H_\nu\}$ is (eventually) upwardly exhausting for every $\text{SH}^{[c,d)}(B)$ with $a \leq c \leq d < 0$.

For $1 \leq s \leq R$, we now define the functions H_ν^s by the equation

$$H_\nu^s(\|z\|) = H_\nu(\|z\|/s) = f_\nu(\|z\|^2/s^2).$$

For these new Hamiltonians the level sets consisting of 1-periodic orbits are defined by the equations

$$(7) \quad f'_\nu(\|z\|^2/s^2) = s^2 k \pi.$$

When k is any integer in $[1, 2^{\nu+3}/\pi s^2)$ each of these equations has two solutions, $\|z\|^2 = s^2 t_{\nu,k}^\pm(s)$, where $t_{\nu,k}^-(s)$ is in $[1 - 1/2^\nu, 1 - 3/2^{\nu+2})$ and $t_{\nu,k}^+(s)$ is in $(1 - 1/2^{\nu+1}, 1 - 1/2^{\nu+2}]$. We also have $t_{\nu,k}^\pm(s) \rightarrow t_{\nu,k}^\pm$ as $s \rightarrow 1$ for all k and ν .

The actions of the orbits on these level sets are equal to

$$\mathcal{A}_{\nu,k}^\pm(s) = f_\nu(t_{\nu,k}^\pm(s)) - t_{\nu,k}^\pm(s) \pi k s^2.$$

As $\nu \rightarrow \infty$, the actions $\mathcal{A}_{\nu,k}^-(s)$ decrease exponentially to negative infinity, and the actions $\mathcal{A}_{\nu,k}^+(s)$ decrease monotonically to $-\pi k s^2$.

Just as above, H_ν^s is in $\mathcal{H}^{a,b}$ for sufficiently large ν , and the only orbits with action in $[a, b)$ are those on the level sets $\|z\|^2 = s^2 t_{\nu,k}^+(s)$ for $k \in [1, -a/\pi s^2)$. The same arguments also show that the functions H_ν^s (eventually) form an upward exhausting sequence for $\text{SH}^{[a,b)}(B_s)$. In particular, this is true for $s = R$.

4.2.4. Proof of Claim 4.5. By the scaling properties of the functions H_ν^s it suffices to prove Claim 4.5 for the case $s = 1$. Recall that the periodic orbits of H_ν which lie on the level sets defined by $\|z\|^2 = t_{\nu,k}^-$ have actions which decrease exponentially as $\nu \rightarrow \infty$. So, when considering bounded intervals of actions and large enough ν , the only relevant periodic orbits are those on the level sets of H_ν defined by $\|z\|^2 = t_{\nu,k}^+$. The actions of these orbits is given by

$$\mathcal{A}_{\nu,k}^+(1) = f_\nu(t_{\nu,k}^+) - t_{\nu,k}^+ \pi k$$

which decreases to $-\pi k$ monotonically. The following result is now obvious.

Lemma 4.10. *Let l be a non-negative integer and let $-(l+1)\pi < \alpha \leq \beta \leq -l\pi$. Then for sufficiently large ν*

$$\text{HF}^{[\alpha, \beta)}(H_\nu) = 0.$$

Now, consider intervals of the form $[-k\pi - \epsilon, -k\pi + \delta)$ where $0 < \epsilon, \delta < \pi$ and $k \in \mathbb{Z}^+$. By the same reasoning as above, for sufficiently large ν , the only level set with action in this interval is $\|z\|^2 = t_{\nu,k}^+$.

Lemma 4.11. *For any $0 < \epsilon, \delta < \pi$ and all ν sufficiently large*

$$\text{HF}_*^{[-k\pi - \epsilon, -k\pi + \delta)}(H_\nu) \cong H_{*+j(k)}(SM, \mathbb{Z}_2),$$

where SM is the sphere bundle of the normal bundle to M in W and

$$j(k) = \frac{1}{2} \dim W + k \operatorname{codim} M - 1.$$

Proof. The assertion, up to the value of the shift j , follows immediately from Theorem 3.4 provided that the set of periodic orbits in $\|z\|^2 = t_{\nu,k}^+$ is Morse–Bott. However, it is straightforward to check that the Morse–Bott condition is equivalent, for the Hamiltonians in questions, to the condition $f''_\nu(t_k^+) \neq 0$, [BPS, Lemma 5.3.2]. The value of $j(k)$ is determined by the standard analysis of the Conley–Zehnder indices for a small generic perturbation of the Hamiltonian. \square

Proceeding with the proof of Claim 4.5, we first show that for any constant $c < a$, where c is not a negative integer multiple of π , the map

$$\Pi: \mathrm{HF}_{n_0}^{[c, b)}(H_\nu) \rightarrow \mathrm{HF}_{n_0}^{[a, b)}(H_\nu)$$

is an isomorphism provided that ν is large enough.

To do this, we consider the following exact sequence which is a part of the exact triangle $\triangle_{c, a, b}(H_\nu)$ (see Section 3.2.4).

$$(8) \quad \mathrm{HF}_{n_0}^{[c, a)}(H_\nu) \rightarrow \mathrm{HF}_{n_0}^{[c, b)}(H_\nu) \xrightarrow{\Pi} \mathrm{HF}_{n_0}^{[a, b)}(H_\nu) \rightarrow \mathrm{HF}_{n_0-1}^{[c, a)}(H_\nu).$$

First, assume that c and a are contained in an interval of the form $(-\pi(k+1), -\pi k]$. Then Lemma 4.10 immediately implies that the first and last terms in (8) vanish for sufficiently large ν and hence Π is an isomorphism.

Next, we look at the case where c and a are in adjacent intervals, i.e.,

$$-\pi(k+1) < c < -\pi k < a \leq -\pi(k-1)$$

for some integer $k \geq 2$ (since $a < -\pi$). By Lemma 4.11, if ν is large enough,

$$\mathrm{HF}_{n_0}^{[c, a)}(H_\nu) = H_{n_0+j(k)}(SM, \mathbb{Z}_2) = 0,$$

because $n_0 + j(1) = \dim SM$ and $j(1) < j(k)$. (The sequence $j(k)$ is increasing.) In a similar manner we obtain

$$\mathrm{HF}_{n_0-1}^{[c, a)}(H_\nu) = H_{n_0+j(k)-1}(SM, \mathbb{Z}_2) = 0,$$

when $k \geq 2$. Hence, in this case, Π is again an isomorphism.

Now, for any $c < a$ such that $c \notin \pi\mathbb{Z}^+$, the map Π can be factored as a composition of isomorphisms for points in adjacent intervals. More precisely, $\Pi = \Pi_l \circ \Pi_{l-1} \circ \cdots \circ \Pi_1$ where

$$\Pi_j: \mathrm{HF}_{n_0}^{[c_j, b)}(H_\nu) \rightarrow \mathrm{HF}_{n_0}^{[c_{j-1}, b)}(H_\nu).$$

The constants c_j are chosen to be in adjacent intervals so that $c_0 = a$ and $c_j \notin -\pi\mathbb{Z}^+$ for $j \geq 1$. The previous argument then implies that each Π_j is an isomorphism for sufficiently large ν . Thus, the same is true for Π .

It remains to show that $\mathrm{HF}_{n_0}^{[c, b)}(H_\nu) = \mathrm{HF}_{n_0}^{[a, b)}(H_\nu) = \mathbb{Z}_2$. The above argument reduces the problem to the calculation of $\mathrm{HF}_{n_0}^{[\alpha, b)}(H_\nu)$ for $-2\pi < \alpha < -\pi$ and ν large enough. By Lemma 4.11, we have

$$\mathrm{HF}_{n_0}^{[\alpha, b)}(H_\nu) = H_{n_0+j(1)}(SM, \mathbb{Z}_2) = H_{\dim SM}(SM, \mathbb{Z}_2) = \mathbb{Z}_2$$

since $n_0 + j(1) = \dim SM$. This completes the proof of Claim 4.5.

4.2.5. Proof of Claim 4.6. Again by the scaling properties of the functions H_ν^s , it suffices to prove Claim 4.6 when $s_0 = 1$ and $1 < s_1 < \sqrt{1 + 1/(k' + 1)}$.

We must first find constants $c < a$ for which the homomorphism

$$(9) \quad \hat{\sigma}_{H_\nu^{s_1} H_\nu}: \mathrm{HF}^{[a, b)}(H_\nu) \rightarrow \mathrm{HF}^{[c, b)}(H_\nu^{s_1})$$

is well-defined. For any $c < a$, the function $H_\nu^{s_1}$ is in $\mathcal{H}^{c, b}$ for all sufficiently large ν . This follows again from the fact that the actions $\mathcal{A}_{\nu, k}^+(s_1)$ decrease monotonically to $-\pi k s_1^2$ as $\nu \rightarrow \infty$. So, it remains for us to find constants $c < a$ such that the Floer chain map $\sigma_{H_\nu^{s_1} H_\nu}$ satisfies

$$(10) \quad \sigma_{H_\nu^{s_1} H_\nu}(\mathrm{CF}^a(H_\nu)) \subset \mathrm{CF}^c(H_\nu^{s_1})$$

(see Section 3.1.3).

In fact, condition (10) only makes sense for generic perturbations of H_ν and $H_\nu^{s_1}$ whose closed orbits, with period equal to one and negative action, are nondegenerate. Let \tilde{H}_ν and $\tilde{H}_\nu^{s_1}$ be such perturbations of H_ν and $H_\nu^{s_1}$, respectively. Then it suffices to find constants $c < a$ for which the condition

$$(11) \quad \sigma_{\tilde{H}_\nu^{s_1} \tilde{H}_\nu}(\text{CF}^a(\tilde{H}_\nu)) \subset \text{CF}^c(\tilde{H}_\nu^{s_1})$$

is satisfied whenever \tilde{H}_ν (respectively, $\tilde{H}_\nu^{s_1}$) is sufficiently C^∞ -close to H_ν (respectively, $H_\nu^{s_1}$).

Consider first the case when $a \in (-k'\pi, -(k'+1)\pi)$. For any $\epsilon > 0$, we can choose ν sufficiently large and $\|H_\nu - \tilde{H}_\nu\|_{C^\infty}$ sufficiently small, so that the actions of the periodic orbits for \tilde{H}_ν are all within ϵ of some $\mathcal{A}_{\nu,k}^\pm(1)$. Similarly, we can choose ν and $\tilde{H}_\nu^{s_1}$ so that the actions of the periodic orbits for $\tilde{H}_\nu^{s_1}$ are within ϵ of some $\mathcal{A}_{\nu,k}^\pm(s_1)$.

Let x be any periodic orbit of \tilde{H}_ν with period equal to one and action less than a , i.e., $x \in \mathcal{P}^a(\tilde{H}_\nu)$. By choosing $\epsilon < -k'\pi - a$, we have

$$(12) \quad \mathcal{A}_{\tilde{H}_\nu}(x) \leq -(k'+1)\pi + \epsilon.$$

Now, the Floer chain map $\sigma_{\tilde{H}_\nu^{s_1} \tilde{H}_\nu}$ is defined using a homotopy \tilde{H}_ν^s from \tilde{H}_ν to $\tilde{H}_\nu^{s_1}$. We can choose this homotopy so that $\|\partial_s \tilde{H}_\nu^s - \partial_s H_\nu^s\|_{L^\infty}$ is arbitrarily small.

Assume that $y \in \mathcal{P}^a(\tilde{H}_\nu^{s_1})$ appears in $\sigma_{\tilde{H}_\nu^{s_1} \tilde{H}_\nu}(x)$ with nonzero coefficient. By the definition of $\sigma_{\tilde{H}_\nu^{s_1} \tilde{H}_\nu}$, this implies the existence of a perturbed pseudo-holomorphic cylinder $u: \mathbb{R} \times S^1 \rightarrow W$ whose ends are mapped to x and y . The energy of this cylinder satisfies

$$E(u) \leq \mathcal{A}_{\tilde{H}_\nu}(x) - \mathcal{A}_{\tilde{H}_\nu^{s_1}}(y) + \sup\{\partial_s \tilde{H}_\nu^s\}.$$

(See, for example, [BPS] Section 4.4.)

Since H_ν^s is monotone and $E(u) \geq 0$, this implies that

$$(13) \quad \mathcal{A}_{\tilde{H}_\nu^{s_1}}(y) \leq \mathcal{A}_{\tilde{H}_\nu}(x) + \|\partial_s \tilde{H}_\nu^s - \partial_s H_\nu^s\|_{L^\infty}.$$

Together, inequalities (12) and (13) yield

$$(14) \quad \mathcal{A}_{\tilde{H}_\nu^{s_1}}(y) \leq -(k'+1)\pi + \epsilon + \|\partial_s \tilde{H}_\nu^s - \partial_s H_\nu^s\|_{L^\infty}.$$

Now, the assumption that $s_1 < \sqrt{1 + 1/(k'+1)}$ implies that

$$-(k'+1)\pi < -k'\pi s_1^2.$$

If we choose ϵ and $\|\partial_s \tilde{H}_\nu^s - \partial_s H_\nu^s\|_{L^\infty}$ to be small enough to satisfy

$$2\epsilon + \|\partial_s \tilde{H}_\nu^s - \partial_s H_\nu^s\|_{L^\infty} < -k'\pi s_1^2 + (k'+1)\pi,$$

then we get

$$\mathcal{A}_{\tilde{H}_\nu^{s_1}}(y) < -k'\pi s_1^2 - \epsilon.$$

This implies that

$$\mathcal{A}_{\tilde{H}_\nu^{s_1}}(y) \leq -(k'+1)\pi s_1^2 + \epsilon.$$

Hence, for $a \in (-k'\pi, -(k'+1)\pi)$, condition (11) holds for all $c \in (a, -(k'+1)\pi s_1^2)$.

Next we consider the case when $a = -(k'+1)\pi$. If ν is sufficiently large and $\|H_\nu - \tilde{H}_\nu\|_{C^\infty}$ is sufficiently small, then any periodic orbit $x \in \text{CF}^a(\tilde{H}_\nu)$ will satisfy

$$\mathcal{A}_{\tilde{H}_\nu}(x) \leq -(k'+2)\pi + \epsilon.$$

This follows from the fact that the actions $\mathcal{A}_{\nu,k}^+$ decrease monotonically to $-k\pi$. Using the same arguments as above, we see that any $y \in CF^a(\tilde{H}_\nu^{s_1})$ which appears in $\sigma_{\tilde{H}_\nu^{s_1} \tilde{H}_\nu}(x)$ with nonzero coefficient satisfies

$$\mathcal{A}_{\tilde{H}_\nu^{s_1}}(y) \leq -(k' + 2)\pi + \|\partial_s \tilde{H}_\nu^s - \partial_s H_\nu^s\|_{L^\infty}.$$

The assumption that $s_1 < \sqrt{1 + 1/(k' + 1)}$ is equivalent to

$$-(k' + 2)\pi < -(k'_1 + 1)\pi s_1^2,$$

so choosing ϵ and $\|\partial_s \tilde{H}_\nu^s - \partial_s H_\nu^s\|_{L^\infty}$ to be small enough we get

$$\mathcal{A}_{\tilde{H}_\nu^{s_1}}(y) \leq -(k' + 2)\pi s_1^2 + \epsilon.$$

Thus, when $a = -(k' + 1)\pi$, condition (11) holds for any $c \in (-(k' + 1)\pi, -(k' + 2)\pi s_1^2)$.

To complete the proof of Claim 4.6 we must prove that $\hat{\sigma}_{H_\nu^{s_1} H_\nu}$ is an isomorphism. This will follow almost immediately from Lemma 3.7 and the fact that for large ν the actions $\mathcal{A}_{\nu,k}^+(s)$ are arbitrarily close to $-k\pi s^2$ for all $s \in [1, s_1]$.

For $a \in (-k'\pi, -(k' + 1)\pi)$, we choose $c \in (a, -(k' + 1)\pi s_1^2)$ so that $\hat{\sigma}_{H_\nu^{s_1} H_\nu}$ is well-defined. (Note that here we can choose c so that $c \notin -\pi s_1^2 \mathbb{Z}^+$.) Then, for ν sufficiently large, it is easy to see that there are continuous families a_s with $a_1 = a$ and $a_{s_1} = c$, such that

$$\mathcal{A}_{\nu,k'+1}^+(s) < a_s < \mathcal{A}_{\nu,k'}^+(s)$$

for all $s \in [1, s_1]$. For any such family a_s , $\tilde{H}_\nu^s \in \mathcal{H}^{a_s, b}$ for all $s \in [1, s_1]$. Lemma 3.7 then implies that $\hat{\sigma}_{H_\nu^{s_1} H_\nu}$ is an isomorphism. A similar argument works for the case $a = -(k' + 1)\pi$.

This completes the proof of Claim 4.6 and hence Proposition 4.2.

5. HOMOLOGICAL SYMPLECTIC CAPACITY

Following [FW] closely, we will now define a certain relative homological capacity. The results of this paper imply the finiteness of this capacity in various cases.

As above, let W be a geometrically bounded symplectically aspherical manifold. For an open subset $U \subset W$, we set

$$\mathrm{SH}^{[a, 0]}(U) = \varinjlim_{b \nearrow 0} \mathrm{SH}^{[a, b]}(U).$$

Furthermore, for a compact subset $Z \subset W$, we define its symplectic homology as

$$\mathrm{SH}^{[a, 0]}(Z) = \varprojlim_{U \searrow Z} \mathrm{SH}^{[a, 0]}(U),$$

where the limit is taken over all open sets $U \supset Z$. By the definition of the inverse limit, we have the projection map

$$\phi_U^a : \mathrm{SH}^{[a, 0]}(Z) \rightarrow \mathrm{SH}^{[a, 0]}(U)$$

for every open set $U \supset Z$.

Example 5.1. Let M be a closed symplectic submanifold of W and let B_R be the tubular neighborhoods of M constructed in Section 4.1. As readily follows from Proposition 4.2, $\mathrm{SH}_{n_0}^{[a, 0]}(M) = \mathbb{Z}_2$, where $n_0 = \frac{3}{2} \dim W - \mathrm{codim} M - 1$. Moreover,

$\phi_{B_R}^a : \mathrm{SH}_{n_0}^{[a, 0]}(M) \rightarrow \mathrm{SH}_{n_0}^{[a, 0]}(B_R)$ is an isomorphism, provided that $a < -\pi R^2$. Furthermore, $\mathrm{SH}^{[a, 0]}(B_R) = 0$ if $-\pi R^2 < a < 0$.

Define the *relative homological capacity* of (U, Z) as

$$\mathrm{c}_{\mathrm{SH}}(U, Z) = \inf\{-a \mid \phi_U^c \neq 0 \text{ for all } c < a\}.$$

Clearly, c_{SH} is an invariant of symplectomorphisms of the ambient manifold W . The fact that c_{SH} is a symplectic capacity (defined only on submanifolds of W) can be easily verified using Proposition 4.2. In other words, we have

Theorem 5.2.

- (1) [Invariance]. *The relative capacity c_{SH} is an invariant of symplectomorphisms of W .*
- (2) [Monotonicity]. *Let $Z' \subset Z \subset U \subset U'$. Then $\mathrm{c}_{\mathrm{SH}}(U, Z) \leq \mathrm{c}_{\mathrm{SH}}(U', Z)$ and $\mathrm{c}_{\mathrm{SH}}(U, Z) \leq \mathrm{c}_{\mathrm{SH}}(U, Z')$.*
- (3) [Homogeneity]. *For any constant $a > 0$,*

$$\mathrm{c}_{\mathrm{SH}}(U, Z, a\omega) = a \mathrm{c}_{\mathrm{SH}}(U, Z, \omega).$$

- (4) [Normalization]. *Assume that M is a closed symplectic submanifold of a geometrically bounded symplectically aspherical manifold W and let B_R be a symplectic tubular neighborhood of M in W of radius $R > 0$ (see Section 4.1). Then $\mathrm{c}_{\mathrm{SH}}(B_R, M) = \pi R^2$.*

Here the first three assertions are obvious. To prove the last assertion note that $\mathrm{c}_{\mathrm{SH}}(B_R, M) \leq \pi R^2$ by Proposition 4.2. On the other hand, $\mathrm{SH}^{[a, 0]}(B_R) = 0$ when $-\pi R^2 < a < 0$ as readily follows from the analysis carried out in Section 4.2.3 (see Lemma 4.10). Thus, $\mathrm{c}_{\mathrm{SH}}(B_R, M) \geq \pi R^2$, which completes the proof of the last assertion.

The capacity $\mathrm{c}_{\mathrm{SH}}(U, \text{point})$ is essentially the homological capacity introduced in [FW]. Once, $\mathrm{c}_{\mathrm{SH}}(U, Z) < \infty$, the symplectic homology of V is non-zero for any open subset V of U such that $Z \subset V \subset U$. Then a suitable version of Proposition 3.13 implies the nearby existence theorem in (U, Z) , i.e., the existence of contractible periodic orbits on a dense of levels of any function attaining an isolated minimum on Z . Yet, the finiteness of this capacity does not seem to imply the existence of periodic orbits for almost all energy levels as does the Hofer–Zehnder capacity. A different version of a relative capacity, a relative analogue of the Hofer–Zehnder capacity, which is sufficient for proving a version of the almost existence theorem is introduced and analyzed in detail in [GG].

Finally, we note that the relative homological capacity c_{SH} is different from, although apparently related to, the one introduced in [BPS].

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