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The Master Ward Identity and Generalized Schwinger-Dyson Equation in Classical Field Theory

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Abstract

In the framework of perturbative quantum field theory a new, universal renormalization condition (called Master Ward Identity) was recently proposed by one of us (M.D.) in a joint paper with F.-M. Boas. The main aim of the present paper is to get a better understanding of the Master Ward Identity by analyzing its meaning in classical field theory. It turns out that it is the most general identity for classical local fields which follows from the field equations. It is equivalent to a generalization of the Schwinger-Dyson Equation and is closely related to the Quantum Action Principle of Lowenstein and Lam.

The validity of the Master Ward Identity makes possible a local construction of quantum gauge theories.

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As a byproduct we complete Peierls' manifestly covariant definition of the Poisson bracket.
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Contents

1 Introduction

The hard question in the renormalization of a perturbative quantum field theory (QFT) is whether the symmetries of the underlying classical theory can be maintained in the process of renormalization. The difficulties are connected with the singular character of quantized fields which forbids a straightforward transfer of the arguments valid for the classical theory.

Traditionally, the impact of symmetries of the classical theory on the structure of quantum theory was analyzed in terms of the functional formulation of QFT (see, e.g. [24]). In this formalism, the algebraic properties of the interacting fields are not directly visible. In causal perturbation theory à la Bogoliubov-Epstein-Glaser [2, 11], on the other hand, the algebras of observables of the interacting theory can be constructed directly [3], and it is desirable to have a general method by which the structure of the classical theory can be transferred into quantum theory.

Typically the various symmetries which one wants to be present in the quantized theories are implied by certain identities (the Ward identities) which one imposes as renormalization conditions. A universal formulation of these identities is given by the Master Ward Identity (MWI) [6]. In [6] it was shown that the MWI implies field equations, energy momentum conservation, charge conservation and a rigorous substitute for equal-time commutation relations of quark currents. Application of the MWI to the ghost- and to the BRS-current of non-Abelian gauge theories yields ghost number conservation and the 'Master BRST Identity' [6]. These symmetries contain the information which is needed for a local construction of the algebra of observables, i.e. the elimination of the unphysical fields and the construction of physical states in the presence of an adiabatically switched off interaction (see [4], [6]).

In [6] the MWI was obtained in the following way: the difference between different orders of differentiation and time-ordering,

$$\partial^\nu_{x_1}\tilde{T}(W_1,...,W_n)(x_1,...,x_n)-\tilde{T}(\partial^\nu W_1,...,W_n)(x_1,...,x_n) \tag{1}$$

($\tilde{T}(W_1,...,W_n)(x_1,...,x_n)$ denotes the time-ordered product of the Wick polynomials $W_1(x_1),...,W_n(x_n)$ in free fields), is formally computed by means of the Feynman rules and the causal Wick expansion (see Sect. 4 of [11]) (or equivalently the normalization condition **(N3)** [4]). The MWI requires then that renormalization has to be done in such a way that this heuristically derived result is preserved. The main motivations for imposing this condition were, on the one hand side, the many, important and far-reaching consequences of the MWI, and on the other hand side, the experience that the MWI can nearly always be fulfilled.

In this paper we give a further important argument in favor of the MWI: *it is the straightforward generalization to QFT of the most general classical identity for local fields which can be obtained from the field equations and the fact that classical fields may be multiplied point-wise* (see (10)). Since quantum fields are distributions which cannot, in general, be multiplied pointwise, the derivation of the MWI in classical field theory is not transferable to quantum field theory. There, the MWI is a highly non-trivial normalization condition which contains much more information than merely the field equations.

We will start our study of the MWI with another equation, which will turn out to be equivalent to the MWI. Namely we first formulate the most general identity which follows in classical field theory from the field equations. Due to formal similarity we call it the *Generalized Schwinger-Dyson Equation* (GSDE). In this form it does not depend on a splitting of the Lagrangian into a free and an interaction part. We then introduce such a splitting and obtain the perturbative version of the Generalized Schwinger-Dyson Equation which can be imposed as renormalization condition.

It turns out that it is appropriate to use an off shell formalism where the entries of time-ordered products are classical fields not subject to any field equation as advocated by Stora. *The MWI then gives a formula for time-ordered products of fields where one of the entries vanishes if the free field equations are imposed.* In the traditional version of causal perturbation theory all calculations are done in terms of Wick products of free fields. There the same identity becomes visible as a non-commutativity of differentiation and time-ordering (1). To understand the connection between both formalisms we introduce a map σ which associates free fields to general (off shell) fields. The time-ordered products T of off shell fields in this paper are related to the time-ordered products $\tilde{T}$ (1) of on shell fields by

$$\tilde{T}(W_1(x_1),\dots,W_n(x_n)) = T(\sigma(W_1)(x_1),\dots,\sigma(W_n)(x_n)) \ .$$

In contrast to $\tilde{T}$, there is no reason why T should not commute with derivatives. Therefore, we adopt here the proposal of Stora [29] and postulate that T can be freely commuted with derivatives. Stora calls this the Action Ward Identity (AWI), because it means that the interacting fields as well as the

S-matrix depend on the interaction Lagrangian only via its contribution to the action. A caveat has to be added here: while no anomaly of the AWI is known, there does not yet exist a proof that anomalies are always absent.[1]

With that the non-commutativity of $\tilde{T}$ with derivatives is traced back to the non-commutativity of σ with derivatives. So, the MWI of [6] can be formulated in terms of time ordered products where one of the entries is of the form $[\partial_\mu, \sigma](W)$. The latter expression vanishes if the free field equations are imposed, hence the MWI of [6] is a special case of the MWI proposed in this paper. Actually, under a natural condition on the choice of σ, the two formulations are even equivalent. The freedom in the choice of parameters in the Feynman propagators of derivated fields (see [8] and [6]) is converted in the present formalism into the freedom in the choice of σ.

The use of off shell fields has another advantage: it facilitates the introduction of auxiliary fields which in the presence of derivative couplings or in the definition of the BRS transformation may lead to a more elegant formulation. On the other hand, the use of auxiliary fields introduces more free parameters in the choice of σ.

Our analysis might be compared with the formulation of the Quantum Action Principle of Lowenstein [19, 20] and Lam [17, 18]. These authors showed in the framework of BPHZ renormalization how classical symmetries can be transferred into renormalized perturbation theory. In contrast to these works, we emphasize the structural similarity of classical and quantum perturbative field theory. As a consequence, our arguments do not rely on the rather involved combinatorics of BPHZ renormalization. However, we did not yet investigate the structure of anomalies of the MWI. Another difference is that the formalism of Lam seems to be inconsistent for vertices containing higher than first derivatives of the basic fields, see Sect. V of [17]. We overcome these difficulties by means of the map σ (cf. the Example (111)).

Another, more recent approach to a general treatment of symmetries in renormalized perturbation theory is the Quantum Noether Condition (QNC) of Hurth and Skenderis [15]. In case of the BRS-current it may be understood as a reformulation of the 'perturbative gauge invariance' of [8], see the last Remark in Sect. 4.5.2 of [6] (published version). Similar to that condition it has two different kinds of implications: (i) The QNC for tree diagrams

[1] It might be that Lemma 1 in [20] or Lemma 1 in [17] actually implies the AWI, but due to the rather different formalisms this conjecture could not yet be verified.

is simply the conservation of the *classical* symmetry current[2] in presence of an adiabatically switched off interaction. This yields restrictions on the interaction. A rigorous and enlarged version of that analysis is given (as an application of MWI/AWI) in Appendix B for the example of the BRS-current. (ii) Provided the interaction is such that the (Q)NC is classically fulfilled (i.e. for all tree diagrams), it makes sense to impose it for loop diagrams. There it is a renormalization condition which is less general than the MWI/AWI (cf. again the above mentioned Remark in [6]).

The paper is organized as follows: in Sect. 2.1 we study the canonical structure of classical field theory. We use Peierls' covariant definition of the Poisson bracket [23] which does not rely on a Hamiltonian formalism, and give a direct proof that it satisfies the Jacobi identity. In Sect. 2.2 we determine the perturbative expansion of the classical fields as formal power series. The coefficients of this expansion are the retarded products. We prove that they satisfy the GLZ relations [12]. We briefly discuss the possibility of eliminating derivative couplings by introducing auxiliary fields in Sect. 2.3.

In Sect. 3 we formulate the GSDE and the MWI, introduce the map σ and discuss the relation to the formulation of the MWI in [6].

In Sect. 4 we introduce the perturbative expansion of the interacting quantum fields (as formal power series) by the principle that *as much as possible of the classical structure is maintained in the process of quantization*, in particular the GLZ relation, the AWI and the MWI. All these conditions are formulated in terms of the retarded products, because in this formulation the conditions have the same form in the classical theory as well as in the quantum theory. In quantum theory, on the other hand, one can equivalently formulate everything in terms of the more familiar time-ordered products. The resulting formalism is not completely equivalent to the one given in [6]. We clarify the significance of the difference.

In Sect. 5 we derive the 'Master BRST Identity' [6] (which results from the application of the MWI and AWI to the free BRS-current) in the formalism of this paper. We also use the MWI and AWI to determine the admissible interaction of a BRS-invariant local gauge theory. By a modification of this procedure one can derive the conditions which are used in [6] to express BRS-invariance of the interaction from more fundamental principles. This is done in Appendix B. As a byproduct this will clarify the relation to perturbative gauge invariance (in the sense of [8]).

[2]Some confusion could be avoided by omitting the 'Q' of 'QNC' in these calculations.

Appendix A gives an explicit formula for the map σ and shows its uniqueness in a particular framework.

2 Classical field theory for localized interactions

2.1 Retarded product and Poisson bracket

To keep the notations simple we consider only one real scalar field φ (on the d-dimensional Minkowski space $\mathbb{M}$, $d > 2$) and Lagrangians

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \tag{2}$$

where the free part $\mathcal{L}_0$ is fixed. We will vary the interaction part

$$\mathcal{L}_{\text{int}} = -gP(\varphi, \partial_\mu \varphi) \stackrel{\text{def}}{=} : -gL_{\text{int}}, \tag{3}$$

which is a polynomial $P = L_{\text{int}}$ in φ and $\partial_\mu \varphi$ (later we will also allow for higher derivatives of φ) multiplied by a test function $g \in \mathcal{D}(\mathbb{M})$. The latter is interpreted as a space-time dependent coupling constant. We assume that the Cauchy problem is well posed for all Lagrangians in our class. For simplicity we restrict our formalism to smooth solutions. In non-linear theories, classical fields which are initially smooth may get singularities. But, in this paper, we are mainly interested in perturbation theory. It follows from the analysis in Sect. 2.2 that there is a unique smooth perturbative solution, if the given incoming free solution is smooth.

Let $\mathcal{C}_\mathcal{L}$ be the set of smooth solutions $f : \mathbb{M} \to \mathbb{R}$ of the Euler-Lagrange equation

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \frac{\partial \mathcal{L}}{\partial \varphi} \ , \tag{4}$$

with compactly supported Cauchy data. We consider $\mathcal{C}_\mathcal{L}$ as the classical phase space. (This is equivalent to the traditional point of view in which an element of the phase space is the set of the corresponding Cauchy data, e.g. the functions $(f, \dot{f})$, $f \in \mathcal{C}_\mathcal{L}$, restricted to the time $x^0 = 0$.)

We interpret the field φ as the evaluation functional on $\mathcal{C} \stackrel{\text{def}}{=} \mathcal{C}^\infty(\mathbb{M}, \mathbb{R})$: [3]

$$\varphi(x)(f) \stackrel{\text{def}}{=} f(x), \qquad f \in \mathcal{C}^\infty(\mathbb{M}, \mathbb{R}) \ . \tag{5}$$

[3] A *complex* scalar field is the analogous evaluation functional on $\mathcal{C}^\infty(\mathbb{M}, \mathbb{C})$ and we define $\varphi^*(x)(f) \equiv f^*(x)$.

Functionals of the field

$$F(\varphi) \equiv \sum_{n=0}^{N} \int dx_1 ... dx_n \, \varphi(x_1) \ldots \varphi(x_n) t_n(x_1, ..., x_n), \quad N < \infty, \tag{6}$$

then lead in a natural way to functionals on $\mathcal{C}$,

$$F(\varphi)(f) \stackrel{\text{def}}{=} F(f), \quad f \in \mathcal{C}. \tag{7}$$

Here $t_0 \in \mathbb{C}$ and the t_n are suitable test functions where we admit also certain distributions with compact support, in particular $\delta^{4(n-1)}(x_1 - x_n, ..., x_{n-1} - x_n) f(x_n)$, $f \in \mathcal{D}(\mathbb{M})$. More precisely, we admit all distributions with compact support whose Fourier transform decays rapidly outside of the hyperplane $\{(k_1, \ldots, k_n), \sum_i k_i = 0\}$. We denote the algebra of functionals of the form (6) by $\mathcal{F}(\mathcal{C})$ and, when restricted to $\mathcal{C}_{\mathcal{L}}$, by $\mathcal{F}(\mathcal{C}_{\mathcal{L}})$.

The field $\varphi_{\mathcal{L}}$ which satisfies the field equation (4) is obtained as the restriction of φ to $\mathcal{C}_{\mathcal{L}}$,

$$\varphi_{\mathcal{L}} \stackrel{\text{def}}{=} \varphi|_{\mathcal{C}_{\mathcal{L}}}, \tag{8}$$

This restriction induces a homomorphism of the algebras of functionals F

$$F(\varphi) \to F(\varphi)_{\mathcal{L}} = F(\varphi_{\mathcal{L}}). \tag{9}$$

In particular, the factorization property

$$(AB)_{\mathcal{L}}(x) = A_{\mathcal{L}}(x) B_{\mathcal{L}}(x) \tag{10}$$

holds for polynomials A, B in φ and their partial derivatives. (The algebra of these polynomials will be denoted by $\mathcal{P}$,

$$\mathcal{P} = \bigvee \{\partial^a \varphi, a \in \mathbb{N}_0^d\} \; . \;) \tag{11}$$

It is a main difficulty of quantum field theory that the factorization property (10) is no longer valid.

To compare theories with different Lagrangians we use the fact that by the assumed uniqueness of the solution of the Cauchy problem there exists to each $f_2 \in \mathcal{C}_{\mathcal{L}_2}$ precisely one $f_1 \in \mathcal{C}_{\mathcal{L}_1}$ which coincides with f_2 outside of the future of the region where the respective Lagrangians differ, $f_1(x) =$

$f_2(x)\,\forall x \notin (\text{supp}\,(\mathcal{L}_1 - \mathcal{L}_2) + \overline{V}_+)$.[4] We denote the corresponding map (the 'wave operator') by $r_{\mathcal{L}_1,\mathcal{L}_2}$:

$$r_{\mathcal{L}_1,\mathcal{L}_2} : \mathcal{C}_{\mathcal{L}_2} \to \mathcal{C}_{\mathcal{L}_1}\ ,\ f_2 \mapsto f_1. \tag{12}$$

Obviously it holds $r_{\mathcal{L}_1,\mathcal{L}_2} \circ r_{\mathcal{L}_2,\mathcal{L}_3} = r_{\mathcal{L}_1,\mathcal{L}_3}$. Analogously we define $a_{\mathcal{L}_1,\mathcal{L}_2} : \mathcal{C}_{\mathcal{L}_2} \to \mathcal{C}_{\mathcal{L}_1}, f_2 \mapsto f_1$ by requiring that f_1 and f_2 agree in the distant future.

This bijection between the spaces of solutions can be used to express the interacting fields as functionals on the space of *free* solutions. We call

$$A^{\text{ret}}_{\mathcal{L}_{\text{int}}}(x) \stackrel{\text{def}}{=} A(x) \circ r_{\mathcal{L}_0+\mathcal{L}_{\text{int}},\mathcal{L}_0} : \mathcal{C}_{\mathcal{L}_0} \to \mathbb{R} \tag{13}$$

for $A \in \mathcal{P}$ the 'retarded field'. The retarded field is a functional on the free solutions which solves the interacting field equation. We will define the perturbation expansion of classical interacting fields as the Taylor series of the retarded fields as functionals of the interaction Lagrangian,

$$A^{\text{ret}}_{\mathcal{L}_{\text{int}}}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} R_{n,1}(\mathcal{L}_{\text{int}}^{\otimes n}, A(x))\ . \tag{14}$$

The retarded products $R_{n,1}$ will be constructed in Sect. 2.2. Note

$$\partial^\mu_x\, A^{\text{ret}}_{\mathcal{L}_{\text{int}}}(x) = (\partial^\mu A)^{\text{ret}}_{\mathcal{L}_{\text{int}}}(x) \tag{15}$$

and that the factorization property (10) holds for the retarded fields, too.

Besides the commutative and associative product (10) there is a second product for classical fields: the Poisson bracket. Peierls [23] has given a definition of the Poisson bracket without recourse to a Hamiltonian formalism. We now review his procedure. It is convenient to generalize our formalism somewhat: we admit also non-local interactions, i.e. the interaction part of the action S does not need to be of the form $S_{\text{int}} = \int dx\, \mathcal{L}_{\text{int}}(x)$, but may be replaced by an arbitrary functional $F \in \mathcal{F}(\mathcal{C})$. The field equations are still obtained by the principle of least action, but in contrast to (4) they may involve non-local terms. The classical phase space $\mathcal{C}_{\mathcal{L}}$, the field $\varphi_{\mathcal{L}}$, $F(\varphi)_{\mathcal{L}}$ and the maps $r_{\mathcal{L}_1,\mathcal{L}_2}$, $a_{\mathcal{L}_1,\mathcal{L}_2}$ (12) are defined in the same way as before, but now denoted by $\mathcal{C}_S$, φ_S, $F(\varphi)_S$ and r_{S_1,S_2}, a_{S_1,S_2}. (9) and the factorization (10) hold still true. We will not discuss whether solutions of the general Cauchy

[4] $V_\pm$ denote as usual the forward and backward light-cones, respectively, and $\overline{V}_\pm$ their closures.

problem for these non-local actions exist. It is sufficient for our purpose that perturbative solutions always exist and are unique.

Let $F \equiv F(\varphi)$ and $G \equiv G(\varphi)$ be functionals from $\mathcal{F}(\mathcal{C})$. We introduce the retarded product $R_S(F,G)$ and the advanced product $A_S(F,G)$,

$$R_S(F,G) \stackrel{\text{def}}{=} \frac{d}{d\lambda}|_{\lambda=0} G \circ r_{S+\lambda F,S} \ , \tag{16}$$

$$A_S(F,G) \stackrel{\text{def}}{=} \frac{d}{d\lambda}|_{\lambda=0} G \circ a_{S+\lambda F,S} \tag{17}$$

which are functionals on $\mathcal{C}_S$. Note that the entries of the retarded and advanced products are unrestricted functionals of the field. In general it is not possible to replace them by their restriction to the space of solutions, as the following example shows: let $S = \int dx\, \mathcal{L}_0(x)$ with $\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi$, $F = \int dx\, g(x)\Box\varphi(x)$ and $G = \varphi(y)$. Then $\varphi(y) \circ r_{S+\lambda F,S}(f) = f(y) + \lambda g(y)$ and hence $R_S(F,\varphi(y)) = g(y)$, but $F_S = 0$.

The retarded and advanced products have the following important properties:

Proposition 1. (a) *The retarded product can be expressed in terms of the retarded Green function,*

$$R_S(F,G) = -\int dx\, dy\, \Big(\frac{\delta G}{\delta\varphi(x)} \Delta_S^{\text{ret}}(x,y) \frac{\delta F}{\delta\varphi(y)}\Big)_S \ . \tag{18}$$

(b) *The advanced and retarded products are related by*

$$A_S(F,G) = R_S(G,F) \ . \tag{19}$$

Here Δ_S^{ret} is the unique retarded Green function of $\frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)}$ considered as an integral operator, i.e. it satisfies the equations

$$\int dy\, \Delta_S^{\text{ret}}(x,y) \frac{\delta^2 S}{\delta\varphi(y)\varphi(z)} = \delta(x-z) = \int dy\, \frac{\delta^2 S}{\delta\varphi(x)\varphi(y)} \Delta_S^{\text{ret}}(y,z), \tag{20}$$

and (in case S is local)

$$\text{supp}\, \Delta_S^{\text{ret}} \subset \{(x,y) \,|\, x \in y + \overline{V}_+\}. \tag{21}$$

For non-local interactions with compact support we may construct the retarded Green function (and also the retarded product) in the sense of formal power series.

Since $\frac{\delta^2 S}{\delta\varphi(x)\varphi(y)}$ is symmetric, it follows that

$$\Delta_S^{adv}(x,y) \stackrel{\text{def}}{=} \Delta_S^{ret}(y,x) \tag{22}$$

is the advanced Green function. Similarly to the proof of (a) one finds that the advanced product $A_S(F,G)$ fulfills (18) with Δ_S^{ret} replaced by Δ_S^{adv}. This and (22) immediately imply (b).

Example: The abstract formalism may be illustrated by the example of a real Klein Gordon field with a polynomial interaction, $S = \int dx\, \frac{1}{2}[\partial^\mu\varphi\partial_\mu\varphi - m^2\varphi^2 - gP(\varphi)]$. We obtain $\frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)}(f) = -(\Box + m^2 + g(x)P''(f(x)))\delta(x-y)$, and

$$\Delta_S^{ret}(x,y)(f) = -\Delta^{ret}(x,y;gP''(f)), \tag{23}$$

is the unique retarded Green function of the Klein-Gordon operator with a potential $gP''(f)$, where f is the classical field configuration on which the functionals are evaluated.

We will use the formula (18) as definition of the retarded product outside of the space of solutions $f \in \mathcal{C}_S$ (and analogously for the advanced products).

Proof. It remains to prove (a).
Let $f \in \mathcal{C}_S$ and $r_{S+\lambda F,S}(f) = f + \lambda h + \mathcal{O}(\lambda^2)$. Then

$$0 = \frac{d}{d\lambda}|_{\lambda=0} \frac{\delta(S+\lambda F)}{\delta\varphi(y)}(f+\lambda h) = \frac{\delta F}{\delta\varphi(y)}(f) + \int dz\, \frac{\delta^2 S}{\delta\varphi(y)\varphi(z)}(f)h(z) \tag{24}$$

and (in the case of a local action S) $h(z) = 0$ if $z \notin \text{supp}\,(F) + \bar{V}_+$. Hence,

$$R_S(F,\varphi(x))(f) = h(x) = -\int dy\, \Delta_S^{ret}(x,y)\frac{\delta F}{\delta\varphi(y)}(f), \tag{25}$$

and by means of

$$R_S(F,G)(f) = \frac{d}{d\lambda}|_{\lambda=0} G(\varphi)(f+\lambda h) = \int dx\, \frac{\delta G}{\delta\varphi(x)}(f)h(x) \tag{26}$$

we obtain the assertion (18). (In the case of a non-local action the condition on the support of h has to be appropriately modified.) $\Box$

Definition 1. *The* **Peierls bracket** *associated to an action S is a product on $\mathcal{F}(\mathcal{C})$ with values in $\mathcal{F}(\mathcal{C}_S)$ defined by*

$$\{F,G\}_S \stackrel{\text{def}}{=} R_S(F,G) - R_S(G,F) = R_S(F,G) - A_S(F,G). \tag{27}$$

The Peierls bracket depends only on the restriction of the functionals to the space of solutions,

$$\{F,G\}_S = \{F',G\}_S \text{ if } F_S = F'_S \ . \tag{28}$$

Namely, let F be a functional which vanishes on the space of solutions. This is the case if F is of the form[5]

$$F = \int dx\, G(x) \frac{\delta S}{\delta\varphi(x)} \ . \tag{29}$$

Then the retarded product with a functional H is

$$R_S(F,H) = \\ \int dx\, dy\, dz\, \Big(\frac{\delta G(x)}{\delta\varphi(y)} \frac{\delta S}{\delta\varphi(x)} + G(x) \frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)} \Big) \Delta_S^{\mathrm{ret}}(y,z) \frac{\delta H}{\delta\varphi(z)} \ . \tag{30}$$

The first term vanishes on the space of solutions, therefore we obtain

$$R_S(F,H) = \int dz\, G(z) \frac{\delta H}{\delta\varphi(z)} \ . \tag{31}$$

The same expression is obtained for the advanced product, thus the Peierls bracket of F and H vanishes. We may therefore define the Peierls bracket for functionals on the space of solutions by

$$\{F_S, G_S\} = \{F,G\}_S \ .$$

It is easy to see that for the example of the Klein Gordon field with a polynomial interaction without derivatives the Peierls bracket coincides with the Poisson bracket obtained from the Hamiltonian formalism. The Peierls

[5] We tacitly assume here that the ideal $\mathcal{J}_S$ generated by the field equation (i.e. the set of functionals of the form (29)) is identical with the set of functionals which vanish on the space of solutions. This seems to be true in relevant cases. Otherwise, we have to replace the restriction map $F \mapsto F_S$ by the quotient map with respect to $\mathcal{J}_S$.

bracket, however is defined also for derivative couplings and even for non-local interactions where the Hamiltonian formalism has problems. Moreover, it is manifestly covariant and does not use a splitting of space-time into space and time.

We now want to show that the Peierls bracket fulfils in general the usual properties of a Poisson bracket. Antisymmetry, linearity and the Leibniz rule are obvious [6], but the Jacobi identity is non-trivial (actually it is not discussed in the paper of Peierls). We will see that the Jacobi identity follows from the fact that r_{S_2,S_1} commutes with the Peierls bracket (hence it is a canonical transformation).

Proposition 2. (a) *The retarded wave operator* r_{S_2,S_1} *preserves the Peierls bracket (27),*

$$\{F \circ r_{S_2,S_1}, G \circ r_{S_2,S_1}\}_{S_1} = \{F, G\}_{S_2} \circ r_{S_2,S_1} \tag{32}$$

and the same statement holds for a_{S_2,S_1}.
(b) *The Peierls bracket (27) satisfies the conditions which are required for a Poisson bracket, in particular the Jacobi identity*

$$\{F_S, \{H_S, G_S\}\} + \{G_S, \{F_S, H_S\}\} + \{H_S, \{G_S, F_S\}\} = 0 \ . \tag{33}$$

Proof. It suffices to prove (32) for an infinitesimal change of the interaction: setting $S_1 = S$ and $S_2 = S + \lambda H$, the infinitesimal version of (32) reads

$$\begin{aligned} &\{R_S(H, F), G_S\} + \{F_S, R_S(H, G)\} = \\ &\qquad R_S(H, \{F, G\}) + \frac{d}{d\lambda}|_{\lambda=0}(R_{S+\lambda H}(F, G) - A_{S+\lambda H}(F, G)) \end{aligned} \tag{34}$$

where we used in the last term the extended definition of the retarded and advanced products outside of the space of solutions introduced after Proposition 1.

We now insert the formula for the retarded product of Proposition 1 everywhere in (34). Applying $\frac{\delta}{\delta\varphi}$ to (20) we find

$$\frac{\delta}{\delta\varphi(z)}\Delta_S^{\mathrm{adv}}(x, y) = -\int dv\, dw\, \Delta_S^{\mathrm{adv}}(x, v)\frac{\delta^3 S}{\delta\varphi(v)\delta\varphi(w)\delta\varphi(z)}\Delta_S^{\mathrm{adv}}(w, y), \tag{35}$$

[6] Linearity and the Leibniz rule hold already for the retarded and advanced products.

and analogously

$$\frac{d}{d\lambda}|_{\lambda=0}\Delta^{\mathrm{a}dv}_{S+\lambda H}(x,y) = -\int dv\,dw\,\Delta^{\mathrm{a}dv}_{S}(x,v)\frac{\delta^2 H}{\delta\varphi(v)\delta\varphi(w)}\Delta^{\mathrm{a}dv}_{S}(w,y). \quad (36)$$

With that (34) can be verified by a straightforward calculation.

By an analogous calculation we prove that the advanced transformation a_{S_2,S_1} is a canonical transformation. The infinitesimal version is (34) where in the first three terms R_S is replaced by A_S and where the last term, the term with the λ-derivative, is unchanged. Hence, considering the difference of these two versions of (34), the latter term drops out, and we obtain the Jacobi identity (33). □

2.2 Higher order retarded products and perturbation theory

In analogy to equation (16) we define the higher order retarded products by

$$R_S(F^{\otimes n}, G) = \frac{d^n}{d\lambda^n}|_{\lambda=0} G \circ r_{S+\lambda F,S}\ . \quad (37)$$

They have a unique extension to $(n+1)$-linear functionals on $\mathcal{F}(\mathcal{C})$ which are symmetric in the first n variables. With that the perturbative expansion of $G \circ r_{S+\lambda F,S}$ in λ reads

$$G \circ r_{S+\lambda F,S} \simeq \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} R_S(F^{\otimes n}, G) \equiv: R_S(e_{\otimes}^{\lambda F}, G) \quad (38)$$

in the sense of formal power series. If S is the free part of the action, this is the perturbative expansion of the retarded fields (13) in terms of free fields.

In the first paper of [5], equation (71), we gave an explicit formula for R in the case where all functionals are local and where F and G do not contain derivatives. It took the form of a retarded multi-Poisson bracket. In case of derivative couplings this can no longer be true, as we already know from the discussion of the retarded product of two factors, cf. the counter example after equation (17).

The *general* case where S, F and G might be non-local can be obtained from the formula

$$\frac{d}{d\lambda} R_S(e_{\otimes}^{\lambda F}, G) = R_S(e_{\otimes}^{\lambda F}, R_{S+\lambda F}(F,G)) \quad (39)$$

which is the Taylor series expansion of

$$\frac{d}{d\lambda} G \circ r_{S+\lambda F,S} = R_{S+\lambda F}(F,G) \circ r_{S+\lambda F,S} \ , \tag{40}$$

the latter identity following directly from the definition of the retarded products. On the r.h.s. of (39) we understand $R_{S+\lambda F}(F,G)$ as an unrestricted functional, i.e. $R_{S+\lambda F}(F,G) \in \mathcal{F}(\mathcal{C})$. Comparing the coefficients on both sides of (39) yields a recursion relation:

$$R_S(F^{\otimes(n+1)},G) = -\sum_{l=0}^{n} \binom{n}{l} R_S\Big(F^{\otimes l}, \int dx\, dy\, \frac{\delta F}{\delta\varphi(x)} \Delta^{\mathrm{adv}\,(n-l)}_{S+\lambda F}(x,y) \frac{\delta G}{\delta\varphi(y)}\Big), \tag{41}$$

where

$$\Delta^{\mathrm{adv}\,(k)}_{S+\lambda F}(x,y) \overset{\mathrm{def}}{=} \frac{d^k}{d\lambda^k}|_{\lambda=0} \Delta^{\mathrm{adv}}_{S+\lambda F}(x,y)$$
$$= (-1)^k k! \int dv_1 ... dv_k\, dz_1 ... dz_k\, \Delta^{\mathrm{adv}}_S(x,v_1)$$
$$\cdot \frac{\delta^2 F}{\delta\varphi(v_1)\delta\varphi(z_1)} \Delta^{\mathrm{adv}}_S(z_1,v_2) ... \frac{\delta^2 F}{\delta\varphi(v_k)\delta\varphi(z_k)} \Delta^{\mathrm{adv}}_S(z_k,y)\ .$$

This shows the existence of solutions in the sense of formal power series.

Peierls' formula for the Poisson bracket together with the fact that the retarded wave operators r_{S,S_0} are canonical transformations lead to an interesting relation between the higher order retarded products.

Let F, G and S_1 be functionals from $\mathcal{F}(\mathcal{C})$ and let $S = S_0 + \lambda S_1$. Then we have

$$\{F \circ r_{S,S_0}, G \circ r_{S,S_0}\}_{S_0} = (R_S(F,G) - R_S(G,F)) \circ r_{S,S_0}\ . \tag{42}$$

According to the definition of the retarded products it holds

$$R_S(F,G) \circ r_{S,S_0} = \frac{d}{d\mu}|_{\mu=0} G \circ r_{S+\mu F, S_0}$$

where we used the composition property of the wave operators. If we now take the n-th derivative with respect to λ on both sides of (42) we find

$$\sum_{I \subset \{1,...,n\}} \{R_{S_0}(\otimes_{i\in I} H_i, F), R_{S_0}(\otimes_{j\notin I} H_j, G)\}_{S_0} =$$
$$R_{S_0}(\otimes_{i=1}^n H_i \otimes F, G) - R_{S_0}(\otimes_{i=1}^n H_i \otimes G, F) \tag{43}$$

with $H_i \in \mathcal{F}(\mathcal{C})$. This relation is in quantum field theory known as the GLZ-Relation (see below). It plays an important role in renormalization.

In case S and F are *local*, we can find an elegant expression for the retarded products. We introduce the following differential operator on the space of functionals $\mathcal{F}(\mathcal{C})$

$$\mathcal{R}(x) := -\int dy\, \Big(\frac{\delta F}{\delta\varphi(x)}\Delta_S^{\mathrm{ret}}(y,x)\frac{\delta}{\delta\varphi(y)}\Big)\ . \tag{44}$$

Note that $\mathcal{R}(x)$ is smooth in x since Δ_S^{ret} maps smooth functions with compact support onto smooth functions. According to (18) we have

$$R_S(F,G) = \int dx\, (\mathcal{R}(x)G)_S\ . \tag{45}$$

The n-th order case looks quite similar:

Proposition 3. *The n-th order retarded product is given by the formula*

$$R_S(F^{\otimes n},G) = n! \int_{x_1^0\leq\ldots\leq x_n^0} dx_1\ldots dx_n(\mathcal{R}(x_1)\cdots\mathcal{R}(x_n)G)_S \tag{46}$$

Proof. We first show that the power series defined by the r.h.s. of formula (46)

$$G\mapsto G(\lambda) = R_S^0(\exp_\otimes \lambda F, G) \tag{47}$$

defines a homomorphism on the algebra of functionals $G\in\mathcal{F}(\mathcal{C})$. This means that for two functionals G and H we have the factorization

$$R_S^0(F^{\otimes n},GH) = \sum_{k=0}^{n}\binom{n}{k}R_S^0(F^{\otimes k},G)R_S^0(F^{\otimes n-k},H)\ . \tag{48}$$

We use the fact that the operators $\mathcal{R}(x)$ are functional derivatives of first order. Hence from Leibniz' rule we get

$$\mathcal{R}(x_1)\cdots\mathcal{R}(x_n)GH = \sum_{I\subset\{1,\ldots,n\}}(\prod_{i\in I}\mathcal{R}(x_i)G)(\prod_{j\notin I}\mathcal{R}(x_j)H)\ . \tag{49}$$

It remains to check the time ordering prescription. For any n-tupel of times $t=(x_1^0,\ldots,x_n^0)$ we choose a permutation π_t with $x^0_{\pi_t(1)}\leq\ldots\leq x^0_{\pi_t(n)}$. We obtain

$$R_S^0(F^{\otimes n},G) = \int d^nx(\mathcal{R}(x_{\pi_t(1)})\ldots\mathcal{R}(x_{\pi_t(n)})G)_S\ . \tag{50}$$

If we insert (49) into (50) we see that the permutation π_t restricted to I as well as to the complement of I yields the correct time ordering. The n-fold integral factories, and the integrals over $(x_i, i \in I)$ and $(x_j, j \notin I)$ do not depend on the choice of I, but only on the cardinality of I. This proves (48).

We now show that the formal power series for the retarded field (47) with action $S + \lambda F$ satisfies the field equation $\frac{\delta}{\delta\varphi(x)}(S + \lambda F) = 0$ and thus coincides with $R_S(\exp_\otimes \lambda F, G)$ because of (48) and the uniqueness of the retarded solutions. We have to show $R_S^0(\exp_\otimes \lambda F, \frac{\delta(S+\lambda F)}{\delta\varphi(x)}) = 0$. So we insert $G = \frac{\delta}{\delta\varphi(y)} S$ into (47), use

$$\mathcal{R}(x)\frac{\delta}{\delta\varphi(y)}S = -\frac{\delta F}{\delta\varphi(x)}\int dz\, \Delta_S^{\mathrm{ret}}(z,x)\frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(z)} = -\frac{\delta F}{\delta\varphi(x)}\delta(x-y)\ . \tag{51}$$

and obtain the wanted field equation where we exploit the fact that $\mathcal{R}(y)\frac{\delta F}{\delta\varphi(x)} = 0$ if y is not in the past of x. □

Since we are mainly interested in local functionals, we change our point of view somewhat. Let $\mathcal{P}$ denote as before the set of polynomials of φ and its derivatives (11). Each field $A \in \mathcal{P}$ defines a distribution with values in $\mathcal{F}(\mathcal{C})$,

$$A(f) = \int dx\, A(x) f(x)\ ,\ f \in \mathcal{D}(\mathbb{M})\ .$$

We fix a local action S which later will be the free action. We now define the retarded products of fields as $\mathcal{F}(\mathcal{C}_S)$ valued distributions in several variables by

$$\begin{gathered} R_{n,1}(A_1 \otimes \cdots \otimes A_n, B)(f_1 \otimes \cdots \otimes f_n, g) \equiv \\ \int d(x,y)\, R_{n,1}(A_1(x_1), \ldots, A_n(x_n); B(y)) f_1(x_1)\cdots f_n(x_n) g(y) \stackrel{\mathrm{def}}{=} \\ R_S(A_1(f_1) \otimes \cdots \otimes A_n(f_n), B(g)) \end{gathered} \tag{52}$$

The retarded products $R_{n,1}$ are multi-linear functionals on $\mathcal{P}$ with values in the space of $\mathcal{F}(\mathcal{C}_S)$ valued distributions. We may equivalently consider them as distributions on the space of $\mathcal{P}^{\otimes n+1}$ valued test functions $\mathcal{D}(\mathbb{M}^{n+1}, \mathcal{P}^{\otimes n+1}) = (\mathcal{P} \otimes \mathcal{D}(\mathbb{M}))^{\otimes n+1}$, i.e. we sometimes write $R_S(A_1 f_1 \otimes \cdots \otimes A_n f_n, Bg)$

In Sect. 4 we will define perturbative quantum fields by the principle that as much as possible of the structure of perturbative classical fields is maintained in the process of quantization. For this purpose we are going to work out main properties of the retarded products $R_{n,1}$ (52).

- The **causality** of the retarded fields,

$$B_{S+A(f)}(x) = B_S(x), \quad \text{if} \quad \text{supp}\, f \cap (x + \overline{V_-}) = \emptyset \tag{53}$$

(where $f \in \mathcal{D}(\mathbb{M}), A, B \in \mathcal{P}$) translates into the support property:

$$\text{supp}\, R_{n,1} \subset \{(x_1, ..., x_n, x) | x_l \in x + \overline{V_-}, \forall l = 1, ..., n\}. \tag{54}$$

- A deep property of the retarded products $R_{n,1}$ is the **GLZ-Relation**. In (43) we formulated it for general retarded products. For the retarded products of local fields the GLZ-Relation reads

$$\sum_{I \subset \{1,...,n\}} \{R_{|I|,1}(\otimes_{i\in I} f_i; f), R_{|I^c|,1}(\otimes_{k\in I^c} f_k; g)\} =$$
$$R_{n+1,1}(f_1 \otimes ... \otimes f_n \otimes f; g) - R_{n+1,1}(f_1 \otimes ... \otimes f_n \otimes g; f) \tag{55}$$

for $f_1, ..., f_n, f, g \in \mathcal{P} \otimes \mathcal{D}(\mathbb{M})$.

Glaser, Lehmann and Zimmermann (GLZ) [12] found this formula in the framework of non-perturbative QFT for the retarded products introduced by Lehmann, Symanzik and Zimmermann [22]. In causal perturbation theory [11] the GLZ-relation is a consequence of Bogoliubov's definition of interacting fields [2], see Proposition 2 in [4]. The important point is that the retarded products on the l.h.s. in (55) are of lower orders, $|I|, |I^c| < n + 1$.

- From their definition (52) it is evident that the **retarded products** $R_{n,1}$ **commute with partial derivatives**

$$\partial^\mu_{x_l} R_{n,1}(\ldots, A_l(x_l), \ldots) = R_{n,1}(\ldots, \partial^\mu A_l, \ldots)\ , \ l = 1, \ldots, n+1\ , \tag{56}$$

where $A_1, \ldots, A_{n+1} \in \mathcal{P}$. Note that the kernel of the linear map

$$\mathcal{P} \otimes \mathcal{D}(\mathbb{M}) \longrightarrow \mathcal{F}(\mathcal{C}) : \ A \otimes g \mapsto A(g) \tag{57}$$

is precisely the linear span of $\{\partial^\mu A \otimes g + A \otimes \partial^\mu g \mid A \in \mathcal{P}, g \in \mathcal{D}(\mathbb{M})\}$. (56) expresses the fact that the retarded products $R_{n,1}$ depend on the functionals (i.e. the images of the map (57)) only. This can be interpreted in physical terms: Lagrangians which give the same action yield the same physics. This is the motivation for Raymond Stora to require (56) for the retarded (or equivalently: time ordered) products of QFT, and he calls this the **Action Ward Identity** (AWI) [29], see Sect. 4.

- We now assume that S is at most quadratic in the fields. Then the second derivative is independent of the fields, and therefore also the Green functions. We set

$$\Delta_S(x,y) \stackrel{\mathrm{def}}{=} -\Delta_S^{ret}(x,y) + \Delta_S^{adv}(x,y) \ . \tag{58}$$

We look at the **Poisson bracket of a retarded product with a free field**. Let $A, B \in \mathcal{P}$ and $f, g, h \in \mathcal{D}(\mathbb{M})$. We are interested in

$$\{R_{n,1}(A^{\otimes n}, B)(f^{\otimes n}, g), \varphi(h)\}_S \ . \tag{59}$$

By definition of the retarded products of local fields this is equal to

$$\{R_S(A(f)^{\otimes n}, B(g)), \varphi(h)\}_S \tag{60}$$

We apply Proposition 1 to compute the Poisson bracket. From Proposition 3 it follows that R_S commutes with functional derivatives if S is a quadratic functional. Hence we obtain

$$\int dx \int dy \Big(R_S(n \frac{\delta A(f)}{\delta \varphi(x)} \otimes A(f)^{\otimes n-1}, B(g)) + R_S(A(f)^{\otimes n}, \frac{\delta B(g)}{\delta \varphi(x)}) \Big) \Delta_S(x,y) h(y) \tag{61}$$

Using the formula

$$\int dx \, \frac{\delta A(f)}{\delta \varphi(x)} k(x) = \sum_a \int dx \, \frac{\partial A}{\partial(\partial^a \varphi)}(x) f(x) \partial^a k(x) \tag{62}$$

for the functional derivative of a local functional, we finally arrive at the formula

$$\{R_{n,1}(f_1, \ldots, f_{n+1}), \varphi(h)\}_S = \sum_{k=1}^{n+1} \sum_a (R_{n,1}(f_1, \ldots, \frac{\partial f_k}{\partial(\partial^a \varphi)} \partial^a \Delta_S h, \ldots, f_{n+1}) \tag{63}$$

where $f_1, \ldots, f_{n+1} \in \mathcal{D}(\mathbb{M}, \mathcal{P})$, $h \in \mathcal{D}(\mathbb{M})$ and where Δ_S was considered as an integral operator acting on h.

The requirement that this relation holds also in perturbative QFT plays an important role in the inductive construction of perturbative quantum fields (see Sect. 4).

- **Symmetries:** there are natural automorphic actions α_L and β_L of the Poincare group ($L \in \mathcal{P}_+^\uparrow$) on $(\mathcal{P}\otimes\mathcal{D}(\mathbb{M}))^{\otimes n+1}$ and on $\mathcal{F}(\mathcal{C})$, respectively. The retarded products are **Poincare covariant**: $R_{n,1}\circ\alpha_L = \beta_L \circ R_{n,1}$, provided S is invariant. A universal formulation of all symmetries which can be derived from the field equations in classical field theory is given by the **MWI**, see Sect. 3.

- The **factorization** $(AB)^{\rm ret}_{\mathcal{L}_{\rm int}}(x) = A^{\rm ret}_{\mathcal{L}_{\rm int}}(x)B^{\rm ret}_{\mathcal{L}_{\rm int}}(x)$ and the Leibniz rule for the retarded product of two factors yield, in general, ill-defined expressions in QFT. It is the Master Ward Identity which allows to implement the consequences of the factorization property of classical field theory into quantum field theory.

2.3 Elimination of derivative couplings

Interaction Lagrangians containing derivatives of fields usually cause complications in the canonical formalism. They also change relations between different fields, as may be seen by the non-linear term in the formula which expresses the field strength $F^{\mu\nu}$ of a Yang-Mills theory in terms of the vector potential A^μ. A convenient way to deal with these complications is the introduction of auxiliary fields.

As an example we consider the Lagrangian

$$\mathcal{L}(\varphi,\partial^\mu\varphi) = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{m^2}{2}\varphi^2 + \mathcal{L}_{\rm int}(\varphi,\partial^\mu\varphi) \tag{64}$$

of a real scalar field φ with the Euler-Lagrange equation

$$(\Box + m^2)\varphi = \frac{\partial\mathcal{L}_{\rm int}(\varphi,\partial^\mu\varphi)}{\partial\varphi} - \partial^\nu\frac{\partial\mathcal{L}_{\rm int}(\varphi,\partial^\mu\varphi)}{\partial\,\partial^\nu\varphi}. \tag{65}$$

To eliminate $\partial^\mu\varphi$ in $\mathcal{L}_{\rm int}$ we introduce a vector field φ^μ and a Lagrangian

$$\mathcal{L}(\varphi,\partial^\mu\varphi,\varphi^\mu) = -\frac{1}{2}\varphi^\mu\varphi_\mu + \varphi_\mu\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 + \mathcal{L}_{\rm int}(\varphi,\varphi^\mu). \tag{66}$$

with the Euler-Lagrange equations:

$$0 = -\varphi_\mu + \partial_\mu\varphi + \frac{\partial\mathcal{L}_{\rm int}(\varphi,\varphi^\mu)}{\partial\varphi^\mu}, \tag{67}$$

$$\partial_\mu\varphi^\mu = -m^2\varphi + \frac{\partial\mathcal{L}_{\rm int}(\varphi,\varphi^\mu)}{\partial\varphi}, \tag{68}$$

which are equivalent to (65). We see that precisely in the case when the interaction Lagrangian depends on $\partial^\mu\varphi$ the interacting field φ^μ differs from $\partial^\mu\varphi$.

Example: By explicit calculation we are going to show that the retarded products $R_{\mathcal{L}_0}(\varphi^\nu(y),\partial^\mu\varphi(x))$ and $R_{\mathcal{L}_0}(\varphi^\nu(y),\varphi^\mu(x))$ are different, although $\partial^\mu\varphi_{\mathcal{L}_0} = \varphi^\mu_{\mathcal{L}_0}$ (where $\mathcal{L}_0$ is the free part of the Lagrangian (66)), so we see again that the entries of the retarded products must not be replaced by their restriction to the space of solutions. The fastest way to compute these retarded products is to use Proposition 1(a), as in (23). However, we find it more instructive to go back to Peierls' definition of retarded products (16): by definition $r_{\mathcal{L}_0+\lambda\delta_y\varphi^\nu,\mathcal{L}_0}(f_0,h_0)$ is the solution (f,h) of (67)-(68) with $\mathcal{L}_{\rm int}(z)=\lambda\delta(z-y)\varphi^\nu(z)$ which agrees in the distant past with (f_0,h_0). We obtain

$$\begin{gathered}\varphi(x)\circ r_{\mathcal{L}_0+\lambda\delta_y\varphi^\nu,\mathcal{L}_0}(f_0,h_0)=f(x)=f_0(x)-\lambda\partial^\nu\Delta^{\rm ret}(x-y),\\ \varphi^\mu(x)\circ r_{\mathcal{L}_0+\lambda\delta_y\varphi^\nu,\mathcal{L}_0}(f_0,h_0)=h^\mu(x)\\ =h_0^\mu(x)-\lambda(\partial^\mu\partial^\nu\Delta^{\rm ret}(x-y)-g^{\mu\nu}\delta(x-y)).\end{gathered}\tag{69}$$

Hence,

$$R_{\mathcal{L}_0}(\varphi^\nu(y),\partial^\mu\varphi(x))=\partial^\mu_x R_{\mathcal{L}_0}(\varphi^\nu(y),\varphi(x))=-\partial^\nu\partial^\mu\Delta^{\rm ret}(x-y),\tag{70}$$

but

$$R_{\mathcal{L}_0}(\varphi^\nu(y),\varphi^\mu(x))=-(\partial^\nu\partial^\mu\Delta^{\rm ret}(x-y)-g^{\nu\mu}\delta(x-y)).\tag{71}$$

3 The Master Ward Identity

3.1 Generalized Schwinger-Dyson Equation

It is an immediate consequence of the field equations $\left(\frac{\delta S}{\delta\varphi}\right)_S=0$ for a given local action S that all functionals of the form

$$\left(A\frac{\delta S}{\delta\varphi}\right)(h)\tag{72}$$

(by which we mean the point-wise product of an arbitrary classical field $A\in\mathcal{P}$ with $\frac{\delta S}{\delta\varphi}$ smeared out with the test function h) vanish on the space of solutions $\mathcal{C}_S$.

If we set $S = S_0 + \lambda S_1$, and differentiate with respect to λ at $\lambda = 0$ we obtain the identity

$$R_{S_0}\Big(S_1, A\frac{\delta S_0}{\delta\varphi}(h)\Big) + \Big(A\frac{\delta S_1}{\delta\varphi}(h)\Big)_{S_0} = 0. \tag{73}$$

For $A \equiv 1$ this equation looks similar to the Schwinger-Dyson equation

$$\frac{i}{\hbar}\langle TS_1\frac{\delta S_0}{\delta\varphi}\rangle + \langle\frac{\delta S_1}{\delta\varphi}\rangle = 0, \tag{74}$$

which holds for the vacuum expectation values of time ordered products for a quantum field theory with action S_0 (see, e.g. [14]). (Note that the factor $\frac{i}{\hbar}$ is absorbed in (73) in the retarded part of the Poisson bracket.) For this reason we call (73) the **retarded Schwinger-Dyson Equation** and the vanishing of (72) the **generalized Schwinger-Dyson Equation** (GSDE). Note that the retarded Schwinger-Dyson Equation has the same form in classical physics as in quantum physics.

In the retarded Schwinger-Dyson equation we may permute the two entries in the retarded product. Namely, the difference is just the Poisson bracket which vanishes if one of the entries vanishes on the space of solutions.

For the retarded products of local fields we obtain the perturbative version of the generalized Schwinger-Dyson Equation,

$$R_{n,1}(f_1,\ldots,f_n,h\frac{\delta S_0}{\delta\varphi}) + \sum_{l=1}^{n} R_{n-1,1}(f_1,\ldots,f_{l-1},f_{l+1},\ldots,f_n,h\frac{\delta f_l}{\delta\varphi}) = 0 \tag{75}$$

with $f_i, h \in \mathcal{D}(\mathbb{M},\mathcal{P}), i = 1,\ldots,n$, and where the functional derivative of $f \in \mathcal{D}(\mathbb{M},\mathcal{P})$ is defined by

$$\frac{\delta f}{\delta\varphi}(x) = \frac{\delta\int dy\, f(y)}{\delta\varphi(x)}\,, \tag{76}$$

i.e.

$$\frac{\delta f}{\delta\varphi} = \sum_a (-1)^{|a|}\partial^a\frac{\partial f}{\partial(\partial^a\varphi)}\ . \tag{77}$$

Proceeding by induction on n and using the GLZ-Relation (55), we obtain an equivalent formula for the case that the term $h\frac{\delta S_0}{\delta\varphi}$ is one of the first n entries of $R_{n,1}$, namely

$$R_{n,1}(f_1,\ldots,f_{n-1},h\frac{\delta S_0}{\delta\varphi},f_n)+\\ \sum_{l=1}^{n} R_{n-1,1}\big(f_1,\ldots,f_{l-1},h\frac{\delta f_l}{\delta\varphi},f_{l+1},\ldots,f_n\big)=0. \quad (78)$$

The equations (75) and (78) remain meaningful in perturbative QFT. Also there they are equivalent since the GLZ-Relation still holds. We require either of them as a renormalization condition (see Sect. 4). Equivalently, an analogous identity may be postulated for the time ordered products (where the two versions above coincide in view of the symmetry of time ordered products). It is a generalization of the condition (N4) in [4].[7] But there, following the tradition in causal perturbation theory, we considered the entries of time ordered or retarded products as Wick polynomials of the free field. Therefore, time ordering and partial derivatives did not commute, and the formulation of identities involving derivatives of fields contained many free parameters. A consistent choice of these parameters was made possible by the MWI proposed in [6].

The QFT version of (75) or (78) seems to correspond to the 'broomstick identity' of Lam given in Fig. 8 of [18]. We think that Lam is unable to write down this identity as an *equation* because the arguments of his time-ordered products are on shell fields; and compared to [6] (which uses also an on shell formalism, cf. Sect. 4) he is not equipped with the 'external derivative'.

We will see that the MWI is equivalent to the generalized Schwinger-Dyson Equation (75). Therefore, the *MWI can be interpreted as a quantum version of all identities for local fields which follow in classical field theory from the field equations.*

3.2 Definition of a map σ from free fields to unrestricted fields

To keep the formulas simple, we consider the case of one real scalar field φ. The procedure, however, applies to a general model. Let $\mathcal{J}$ denote the ideal

[7] A first generalization (in the framework of causal perturbation theory) of the renormalization condition (N4) was given in an unpublished preversion of [25].

in the algebra $\mathcal{P}$ of polynomials of fields which is generated from the free field equation,

$$\mathcal{J} = \{\sum_a A_a \partial^a (\Box + m^2)\varphi,\ A_a \in \mathcal{P}, a \in \mathbb{N}_0^d\}\ , \tag{79}$$

let $\mathcal{P}_0 = \mathcal{P}/\mathcal{J}$ be the algebra of free fields and let $\pi : \mathcal{P} \to \mathcal{P}_0$ be the canonical surjection. Since $\mathcal{J}$ is translation invariant, we may define derivatives with respect to space-time coordinates in $\mathcal{P}_0$ by $\partial^\mu \pi(B) \stackrel{\text{def}}{=} \pi(\partial^\mu B)$, and in this sense the free field equation holds true for $\pi\varphi$.

The wanted map σ is a section $\sigma : \mathcal{P}_0 \to \mathcal{P}$. In contrast to the surjection π, the section σ is not canonically given. We restrict its choice by the following requirements:

(i) $\pi \circ \sigma = \text{id}$, i.e. σ is a section.

(ii) σ is an algebra homomorphism.[8]

(iii) The Lorentz transformations commute with $\sigma\pi$.

(iv) $\sigma\pi(\mathcal{P}_1) \subset \mathcal{P}_1$, where $\mathcal{P}_1$ is the subspace of fields which are linear in $\partial^a\varphi$.

(v) $\sigma\pi$ does not increase the mass dimension of the fields, i.e. $\sigma\pi(B)$ is a sum of terms with mass dimension $\leq \dim(B)$. In particular we find $\sigma\pi(\varphi) = \varphi$.

(vi) $\mathcal{P}$ is generated by fields in the image of σ and their derivatives. In the present case (one real scalar field), this condition is automatically satisfied.

By (i) $\sigma\pi : \mathcal{P} \to \mathcal{P}$ is a projection: $\sigma\pi\sigma\pi = \sigma\pi$. The linearity of σ and condition (i) imply $\ker \sigma\pi = \ker \pi = J$, and hence

$$\sigma\pi\Box\varphi = -m^2 \sigma\pi\varphi\ . \tag{80}$$

We are now looking for the most general explicit formula for $\sigma\pi$ which satisfies the above requirements. Due to (ii) it suffices to determine $\sigma\pi(\partial^a\varphi)$. By definition of π and σ it must hold

$$\sigma\pi(A) - A \in J \qquad \forall A \in \mathcal{P} \tag{81}$$

[8] In case of complex fields we additionally require $\sigma(A^*) = \sigma(A)^*$.

and hence

$$\sigma\pi\partial^a\varphi = \partial^a\varphi + \sum_b c^a_b \partial^b(\Box + m^2)\varphi \tag{82}$$

with constants $c^a_b \in \mathbb{R}$.

The determination of an admissible section σ satisfying conditions (i) to (v) is equivalent to the determination of a complementary subspace $K = \sigma\pi(\mathcal{P}_1)$ of $\mathcal{J}_1 \stackrel{\text{def}}{=} \mathcal{J} \cap \mathcal{P}_1$ which is Lorentz invariant and satisfies the condition that already the subspaces with mass dimension $\leq n$ are complementary,

$$K^{(n)} + \mathcal{J}_1^{(n)} = \mathcal{P}_1^{(n)} \; . \tag{83}$$

Since the finite dimensional representations of the Lorentz group are completely reducible, this is always possible. Namely, for the lowest mass dimension $n_0 = (d-2)/2$ the subspace generated by the field equation is zero, thus $K^{(n_0)} = \mathcal{P}_1^{(n_0)} = \mathbb{R}\varphi$. From this the existence of $K^{(n+1)}$ may be proved by induction on n. One just has to choose a Lorentz invariant complementary subspace $L^{(n+1)}$ of $K^{(n)} + \mathcal{J}_1^{(n+1)}$ in $\mathcal{P}_1^{(n+1)}$ and to set $K^{(n+1)} = K^{(n)} + L^{(n+1)}$.

The arbitrariness in the choice of $L^{(n)}$ depends on the multiplicity in which the irreducible subrepresentations of the Lorentz group occur in the respective subspaces. In the present case it turns out that σ is unique (see the second part of Appendix A).

In case one introduces the auxiliary field φ^μ, the choice of σ involves free parameters. A special choice for σ is given in the first part of Appendix A. For the lowest derivatives we obtain the following general solution of the requirements (i)-(vi):

$$\sigma\pi(\varphi) = \varphi, \tag{84}$$

$$\sigma\pi(\partial^\mu\varphi) = \sigma\pi(\varphi^\mu) = \gamma\varphi^\mu + (1-\gamma)\partial^\mu\varphi, \quad \gamma \in \mathbb{R}\setminus\{0\}, \tag{85}$$

$$\sigma\pi(\partial^\mu\partial^\nu\varphi) = \sigma\pi(\partial^\nu\varphi^\mu) = (1+2\alpha)\partial^\mu\partial^\nu\varphi$$

$$-\alpha(\partial^\mu\varphi^\nu + \partial^\nu\varphi^\mu) - \frac{1+2\alpha}{d} g^{\mu\nu}\Box\varphi - \frac{1}{d} g^{\mu\nu} m^2\varphi + \frac{2\alpha}{d} g^{\mu\nu}\partial^\sigma\varphi_\sigma, \tag{86}$$

where γ and $\alpha \in \mathbb{R}$ are free parameters. The condition $\gamma \neq 0$ is necessary and sufficient for (vi) provided (i)-(v) are satisfied. A preferred choice is $\gamma = 1$.

3.3 The Master Ward identity

Let A be a functional of the field which vanishes according to the field equation derived from the action S_0, i.e. $A_{S_0} = 0$. A is of the form (cf. the remark in footnote 3)

$$A = \int dx\, G(x) \frac{\delta S_0}{\delta\varphi(x)}\ , \tag{87}$$

with $G(x) \in \mathcal{F}(\mathcal{C})$. (This formula states that A is an arbitrary element of the ideal $\mathcal{J}_{S_0}$ generated by the field equation belonging to S_0.) We may introduce the derivation

$$\delta_A = \int dx\, G(x) \frac{\delta}{\delta\varphi(x)} \tag{88}$$

on $\mathcal{F}(\mathcal{C})$. The GSDE imply

$$(\mathbf{MWI}) \qquad R_{S_0}(e_\otimes^{S_1}, A) = -R_{S_0}(e_\otimes^{S_1}, \delta_A(S_1)) \quad \forall S_1 \in \mathcal{F}(\mathcal{C}) \tag{89}$$

and, by using the GLZ equation

$$R_{S_0}(e_\otimes^{S_1} \otimes A, B) = -R_{S_0}(e_\otimes^{S_1} \otimes \delta_A(S_1), B) - R_{S_0}(e_\otimes^{S_1}, \delta_A(B)) \quad \forall S_1, B \in \mathcal{F}(\mathcal{C})\ . \tag{90}$$

This equation (in a different form, see (109) below) was proposed by Boas and Dütsch [6] under the name **Master Ward Identity** (MWI) as a universal renormalization condition in perturbative QFT. In the present formulation it is evidently equivalent to the GSDE. Note that up to now (in contrast to [6]) the formulation of the MWI makes sense also in the case that S_0 is not a quadratic functional of the field. Similarly to the GSDE, the MWI holds also non-perturbatively:

$$A_{S_0+S_1} = -\big(\delta_A(S_1)\big)_{S_0+S_1}, \quad A \in \mathcal{J}_{S_0},\ S_1 \in \mathcal{F}(\mathcal{C}). \tag{91}$$

Example: As a typical application let us look at the free complex scalar field with the conserved current

$$j_\mu = \frac{1}{i}(\varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^*)\ . \tag{92}$$

Let $A = \langle \partial j, g\rangle \equiv \int dx\, \partial^\mu j_\mu(x) g(x)$ with $g \in \mathcal{D}(\mathbb{M})$. We have

$$A = \frac{1}{i}\left(\langle g\varphi^*, \frac{\delta S_0}{\delta\varphi^*}\rangle - \langle g\varphi, \frac{\delta S_0}{\delta\varphi}\rangle\right) \tag{93}$$

and hence

$$\delta_A = \frac{1}{i}\left(\langle g\varphi^*, \frac{\delta}{\delta\varphi^*}\rangle - \langle g\varphi, \frac{\delta}{\delta\varphi}\rangle\right)\ . \tag{94}$$

If $g \equiv 1$ on the localization region of $F \in \mathcal{F}(\mathcal{C})$, $\delta_A F$ is the infinitesimal gauge transformation of F, and inserting A into the MWI yields the well known Ward identity of the model.

A problem with the MWI in the form presented above is the non-uniqueness of the derivation δ_A for a given A, e.g. for

$$A = \int dx\, h(x)((\Box + m^2)\varphi^*(x))(\Box + m^2)\varphi(x) \tag{95}$$

in case of the complex scalar field. We therefore turn now to the free field case and use the techniques and conventions developed in the preceding section.

We will give a unique prescription to write any $A = \int dx\, h(x) B(x)$ with $B \in \mathcal{J} \subset \mathcal{P}$ and $h \in \mathcal{D}(\mathbb{M})$ (i.e. $A \in \mathcal{J}_{S_0}$) in the form $A = \int dx\, h(x) \sum_j B_j(x) b_j(x)$ with $b_j \in \mathcal{P}_1 \cap \mathcal{J}$. Then we may set

$$\delta_A = \int dx\, h(x) \sum_j B_j(x) \delta_{b_j(x)} \tag{96}$$

with

$$\delta_{b_j(x)}(F) = -R_{S_0}(b_j(x), F)\ , \quad F \in \mathcal{F}(\mathcal{C})\ , \tag{97}$$

where we used (30) and the fact that for terms which are linear in the field the first term on the right hand side vanishes, such that (31) holds everywhere, not only on the space of solutions.

To give the mentioned prescription let $B \in \mathcal{J}$. Then $B = p(B)$ where $p = 1 - \sigma\pi$ is a projection from $\mathcal{P}$ onto $\mathcal{J}$. Since $\sigma\pi$ is an algebraic homomorphism we find

$$p(B_1 B_2) = B_1 p(B_2) + p(B_1)\sigma\pi(B_2)\ . \tag{98}$$

Hence, we may write every $B \in \mathcal{J}$ in the form

$$B = \sum_{\chi \in \mathcal{G}} B_\chi p(\chi) \tag{99}$$

where we introduced the set $\mathcal{G}$ of generators of the field algebra $\mathcal{P}$,

$$\mathcal{G} \stackrel{\mathrm{def}}{=} \{\partial^a \varphi \mid a \in \mathbb{N}_0^d\} \tag{100}$$

which is a vector space basis of $\mathcal{P}_1$. The coefficients B_χ can be found in the following way: Any $B \in \mathcal{P}$ is a polynomial P in finitely many different elements $\chi_1, \ldots, \chi_n \in \mathcal{G}$, $B = P(\vec{\chi}), \vec{\chi} = (\chi_1, \ldots, \chi_n)$. Since $B \in \mathcal{J}$ we have

$$B = B - \sigma\pi(B) = P(\vec{\chi}) - P(\sigma\pi(\vec{\chi})) \tag{101}$$

$$= \int_0^1 d\lambda \frac{d}{d\lambda} P(\lambda\vec{\chi} + (1-\lambda)\sigma\pi(\vec{\chi})) \tag{102}$$

$$= \sum_{i=1}^{n} P_i(\vec{\chi}, \sigma\pi(\vec{\chi}))p(\chi_i) \tag{103}$$

with

$$P_i(\vec{\chi}, \sigma\pi(\vec{\chi})) = \int_0^1 d\lambda \partial_i P(\lambda\vec{\chi} + (1-\lambda)\sigma\pi(\vec{\chi})). \tag{104}$$

Hence we may set $B_{\chi_i} = P_i(\vec{\chi}, \sigma\pi(\vec{\chi}))$, $i = 1, \ldots, n$ and $B_\chi = 0$ if $\chi \notin \{\chi_1, \ldots \chi_n\}$.

Example: Let φ be a real scalar field. For $\chi = \varphi$, $\partial^\mu\varphi$ the expression $p(\chi)$ vanishes. But for $\chi = \partial^\mu\partial^\nu\varphi$ we obtain $p(\chi) = \frac{g^{\mu\nu}}{d}(\Box + m^2)\varphi = \frac{g^{\mu\nu}}{d}\frac{\delta S_0}{\delta\varphi}$ and hence

$$\delta_{p(\chi)(x)}(F) = \frac{g^{\mu\nu}}{d}\frac{\delta F}{\delta\varphi} \ . \tag{105}$$

In many applications one wants to compare derivatives in the free theory with those in the interacting theory. One is therefore interested in expressions of the form

$$A = \int dx\, h(x)(\partial_\mu\sigma\pi(B)(x) - \sigma\pi(\partial_\mu B)(x)) \tag{106}$$

with $B \in \mathcal{P}$ and $h \in \mathcal{D}(\mathbb{M})$. To get a more general identity we even admit $h \in \mathcal{D}(\mathbb{M}, \mathcal{P})$. Clearly, $[\partial_\mu, \sigma\pi](B) \in \mathcal{J}$, hence $A \in \mathcal{J}_{S_0}$. Using the Leibniz' rule $\partial^\mu B = \sum_{\chi\in\mathcal{G}} \frac{\partial B}{\partial\chi}\partial_\mu\chi$ we find

$$[\partial_\mu, \sigma\pi](B) = \sum_{\chi\in\mathcal{G}} \sigma\pi(\frac{\partial B}{\partial\chi})[\partial_\mu, \sigma\pi](\chi) \tag{107}$$

and get the formula

$$\delta_A = \int dx\, h(x) \sum_{\chi\in\mathcal{G}} \sigma\pi(\frac{\partial B}{\partial\chi})(x)\delta_{[\partial_\mu,\sigma\pi](\chi)(x)} \ . \tag{108}$$

In terms of the retarded fields (13) we end up with

$$(\mathbf{MWI}') \qquad \big(h([\partial^\mu,\sigma\pi]B)\big)_{\mathcal{L}_{\rm int}} = \sum_{\chi,\psi\in\mathcal{G}} \Big(h\,\sigma\pi\Big(\frac{\partial B}{\partial\chi}\Big)\,\delta^\mu_{\chi,\psi}\frac{\partial\mathcal{L}_{\rm int}}{\partial\psi}\Big)_{\mathcal{L}_{\rm int}}, \quad (109)$$

where the differential operator $\delta^\mu_{\chi,\psi}$ is defined by

$$\delta^\mu_{\chi,\psi}f(x) = -\int dy\,R_{S_0}\big([\partial^\mu,\sigma\pi](\chi)(x),\psi(y)\big)f(y)\ ,\ f\in\mathcal{D}(\mathbb{M},\mathcal{P})\ . \quad (110)$$

Note that $R_{S_0}\big([\partial^\mu,\sigma\pi](\chi)(x),\psi(y)\big)$ is a linear combination of partial derivatives of $\delta(x-y)$.

Example: Let $\chi=\partial^\nu\varphi$. Then $[\partial^\mu,\sigma\pi](\chi)=\frac{g^{\mu\nu}}{d}(\square+m^2)\varphi$. Hence $\delta_{[\partial^\mu,\sigma\pi](\chi)}=\frac{g^{\mu\nu}}{d}\frac{\delta}{\delta\varphi}$, therefore one obtains for the difference between derivatives of free or interacting fields

$$\partial^\mu(\partial^\nu\varphi)_{S_0+S_1}-(\sigma\pi\partial^\mu\partial^\nu\varphi))_{S_0+S_1} = \frac{g^{\mu\nu}}{d}\left(\frac{\delta S_1}{\delta\varphi}\right)_{S_0+S_1}\ . \quad (111)$$

We think that the non-vanishing of $[\partial^\mu,\sigma\pi](\partial^\nu\varphi)$ compared to $[\partial^\mu,\sigma\pi](\varphi)=0$ explains why the formalism of Lam becomes inconsistent for vertices containing higher than first derivatives of the basic fields (cf. Sect. V of [17]).

We derived (109) as a consequence of the MWI (91). It is even equivalent if $\mathcal{J}\cap\mathcal{P}_1$ is spanned by fields of the form $[\partial^\mu,\sigma\pi](\psi)$ and their derivatives, with $\psi\in\mathcal{P}_1$, since then the l.h.s. of the MWI (89) can be written as a linear combination of fields of the form of the l.h.s. of (109). In the case of the free scalar field this condition is clearly fulfilled, since $[\partial^\mu,\sigma\pi](\partial_\mu\varphi)=(\square+m^2)\varphi$. In the enlarged model with the auxiliary field φ_μ we find $[\partial^\mu,\sigma\pi](\varphi)=\gamma(\partial^\mu\varphi-\varphi^\mu)$ and $[\partial^\mu,\sigma\pi](\varphi_\mu)=(\partial^\mu\varphi_\mu+m^2\varphi)+(\gamma-1)\partial^\mu(\varphi_\mu-\partial_\mu\varphi)$, hence here it follows from $\gamma\neq 0$. Also in general it follows from condition (vi) on σ in Sect. 3.2. Namely, let $\chi\in\mathcal{J}\cap\mathcal{P}_1$. According to (vi) and (iv), χ is of the form

$$\chi=\sum_{k,a}\partial^a\sigma\pi(\psi_{k,a})\quad \psi_{k,a}\in\mathcal{P}_1\ .$$

Hence,

$$\chi=\chi-\sigma(\pi\chi)=\sum_{k,a}[\partial^a,\sigma\pi]\sigma\pi(\psi_{k,a})\ .$$

The result now follows from the derivation property of the commutator

$$[\partial_{\mu_1}\cdots\partial_{\mu_n},\sigma\pi]=\sum_k \partial_{\mu_1}\cdots\partial_{\mu_{k-1}}[\partial_{\mu_k},\sigma\pi]\partial_{\mu_{k+1}}\cdots\partial_{\mu_n}\ .$$

We point out that the MWI (in the original form (89) or (90) as well as in the second form (109)) is well defined in perturbative QFT, too. This is a main ingredient of the next Section.

4 Quantization: defining properties of perturbative quantum fields

The structure of perturbative classical field theory which was analyzed in this paper can to a large degree be preserved during quantization. The main change is the replacement of the commutative product of functionals $F\in\mathcal{F}(\mathcal{C})$ by a $\hbar$-dependent non-commutative associative product, and by the replacement of the Poisson bracket by $\frac{1}{i\hbar}$ times the commutator. The definition of the product can be read off from Wick's Theorem,

$$F*_\hbar G=\sum_{n=0}^\infty\frac{\hbar^n}{n!}\int d(x,y)\frac{\delta^n F}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}\prod_{i=1}^n\Delta_+(x_i-y_i)\frac{\delta^n G}{\delta\varphi(y_1)\cdots\delta\varphi(y_n)} \tag{112}$$

where Δ_+ denotes the positive frequency fundamental solution of the Klein Gordon equation. This abstract algebra (which we still denote by $\mathcal{F}(\mathcal{C})$) can be represented on Fock space by Wick polynomials

$$\pi(F)=\sum_n\frac{1}{n!}\int d^nx\,\frac{\delta^n F}{\delta\varphi(x_1)...\delta\varphi(x_n)}|_{\varphi=0}:\varphi(x_1)...\varphi(x_n):, \tag{113}$$

the kernel of this representation being the set of functionals F vanishing on $\mathcal{C}_{\mathcal{L}_0}$, cf. [5].

A direct construction of solutions of the field equations in the case of local interactions is, in general, not possible because of ultraviolet divergences. One may, however, start from the ansatz

$$B_{\mathcal{L}_{\rm int}}(f)=\sum_{n=0}^\infty\frac{1}{n!}R_{n,1}(\mathcal{L}_{\rm int}^{\otimes n},Bf) \tag{114}$$

in analogy to (14) and try to determine the retarded products $R_{n,1}$ as polynomials in $\hbar$ such that they satisfy the following properties: They are $(n+1)$-linear continuous functionals on $\mathcal{D}(\mathbb{M},\mathcal{P})$ with values in $\mathcal{F}(\mathcal{C})$ which are symmetric in the first n entries, have retarded support (54) and satisfy the GLZ-Equation (55). Moreover they have to fulfil the unitarity condition

$$R_{n,1}(f_1 \otimes ... \otimes f_n, f)^* = R_{n,1}(f_1^* \otimes ... \otimes f_n^*, f^*). \tag{115}$$

It turns out, that already by these properties, the retarded products $R_{n,1}$ are uniquely determined outside of the total diagonal $x_1 = \ldots = x_{n+1}$ in terms of the lower order retarded products where the lowest order is defined by $R_{0,1}(Bf) = B(f)|_{\mathcal{C}_{\mathcal{L}_0}}$. *Renormalization then means the extension of the retarded products to the diagonal.* This is a variant of the Bogoliubov-Epstein-Glaser renormalization method and, in the adiabatic limit $g \equiv 1$, it has been worked out by Steinmann [27]. A modernized and local version of the procedure will be presented in [7].

The main work which remains to be done is the so-called finite renormalization, i.e., the analysis of the ambiguities in the extension process.

In a first step, the condition (63) (condition (N3) of [4]) can be used to reduce the extension problem to a problem for numerical distributions. By requiring translation invariance these numerical distributions depend only on the differences of coordinates, thus one has to study the mathematical problem of extending a distribution which is defined outside of the origin to an everywhere defined distribution. The possible extensions can be classified in terms of Steinmann's [27] scaling degree, and it is a natural requirement that the scaling degree should not increase during the extension process. In addition one can show that the extension can always be done such that the retarded products are Lorentz covariant.

The steps described above can always be performed and leave, for every numerical distribution, a finite set of parameters undefined. The proposal is now to add two further conditions.

One is the Action Ward Identity (56) proposed by Stora [29]. At present, no example is known where this identity cannot be fulfilled, so we assume that it holds. We require then the Master Ward Identity in the form (89) or (90), or, equivalently, the generalized Schwinger-Dyson equation in the form (75) or (78). Here anomalies may occur, and one has to check in a given model whether these identities can be satisfied. Fortunately, for a typical application, one needs these identities only for special cases which may be characterized by the polynomial degree of the fields and the number of derivatives which are involved.

Without the validity of the AWI, one had to require the MWI or, equivalently, the GSDE, also for all derivatives of the fields which appear in the formulas, i.e. for $G\partial^a \frac{\delta S_0}{\delta\varphi}$, $\forall a \in \mathbb{N}_0^d$, as well as for derivatives of the retarded products.

Remark: The Quantum Noether Condition (QNC) [15] (already mentioned in the introduction) is an alternative, less general normalization condition for causal perturbation theory. It is not compatible with the normalization condition (N3) (see the last Remark in Sect. 4.5.2 of [6], published version) which makes its application somewhat cumbersome.

The formalism given here is not completely equivalent to the one of [6]. The algebras of *symbols* $\mathcal{P}$ and $\mathcal{P}_0$ given in [6] may be identified with our algebras $\mathcal{P}$ and $\mathcal{P}_0$ of *classical fields*. The main difference is the absence of the Action Ward Identity, thus derivatives could not freely be shifted from fields to test functions. Therefore, in [6] an 'external derivative' $\tilde{\partial}^\mu$ on $\mathcal{P}_0$ was introduced which generates new symbols $\tilde{\partial}^a A$ ($A \in \mathcal{P}_0$, $a \in \mathbb{N}_0^d$). The argument of the retarded product of [6] (we denote it here by $\tilde{R}_{n,1}$) is an element of

$$(\tilde{\mathcal{P}}_0 \otimes \mathcal{D}(\mathbb{M}))^{\otimes(n+1)} \quad \text{where} \quad \tilde{\mathcal{P}}_0 \stackrel{\text{def}}{=} \bigvee \{\tilde{\partial}^a A \,|\, A \in \mathcal{P}_0, a \in \mathbb{N}_0^4\} \tag{116}$$

(for details see [6]). The retarded products $\tilde{R}_{n,1}$ of symbols with external derivative(s) can be defined in terms of retarded products without external derivative (normalization condition $(\tilde{\mathbf{N}})$). The MWI is then expressed by a further normalization condition $(\mathbf{N})$, combined with $(\tilde{\mathbf{N}})$. In particular $(\tilde{\mathbf{N}})$ and $(\mathbf{N})$ imply

$$\tilde{R}_{n,1}(W_1 g_1, \ldots, (\tilde{\partial}^\nu W_l) g_l + W_l \partial^\nu g_l, \ldots, W_{n+1} g_{n+1}) = 0 \;, W_1, \ldots W_{n+1} \in \mathcal{P}_0 \;. \tag{117}$$

To clarify the relation of the two formalisms we extend σ to a map $\tilde{\sigma}$: $\tilde{\mathcal{P}}_0 \to \mathcal{P}$ by setting

$$\tilde{\sigma}(\tilde{\partial}^a A) \stackrel{\text{def}}{=} \partial^a \sigma(A), \quad A \in \mathcal{P}_0, \tag{118}$$

and requiring that $\tilde{\sigma}$ is an algebra $*$-homomorphism, similarly to [6]. The defining property (vi) of σ means that $\tilde{\sigma}$ is surjective. But $\tilde{\sigma}$ is not injective, even if the auxiliary field φ^μ is introduced. So, from the retarded product

$R_{n,1}$ of this paper, we can construct a retarded product $\tilde{R}_{n,1}$ in the sense of [6], by defining

$$\tilde{R}_{n,1}(W_1 g_1, \ldots, W_{n+1} g_{n+1}) \stackrel{\mathrm{def}}{=} R_{n,1}(\tilde{\sigma}(W_1) g_1, \ldots, \tilde{\sigma}(W_{n+1}) g_{n+1}). \qquad (119)$$

But, in general, it might happen that $\tilde{R}_{n,1}$ does not vanish if one entry is in $(\ker \tilde{\sigma})$, and then it would be impossible to construct $R_{n,1}$ from $\tilde{R}_{n,1}$. If the $R_{l,1}$, $l \leq n$, satisfy all defining properties given here (including the AWI and the MWI), then the corresponding $\tilde{R}_{l,1}$, $l \leq n$, (119) fulfill the requirements on a retarded product given in [6], in particular $(\tilde{\mathbf{N}})$ and $(\mathbf{N})$.

We see, that the normalization conditions on $\tilde{R}_{n,1}$ are weaker than the normalization conditions on $R_{n,1}$: (117) can always be fulfilled by definition (see [6]), even for $W_1, ..., W_{n+1} \in \tilde{\mathcal{P}}_0$. On the other side it is still unclear whether one can always satisfy the 'Action Ward Identity'. It seems, however, *that the formalism given here is the natural one when departing from classical field theory.*

Remark: In the formalism of [6] and in [8] the Feynman (or retarded) propagators of perturbative QFT contain undetermined parameters, if there are at least two derivatives present (in $d = 4$ dimensions). On the classical side (retarded) fields and their perturbative expansion are unique. The non-uniqueness is located in the choice of the map σ: *the free parameters in σ can be identified with the free parameters in the Feynman propagators of QFT.* This is obvious from

$$\Delta^{\mathrm{ret}}_{\varphi_0,\chi_0}(x-y) \stackrel{\mathrm{def}}{=} \tilde{R}_{1,1}(\chi_0(y); \varphi_0(x)) = R_{1,1}(\tilde{\sigma}(\chi_0)(y), \tilde{\sigma}(\varphi_0)(x)). \qquad (120)$$

In the non-enlarged formalism (without φ^μ), in which σ is unique, a particular choice of the parameters in the Feynman propagators of [6] is done.

5 Application to BRS-Symmetry

5.1 Motivation

The canonical formalism as developed in this paper cannot directly be applied to gauge theories because there the Cauchy problem is ill posed due to the existence of time dependent gauge transformations. As usual, one may add a gauge fixing term as well as a coupling to ghost and antighost fields to the

Lagrangian such that the Cauchy problem becomes well posed. The algebra of observables is then obtained as the cohomology of the BRS transformation s [1] which is a graded derivation which is implemented by the BRS charge Q. In QFT one finally constructs the space of physical states as the cohomology of Q (see e.g. Sects. 4.1-4.2 of [4]).

The implementation of this program in the case of perturbative gauge field theory meets the problem that in general the BRS operator Q is changed due to the interaction [4]. It is a major problem to exhibit the corresponding Ward identities which generalize the Slavnov Taylor identities to the case of *couplings of compact support.*

In the case of a purely massive theory one may adopt a formalism due to Kugo and Ojima [16] who use the fact that in these theories the BRS charge Q can be identified with the incomimg (free) BRS charge, which we denote by Q_0. For the S-matrix to be a well defined operator on the physical Hilbert space of the free theory one then has to require

$$[Q_0, T((gL)^{\otimes n})]|_{\mathrm{ker}Q_0} \to 0 \tag{121}$$

in the adiabatic limit $g \to 1$, see e.g. [10, 13]. This is the motivation to require 'perturbative gauge invariance' [8, 9, 26], which is a somewhat stronger condition than (121) but has the advantage that it is well defined independent of the adiabtic limit. The condition (121) (or perturbative gauge invariance) can be satisfied if additional scalar fields (corresponding to Higgs fields) are included. Unfortunately, in the massless case, it is unlikely that the adiabatic limit exists.[9]

So, in the general case an S-matrix formalism is problematic. One should better rely on the construction of local observables in terms of couplings with compact support. But then Q is a formal power series with zeroth order term Q_0, and it is not obvious which conditions one should put on the retarded (or time ordered) products.

The difficulty is that one has to formulate symmetry conditions for the perturbed fields which themselves are deformed due to the interaction. But using the formalism of the present paper we can disentangle these two problems. Namely, we first use the MWI together with the AWI to compute the commutator of the free BRS charge Q_0 with the retarded (or time ordered)

[9]To motivate perturbative gauge invariance in that case one can derive it from a suitable form of conservation of the BRS-current. To lowest orders this results (as a byproduct) from Appendix B.

products. The resulting family of identities is called the Master BRST Identity and may be used as a renormalization condition in its own right. One then can formulate conditions on the interaction which ensure that the Master BRST Identity implies BRS-invariance of the interacting theory.

5.2 The free BRS-transformation

We illustrate the general ideas on the example of N massless gauge fields A_a^μ, $a = 1, ..., N$, each of them accompanied by a pair of fermionic ghost fields $\tilde{u}_a$, u_a. We may also introduce auxiliary fields B_a (the Nakanishi-Lautrup fields [21]). We work in Feynman gauge, in which the free field equations read

$$\Box A_a^\mu = 0, \qquad \Box u_a = 0 = \Box \tilde{u}_a, \qquad \forall a \tag{122}$$

together with the equation for the auxiliary field

$$\partial_\mu A_a^\mu = B_a \ . \tag{123}$$

We omit in what follows the colour index a by using matrix notation.

The free BRS-current

$$j^\mu \stackrel{\text{def}}{=} B\partial^\mu u - (\partial^\mu B)u \tag{124}$$

is conserved due to the free field equations, $\partial_\mu j^\mu = B\Box u - u\Box B$. The corresponding charge

$$Q_0 \stackrel{\text{def}}{=} \left(\int_{x^0=\text{const.}} d^3x j^0(x) \right)_{S_0} , \tag{125}$$

is nilpotent, i.e. $Q_0^2 = 0$ and

$$\{Q_0, Q_0\}_{S_0} = 0 \ , \tag{126}$$

where we introduced a grading into our Poisson bracket corresponding to ghost number.

Using current conservation as well as the GLZ relation we find for the Poisson bracket of Q_0 with a retarded product

$$\{Q_0, R_{S_0}(F_1, \ldots, F_n)\}_{S_0} = -R_{S_0}(\langle \partial j, h \rangle, F_1, \ldots, F_n) \ , \tag{127}$$

where $h \equiv 1$ on a causally complete open region $\mathcal{O}$ containing the localization regions of all $F_i \in \mathcal{F}(\mathcal{C})$, $i = 1, \ldots, n$ (cf. the analogous argument for time ordered products in [5]).[10] To avoid signs which are due to ferminoic permutations, we assume that all (local) functionals $F_1, ..., F_n$ are bosonic, i.e. a field polynomial with an odd ghost number is smeared out with a Grassmann valued test function.

The free field equations are derived from the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu A_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) + \partial_\mu \tilde{u}\, \partial^\mu u - B\partial_\mu A^\mu + \frac{1}{2}B^2 \tag{130}$$

hence

$$\Box u = -\frac{\delta S_0}{\delta \tilde{u}} \ , \ \Box B = \partial_\mu \frac{\delta S_0}{\delta A_\mu} \tag{131}$$

with $S_0 = \int \mathcal{L}_0$. Thus we obtain

$$\delta_{\partial j(x)} = -B(x)\frac{\delta}{\delta \tilde{u}(x)} - u(x)\partial_\mu \frac{\delta}{\delta A_\mu(x)} \stackrel{\mathrm{def}}{=} \tilde{s}_0(x) \ , \tag{132}$$

hence

$$\delta_{\langle \partial j, h\rangle} = \int dx\, h(x)\tilde{s}_0(x) \stackrel{\mathrm{def}}{=} s_0 \tag{133}$$

on fields localized in the region where $h \equiv 1$, i.e. we obtain the free BRS transformation

$$s_0(\tilde{u}) = -B \ , \ s_0(B) = 0 \ , \ s_0(A_\mu) = \partial_\mu u \ , \ s_0(u) = 0 \ , \tag{134}$$

[10] It is instructive to derive (127) in terms of retarded products: let $\partial^\mu h = b^\mu - a^\mu$ with $\mathrm{supp}\, b^\mu \cap (\overline{V}_+ + \mathcal{O}) = \emptyset$ and $\mathrm{supp}\, a^\mu \cap (\overline{V}_- + \mathcal{O}) = \emptyset$. Since $R_{S_0}(F_1, \ldots, F_n)$ is localized in $\mathcal{O}$, we may vary b^μ in the spacelike complement of $\overline{\mathcal{O}}$ without affecting $\{\langle j_\mu, b^\mu\rangle_{S_0}, R_{S_0}(F_1, \ldots, F_n)\}$. In this way and by using $(\partial^\mu j_\mu(x))_{S_0} = 0$ we find

$$\{Q_0, R_{S_0}(F_1, \ldots, F_n)\}_{S_0} = \{\langle j_\mu, b^\mu\rangle_{S_0}, R_{S_0}(F_1, \ldots, F_n)\}. \tag{128}$$

By means of the support property (54) of the retarded products and the GLZ-Relation (55) we obtain for the r.h.s. of (128)

$$\begin{aligned} &= \sum_{I \subset \{1,\ldots,n-1\}} \{R_{S_0}((F_l)_{l\in I}, \langle j, b\rangle), R_{S_0}((F_k)_{k\in I^c}, F_n)\} \\ &= R_{S_0}(\langle j, b\rangle, F_1, \ldots, F_n) - R_{S_0}(F_1, \ldots, F_n, \langle j, b\rangle) \\ &= R_{S_0}(\langle j, b\rangle, F_1, \ldots, F_n) = -R_{S_0}(\langle \partial j, h\rangle, F_1, \ldots, F_n) \ . \end{aligned} \tag{129}$$

which is obviously nilpotent. Note that the 'local' free BRS-transformation $\tilde{s}_0(x)\int dy\, f(y)$ (where $f \in \mathcal{P} \otimes \mathcal{D}(\mathbb{M})$) differs from $s_0 f(x)$ by a sum of divergences. We now apply the MWI to (127) and find the identity

$$\{Q_0, R_{S_0}(F_1,\ldots,F_n)\}_{S_0} = \sum_{k=1}^{n} R_{S_0}(F_1,\ldots,s_0(F_k),\ldots,F_n)\ . \tag{135}$$

In [6] this identity (in a somewhat different form, see (141) and (144) below) is called **'Master BRST Identity'**.

We may ask how the Master BRST Identity (135) changes if one eliminates the Nakanishi-Lautrup field B by using the field equation $B = \partial A$. The problem is that the ideal $\mathcal{J}_B$ generated from $B - \partial A$ in the algebra $\mathcal{P}_B$ of all polynomials in the fields and their derivatives is not stable under s_0. We discuss two possibilities.

- The quotient algebra $\mathcal{P} = \mathcal{P}_B/\mathcal{J}_B$ may be identified with the subalgebra of polynomials in $\mathcal{P}_B$ which do not contain B or its derivatives. Let $\sigma_B : \mathcal{P} \to \mathcal{P}_B$ denote this identification (i.e. $\sigma_B(\partial^a B + \mathcal{J}_B) = \partial^a \partial_\nu A^\nu$) and $\pi_B : \mathcal{P}_B \to \mathcal{P} : X \to X + \mathcal{J}_B$ the canonical homomorphism. Then we set
$$t = \pi_B s_0 \sigma_B \tag{136}$$
i.e.
$$t(\tilde{u}) = \partial A\ ,\ t(A_\mu) = \partial_\mu u\ , t(u) = 0\ . \tag{137}$$
This choice has the disadvantage that $t^2 \neq 0$. On the other hand, it has the advantage that it commutes with derivatives. Another advantage is that the arising form of the Master BRST identity is equally simple as in the model with the auxiliary field B. Namely, let F be a functional of the fields $A_\mu, u, \tilde{u}$. The image under s_0 might depend on B, but
$$s_0(F) - t(F) = \int dx\, (B(x) - \partial A(x)) \frac{\delta F}{\delta \tilde{u}(x)} \tag{138}$$
and $B - \partial A = \frac{\delta S_0}{\delta B}$. Hence if we replace in the Master BRST identity s_0 by t the correction terms due to the MWI involve derivatives with respect to B. Hence if none of the functionals F_i depends on B, we obtain the Master BRST Identity (second form)
$$\{Q_0, R_{S_0}(F_1,\ldots,F_n\}_{S_0} = \sum_{k=1}^{n} R_{S_0}(F_1,\ldots,t(F_k),\ldots,F_n) \tag{139}$$

where now the field B has been eliminated.

- Another possibility is to use the fact that on the algebra $\mathcal{P}_0 = \mathcal{P}/\mathcal{J}$, where $\mathcal{J}$ is the ideal of $\mathcal{P}$ which is generated by the free field equations, the BRS transformation $\hat{s}_0$ is well defined e.g. by the adjoint action of Q_0 (w.r.t. the Poisson bracket). Using the section $\sigma : \mathcal{P}_0 \to \mathcal{P}$ of Sect. 3.2 [11] we set

$$\hat{t} \stackrel{\mathrm{def}}{=} \sigma \hat{s}_0 \pi = \sigma \pi t, \tag{140}$$

π denoting the canonical homomorphism $\mathcal{P} \to \mathcal{P}_0$ (as in Sect. 3.2). $\hat{t}$ has vanishing square but does not commute with derivatives. It naturally occurs if one considers the entries of retarded (and time-ordered) products as functionals of the free fields, as traditionally done in causal perturbation theory. The price to be paid is a more complicated form of the Master BRST Identity. Namely, $(t-\hat{t})(F) = (1-\sigma\pi)t(F) \in \mathcal{J}_{S_0}$, hence from the MWI we find the Master BRST identity (third form)

$$\{Q_0, R_{S_0}(F_1, \dots, F_n)\}_{S_0} = \sum_{k=1}^{n} R_{S_0}(F_1, \dots, \hat{t}(F_k), \dots, F_n)$$
$$- \sum_{k \neq l} R_{S_0}((F_i)_{i<\max(l,k)}, \delta_{(t-\hat{t})F_k} F_l, (F_j)_{j>\max(l,k)}) \ . \tag{141}$$

In many applications $\hat{t}(P)$, $P \in \mathcal{P}$, can be written as the divergence of another field polynomial $P'^{\nu} \in \mathcal{P}$ by using the free field equations, i.e. $\pi\hat{t}(P)\big(\equiv \pi t(P)\big) = \pi(\partial_\nu P'^{\nu})$, or equivalently

$$\hat{t}(P) = \sigma\pi(\partial_\nu P'^{\nu}). \tag{142}$$

In the next Subsect. we will see that an admissible interaction L_{int} must fulfil this property. So let us assume that in (141)

$$F_k = f_k P_k \quad \text{with} \quad \hat{t}(P_k) = \sigma\pi(\partial_\nu P'^{\nu}_k) \ , \quad f_k \in \mathcal{D}(\mathbb{M}), \quad k = 1, ..., n. \tag{143}$$

In $R_{S_0}(F_1, \dots, f_k \sigma\pi(\partial_\nu P'^{\nu}_k), \dots, F_n)$ we would then like to move the derivative to the test function f_k (i.e. outside of the unsmeared retarded

[11] The avoidance of the field B contradicts the requirement (vi) on σ, but this is no harm. To do so we choose $\sigma\pi(\partial_\nu A^\nu) = \partial_\nu A^\nu$, cf. (85).

product). This produces corrections which can directly be read off from the second formulation of the MWI (109):

$$\{Q_0, R_{S_0}(f_1P_1,\ldots,f_nP_n)\}_{S_0} =$$
$$-\sum_{k=1}^{n} R_{S_0}(f_1P_1,\ldots,(\partial_\nu f_k)\sigma\pi(P_k'^{\nu}),\ldots,f_nP_n)$$
$$+\sum_{k\neq l} R_{S_0}((f_iP_i)_{i<\max(l,k)}, G((P_k,P_k')f_k, P_lf_l), (f_jP_j)_{j>\max(l,k)}) \;, \quad (144)$$

where

$$G((P_1,P_1')f_1,P_2f_2) \stackrel{\mathrm{def}}{=} -\delta_{(t-\hat{t})f_1P_1}(f_2P_2) - \sum_{\chi,\psi\in\mathcal{G}} f_1\sigma\pi\Big(\frac{\partial P_{1\nu}'}{\partial\chi}\Big)\delta^{\nu}_{\chi,\psi}\frac{\partial(f_2P_2)}{\partial\psi}\,. \quad (145)$$

Obviously, there is also a mixed formula in which the step from (135) (or (139), or (141)) to (144) is done for some of the factors F_k only (not for all). In the forms (141) and (144) the Master BRST Identity was found in [6] with $\delta_{(t-\hat{t})F_k}F_l$ corresponding to the terms $G^{(1)}(F_k,F_l)$. ($G(\ldots)$ (145) denotes the same terms.)

Note that the Master BRST Identity (in either form) is independent of the choice of an interaction and is therefore well suited for the formalism of causal perturbation theory where one aims at finding the retarded (or time ordered) products not only for the interaction Lagrangian itself but for a whole class of fields.

Given the free (quantum) gauge fields, requirements on the interaction are formulated in [6], in particular a suitable form of BRS-invariance. These requirements determine the interaction to a far extent [28, 9, 26]. Then it is demonstrated that for such an interaction the validity of particular cases of the Master BRST Identity and of ghost number conservation (which is another consequence of the MWI) suffices for a construction of the net of local algebras of observables. This construction yields also a space of physical states and an explicit formula for the computation of the BRS-transformation of an arbitrary quantum field.

In that reference the requirements expressing BRS-invariance of the interaction have been motivated by the particular case of purely massive gauge models (121) (in which the adiabatic limit exists, see e.g. [10, 13, 8]), and by what has been used in the construction and holds true in the most important examples. However, it is desirable to derive these conditions from more fundamental principles without using the adiabatic limit. The MWI and AWI are well suited tools for such a derivation, as it is demonstrated in Appendix B. However, in the next Subsection we determine the admissible interaction of a *local* gauge theory independently of the corresponding procedure in [6]. This is a further important application of the MWI and AWI.

5.3 Admissible interaction

By an admissible interactions we understand an interaction for which a deformed BRS charge Q exists[12]. Let the free action S_0 and the free BRS-current be given. We make the ansatz

$$S_{\text{int}} = \sum_{n \geq 1} S_n \lambda^n \tag{146}$$

with $S_n = \int dx g(x)^n \mathcal{L}_n(x)$, $\mathcal{L}_n \in \mathcal{P}$, $g \in \mathcal{D}(\mathbb{M})$, $g \equiv 1$ on some causally complete open region $\mathcal{O}_1$. We want to find a conserved current of the interacting theory

$$j_\mu = \sum_{n \geq 0} j_\mu^{(n)} \lambda^n \tag{147}$$

where $j_\mu^{(0)}$ is the free BRS current. The BRS-transformation $s : \mathcal{P} \to \mathcal{P}$, will be constructed in the form

$$s = \sum_{n \geq 0} \int dx \, \tilde{s}_n(x) \lambda^n \tag{148}$$

with $\tilde{s}_n(x)$ a local (graded) derivation, and $\tilde{s}_0(x)$ given by (132). In addition we require that s is nilpotent on the space of solutions and fulfills

$$(s(F))_S = \hat{s}(F_S) \ , \tag{149}$$

where $\hat{s}$ is defined in terms of the BRS-current j (147) by

$$\hat{s}(F_S) = \{Q, F_S\} \quad , \ Q \stackrel{\text{def}}{=} \langle j, b \rangle_S \quad , \forall \, \text{local } F \text{ with supp } F \subset \mathcal{O} \tag{150}$$

[12]For simplicity we do not investigate the existence of Q as an *operator* (which is used for the construction of physical states in [4] and [6]). This existence involves an infrared problem which can be avoided by a spatial compactification [4]. Here, we only require that Q implements a nilpotent (graded) derivation on the interacting fields (149)-(150), which is a deformation of the free BRS-transformation s_0 (133).

(with $S = S_0 + S_{\rm int}$). Thereby, $\mathcal{O}$ is a causally complete open region with $\overline{\mathcal{O}} \subset \mathcal{O}_1$. Due to current conservation there is a rather large freedom in the choice of $b = (b_\mu)$. It only needs to be a smooth version of a delta function on a Cauchy surface of $\mathcal{O}$ with $\mathrm{supp}\, b \subset \mathcal{O}_1$. For later purpose we choose b in the following way: let $h \in \mathcal{D}(\mathcal{O}_1)$ with $h \equiv 1$ on $\mathcal{O}$. Then the b_μ which we will use later on is obtained from $\partial_\mu h$ by the same causal splitting as in (128). Note that for an arbitrary given local $F \in \mathcal{F}(\mathcal{C})$ the regions $\mathcal{O}$, $\mathcal{O}_1$ as well as h and b can be suitably adjusted.

We require current conservation within the region $\mathcal{O}_1$ (where g is constant) only,

$$R_{S_0}(e_\otimes^{S_{\rm int}}, \partial j(x)) = 0 \ , \ \forall x \in \mathcal{O}_1 \ . \tag{151}$$

To zeroth order in λ this is simply the condition that the free BRS current is conserved in the free theory

$$\partial j^{(0)} \equiv: G_0 \in \mathcal{J} \tag{152}$$

and apply the MWI,

$$0 = R_{S_0}(e_\otimes^{S_{\rm int}}, \tilde{s}_0(x) S_{\rm int} - \sum_{n \geq 1} \partial j^{(n)}(x) \lambda^n) \ , \quad x \in \mathcal{O}_1, \tag{153}$$

where $\tilde{s}_0(x) = \delta_{G_0(x)}$ (132).

To first order we find the requirement

$$\tilde{s}_0(x) S_1 - \partial j^{(1)}(x) \equiv: -G_1(x) \in \mathcal{J} \ . \tag{154}$$

Therefore we can apply again the MWI and obtain

$$0 = R_{S_0}(e_\otimes^{S_{\rm int}}, \tilde{s}_1(x) S_{\rm int} + \tilde{s}_0(x) \sum_{n \geq 2} S_n \lambda^n - \sum_{n \geq 2} \partial j^{(n)}(x) \lambda^n) \ , \quad x \in \mathcal{O}_1, \tag{155}$$

with $\tilde{s}_1(x) = \delta_{G_1(x)}$. Iterating the procedure we obtain the conditions

$$\tilde{s}_0(x) S_n + \tilde{s}_1(x) S_{n-1} + \ldots + \tilde{s}_{n-1}(x) S_1 - \partial j^{(n)}(x) \equiv: -G_n(x) \in \mathcal{J} \ , \tag{156}$$

and set $\tilde{s}_n(x) := \delta_{G_n(x)}$. We see that at every order, S_n must be chosen such that

$$\sum_{k=0}^{n-1} \tilde{s}_k(x) S_{n-k} \in \mathcal{P}_{\rm div} + \mathcal{J} \tag{157}$$

where $\mathcal{P}_{\rm div} = \{\partial^\mu f_\mu\ ,\ f_\mu \in \mathcal{P}\}$. This inductive determination of s and $S_{\rm int}$ by requiring $\partial j = 0$ has some similarity with the procedure in [15], cf. Appendix B. Since $G_n(x) = \delta_{G_n(x)} S_0 = \tilde{s}_n(x) S_0$ the relation (156) and current conservation imply 'local' BRS-invariance of the action $S = S_0 + S_{\rm int}$ within $\mathcal{O}_1$:

$$\tilde{s}(x)S = \sum_{n\geq 0} \lambda^n \sum_{k=0}^{n} \tilde{s}_k(x) S_{n-k} = \partial j(x)\ , \quad x \in \mathcal{O}_1\ , \tag{158}$$

and hence

$$\big(\tilde{s}(x)S\big)_S = 0\ , \qquad \forall x \in \mathcal{O}_1\ . \tag{159}$$

It remains to verify that the so constructed BRS-transformation s (148) satisfies (149) and the nilpotency

$$0 = (s^2(F))_S = \hat{s}^2(F_S) \quad \forall F\ . \tag{160}$$

To prove the first property we choose h and b as in (128). Analogously to (129) we find

$$\{Q, F_S\} = -R_S(\langle \partial j, h\rangle, F)\ . \tag{161}$$

Because $\langle \partial j, h\rangle \in \mathcal{J}_S$ we can apply the MWI:

$$\{Q, F_S\} = (\delta_{\langle \partial j, h\rangle} F)_S = (s(F))_S\ . \tag{162}$$

In the last step we have used (158) as well as $\mathrm{supp}\, h \subset \mathcal{O}_1$ and $h \equiv 1$ on $\mathrm{supp}\, F$.

Finally, we want to check the nilpotency of $\hat{s}$. Using the Jacobi identity we find

$$\begin{aligned}(s^2(F))_S &= \frac{1}{2}\{Q(b), \{Q(b'), F_S\}\} + \frac{1}{2}\{Q(b'), \{Q(b), F_S\}\} \\ &= \{\{Q(b'), Q(b)\}, F_S\}\end{aligned} \tag{163}$$

for all admissible test functions b, b' (depending on the support of F). We may now choose b, b' such that b' satisfies the conditions also with respect to the support of b (and of course $\mathrm{supp}\, b' \subset \mathcal{O}_1$). Then

$$(s^2(F))_S = \{s(\langle j, b\rangle)_S, F_S\}\ . \tag{164}$$

We now assume that

$$s(j_\mu) = \partial^\nu C_{\mu\nu} + H_\mu \tag{165}$$

with an antisymmetric tensor field $C_{\mu\nu} \in \mathcal{P}$ and $H_\mu \in \mathcal{J}_\mathcal{S}$. Then

$$(s^2(F))_S = \{\langle\partial^\nu C_{\mu\nu}, b^\mu\rangle_S, F_S\} = \frac{1}{2}\{\langle C_{\mu\nu}, \partial^\mu b^\nu - \partial^\nu b^\mu\rangle_S, F_S\} = 0 \qquad (166)$$

since the support of $(\partial^\nu b^\mu - \partial^\mu b^\nu)$ is spacelike to the support of F.[13]

For massless gauge fields without matter fields (i.e. the model studied in the preceding Subsect.) the usual expression for the BRS-current

$$j^\mu = B \cdot D^\mu u - \partial^\mu B \cdot u + \frac{1}{2}\partial^\mu \tilde{u} \cdot (u \times u) \qquad (167)$$

(where $(D^\mu u)_a = \partial^\mu u_a + f_{abc} A_b^\mu u_c$) is BRS-invariant: $s(j^\mu) = 0$. So the assumption (165) is trivially satisfied.

In cases where the condition (165) cannot be directly checked one may use a perturbative formulation. Set

$$H \stackrel{\text{def}}{=} s(j) - \partial C \qquad (168)$$

for some choice of $C = (C_{\mu\nu})$. The condition $H \in \mathcal{J}_\mathcal{S}$ means that

$$R_{S_0}(e_\otimes^{S_{\text{int}}}, H) = 0\ . \qquad (169)$$

for all λ. In zeroth order we find that $H^{(0)} \in \mathcal{J}_{S_0}$. Set $K^{(0)} = -H^{(0)}$. We apply the MWI and get

$$R_{S_0}(e_\otimes^{S_{\text{int}}}, \delta_{K^{(0)}} S_{\text{int}} + H^{(n\geq 1)}) = 0\ . \qquad (170)$$

In lowest order this implies

$$K^{(1)} \stackrel{\text{def}}{=} -\delta_{K^{(0)}} S_1 - H^{(1)} \in \mathcal{J}_{S_0}\ . \qquad (171)$$

We now define recursively

$$K^{(n)} \stackrel{\text{def}}{=} -\sum_{k=1}^{n} \delta_{K^{(n-k)}} S_k - H^{(n)} \qquad (172)$$

and prove by induction that

$$R_{S_0}(e_\otimes^{S_{\text{int}}}, \sum_{k=1}^{n} \delta_{K^{(n-k)}} S_{l\geq k} + H^{(l\geq n)}) = 0\ . \qquad (173)$$

[13] In supp $F + \bar{V}_\pm$ we have $b = 0$ or $b^\mu = \partial^\mu h$.

The lowest order term of (173) is $(-K^{(n)})_{S_0}$, hence $K^{(n)} \in \mathcal{J}_{S_0}$. The recursion problem can be solved if for every n there exists an antisymmetric tensor field $C^{(n)}$ and a vector field $K^{(n)}_\mu \in \mathcal{J}_{S_0}$ such that

$$\sum_{k=1}^{n} \delta_{K^{(n-k)}} S_k + \sum_{k=0}^{n} s_{n-k}(j^{(k)}) = \partial C^{(n)} - K^{(n)} \ . \tag{174}$$

In the given derivation we have used various cases of the MWI. In QFT it may therefore happen that the appearance of anomalies restricts the set of admissible interactions further, e.g. models with (non-compensated) axial anomalies must be excluded.

The conditions on a gauge interaction found here differ somewhat from the corresponding conditions in [6] (cf. Appendix B) or in [26]. For example: to ensure renormalizability it is required $S_l = 0$ for all $l \geq 3$ in [6]. And the condition (189) (which is the input for the derivation of the conditions of [6] given in Appendix B) is stronger than (151). However, for the class of renormalizable (by power counting) interactions we expect that the requirements derived here and the ones given in [6] have precisely the same solutions.

6 Appendix A: Construction of the map σ

We work in $d = 4$ dimensions. In the first part we construct recursively a particular σ in the enlarged model (i.e. with the field φ^μ). This construction applies also to the non-enlarged model. For the latter we prove that σ is unique (in the second part of this Appendix).

6.1 Particular solution for σ

We define

$$H^{\mu_1...\mu_s}_{s,n} \stackrel{\text{def}}{=} \square^n \partial^{\mu_1}...\partial^{\mu_s}\varphi - (-m^2)^n \sigma\pi(\partial^{\mu_1}...\partial^{\mu_s}\varphi), \tag{175}$$

which is obviously an element of the ideal $\mathcal{J}$ (79) and totally symmetrical in $\mu_1, ..., \mu_s$. Hence, these properties hold true also for

$$F^{\nu_1...\nu_r}_{r,n} = \frac{(-1)^n}{N_{r,n}} \sum_{1\leq j_1<...<j_{2n}\leq r} \sum_{\pi\in\mathcal{S}_{2n}} \frac{1}{2^n n!} g^{\nu_{j_{\pi 1}}\nu_{j_{\pi 2}}}...$$

$$...g^{\nu_{j_{\pi(2n-1)}}\nu_{j_{\pi 2n}}} H^{\nu_1...\hat{j}_1...\hat{j}_{2n}...\nu_r}_{r-2n,n}, \quad n = 1, ..., \left[\frac{r}{2}\right], \tag{176}$$

(the hat means that the corresponding index is omitted, and $[r/2] = \frac{r}{2}$ if r even and $[r/2] = \frac{r-1}{2}$ if r odd) where

$$N_{r,n} = \prod_{l=1}^{n}(2 - 2l + 2r). \tag{177}$$

We now claim that

$$\sigma\pi(\partial^{\nu_1}...\partial^{\nu_r}\varphi) = \sigma\pi(\partial^{\nu_1}...\partial^{\nu_{r-1}}\varphi^{\nu_r}) = \partial^{\nu_1}...\partial^{\nu_r}\varphi + \sum_{n=1}^{[\frac{r}{2}]} F_{r,n}^{\nu_1...\nu_r} \tag{178}$$

yields a particular solution for $\sigma\pi(\partial^{\nu_1}...\partial^{\nu_r}\varphi)$ for $r \geq 2$ by recursion with respect to r. Together with the formulas (84)-(85) for $r = 0, 1$ this determines a map σ completely.

Proof: The only non-trivial point is to verify $\sigma\pi(\partial^a\partial_\mu\varphi^\mu) = -m^2\sigma\pi(\partial^a\varphi)$. Because the r.h.s. of (178) is totally symmetrical in $\nu_1, ..., \nu_r$, it suffices to show

$$g_{\nu_1\nu_2}\sigma\pi(\partial^{\nu_1}...\partial^{\nu_r}\varphi) = -m^2\sigma\pi(\partial^{\nu_3}...\partial^{\nu_r}\varphi). \tag{179}$$

For this purpose we set $N_{r,0} \stackrel{\text{def}}{=} 1$ and

$$G_{r,s}^{\nu_3...\nu_r} \stackrel{\text{def}}{=} \sum_{3\leq j_1<...<j_{2s}\leq r} \sum_{\pi\in\mathcal{S}_{2s}} \frac{1}{2^s s!} g^{\nu_{j_{\pi 1}}\nu_{j_{\pi 2}}}... \\ ...g^{\nu_{j_{\pi(2s-1)}}\nu_{j_{\pi 2s}}} H_{r-2(s+1),s+1}^{\nu_3...\hat{j}_1...\hat{j}_{2s}...\nu_r} \tag{180}$$

for $0 \leq 2s \leq r-2$, and $G_{r,s} \stackrel{\text{def}}{=} 0$ for $2s > r-2$. By inserting the definitions we find that the identity

$$g_{\nu_1\nu_2}F_{r,n}^{\nu_1...\nu_r} = \frac{(-1)^n}{N_{r,n-1}}G_{r,n-1}^{\nu_3...\nu_r} + \frac{(-1)^n}{N_{r,n}}G_{r,n}^{\nu_3...\nu_r}, \quad n = 1, ..., \Big[\frac{r}{2}\Big], \tag{181}$$

implies the assertion (179).

To prove (181) we write $F^{\nu_1...\nu_r}_{r,n}$ in the following form:

$$F^{\nu_1...\nu_r}_{r,n} = \frac{(-1)^n}{N_{r,n-1}(2-2n+2r)}\Big[g^{\nu_1\nu_2}G^{\nu_3...\nu_r}_{r,n-1}+$$
$$\sum_{3\leq l<k\leq r}(g^{\nu_1\nu_l}g^{\nu_2\nu_k}+g^{\nu_1\nu_k}g^{\nu_2\nu_l})\sum g^{\cdots}...H_{r-2n,n}+$$
$$\sum_{3\leq l\leq r}[g^{\nu_1\nu_l}\sum g^{\cdots}...H^{\nu_2...}_{r-2n,n}+(\nu_1\leftrightarrow\nu_2)]\Big]$$
$$+\frac{(-1)^n}{N_{r,n}}\sum g^{\cdots}...H^{\nu_1\nu_2...}_{r-2n,n}. \tag{182}$$

Contraction of the second (third respectively) line with $g_{\nu_1\nu_2}$ gives $2(n-1)G^{\nu_3...\nu_r}_{r,n-1}$ (and $2(r-2n)G^{\nu_3...\nu_r}_{r,n-1}$ resp.). In the last line we use

$$g_{\mu_1\mu_2}H^{\mu_1...\mu_s}_{s,n} = H^{\mu_3...\mu_s}_{s-2,n+1}, \tag{183}$$

and end up with

$$g_{\nu_1\nu_2}F^{\nu_1...\nu_r}_{r,n} = \frac{(-1)^n}{N_{r,n-1}(2-2n+2r)}(4+2(n-1)+2(r-2n))G^{\nu_3...\nu_r}_{r,n-1}$$
$$+\frac{(-1)^n}{N_{r,n}}G^{\nu_3...\nu_r}_{r,n}. \qquad \square \tag{184}$$

6.2 Uniqueness of σ for a single real Klein-Gordon field

We will prove that the map σ is unique for $\mathcal{P}$ and $\mathcal{J}$ given by (11) and (79). Obviously, the relations $\sigma\pi(\varphi)=\varphi$, $\sigma\pi(\partial^\mu\varphi)=\partial^\mu\varphi$ and the recursive formula (178) give a solution for σ which we denote by σ_0. Analogously to (82) and by the requirements (iii) and (v) on σ we conclude that the most general solution for σ is of the form:

$$\sigma\pi(\partial^{\nu_1}...\partial^{\nu_r}\varphi) = \sigma_0\pi(\partial^{\nu_1}...\partial^{\nu_r}\varphi)+M^{\nu_1...\nu_r},$$
$$M^{\nu_1...\nu_r} \stackrel{\text{def}}{=} \frac{1}{r!}\sum_{\pi\in\mathcal{S}_r}\sum_{l=1}^{[r/2]}\sum_{j=1}^{l} a_{l,j}g^{\nu_{\pi1}\nu_{\pi2}}...g^{\nu_{\pi(2l-1)}\nu_{\pi2l}}\partial^{\nu_{\pi(2l+1)}}...\partial^{\nu_{\pi r}}\square^{(l-j)}(\square+m^2)\varphi, \tag{185}$$

where $a_{l,j}\in\mathbb{R}$ is a constant. An ϵ-tensor is excluded in $M^{\nu_1...\nu_r}$ by the total symmetry in $\nu_1,...,\nu_r$. We are now going to show that $g_{\nu_{r-1}\nu_r}M^{\nu_1...\nu_r}=0$

(see (179)) yields $a_{l,j} = 0$, $\forall l, j$. We use the equation

$$g_{\nu_{r-1}\nu_r} \frac{1}{r!} \sum_{\pi \in \mathcal{S}_r} g^{\nu_{\pi 1}\nu_{\pi 2}} ... g^{\nu_{\pi(2l-1)}\nu_{\pi 2l}} \partial^{\nu_{\pi(2l+1)}} ... \partial^{\nu_{\pi r}} =$$

$$\frac{1}{(r-2)!} \sum_{\pi \in \mathcal{S}_{r-2}} \{ N_{r,l} g^{\nu_{\pi 1}\nu_{\pi 2}} ... g^{\nu_{\pi(2l-3)}\nu_{\pi(2l-2)}} \partial^{\nu_{\pi(2l-1)}} ... \partial^{\nu_{\pi(r-2)}}$$
$$+ M_{r,l} g^{\nu_{\pi 1}\nu_{\pi 2}} ... g^{\nu_{\pi(2l-1)}\nu_{\pi 2l}} \partial^{\nu_{\pi(2l+1)}} ... \partial^{\nu_{\pi(r-2)}} \square \} \tag{186}$$

where $N_{r,l}$ and $M_{r,l}$ are non-vanishing combinatorical factors, except $M_{r,[r/2]} = 0$. The requirement $g_{\nu_{r-1}\nu_r} M^{\nu_1 ... \nu_r} = 0$ gives the following chains of equations

$$a_{s,s} N_{r,s} = 0, \qquad a_{l,s} M_{r,l} + a_{l+1,s} N_{r,l+1} = 0 \quad \forall l \in \{s, s+1, ..., [r/2]-1\}, \tag{187}$$

where the chains are labeled by $s = 1, 2, ..., [r/2]$. We find indeed $a_{l,j} = 0$, $\forall l, j$. $\square$

7 Appendix B: Formulation of BRS-invariance of the interaction used in [6]

To derive the conditions which are used in [6] to express BRS-invariance of the interaction[14] we only require current conservation (151) but generalized to all $x \in \mathbb{M}$. The latter makes possible an integration over $x \in \mathbb{M}$, which removes $\partial j^{(n)}$ from (156) and replaces $\tilde{s}_0(x)$ by s_0. This generalization is done by admitting that current conservation at x is violated by a term of the form $M_\nu(x) \partial^\nu g(x)$. Since we want to obtain conditions on the interaction which are expressable in terms of *free* fields, we require that M_ν is in the range of σ: $M_\nu = \sigma\pi(K_\nu)$ for some K_ν. In detail our input is the requirement that there exist an interaction S_{int} (146), a BRS-current j_μ (147) (with compact support[15] of $j_\mu^{(n)}$ $\forall n \geq 1$) and a formal power series

$$K_\nu(x) = \sum_{n \geq 1} \lambda^n K_\nu^{(n)}(x), \qquad K_\nu^{(n)} \in \mathcal{P} \otimes \mathcal{C}(\mathbb{M}) \ , \tag{188}$$

[14] We shall not obtain that conditions in full generality. However, as far as we know, the particular version which we shall obtain determines the interaction to the same extent, see below.

[15] Usually we expect supp $j_\mu^{(n)} \subset$ supp g for $n \geq 1$.

such that

$$R_{S_0}(e_\otimes^{S_{\rm int}}, \partial j(x) - \sigma\pi(K_\nu)(x)\partial^\nu g(x)) = 0 \ , \ \forall x \in \mathbb{M} \ , \ \forall g \in \mathcal{D}(\mathbb{M}) \ . \qquad (189)$$

So, in a formalism with auxiliary fields, $S_{\rm int}$ and j depend on the choice of σ, but this dependence drops out when the auxiliary fields are eliminated.[16] The condition (189) corresponds to the 'Quantum Noether condition in terms of interacting fields' [15], and the following derivation may be viewed as a rigorous and enlarged version of the 'off shell formalism' given there.

In working out the consequences of (189) we will frequently use the AWI and the MWI (in particular various formulations of the Master BRST Identity) without mentioning it. As in [6] we use a formalism without B-field; so we may replace s_0 by t due to (138). Similarly to (156) our requirement (189) is equivalent to the sequence of conditions

$$\tilde{s}_0(x)S_n + \tilde{s}_1(x)S_{n-1} + \ldots + \tilde{s}_{n-1}(x)S_1 - \partial j^{(n)}(x) + \sigma\pi(K_\nu^{(n)})(x)\partial^\nu g(x)$$
$$\equiv: -G_n(x) \in \mathcal{J} \ , \ n \geq 1 \ , \qquad (190)$$

and (152) for $n = 0$, where $\tilde{s}_n(x) \stackrel{\rm def}{=} \delta_{G_n(x)}$. We will proceed in the following way: first we insert formulas which are inductively known into $\tilde{s}_1(x)S_{n-1} + \ldots + \tilde{s}_{n-1}(x)S_1$. Then we integrate the resulting equation over $x \in \mathbb{M}$. Thereby, we take into account that $j^{(n)}$ has compact support.

- To first order we obtain

$$(tS_1)_{S_0} = (\partial^\nu \, \sigma\pi(K_\nu^{(1)}))(g)_{S_0} = \sigma\pi(\partial^\nu K_\nu^{(1)})(g)_{S_0} \ . \qquad (191)$$

  Since g is arbitrary we end up with the condition that there must exist $\mathcal{L}_1, \ K_\nu^{(1)} \in \mathcal{P}$ with

$$\hat{t}\mathcal{L}_1 = \sigma\pi(\partial^\nu K_\nu^{(1)}) \ . \qquad (192)$$

- To second order $n = 2$ we multiply equation (190) by 2. Then we insert once

$$(\tilde{s}_1(x)S_1)_{S_0} = -R_{S_0}(G_1(x), S_1)$$
$$= -R_{S_0}(\partial j^{(1)}(x), S_1) + R_{S_0}(\tilde{s}_0(x)S_1, S_1) + R_{S_0}(\sigma\pi(K_\nu^{(1)})(x)\partial^\nu g(x), S_1) \qquad (193)$$

[16] We explain this for the 4-gluon interaction in the formalism of [6]. For a σ corresponding to $C_A = -\frac{1}{2}$ this coupling is generated by S_1, but for $C_A = 0$ it appears in S_2, cf. the Remark at the end of Sect. 4.

and once $(\tilde{s}_1(x)S_1)_{S_0} = -R_{S_0}(S_1, G_1(x)) = \ldots$. Integration yields

$$\{Q_0, R^N_{S_0}(S_1, S_1)\} = -R^N_{S_0}(\sigma\pi(K^{(1)}_\nu)\partial^\nu g, S_1) - R^N_{S_0}(S_1, \sigma\pi(K^{(1)}_\nu)\partial^\nu g) \ , \tag{194}$$

where

$$R^N_{S_0}(S_1, S_1) \stackrel{\text{def}}{=} R_{S_0}(S_1, S_1) + 2(S_2)_{S_0} \tag{195}$$

and

$$\begin{aligned}(R^N_{S_0} - R_{S_0})(\sigma\pi(K^{(1)}_\nu)f, S_1) :&\stackrel{\text{def}}{=} \sigma\pi(K^{(2)}_\nu)(f)_{S_0} \\ &\stackrel{\text{def}}{=} :(R^N_{S_0} - R_{S_0})(S_1, \sigma\pi(K^{(1)}_\nu)f) \ .\end{aligned} \tag{196}$$

are shorthand notations which we will interpret below. We find the additional requirement that there must exist $\mathcal{L}_2 \in \mathcal{P}$, $K^{(2)}_\nu \in \mathcal{P} \otimes \mathcal{C}(\mathbb{M})$ with

$$\hat{t}\mathcal{L}_2(g^2) + \sigma\pi G((\mathcal{L}_1, K^{(1)})g, \mathcal{L}_1 g) + \sigma\pi(K^{(2)}_\nu)(\partial^\nu g) = 0 \ . \tag{197}$$

- To third order $n = 3$ we multiply equation (190) by $3! = 6$. Then we insert twice $(\tilde{s}_1(x)S_2)_{S_0} = -R_{S_0}(S_2, G_1(x)) = \ldots$ (cf. (193)) and

$$\begin{aligned}(\tilde{s}_2(x)S_1)_{S_0} &= -R_{S_0}(G_2(x), S_1) = -R_{S_0}(\partial j^{(2)}(x), S_1) \\ +R_{S_0}(\tilde{s}_0(x)S_2, S_1) &+ R_{S_0}(\delta_{G_1(x)}S_1, S_1) + R_{S_0}(\sigma\pi(K^{(2)}_\nu)(x)\partial^\nu g(x), S_1) \ ,\end{aligned} \tag{198}$$

as well as four times $(\tilde{s}_1(x)S_2)_{S_0} = -R_{S_0}(G_1(x), S_2) = \ldots$ and $(\tilde{s}_2(x)S_1)_{S_0} = -R_{S_0}(S_1, G_2(x)) = \ldots$. Next we use

$$\begin{aligned}R_{S_0}(\delta_{G_1(x)}S_1, S_1) + R_{S_0}(S_1, \delta_{G_1(x)}S_1) &= -R_{S_0}(G_1(x), S_1, S_1) \\ = -R_{S_0}(\partial j^{(1)}(x), S_1, S_1) &+ R_{S_0}(\tilde{s}_0(x)S_1, S_1, S_1) \\ &+ R_{S_0}(\sigma\pi(K^{(1)}_\nu)(x)\partial^\nu g(x), S_1, S_1)\end{aligned} \tag{199}$$

and $R_{S_0}(S_1, \delta_{G_1(x)}S_1) = -\frac{1}{2}R_{S_0}(S_1, S_1, G_1(x)) = \ldots$. By integration we obtain

$$\begin{aligned}&\{Q_0, R^N_{S_0}(S_1, S_1, S_1)\} = \\ -2R^N_{S_0}(\sigma\pi(K^{(1)}_\nu)\partial^\nu g, S_1, S_1) &- R^N_{S_0}(S_1, S_1, \sigma\pi(K^{(1)}_\nu)\partial^\nu g) \ ,\end{aligned} \tag{200}$$

where

$$R^N_{S_0}(S_1,S_1,S_1) \stackrel{\rm def}{=} R_{S_0}(S_1,S_1,S_1)+2R_{S_0}(S_2,S_1)+4R_{S_0}(S_1,S_2)+6(S_3)_{S_0} \tag{201}$$

and

$$\begin{aligned}(R^N_{S_0}-R_{S_0})(\sigma\pi(K^{(1)}_\nu)f,S_1,S_1) : \stackrel{\rm def}{=} 2R_{S_0}(\sigma\pi(K^{(1)}_\nu)f,S_2)\\ +R_{S_0}(\sigma\pi(K^{(2)}_\nu)f,S_1)+R_{S_0}(S_1,\sigma\pi(K^{(2)}_\nu)f)+2\sigma\pi(K^{(3)}_\nu)(f)\ ,\\ (R^N_{S_0}-R_{S_0})(S_1,S_1,\sigma\pi(K^{(1)}_\nu)f) : \stackrel{\rm def}{=}\\ 2R_{S_0}(S_2,\sigma\pi(K^{(1)}_\nu)f)+2R_{S_0}(S_1,\sigma\pi(K^{(2)}_\nu)f)+2\sigma\pi(K^{(3)}_\nu)(f)\ .\end{aligned} \tag{202}$$

With $H \stackrel{\rm def}{=} tS_2+G((\mathcal{L}_1,K^{(1)})g,\mathcal{L}_1 g)+\sigma\pi(K^{(2)}_\nu)(\partial^\nu g)$ and by using also a mixed version of the Master BRST Identity,

$$\begin{aligned}\{Q_0,R_{S_0}(S_1,S_2)\} = -R_{S_0}(\sigma\pi(K^{(1)}_\nu)\partial^\nu g,S_2)\\ +R_{S_0}(S_1,tS_2)+G((\mathcal{L}_1,K^{(1)})g,S_2)_{S_0}\ ,\end{aligned} \tag{203}$$

we find the following additional requirement: there must exist $\mathcal{L}_3,\ K^{(3)}$ with

$$(t\mathcal{L}_3(g^3))_{S_0}+G((\mathcal{L}_1,K^{(1)})g,\mathcal{L}_2 g^2)_{S_0}+\sigma\pi(K^{(3)}_\nu)(\partial^\nu g)_{S_0}-(\delta_H S_1)_{S_0}=0\ , \tag{204}$$

where we have used $H\in\mathcal{J}_{S_0}$ (197). For the models studied in [6] it holds $G((\mathcal{L}_1,K^{(1)})g,\mathcal{L}_2 g)\in \text{ran}\,\sigma$ and it is possible to fulfill (204) with $\mathcal{L}_3=0=K^{(3)}$. With that we may write $H=(t-\hat{t})S_2$ by using (197), and the condition (204) reduces to

$$-\pi\,\delta_{(t-\hat{t})S_2}S_1+\pi\,G((\mathcal{L}_1,K^{(1)})g,\mathcal{L}_2 g^2)=0\ . \tag{205}$$

- In general it may happen that the higher orders $n\geq 4$ of (189) restrict $\mathcal{L}_1,\mathcal{L}_2$ and $\mathcal{L}_3$ further and non-vanishing expressions for $\mathcal{L}_j$ and $K^{(j)}$ are possible for arbitrary high j. We do not work this out here. Because for the models treated in [6] any solution $(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3=0,K^{(1)},K^{(2)},K^{(3)}=0)$ of (192), (197) and (205) fulfills (189) to all higher orders $n\geq 4$ with $\mathcal{L}_l=0=K^{(l)}$, $\forall l\geq 4$ (up to anomalies, i.e. violations of the MWI and the AWI).

In [6] *generalizations* of (192), (197) and (205) have been used to determine the interaction, namely (141)[17], (209) and (216) in the published version. However, at least for the models studied in [6] it seems that any solution of (192), (197) and (205) with $\mathcal{L}_3 = 0 = K^{(3)}$ satisfies also these generalizations.

So far this Appendix applies to classical field theory and QFT. The following discussions are restricted to QFT. The definition $R^N_{S_0}(h) \stackrel{\text{def}}{=} R_{S_0}(h)$ for $h = S_1, \sigma\pi(K^{(1)}_\nu)f$ (zeroth order), and the above given first order (195)-(196) and second order definitions (201)-(202) of $R^N_{S_0}$ are compatible with the main properties of a retarded product. By the latter we mean linearity, symmetry of $R_{n,1}$ in the first n entries, causal support (54) and in particular the recursion given by the GLZ-relation. The continuation (by analogy) of our inductive evaluation of (189) yields the definitons of $R^N_{S_0}(S_1^{\otimes n}, S_1)$, $R^N_{S_0}(S_1^{\otimes n}, \sigma\pi(K^{(1)}_\nu)f)$ and $R^N_{S_0}(\sigma\pi(K^{(1)}_\nu)f \otimes S_1^{\otimes n-1}, S_1)$ for all $n \geq 3$. We expect that these whole sequences are compatible with the (just mentioned) main properties of a retarded product. If this holds true we can proceed similarly to formula (214) in [6] (which is a particular simple case of Theorem 3.1 in [25]): to all orders we extend $R^N_{S_0}$ to arbitrary factors such that $R^N_{S_0}$ agrees with R_{S_0} as far as possible and that $R_{S_0} \longrightarrow R^N_{S_0}$ is an *admissible finite renormalization* (i.e. the main properties of a retarded product are preserved in this replacement). However, in general the $R^N_{S_0}$ violate some normalization conditions, in particular (63) (i.e. (N3)), the AWI and the MWI, see [6]. With $R^N_{S_0}$ being an admissible retarded product, the equations (194) and (200) are *perturbative gauge invariance* (in the sense of [8, 9, 26]) to second and third order. So, we have shown to lowest orders that our condition (189) implies that the interaction is such that perturbative gauge invariance can be fulfilled by admissible finite renormalizations. From the requirement that the latter property holds true one can derive the interaction (including the Lie-algebraic structure [28]) of massless and massive spin-1 gauge fields and the couplings of spin-2 gauge theories (for an overview see [26]).

To interpret the $R^N_{S_0}$-products in terms of simple equations we go over to the corresponding time-ordered products[18] T^N. As far as they are determined

[17]More precisely we mean here (141) and the antisymmetry of the $\mathcal{L}^{\mu\nu}_2$ which appears in that formula.

[18]We are not aware of a good notion of 'time-ordered products' in classical field theory. But in QFT the retarded products $\{R_{n,1} \mid 0 \leq n \leq N\}$ determine uniquely the corresponding time-ordered products $\{T_n \mid 1 \leq n \leq N+1\}$ and vice versa, see e.g. [11].

by (195)-(196) and (201)-(202) they fulfil

$$T(e_{\otimes}^{S_{\rm int}}) = T^N(e_{\otimes}^{S_1}) \ ,$$
$$T(e_{\otimes}^{S_{\rm int}} \otimes \sum_{n\geq 1} \lambda^n \sigma\pi(K_\nu^{(n)})f) = T^N(e_{\otimes}^{S_1} \otimes \sigma\pi(K_\nu^{(1)})f) \qquad (206)$$

as formal power series in λ, and we expect that this holds true also for the higher orders. But understood as series with respect to the order n of T_n and T_n^N, the equations (206) express a reordering: the contributions to T_n of the terms S_l, $\sigma\pi(K_\nu^{(l)})$ with $l \geq 2$, appear on the r.h.s. in time ordered products T_m^N of higher orders: $(m-n) \geq (l-1)$. The $R_{S_0}^N(S_1^{\otimes n}, \sigma\pi(K_\nu^{(1)})f)$ satisfy

$$R_{S_0}(e_{\otimes}^{S_{\rm int}}, \sum_{n\geq 1} \lambda^n \sigma\pi(K_\nu^{(n)})f) = R_{S_0}^N(e_{\otimes}^{S_1}, \sigma\pi(K_\nu^{(1)})f) \ , \qquad (207)$$

however the corresponding relations for $R_{S_0}^N(S_1^{\otimes n}, S_1)$ and $R_{S_0}^N(\sigma\pi(K_\nu^{(1)})f \otimes S_1^{\otimes n-1}, S_1)$ are more involved.

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