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Fractals, Multifunctions and Markov Operators

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Abstract. We show that attractors of multifunctions have many properties similar to fractals and we introduce the notion of a semiattractor and a semifractal. Further we study the relationship between the multifunctions and transition functions appearing in the theory of Markov operators. We also discuss some properties of a new dimension of measures defined by a use of the Lévy concentration function.

1. Introduction

The main purpose of this lecture is to show a relationship between the dynamics of sets and dynamics of measures. In particular given a metric space X we can construct fractals in two different ways. Using the first one, we define a fractal A_* as the common limit of sequences of sets $(F^n(A))$ where $A \subset X$ and F is a multifunction described by a finite family of transformations $w_i : X \to X$, $i \in I$. In the second method we construct a Markov operator P acting on the space of Borel measures using the same family of transformations (w_i) and a probability vector (p_i), $i \in I$. The operator P and the multifunction F are related by the formula

$$F(x) = \operatorname{supp} P\delta_x \tag{1.1}$$

where δ_x is a probability measure concentrated at x. If for every probability measure μ the sequence $(P^n\mu)$ converges to the same measure μ_* we define the corresponding fractal as the support of μ_*. In the case when all transformations w_i are contractive we have $A_* = \operatorname{supp} \mu_*$ and both definitions of a fractal are equivalent. These classical results will be recalled in Section 3 (see also [1] and [8]).

However, conditions which imply the convergence of $(P^n\mu)$ to the unique measure μ_* are, in general, less restrictive than analogous conditions for the convergence of $(F^n(A))$ and the second method produces a new class of sets of the form $\operatorname{supp} \mu_*$. It is interesting that these sets can be also constructed by a use of topological limits of sequences of sets without any probabilistic tools. We call these sets semifractals.

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In Sections 4-6 we show that this interdependence between transformations of sets and transformations of measures can be extended to a larger class of operators. Namely, it is easy to verify that for every Markov operator P the multifunction F given by formula (1.1) is measurable and closed valued. If in addition P is Fellerian, then the multifunction F is lower semicontinuous. Vice versa, it can be proved that for every measurable closed valued multifunction F there exists a Markov operator P such that condition (1.1) is satisfied. Moreover, for a lower semicontinuous F it is possible to construct a Markov Fellerian operator P satisfying (1.1). These constructions based on the selections theorems of Kuratowski–Ryll Nardzewski and Michael will be shown in Section 6.

In the case when a Fellerian Markov operator P is asymptotically stable, the corresponding multifunction F has some specific properties which are called the asymptotic semistability. In particular if μ_* is the unique P–invariant probability measure then the set $A_* = \operatorname{supp} \mu_*$ is F–invariant and it has many features of a semifractal.

The asymptotic stability of P is a quite strong condition. In Section 7 we only assume that the Markov operator P has a unique normalized invariant measure μ_*. We prove that in this case every F invariant set has μ_* measure either zero or one.

Finally, in Section 8 we introduce a new dimension of probability measures related with the Lévy concentration function. We show some bounds of this dimension for measures invariant with respect to the Markov operators generated by iterated function systems.

2. Preliminaries

Let (X, ρ) be a metric space and let $\mathcal{F}$ be the space of all nonempty closed subsets of X. By $B(x, r)$ we denote the closed ball with center at x and radius r. For a subset A of X, $\operatorname{cl} A$ stands for the closure of A and $\operatorname{diam} A$ for the diameter of A. By $\mathbb{R}$ we denote the set of all reals and by $\mathbb{N}$ the set of all positive integers.

Let (A_n) be a sequence of subsets of X. The *lower bound* $\operatorname{Li} A_n$ and the *upper bound* $\operatorname{Ls} A_n$ are defined by the following conditions. A point x belongs to $\operatorname{Li} A_n$, if for every $\varepsilon > 0$ there is an integer n_0 such that $A_n \cap B(x, \varepsilon) \neq \emptyset$ for $n \geq n_0$. A point x belongs to $\operatorname{Ls} A_n$ if for every $\varepsilon > 0$ the condition $A_n \cap B(x, \varepsilon) \neq \emptyset$ is satisfied for infinitely many n. If $\operatorname{Li} A_n = \operatorname{Ls} A_n$, we say that the sequence (A_n) is topologically convergent and we denote this common limit by $\operatorname{Lt} A_n$. It is called the *topological* (or *Kuratowski*) *limit* of the sequence (A_n) (see [7]). Observe that $\operatorname{Li} A_n$ and $\operatorname{Ls} A_n$ are always closed sets. The basic properties of topological limit can be found in [7]. Here we recall that $\operatorname{Li} A_n = \operatorname{Li}(\operatorname{cl} A_n)$, $\operatorname{Ls} A_n = \operatorname{Ls}(\operatorname{cl} A_n)$ and $\operatorname{Li} A_n \subset B$ provided $A_n \subset B$ for sufficiently large n and B is closed. Moreover, every increasing sequence of sets (A_n) is topologically convergent and $\operatorname{Lt} A_n = \operatorname{cl} \bigcup_{n=1}^{\infty} A_n$.

Now let X and Y be metric spaces. A *multifunction* $F : X \to Y$ is a subset of $X \times Y$ such that for every $x \in X$ the set $F(x) = \{y : (x, y) \in F\}$ is nonempty.

The set $F(x)$ is called the value of the multifunction F at point x. For $A \subset X$ and $B \subset Y$ we define

$$F(A) = \bigcup_{x \in A} F(x) \quad \text{and} \qquad F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

A multifunction $F : X \to Y$ is called *Borel measurable* (or simply *measurable*) if $F^{-}(G)$ is a Borel subset of X for every open subset G of Y.

A multifunction F is called lower *semicontinuous* (shortly l.s.c) if $F^{-}(G)$ is open in X for every open subset G of Y.

For the convenience of the reader we recall some well known properties of lower semicontinuous multifunctions.

Proposition 2.1. *Let $F : X \to Y$ be a multifunction. Then the following conditions are equivalent:*

(i) *F is l.s.c.*
(ii) *$F(\mathrm{cl}\, A) \subset \mathrm{cl}\, F(A)$ for every $A \subset X$.*
(iii) *For every sequence $(x_n) \subset X$ we have*

$$\lim x_n = x \quad \text{implies} \qquad F(x) \subset \mathrm{Li}\, F(x_n).$$

(iv) *For every sequence $(x_n) \subset X$ we have*

$$\lim x_n = x \quad \text{implies} \qquad F(x) \subset \mathrm{Ls}\, F(x_n).$$

A set $A \subset X$ is called *subinvariant* (resp. *invariant*) with respect to a multifunction $F : X \to X$ if $F(A) \subset A$ (resp. $F(A) = A$).

We say that a multifunction $F : X \to X$ is *asymptotically stable* if there exists a closed subset A_* of X such that the following two conditions are satisfied:

(i) $\mathrm{cl}\, F(A_*) = A_*$;
(ii) $\mathrm{Lt}\, F^n(A) = A_*$ for every bounded nonempty subset A of X.

By $\mathcal{B}$ we denote the σ-algebra of Borel subsets of X and by $\mathcal{M}$ the family of all finite Borel measures on X. By $\mathcal{M}_1$ we denote the space of all $\mu \in \mathcal{M}$ such that $\mu(X) = 1$.

As usually, by $B(X)$ we denote the space of all bounded Borel measurable functions $f : X \to \mathbb{R}$ and by $C(X)$ the subspace of all continuous functions. Both spaces are considered with the supremum norm.

Given $\mu \in \mathcal{M}$ we define the support of μ by the formula

$$\mathrm{supp}\, \mu = \{x \in X : \mu(B(x,r)) > 0 \quad \text{for every} \quad r > 0\}.$$

For $f \in B(X)$ and $\mu \in \mathcal{M}$ we write

$$< f, \mu > = \int_X f(x)\mu(dx).$$

We say that a sequence $(\mu_n) \subset \mathcal{M}$ converges weakly to a measure $\mu \in \mathcal{M}$ if

$$\lim < f, \mu_n > = < f, \mu > \qquad \text{for every} \qquad f \in C(X).$$

Using the Alexandrov theorem it is easy to prove the following

Proposition 2.2. *If a sequence $(u_n) \subset \mathcal{M}$ converges weakly to $\mu \in \mathcal{M}$, then*

$$\mathrm{Li}\,\mathrm{supp}\,\mu_n \supset \mathrm{supp}\,\mu.$$

An operator $P : \mathcal{M} \to \mathcal{M}$ is called a *Markov operator* if it satisfies the following conditions:

(i) $P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2 \quad \text{for} \quad \lambda_1, \lambda_2 \in \mathbb{R}_+; \quad \mu_1, \mu_2 \in \mathcal{M}$.
(ii) $P\mu(X) = \mu(X) \quad \text{for} \quad \mu \in \mathcal{M}$.
(iii) There exists an operator $U : B(X) \to B(X)$ such that

$$< Uf, \mu > = < f, P\mu > \quad \text{for} \quad f \in B(X) \quad \text{and} \quad \mu \in \mathcal{M}.$$

The operator U is called *dual* to P. If in addition $Uf \in C(X)$ for $f \in C(X)$, then the Markov operator P is called *fellerian.*

A mapping $\pi : X \times \mathcal{B} \to [0,1]$ is called a *transition function* if $\pi(x, \cdot)$ is a probability measure for every $x \in X$ and $\pi(\cdot, A)$ is a measurable function for every $A \in \mathcal{B}$.

Having a transition function π we may define the corresponding Markov operator $P : \mathcal{M} \to \mathcal{M}$ by the formula

$$P\mu(A) = \int_X \pi(x, A)\mu(dx) \tag{2.1}$$

and its dual operator $U : B(X) \to B(X)$ by

$$Uf(x) = \int_X f(u)\pi(x, du).$$

Vice versa, having a Markov operator P we may define a function $\pi : X \times \mathcal{B} \to [0,1]$ setting

$$\pi(x, A) = P\delta_x(A). \tag{2.2}$$

Clearly the function π is a transition function such that condition (2.1) is satisfied.

Thus, condition (2.1), (2.2) show the one to one correspondence between the Markov operators and transition functions.

Finally note that Markov operator P is Fellerian if and only if its transition function has the following property:

$$x_n \to x \quad \text{implies} \quad \pi(x_n, \cdot) \to \pi(x, \cdot) \text{ (weakly)} .$$

If this condition is satisfied the transition function π is also called *Fellerian.*

A measure μ is called *invariant* (or *stationary*) with respect to P if $P\mu = \mu$. A Markov operator P is called *asymptotically stable* if there exists a stationary measure $\mu_* \in \mathcal{M}_1$ such that

$$\lim P^n\mu = \mu_* \quad \text{for every} \quad \mu \in \mathcal{M}_1. \tag{2.3}$$

Obviously a measure μ_* satisfying condition (2.3) is unique.

3. Classical results

In this section we assume that (X, ρ) is a Polish space (i.e. a complete, separable metric space).

An Iterated Function System (shortly IFS) is given by a family of continuous transformations

$$w_i : X \to X, \quad i \in I.$$

Assume also that there is given a family of continuous functions

$$p_i : X \to \mathbb{R}, \quad i \in I,$$

satisfying

$$p_i(x) > 0 \quad \text{and} \quad \sum_{i \in I} p_i(x) = 1 \quad \text{for} \quad x \in X.$$

The family $\{(w_i, p_i) : i \in I\}$ is called an IFS *with probabilities*. We assume that the set I of indexes is finite or countable.

Having an IFS $\{w_i : i \in I\}$ we define the corresponding Barnsley–Hutchinson multifunction F by

$$F(x) = \{w_i(x) : i \in I\} \quad \text{for} \quad x \in X \tag{3.1}$$

and having an IFS with probabilities $\{(w_i, p_i) : i \in I\}$ we define the corresponding Markov operator $P : \mathcal{M} \to \mathcal{M}$ by

$$P\mu(A) = \sum_{i \in I} \int_{w_i^{-1}(A)} p_i(x)\mu(dx) = \sum_{i \in I} \int_X 1_A(w_i(x))\mu(dx). \tag{3.2}$$

for $A \in \mathcal{B}$.

It is easy to verify that P is a Feller operator and its dual operator U is given by

$$Uf(x) = \sum_{i \in I} p_i(x) f(w_i(x)) \quad \text{for} \quad f \in C(X), \;\; x \in X.$$

We say that an IFS $\{w_i : i \in I\}$ is *asymptotically stable* if the corresponding multifunction F given by (3.1) is asymptotically stable.

Assume that for every $i \in I$ the function w_i is Lipschitzian with a Lipschitz constant L_i and that the function p_i is constant. The following facts are well known (see [1, 8, 11]).

Theorem 3.1. *If*

$$\sup_{i \in I} L_i < 1$$

then the multifunction F is asymptotically stable, the operator P is asymptotically stable and

$$A_* = \operatorname{supp} \mu_*,$$

where A_* is the attractor of F and μ_* is the invariant measure with respect to P.

Theorem 3.2. *If*

$$\sum_{i\in I} p_i L_i < 1$$

then the operator P is asymptotically stable.

The natural question arises, what are the geometric properties of the set $\operatorname{supp}\mu_*$ when the assumptions of Theorem 3.2 are satisfies. More precisely, we would like to define this set by a use of the transformations w_i without any probabilistic tools. The answer to this question will be given in the next section.

4. Semiattractors given by iterated function systems

Let X be a metric space. We say that an IFS $\{w_i : i \in I\}$ is *regular* if there is a nonempty subset I_0 such that the IFS $\{w_i : i \in I_0\}$ is asymptotically stable. The attractor of the subsystem $\{w_i : i \in I_0\}$ is called a *nucleous* of the system $\{w_i : i \in I\}$.

Regular IFS's have some important properties described by the following theorems (see [10]).

Theorem 4.1. *Let $\{w_i : i \in I\}$ be a regular IFS and let A_0 be a nucleous. Denote by F the corresponding to $\{w_i : i \in I\}$ Barnsley–Hutchinson multifunction. Then the sequence $(F^n(A_0))$ is convergent and its topological limit*

$$A_* = \operatorname{Lt} F^n(A_0) \tag{4.1}$$

does not depend on the choice of A_0.

The set A_* given by formula (4.1) will be called the *semiattractor* (or *semifractal*) corresponding to the regular IFS $\{w_i : i \in I\}$.

Theorem 4.2. *Let $\{w_i : i \in I\}$ be a regular IFS and A_* be the corresponding semiattractor. Then*

(i) $\operatorname{cl}(F(A_*)) = A_*$;
(ii) $A_* = \operatorname{cl}\bigcup_{n=1}^{\infty} F^n(A) = \operatorname{Lt} F^n(A)$ *for every* $A \subset A_*, A \neq \emptyset$;
(iii) A_* *is the smallest nonempty closed set subinvariant with respect to F (i.e. if A is a nonempty closed subset of X such that $F(A) \subset A$, then $A \supset A_*$).*

Theorem 4.3. *Let X be a Polish space. Assume that an IFS with probabilities $\{(w_i, p_i) : i \in I\}$ is asymptotically stable and that the IFS $\{w_i : i \in I\}$ is regular. Then*

$$A_* = \operatorname{supp}\mu_*,$$

where A_* is the semiattractor of $\{w_i : i \in I\}$ and μ_* is the invariant measure of $\{(w_i, p_i) : i \in I\}$.

Theorem 4.1, 4.2 and 4.3 are special cases of more general results which will be given in the next section. Here we only show how they are related with the question posed at the end of Section 3. Namely, assume that the functions $w_i : X \to X$ are Lipschitzian with constants L_i and that $\sum p_i L_i < 1$. Clearly there exist a nonempty set $I_0 \subset I$ such that $\sup_{i \in I_0} L_i < 1$. The IFS $\{w_i : i \in I_0\}$ is asymptotically stable and consequently the IFS $\{w_i : i \in I\}$ is regular. According to Theorem 4.3 the support of the invariant measure of $\{(w_i, p_i) : i \in I\}$ is equal to the semiattractor of $\{w_i : i \in I\}$.

5. Semiattractors of multifunctions

Let X be a metric space. Given a multifunction $F : X \to X$ consider the set

$$C = \bigcap_{x \in X} \operatorname{Li} F^n(x). \tag{5.1}$$

If the set C is nonempty, then the multifunction F is called *asymptotically semistable* and the set C is called the *semiattractor* of F.

Theorem 5.1. *Assume that F is a l.s.c. multifunction asymptotically semistable with the semiattractor C. Then the following conditions hold:*

(i) $C \subset \operatorname{Li} F^n(A)$ *for every* $A \subset X$, $A \neq \emptyset$;
(ii) $\operatorname{cl} F(C) = C$;
(iii) $\operatorname{Lt} F^n(A) = C$ *for every* $A \subset C$, $A \neq \emptyset$;
(iv) $C \subset A$ *for every nonempty closed subset* A *of* X *such that* $F(A) \subset A$.

Proof. Condition (i) is obvious. From (5.1) it follows that

$$F(C) \subset \bigcap_{x \in X} F(\operatorname{Li} F^n(x)).$$

Using Proposition 2.1 and the semicontinuity of F it is easy to verify that

$$F(\operatorname{Li} F^n(x)) \subset \operatorname{Li} F^n(x).$$

Thus we have

$$F(C) \subset C. \tag{5.2}$$

Since C is a closed set, we have also $\operatorname{cl} F(C) \subset C$. To prove the opposite inclusion observe that $F^n(C) \subset F(C)$ for $n \geq 1$ which, in turn implies $\operatorname{Li} F^n(C) \subset \operatorname{cl} F(C)$. Since $C \subset \operatorname{Li} F^n(C)$, this completes the proof of (ii).

To verify (iii) observe that (5.2) implies $\operatorname{Ls} F^n(C) \subset C$. Thus for an arbitrary nonempty set $A \subset C$ we have

$$C \subset \operatorname{Li} F^n(A) \subset \operatorname{Ls} F^n(A) \subset \operatorname{Ls} F^n(C) \subset C.$$

Condition (iv) can be verified as follows. Inclusion $F(A) \subset A$ implies $F^n(A) \subset A$ for $n \geq \mathbb{N}$. Consequently

$$C \subset \operatorname{Li} F^n(A) \subset A.$$

□

Theorem 5.2. *Let $F : X \to X$ be a l.s.c. multifunction. Assume that there exists a l.s.c. and asymptotically semistable multifunction $F_0 : X \to Y$ such that $F_0(x) \subset F(x)$, $x \in X$. Then F is asymptotically semistable and its semiattractor C is given by the formula*

$$C = \mathrm{Lt}\, F^n(C_0) = \mathrm{cl} \bigcup_{n=1}^{\infty} F^n(C_0), \tag{5.3}$$

where C_0 is the semiattractor of F_0.

Proof. Since $C_0 \subset C$ the multifunction F is asymptotically semistable. The first equality in (5.3) follows from Theorem 5.1 (iii) with $A = C_0$. Now, observe that $F^n(C_0) \subset C$ for $n = 1, 2, \ldots$. Hence

$$\mathrm{cl} \bigcup_{n=1}^{\infty} F^n(C_0) \subset C.$$

Using this inclusion and the first equality in (5.3) we obtain the second equality of (5.3). The proof is completed. □

6. Markov multifunctions

Let (X, ρ) be a Polish space. Given a Markov operator P and the corresponding transition function π we define a multifunction $\Gamma : X \to X$ by the formula

$$\Gamma(x) = \mathrm{supp}\, \pi(x, \cdot) = \mathrm{supp}\, P\delta_x.$$

This multifunction will be called the *Markov multifunction* corresponding to P or the *support* of π. It is easy to see that Γ is closed valued and measurable. Vice versa, we have the following

Theorem 6.1. *Let $F : X \to X$ be a measurable, closed valued multifunction. Then there exists a transition function $\pi : X \times \mathcal{B} \to [0,1]$ such that F is the support of π.*

Proof. According to Kuratowski–Ryll Nardzewski Theorem (see [2]) there exists a sequence (f_n) of measurable functions $f_n : X \to X$ such that

$$F(x) = \mathrm{cl}\{f_n(x) : n \in \mathbb{N}\} \quad \text{for} \quad x \in X.$$

We define the function $\pi : X \times \mathcal{B} \to [0,1]$ by

$$\pi(x, A) = \sum_{n=1}^{\infty} p_n \delta_{f_n(x)}(A),$$

where (p_n) is a sequence of positive numbers such that $\sum_{n=1}^{\infty} p_n = 1$ and δ_u stands for the δ–Dirac measure supported at u. A simple calculation shows that π is a transition function and that F is the support of π. □

Theorem 6.2. *Assume that $\pi : X \times \mathcal{B} \to [0,1]$ is a Fellerian transition function. Then the corresponding Markov multifunction Γ is l.s.c.*

Proof. Fix an $x \in X$ and consider a sequence $(x_n) \subset X$ converging to x. Since π is Fellerian, the corresponding sequence of measures $(\pi(x_n,\cdot))$ converges weakly to the measure $\pi(x,\cdot)$. By virtue of Proposition 2.2 we have $\Gamma(x) \subset \mathrm{Li}\,\Gamma(x_n)$. Thus the statement of Theorem 6.2 follows from Proposition 2.1. □

Theorem 6.3. *Assume that $F : X \to X$ is a l.s.c. multifunction with closed values. Then there exists a Fellerian transition function $\pi : X \times \mathcal{B} \to [0,1]$ such that F is the support of π.*

Proof. Consider a multifunction $\Phi : X \to \mathcal{M}_1$ given by the formula

$$\Phi(x) = \{\mu \in \mathcal{M}_1 : \operatorname{supp}\mu \subset F(x)\}$$

Clearly Φ is convex and closed valued. It is easy to verify that Φ is l.s.c. . Observe that $\mathcal{M}_1$ is a convex subset of the linear space $\mathcal{M}_{\mathrm{sig}}$ of all signed Borel measures on X and that $\mathcal{M}_1$ is complete with respect to the Fortet–Mourier metric (see [3]). Thus the conditions of the Michael Selection Theorem (see [15]) are satisfied and consequently there exists a sequence (φ_n) of continuous functions $\varphi_n : X \to \mathcal{M}_1$ such that

$$\Phi(x) = \mathrm{cl}\{\varphi_n(x) : n \in \mathbb{N}\}.$$

Let (p_n) be a sequence of positive numbers such that $\Sigma p_n = 1$. Define $\pi : X \times \mathcal{B} \to [0,1]$ by

$$\pi(x,A) = \sum_{n=1}^{\infty} p_n \varphi_n(x)(A).$$

Obviously π is a transition function. To complete the proof it suffices to verify that F is equal to the support of π. □

In order to prove the next result we need two simple lemmas concerning the support of the measure $P\mu$ (see [9], [11]).

Lemma 6.4. *Let $P : \mathcal{M} \to \mathcal{M}$ be a Fellerian operator. If $\mu_1, \mu_2 \in \mathcal{M}_1$ and $\operatorname{supp}\mu_1 \subset \operatorname{supp}\mu_2$ then $\operatorname{supp}P\mu_1 \subset \operatorname{supp}P\mu_2$.*

Lemma 6.5. *Let $P : \mathcal{M} \to \mathcal{M}$ be a Markov operator corresponding to a Fellerian transition function $\pi : X \times \mathcal{B} \to [0,1]$. Further let Γ be a support of π. Then for every $\mu \in \mathcal{M}$ and $n \in \mathbb{N}$ we have*

$$\operatorname{supp}P^n\mu = \mathrm{cl}\,\Gamma^n(\operatorname{supp}\mu).$$

Theorem 6.6. *If a Fellerian Markov operator P is asymptotically stable, then the corresponding Markov multifunctions Γ is asymptotically semistable and*

$$C = \operatorname{supp}\mu_*$$

where C is the semiattractor of Γ and μ_* is the measure invariant with respect to P.

Proof. Fix an arbitrary $x \in X$ and let $\mu = \delta_x$. Since P is asymptotically stable the sequence $(P^n\mu)$ converges weakly to μ_*. By Proposition 2.2 and Lemma 6.5 we have

$$\operatorname{supp}\mu_* \subset \operatorname{Li}\operatorname{supp} P^n\mu = \operatorname{Li}\Gamma^n(x).$$

This implies that $\operatorname{supp}\mu_* \subset C$.

To prove the opposite inclusion fix a point $z \notin \operatorname{supp}\mu_*$ and choose $\varepsilon > 0$ such that

$$B(z,\varepsilon) \cap \operatorname{supp}\mu_* = \emptyset.$$

Let $x \in \operatorname{supp}\mu_*$ and $\mu = \delta_x$. By Lemma 6.4 and 6.5 we have

$$\Gamma^n(x) \subset \operatorname{supp} P^n\mu \subset \operatorname{supp} P^n\mu_* = \operatorname{supp}\mu_* \quad \text{for} \quad n \in \mathbb{N}.$$

Thus

$$\Gamma^n(x) \cap B(z,\varepsilon) = \emptyset.$$

It follows that $z \notin \operatorname{Li}\Gamma^n(x)$ and consequently $z \notin C$. The proof is complete. □

7. A zero–one theorem

Let P be a Fellerian operator and Γ the corresponding Markov multifunction.

Theorem 7.1. *Assume that P has a unique invariant probability measure μ_*. Then*

$$\mu_*(D) = 0 \quad \text{or} \quad \mu_*(D) = 1 \tag{7.1}$$

for every Borel set $D \subset X$ such that $\Gamma(D) \subset D$.

Proof. Let U be the operator dual to P. Fix a Borel set $D \subset X$ such that $\Gamma(D) \subset D$. Let $x \in D$ be an arbitrary point. Since $\operatorname{supp}\pi(x,\cdot) \subset D$, we have $\pi(x, X \setminus D) = 0$. From this and the equality $U1_A = \pi(\cdot, A)$ it follows that $U1_{X\setminus D}(x) = 0$. Define

$$\mu_0(A) = \mu_*(A \cap D) \quad \text{for} \quad A \in \mathcal{B}.$$

A simple calculation shows that μ_0 is invariant with respect to P. If $\mu_*(D) = 0$ the alternative (7.1) is obviously satisfied. If $\mu_*(D) > 0$ it can be proved that $\mu_0 = \mu_*$ and consequently $\mu_*(D) = \mu_0(X) = 1$. □

Theorem 7.2. *Assume that P has a unique invariant probability measure μ_*. Then*

$$\mu_*(D) = 0 \quad \text{or} \quad \mu_*\Big(\bigcap_{n=0}^{\infty} \Gamma^n(D)\Big) = 1 \tag{7.2}$$

for every Borel set $D \subset X$ satisfying $\Gamma(D) \subset D$.

Proof. Assume that $\mu_*(D) > 0$ (otherwise it is nothing to prove). Let $D_n = \Gamma^n(D)$. To prove (7.2) it suffices to show that $\mu_*(D_n) = 1$ for $n \in \mathbb{N}$. This can be done by an induction argument. Indeed, $\mu_*(D_0) = \mu_*(D) = 1$ by Theorem 7.1. Now assume that $\mu_*(D_n) = 1$ for some fixed $n \in \mathbb{N}$. For arbitrary $x \in D_n$ we have

$$\pi(x, \Gamma(D_n)) \geq \pi(x, \Gamma(x)) = 1.$$

and consequently

$$U1_{\Gamma(D_n)}(x) = 1 \quad \text{for} \quad x \in D_n.$$

Using the fact that μ_* is invariant with respect to P and the last equality we have

$$\begin{aligned} \mu_*(D_{n+1}) &= \mu_*(\Gamma(D_n)) = < 1_{\Gamma(D_n)}, \mu_* > \\ &= < 1_{\Gamma(D_n)}, P\mu_* > = < U1_{\Gamma(D_n)}, \mu_* > \\ &\geq \int_{D_n} U1_{\Gamma(D_n)}(x)\mu_*(dx) = \mu_*(D_n) = 1. \end{aligned}$$

The proof is completed. □

In the case when P is defined by an iterated function system, Theorem 7.2 was proved by J.Goodman in [5]. The general situation was discussed in [13].

8. A concentration dimension of measures

Given a measure $\mu \in \mathcal{M}_1$ we define the *lower* and *upper concentration dimension* of μ by

$$\underline{\dim}_L \mu = \liminf_{r \to 0} \frac{\log Q_\mu(r)}{\log r} \tag{8.1}$$

and

$$\overline{\dim}_L \mu = \limsup_{r \to 0} \frac{\log Q_\mu(r)}{\log r}, \tag{8.2}$$

where Q_μ is the Lévy concentration function (see [6]) given by the formula

$$Q_\mu(r) = \sup\{\mu(B(x,r)) : x \in X\}.$$

If $\underline{\dim}_L \mu = \overline{\dim}_L \mu$, then this common value is called the *concentration dimension* of μ and it is denoted by $\dim_L \mu$. The Hausdorff dimension of a set $A \subset X$ will be denoted by $\dim_H A$.

The concentration dimension has some important properties. First, it is relatively easy to be calculated. Moreover, it is strongly related to the Hausdorff dimension and the mass distribution principle (see [4]. Prop. 2.1). Using this principle it is easy to verify that

$$\dim_H K \geq \underline{\dim}_L \mu$$

for every $K \subset X$ and $\mu \in \mathcal{M}_1$ such that $\operatorname{supp} \mu \subset K$. Further using the Frostman Lemma (see [14], Thm. 8.17) one can prove the following

Theorem 8.1. *If $K \subset X$ is a nonempty, compact set, then*

$$\dim_H K = \sup \underline{\dim}_L \mu,$$

where the supremum is taken over all $\mu \in \mathcal{M}_1$ such that $\operatorname{supp} \mu \subset K$.

The following estimates of the upper and lower concentration dimension for fractal measures are proved in [12].

Theorem 8.2. *Let $\{(w_i, p_i) : i \in I\}$ be an IFS with probabilities having an invariant measure $\mu_* \in \mathcal{M}_1$. Assume that the functions $w_i, i \in I$, are Lipschitzian with Lipschitz constants L_i and the set $J = \{i \in I : L_i < 1\}$ is nonempty. Then*

$$\overline{\dim}_L \mu_* \le \inf_{i \in J} \frac{\log \alpha_i}{\log L_i},$$

where

$$\alpha_i = \inf_{x \in X} p_i(x).$$

To obtain the lower estimate of $\underline{\dim}_L \mu_*$ we need more restrictive assumptions concerning the transformations w_i. Let $I = \bigcup_{j=1}^m I_j$, where $I_1, \dots, I_m$ are nonempty and pairwise disjoint. Further, let $K \subset X$ be a nonempty set. Define

$$K_j = \bigcup_{i \in I_j} w_i(K) \quad \text{for} \quad j = 1, \dots, m.$$

We say that the family $\{w_i : i \in I\}$ satisfies the *mixed Moran condition* with respect to the set K and the partition $I_1, \dots, I_m$ if $K_j \subset K$ for $j = 1, \dots, m$ and

$$\inf\{\rho(x, y) : x \in K_{j_1}, y \in K_{j_2}\} > 0 \quad \text{for} \quad j_1, j_2 \in \{1, \dots, m\}, \quad j_1 \neq j_2.$$

Theorem 8.3. *Let $\{(w_i, p_i) : i \in I\}$ be an IFS with probabilities having an invariant measure $\mu_* \in \mathcal{M}_1$. Assume that the family $\{w_i : i \in I\}$ satisfies the mixed Moran condition with respect to the set $K = \operatorname{supp} \mu_*$ and a partition $I_1 \dots, I_m$. Moreover assume that the functions w_i satisfy the condition*

$$\rho(w_i(x), w_i(y)) \ge l_i \rho(x, y) \quad \text{for} \quad x, y \in X, \quad i \in I, \tag{8.3}$$

where l_i are constants such that

$$0 < \inf_{i \in I_j} l_i < 1 \quad \text{for} \quad j = 1, \dots, m. \tag{8.4}$$

Then

$$\underline{\dim}_L \mu \ge \min_{1 \le j \le m} \frac{\log \beta_j}{\log M_j},$$

where

$$\beta_j = \sum_{i \in I_j} \sup_{x \in X} p_i(x) \quad \text{and} \quad M_j = \inf_{i \in I_j} l_i.$$

Using the last theorem we can obtain an evaluation of the Hausdorff dimension of fractals and semifractals. Let I be an at most countable set of indexes. Consider a family of Lipschitzian transformation $w_i : X \to X$, $i \in I$. Assume that $\inf_{i \in I} L_i < 1$, where L_i is the Lipschitz constant of w_i. Obviously IFS $\{w_i : I \in I\}$ is regular. Let A_* be the corresponding semiattractor. In addition assume that the family $\{w_i : i \in I\}$ satisfies conditions (8.3),(8.4) and the mixed Moran condition with respect to the set A_* and a partition $I_1, \dots, I_m$ of I. Then

$$\dim_H A_* \ge d,$$

where d is the unique positive number given by the condition

$$\sum_{j=1}^{m}(M_j)^d = 1 \quad \text{with} \quad M_j = \inf_{i\in I_j} l_i.$$

Indeed, for $j = \{1,\ldots,m\}$ define $\beta_j = (M_j)^d$. Evidently $0 < \beta_j < 1$. Let $p_i > 0$, $i \in I$ be constants such that $\Sigma_{i\in I_j} p_i = \beta_j$ and $\Sigma p_i L_i < 1$. Obviously the IFS with probabilities $\{(w_i, p_i) : i \in I\}$ has an invariant measure μ_* and $\operatorname{supp}\mu_* = A_*$. From Theorems 8.1 and 8.3 it follows that

$$\dim_H A_* \geq \underline{\dim}_L \mu_* \geq \min_{1\leq j\leq m} \frac{\log \beta_j}{\log M_j} = d.$$

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