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The Dirac Operator on the Fuzzy Sphere

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Abstract

We introduce the Fuzzy analog of spinor bundles over the sphere on which the non-commutative analog of the Dirac operator acts. We construct the complete set of eigenstates including zero modes. In the commutative limit we recover known results.

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1 Introduction

The basic notions of non-commutative geometry were developed and applied to various physical problems [1,2]. The essence of this approach consists in reformulating first the geometry in terms of commutative algebras and modules of smooth functions, and then generalizing them to their non-commutative analogs. Using for this construction the set of $n \times n$ complex matrices, the formalism of non-commutative matrix geometry was developed in [3,5] and applied in [6] to the description of the non-commutative Fuzzy sphere. This approach was extended to non-commutative homogenous spaces in [7] using a coherent state approach, and in particular it was used for the description of the Fuzzy sphere and Fuzzy hyperboloid.

The fields on a standard sphere S^2 provide a basis for field theoretical models such as, e.g. the Thirring model, or the Schwinger model. The Schwinger model on a commutative sphere was described in detail in [8]. First steps to its Fuzzy version were formulated in [9] within the matrix formulation.

In this paper we generalize some of the commutative results presented in [8] to the non-commutative case. We extend the coherent state approach to the Fuzzy sphere [7] by developing the non-commutative version of spinor calculus. Section 2 contains the reformulation of the standard spinor formalism on S^2 to the more algebraic language. The generalization to the non-commutative case is described in Section 3, and a complete set of eigenstates of the Dirac operator on the Fuzzy sphere is found. The results are briefly discussed in Section 4.

2 Commutative case

In this section we describe various forms of spinors on manifolds closely related to the usual sphere S^2. Their relations are presented in the following diagram:

$$\begin{tikzcd}
\mathcal{S}(C_0^2) \arrow[rr] \arrow[dr] \arrow[ddd] & & \tilde{\mathcal{S}}(S^2) \arrow[ddd] \\
& \mathcal{S}(S^3) \arrow[ur] \arrow[d] & \\
& S^3 \arrow[dr] & \\
C_0^2 \arrow[rr] \arrow[ur] & & S^2
\end{tikzcd}$$

where $M = S^2, S^3, C_0^2$ are manifolds in question, and $\mathcal{S}(M)$ denotes the corresponding spinor bundles; $\tilde{\mathcal{S}}(S^2)$ is specified below.

The projection $C_0^2 \to S^3$ denotes a restriction of

$$C_0^2 = \{\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \in \mathbf{C^2}, \chi^+\chi > 0\}$$

to the sphere S^3 with a fixed radius r,

$$S^3 = \{\chi \in C_0^2,\ \chi^+\chi = r = const\},$$

and $S^3 \to S^2$ we choose as follows:

(i) on $U_+ = \{\chi \in S^3, \chi_1 \neq 0\}$ we put

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \mapsto \chi' = \begin{pmatrix} \chi_1' \\ \chi_2' \end{pmatrix} = \left(\frac{\chi_1^*}{\chi_1}\right)^{1/2} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \tag{1}$$

(ii) and on $U_- = \{\chi \in S^3, \chi_2 \neq 0\}$ we put

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \mapsto \chi'' = \begin{pmatrix} \chi_1'' \\ \chi_2'' \end{pmatrix} = \left(\frac{\chi_2^*}{\chi_2}\right)^{1/2} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \tag{2}$$

We interpret χ' and χ'' as local coordinates on the northern hemisphere V_+ of S^2 (i.e. S^2 except south pole), or on the southern hemisphere V_- of S^2 (i.e. S^2 without north pole), respectively. The corresponding cartesian coordinates of S^2 are given as

$$x'_i = \chi'^{+}\sigma_i\chi' \in V_+,\ \ i = 1,2,3,$$

$$x''_i = \chi''^{+}\sigma_i\chi'' \in V_-,\ \ i = 1,2,3, \tag{3}$$

where $\sigma_k = (\sigma^k_{\alpha\beta})$, $k = 1,2,3$, are Pauli matrices. On $V_+\bigcap V_-$ they coincide: $x'_i = x''_i = x_i, i = 1,2,3$, and in fact

$$S^2 = \{(x_1,x_2,x_3) \in \mathbf{R^3}; x_i = \chi^+\sigma_i\chi, \chi \in S^3\}\ .$$

We note, that the transition function between χ' andχ'' is

$$\chi' = e^{i\varphi}\chi'' \text{ on } V_+\bigcap V_-\ . \tag{4}$$

This can be most easily seen if we parametrize χ in terms of Euler angles

$$\chi_1 = r^{1/2}\cos\frac{\theta}{2}e^{-\frac{i}{2}(\varphi+\psi)}\ ,$$

$$\chi_2 = -r^{1/2}\sin\frac{\theta}{2}e^{\frac{i}{2}(\varphi-\psi)}\ . \tag{5}$$

Then

$$\chi' = e^{\frac{i}{2}(\varphi+\psi)}\chi = \begin{pmatrix} r^{1/2}\cos\frac{\theta}{2} \\ r^{1/2}\sin\frac{\theta}{2}e^{i\varphi} \end{pmatrix} \text{ on } V^+\ ,$$

$$\chi'' = e^{-\frac{i}{2}(\varphi-\psi)}\chi = \begin{pmatrix} r^{1/2}\cos\frac{\theta}{2}e^{-i\varphi} \\ r^{1/2}\sin\frac{\theta}{2} \end{pmatrix} \text{ on } V^-\ . \tag{6}$$

The spinor bundle $\mathcal{S}(C^2_0)$ is a trivial one. Its sections have the form

$$\Psi(\chi,\chi^*) = \begin{pmatrix} \Psi_1(\chi,\chi^*) \\ \Psi_2(\chi,\chi^*) \end{pmatrix}\ , \tag{7}$$

where

$$\Psi_\alpha = \sum a^\alpha_{n_1n_2m_1m_2}\chi_1^{*n_1}\chi_2^{*n_2}\chi_1^{m_1}\chi_2^{m_2}\ . \tag{8}$$

The sections of the spinor bundle $\mathcal{S}(S^3)$ are obtained simply by the restriction

$$\chi^+\chi = r = \text{const.} \tag{9}$$

Let us consider differential operators

$$J_k = \frac{1}{2}(\chi_\alpha \sigma_k^{\alpha\beta} \partial_{\chi_\alpha} - \chi_\alpha^* \sigma_k^{*\alpha\beta} \partial_{\chi_\beta^*}) \ ,$$

$$K = \chi_\alpha \partial_{\chi_\alpha} - \chi_\alpha^* \partial_{\chi_\alpha^*} \ , \ k = 1,2,3 \ , \tag{10}$$

acting on sections of $\mathcal{S}(C_0^2)$. As they represent flows on S^3 (they are r independent differential operators in θ, φ and ψ) their action on sections of $\mathcal{S}(S^3)$ remains unchanged.

The sections of $\mathcal{S}(C_0^2)$ and $\mathcal{S}(S^3)$ have a natural grading: subbundles $\mathcal{S}_k(C_0^2)$, or $\mathcal{S}_k(S^3)$, $k \in \mathbf{Z}$, are formed by sections of the form (8) with fixed

$$k = n_1 + n_2 - m_1 - m_2 \ . \tag{11}$$

We stress that the sections of $\mathcal{S}_k(C_0^2)$, or $\mathcal{S}_k(S^3)$, are eigenstates of the operator K:

$$K\Psi = k\Psi, \ \Psi \in \mathcal{S}_k(C_0^2), \ \text{or} \ \mathcal{S}_k(S^3) \ . \tag{12}$$

Obviously, $\mathcal{S}_k(C_0^2)$, or $\mathcal{S}_k(S^3)$, are modules with respect to multiplication by elements of the algebra $\mathcal{A}_0$ formed by functions of the form

$$f(\chi, \chi^*) = \sum a_{n_1 n_2 m_1 m_2} \chi_1^{*n_1} \chi_2^{*n_2} \chi_1^{m_1} \chi_2^{m_2} \ . \tag{13}$$

with $n_1 + n_2 = m_1 + m_2$. Due to relations

$$r = \chi^+\chi, \ x_i = \chi^+\sigma_i\chi \ , \ i = 1,2,3 \ ,$$

the algebra $\mathcal{A}_0$ is for fixed r isomorphic to the algebra of functions on S^2.

The sections of the spinor bundle over S^2 have the form

$$\Psi'(\chi', \chi'^*) \text{ on } V_+ \ ,$$

$$\Psi''(\chi'', \chi''^*) \text{ on } V_-$$

and on $V_+ \bigcap V_-$ the relation

$$\Psi'(\chi', \chi'^*) = h(\chi'', \chi''^*)\Psi''(\chi'', \chi''^*) \ , \tag{14}$$

holds, where $|h(\chi'', \chi''^*)| = 1$. The phase factor h has the following general form

$$h = e^{ik\varphi + i\delta} \ , \ k \in \mathbf{Z} \ , \tag{15}$$

where the angle φ is given by $\chi'/\chi'' = e^{i\varphi}$, and $\delta = \delta(x)$ is a function defined globally on S^2. The winding number $k \in \mathbf{Z}$ classifies spinor bundles over S^2, and by $\mathcal{S}_k(S^2)$ we denote the spinor bundle specified by the transition rule (14) and (15).

Two sections Ψ_0 and Ψ_1 we call equivalent if

$$\Psi_0' = \Psi_1' \ , \ \Psi_0'' = e^{i\delta}\Psi_1'' \ ,$$

where $\delta = \delta(x)$ is a globally defined phase factor. The equivalence class of a given section $\Psi \in \mathcal{S}_k(S^2)$ we denote as $\tilde{\Psi}$: it can be characterized by the representative section given as

$$\tilde{\Psi}'_\alpha(\chi', \chi'^*) = \sum a^\alpha_{n_1 n_2 m_1 m_2} \chi'^{*n_1}_1 \chi'^{*n_2}_2 \chi'^{m_1}_1 \chi'^{m_2}_2 \text{ on } V_+ \ ,$$

$$\tilde{\Psi}''_\alpha(\chi'', \chi''^*) = \sum a^\alpha_{n_1 n_2 m_1 m_2} \chi''^{*n_1}_1 \chi''^{*n_2}_2 \chi''^{m_1}_1 \chi''^{m_2}_2 \text{ on } V_- \ , \tag{16}$$

where $k = n_1 + n_2 - m_1 - m_2$. We stress that the coefficients $a^\alpha_{n_1 n_2 m_1 m_2}$ are the same in both expressions, so that the transition rule

$$\tilde{\Psi}'(\chi', \chi'^*) = e^{-ik\varphi}\tilde{\Psi}''(\chi'', \chi''^*) \tag{17}$$

is satisfied (we shall denote this representative section again by $\tilde{\Psi}$). As $\tilde{\mathcal{S}}_k(S^2)$ we denote the bundle formed by sections of the form (16). The bundle $\tilde{\mathcal{S}}(S^2)$ is defined as the direct Whitney sum: $\tilde{\mathcal{S}}(S^2) = \bigoplus \tilde{\mathcal{S}}_k(S^2)$.

We note that there is a one-to-one mapping between the sections from $\tilde{\mathcal{S}}_k(S^2)$ given by (16) and the sections from $\mathcal{S}_k(C_0^2)$ (or $\mathcal{S}_k(S^3)$) defined in (8) with $k = n_1+n_2-m_1-m_2$: the coefficients $a^{\alpha}_{n_1n_2m_1m_2}$ are the same in both eqs. (8) and (16).

The mapping $\mathcal{S}_k(S^3) \to \tilde{\mathcal{S}}_k(S^2)$ means the following mapping of the section $\Psi \in \mathcal{S}_k(S^3)$ to the section $\tilde{\Psi} \in \tilde{\mathcal{S}}(S^2)$ given by

$$\tilde{\Psi}'(\chi', \chi'^*) = e^{\frac{i}{2}k(\varphi+\psi)}\Psi(\chi, \chi^*) \text{ on } V_+ \ ,$$

$$\tilde{\Psi}''(\chi'', \chi''^*) = e^{\frac{-i}{2}k(\varphi-\psi)}\Psi(\chi, \chi^*) \text{ on } V_- \ . \tag{18}$$

Note: The restriction from $\mathcal{S}_k(S^2)$ to $\tilde{\mathcal{S}}_k(S^2)$ is inessential for problems invariant with respect to local gauge transformations. We stress however, that this restriction do not represent the gauge fixing: local gauge transformations defined on S^2 are still allowed.

The determination of the spectrum of Dirac operator is a local gauge invariant problem. The (free) Dirac operator $\tilde{D}_k : \tilde{\mathcal{S}}(S^2) \to \tilde{\mathcal{S}}(S^2)$ is a differential operator defined as (see [7]):

$$\tilde{D}'_k = r[i\sigma'^{\mu}(\partial'_{\mu} + A'_{\mu}) + \frac{1}{r}] \ \text{ on } V_+ \ ,$$

$$\tilde{D}''_k = r[i\sigma''^{\mu}(\partial''_{\mu} + A''_{\mu}) + \frac{1}{r}] \ \text{ on } V_- \ , \tag{19}$$

where the k-monopole field is given as

$$A'_{\mu} = ik\chi'^{+}\partial'_{\mu}\chi' \ \text{ on } V_+ \ ,$$

$$A''_{\mu} = ik\chi''^{+}\partial''_{\mu}\chi'' \ \text{ on } V_- \ , \tag{20}$$

The term $1/r$ in (19) is the curvature term [9] and guarantees the chiral invariance. The factor r in front makes the Dirac operator dimensionless.

On $V_+ \bigcap V_-$ the fields A'_{μ} and A''_{μ} are related by the gauge transformation

$$A'_{\mu} = A''_{\mu} - ih\partial_{\mu}h^{-1} \ , \ h = e^{ik\varphi} \ , \tag{21}$$

where ∂_μ denotes the derivatives $\partial_0 = \partial_\theta$ and $\partial_1 = \partial_\varphi$ in the local coordinates $\xi^0 = \theta$ and $\xi^1 = \varphi$ on $V_+ \bigcap V_-$. The corresponding coordinate dependent σ-matrices are given as $\sigma^\mu = \sigma^i e_i^\mu$, where $e_i^\mu = J_i \xi^\mu$. We stress, that (21) together with (17) guarantees that the operators $\tilde{D}'_k$ and $\tilde{D}''_k$ coincide on $V_+ \bigcap V_-$.

Note: Explicitely on $V_+ \bigcap V_-$ we have

$$\sigma^\varphi = \begin{pmatrix} 1, \ -\cot\theta e^{-i\varphi} \\ -\cot\theta e^{i\varphi}, \ -1 \end{pmatrix} ,$$

$$\sigma^\theta = \begin{pmatrix} 0, \ -ie^{-i\varphi} \\ ie^{i\varphi}, \ 0 \end{pmatrix} ,$$

and

$$A'_\theta = 0, \ A'_\varphi = \frac{k}{2}(\cos\theta - 1) ,$$
$$A''_\theta = 0, \ A''_\varphi = \frac{k}{2}(\cos\theta + 1) . \tag{22}$$

The eigenvalue problem

$$\tilde{D}'_k \tilde{\Psi}'(\chi', \chi'^*) = \lambda \tilde{\Psi}'(\chi', \chi'^*) \ \text{ on } V_+ ,$$
$$\tilde{D}''_k \tilde{\Psi}''(\chi'', \chi''^*) = \lambda \tilde{\Psi}''(\chi'', \chi''^*) \ \text{ on } V_- , \tag{23}$$

in $\tilde{\mathcal{S}}(S^2)$, can be transformed to a problem in $\mathcal{S}_k(S^3)$

$$D_k \Psi = \lambda \Psi \quad , \Psi \in \mathcal{S}_k(S^3) , \tag{24}$$

by putting

$$\Psi = e^{-\frac{i}{2}k(\varphi+\psi)} \tilde{\Psi}' \text{ on } V_+ ,$$
$$\Psi = e^{\frac{i}{2}k(\varphi-\psi)} \tilde{\Psi}'' \text{ on } V_- , \tag{25}$$

and

$$D_k = e^{-\frac{i}{2}k(\varphi+\psi)} \tilde{D}'_k e^{\frac{i}{2}k(\varphi+\psi)} \text{ on } V_+ ,$$
$$D_k = e^{\frac{i}{2}k(\varphi-\psi)} \tilde{D}''_k e^{-\frac{i}{2}k(\varphi-\psi)} \text{ on } V_- . \tag{26}$$

A straightforward caclculation gives on V_+ and V_- the same expression

$$D_k = \sigma_j (J_j - \frac{k}{2} \frac{x_j}{r}) + 1 \ . \tag{27}$$

The eigenvalue problem (24) is solved, e.g. in [8]. In the next section we shall formulate and solve it in the non-commutative case.

3 Non-commutative version

In the non-commutative case we insert the spinor variables $\chi_\alpha, \chi^*_\alpha, \alpha = 1, 2$, by the annihilation and creation operators satisfying the commutation relations

$$[\chi_\alpha, \chi_\beta] = [\chi^*_\alpha, \chi^*_\beta] = 0 \ ,$$

$$[\chi_\alpha, \chi^*_\beta] = \frac{\lambda}{2} \delta_{\alpha\beta} \ , \tag{28}$$

where λ is a positive constant. The operators $\chi_\alpha, \chi^*_\alpha, \alpha = 1, 2$ act in the Fock space $\mathcal{F}$ spanned by the vectors

$$|n_1, n_2\rangle = \sqrt{\frac{(2/\lambda)^{n_1+n_2}}{n_1! \ n_2!}} {\chi_1}^{*n_1} {\chi_2}^{*n_2} |0\rangle \ , \tag{29}$$

where $n_\alpha = 0, 1, 2, \ldots, \alpha = 1, 2$, and $|0\rangle$ is the vacuum satisfying $\chi_\alpha|0\rangle = 0 \ , \alpha = 1, 2$.

As $\mathcal{S}_k, k \in \mathbf{Z}$, we denote the linear space spanned by the two-component normal products

$$\Psi(\chi, \chi^*) = \begin{pmatrix} \Psi_1(\chi, \chi^*) \\ \Psi_2(\chi, \chi^*) \end{pmatrix} \ ,$$

where

$$\Psi_\alpha = \sum a^\alpha_{n_1 n_2 m_1 m_2} \chi_1^{*n_1} \chi_2^{*n_2} \chi_1^{m_1} \chi_2^{m_2} \ , \tag{30}$$

and the indices n_α and m_β are restricted by the condition that

$$k = n_1 + n_2 - m_1 - m_2 = \text{const.} \ . \tag{31}$$

Obviously, $\mathcal{S}_k$ is a two sided module with respect to the multiplication by normal products of the form

$$\chi_1^{*n_1}\chi_2^{*n_2}\chi_1^{m_1}\chi_2^{m_2}\ , \tag{32}$$

where $n_1+n_2=m_1-m_2$. The symbol $\mathcal{A}$ will denote the associative algebra spanned by the elements (32).

The operators J_k and K act on $\chi_\alpha,\ \chi^*_\alpha, \alpha=1,2$, in the same way as in the commutative case

$$\begin{aligned} J_k\chi_\alpha &= \frac{1}{2}\sigma_k^{\alpha\beta}\chi_\beta\ ,\\ J_k\chi^*_\alpha &= \frac{1}{2}\sigma_k^{*\alpha\beta}\chi^*_\beta\ ,\\ K\chi_\alpha = -\chi_\alpha\ &, K\chi^*_\alpha = \chi^*_\alpha\ , \end{aligned} \tag{33}$$

and their action is extended to $\mathcal{S}_k$ and to $\mathcal{A}$ by the Leibniz rule. Obviously,

$$J_k : \mathcal{S}_k \to \mathcal{S}_k\ ,\ \ K : \mathcal{S}_k \to \mathcal{S}_k\ , \tag{34}$$

and the same is true for $\mathcal{A}$ too.

We restrict the Fock space $\mathcal{F}$ to its subspace

$$\mathcal{F}_n = \{|n_1, n_2\rangle \in \mathcal{F}; n_1+n_2 = n\}\ , n=0,1,2,\dots\ .$$

The operator $\chi^+\chi = \chi^+{}_\alpha\chi_\alpha$ takes in $\mathcal{F}_n$ the value $\lambda l\ , l=n/2$.

In $\mathcal{S}_k$ we define the scalar product as follows: if $k>0$ and $\Psi_1,\Psi_2\in\mathcal{S}_k$ we put

$$(\Psi_1,\Psi_2)_k = \mathrm{Tr}_n(\Psi_1^+\Psi_2)\ ,\ \ k>0\ , \tag{35}$$

where Tr_n denotes the trace in the space $\mathcal{F}_n$. Similarly, for $k<0$ and $\Psi_1,\Psi_2\in\mathcal{S}_k$ we define

$$(\Psi_1,\Psi_2)_k = \mathrm{Tr}_n(\Psi_2\Psi_1^+)\ ,\ \ k<0\ , \tag{36}$$

where the trace we take again in the space $\mathcal{F}_n$. Moreover, we restrict the admissible values of k to $|k|\le n$.

The operators

$$x_i = \chi^+ \sigma_i \chi \ , \ i = 1,2,3 \ , \tag{37}$$

satisfy the commutation relations

$$[x_i, x_j] = i\lambda \varepsilon_{ijk} \ x_k \ . \tag{38}$$

The operator x_i^2 commutes with all operators x_i, $i = 1,2,3$, and in $\mathcal{F}_n$ it takes the value $\lambda^2 l(l+1)$.

In the non-commutative case we take the Dirac operator in $\mathcal{S}_k$ in the same form as before

$$D_k = \sigma_j (J_j - \frac{k}{2}\xi_j) + 1 \ , \tag{39}$$

where

$$\xi_j = \frac{\chi^+ \sigma_j \chi}{\chi^+ \chi} = \frac{x_j}{r} \ . \tag{40}$$

We note that ξ_j commutes with J_j, and $\xi_j J_j = K$ holds. We choose the action of ξ_j as the right multiplication in $\mathcal{S}_k$ for $k > 0$, and as the left one for $k < 0$.

The operator $\sigma_j \xi_j$ satisfies the relation

$$(\sigma_j \xi_j)^2 = 1 + \frac{\lambda}{r}(1 \mp \sigma_j \xi_j) \ , \ \frac{\lambda}{r} = \frac{1}{l} \ ,$$

where the upper sign corresponds to $k < 0$, and the lower one to $k > 0$. For $k < 0$ we have eigenvalues $+1$ and $-1 - \frac{\lambda}{r}$, and for $k > 0$ they are -1 and $+1 + \frac{\lambda}{r}$. The eigenfunctions of the operator $\sigma_j \xi_j$ have in the space $\mathcal{S}_k$, $k < 0$, the general form

$$\Psi^{(+)} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} F(\chi, \chi^*) \ ,$$

and

$$\Psi^{(-)} = \begin{pmatrix} \chi_1^* \\ -\chi_2^* \end{pmatrix} F(\chi, \chi^*) \ ,$$

to the eigenvalues $+1$ and $-1-\frac{\lambda}{r}$, respectively. Similarly, in the space $\mathcal{S}_k,\ k>0$, the general eigenfunctions are

$$\Psi^{(+)} = F(\chi,\chi^*)\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}\ ,$$

and

$$\Psi^{(-)} = F(\chi,\chi^*)\begin{pmatrix} \chi_1^* \\ -\chi_2^* \end{pmatrix}\ ,$$

with the corresponding eigenvalues $+1+\frac{\lambda}{r}$ and -1, respectively. In these formulas the functions $F(\chi,\chi^*)$ are arbitrary but such that $\Psi^{(\pm)}$ belong to the corresponding space $\mathcal{S}_k$.

Zero modes. Putting

$$\Psi^{(+)}_{n_1n_2} = \begin{pmatrix} \chi_1^{n_1+1}\chi_2^{n_2} \\ \chi_1^{n_1}\chi_2^{n_2+1} \end{pmatrix} \in \mathcal{S}_k,\ k=-n_1-n_2-1\ ,$$

$$\Psi^{(-)}_{n_1n_2} = \begin{pmatrix} \chi^{+n_1}_1\chi^{+n_2+1}_2 \\ -\chi^{+n_1+1}_1\chi^{+n_2}_2 \end{pmatrix} \in \mathcal{S}_k,\ k=n_1+n_2+1\ , \tag{41}$$

we obtain

$$\sigma_j\xi_j\Psi^{(\pm)}_{n_1n_2} = \pm\ \Psi^{(\pm)}_{n_1n_2}\ . \tag{42}$$

and

$$\sigma_jJ_j\Psi^{(\pm)}_{n_1n_2} = -\frac{1}{2}(|k|+2)\Psi^{(\pm)}_{n_1n_2}\ , \tag{43}$$

where $|k|=n_1+n_2+1$. From eqs. (42) and (43) it follows that

$$D_k\Psi^{(\pm)}_{n_1n_2} = 0\ . \tag{44}$$

We note, that $\Psi^{(\pm)}_{n_1n_2}$ are the only zero mode eigenfuctions of the Dirac operator.

Taking into account that in $\mathcal{S}_k$ the value of $n_1+n_2=|k|+1$ is fixed, we see that we recovered the well-known result on zero modes of the Dirac operator in the commutative case (see e.g. [8]).

Non-zero modes. An important role is played by the operators

$$I_j = J_j + \frac{1}{2}\sigma_j \ , j = 1, 2, 3 \ , \tag{45}$$

satisfying the relations

$$[I_i,\ I_j] = i\varepsilon_{ijk} I_k \ . \tag{46}$$

They allow to express the operator D_k^2 in the following way

$$D_k^2 = I_j^2 - \frac{1}{4}(k^2 - 1) - \frac{k^2}{4}[1 - (\sigma_k \xi_k)^2] \ , \tag{47}$$

where the last term is proportional to λ and represents a non-commutative correction.

We put $j = (n + m - 1)/2$ and define for $k = n - m < 0$ the operators

$$\Phi_{kjj}^{(+)} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \chi_1^{*n} \chi_2^{m-1} \ ,$$

$$\Phi_{kjj}^{(-)} = \begin{pmatrix} \chi_2^* \\ -\chi_1^* \end{pmatrix} \chi_1^{*n-1} \chi_2^{m} \ , \tag{48}$$

satisfying equations

$$(\sigma_k \xi_k)\Phi_{kjj}^{(+)} = \Phi_{kjj}^{(+)} \ ,$$
$$(\sigma_k \xi_k)\Phi_{kjj}^{(-)} = -(1 + \frac{\lambda}{r})\Phi_{kjj}^{(-)} \ . \tag{49}$$

Similarly, for $k = n - m > 0$ the operators

$$\Phi_{kjj}^{(+)} = \chi_1^{*n} \chi_2^{m-1} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \ ,$$

$$\Phi_{kjj}^{(-)} = \chi_1^{*n-1} \chi_2^{m} \begin{pmatrix} \chi_2^* \\ -\chi_1^* \end{pmatrix} \ , \tag{50}$$

satisfy

$$(\sigma_k \xi_k)\Phi_{kjj}^{(+)} = (1 + \frac{\lambda}{r})\Phi_{kjj}^{(+)} \ ,$$

$$(\sigma_k\xi_k)\Phi^{(-)}_{kjj} = -\Phi^{(-)}_{kjj} \ . \tag{51}$$

Moreover, the operators $\Phi^{(\pm)}_{kjj}$ satisfy relations

$$I_j^2\Phi^{(\pm)}_{kjj} = j(j+1)\Phi^{(\pm)}_{kjj} \ ,$$

$$K\Phi^{(\pm)}_{kjj} = k\Phi^{(\pm)}_{kjj} \ , \tag{52}$$

We see that the eigenvalues of the operator D_k^2 are determined by the equation

$$E^{(\pm)2}_{kj} = (j+\frac{1}{2})^2 - \frac{1}{4}k^2 - \frac{k^2}{4}[1-\varepsilon^{(\pm)2}_{kj}] \ , \tag{53}$$

where $|k| \leq 2j$, and $\varepsilon^{(\pm)}_{kj}$ denotes the eigenvalue of the operator $(\sigma_k\xi_k)$ corresponding to the eigenfunction $\Phi^{(\pm)}_{kjj}$ in question. The eigenvalues $E^{(\pm)2}_{kj}$ are $2j+1$ fold degenerate. As the operators $I_j,\ j=k=1,2,3$, and $\sigma_k\xi_k$ commute, the corresponding eigenfuctions are given as

$$\Phi^{(\pm)}_{kjm} = I_-^{j-m}\Phi^{(\pm)}_{kjj} \ , \ |m| \leq j \ , \tag{54}$$

where $I_- = I_1 - iI_2$. They satisfy the equation

$$I_3\Phi^{(\pm)}_{kjm} = m\Phi^{(\pm)}_{kjj} \ , \ |m| \leq j \ . \tag{55}$$

All functions of the form

$$\Psi_{(\pm)} = \Phi \pm \frac{1}{|E|}D_k\Phi \ , \tag{56}$$

where Φ is one of the functions (54) corresponding to the eigenvalue $E^2 = E^{(\pm)2}_{kj}$, are eigenfunctions of the Dirac operator,

$$D_k\Psi_{(\pm)} = E\Psi_{(\pm)} \ , \tag{57}$$

to the eigenvalue $E = \pm E^{(\pm)}_{kj}$.

In the commutative limit $\lambda \to 0$ all formulas given above reduce to the well-known results found on the standard sphere (see ref. [8]). We stress the additional degeneracy $E^{(+)}_{kj} = E^{(-)}_{kj}$ of the non-zero eigenmodes in the commutative case.

4 Concluding remarks

We introduced a non-commutative analog of a spinor calculus on a Fuzzy sphere. As an application we constructed a complete set of eigenstates of the free Dirac operator on the Fuzzy sphere. In the commutative limit we recovered the known results.

It would be desirable to extend this model describing free fermions, to the model with interaction, e.g. to the Thirring model with four-fermionic interaction. The alternative possibility is to include gauge fields, and to formulate the Fuzzy Schwinger model (the non-commutative Euclidean electrodynamics in two dimensions).

Then it would be interesting to investigate the quantum version of the model in question, and to see the regularization induced by the Fuzzy structure of the underlying space manifold.

As a natural next step one can speculate on generalizations to relativistic versions and to extensions to higher dimensional Fuzzy space-time.

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